Part II

Generalized knight's tours on rectangular chessboards

Since the case $a + b = 3$ (where $a = 1$ and $b = 2$) has been settled by Pósa and discussed in [28], we may assume that $a + b \geq 5$ (so that $s \geq 2$).

Consider vertices in the $(t + 1)$-th row. They are all colored with c_{t+1}. Moreover these vertices are adjacent only to the vertices in the $(a + t + 1)$-th and $(b + t + 1)$-th rows because $0 \leq t \leq a - 1$.

Since $a + t + 1 \leq s + t + 1$ and $b + t + 1 > s + t + 1$, vertices in these two rows are colored with c_{a+t+1}.

Now look at those vertices in the $(a + t + 1)$-th row. They are adjacent only to the vertices in the $(t + 1)$-th, $(2a + t + 1)$-th and $(a + b + t + 1)$-th rows and possibly to the vertices in the $(a + t + 1 - b)$-th row (if $b - a < t + 1$). Vertices in the $(a + b + t + 1)$-th row are colored with c_{t+1} while those in the $(2a + t + 1)$-th row are colored with c_{2a+t+1} if $2a \leq s$ and with $c_{b-a+t+1}$ if $2a > s$. Also, vertices in the $(a + t + 1 - b)$-th row are colored with $c_{a+t+1-b}$.

In other words, vertices in the $(a + t + 1)$-th row, which are colored in c_{a+t+1}, are adjacent only to vertices that are colored in c_{t+1}, c_{2a+t+1}, $c_{b-a+t+1}$ or $c_{a-b+t+1}$. By looking at those vertices in the $(b + t + 1)$-th row, which are colored in c_{a+t+1}, it is seen that they are adjacent only to vertices that are colored in c_{t+1}, $c_{a-b+t+1}$, $c_{b-a+t+1}$ or c_{2a+t+1}.

With these observations, we now show that it is not possible to have a closed (a, b)-knight's tour on B.

Suppose on the contrary that there is a closed (a, b)-knight's tour $C = v_1 v_2 \ldots v_{mn} v_1$. Then any vertex in C colored with c_{t+1} must be sandwiched by two vertices colored in c_{a+t+1}.

We may assume without loss of generality that $v_1, v_3, \ldots, v_{2k+1}$ are colored in c_{a+t+1} and that $v_2, \ldots, v_{2k}$ are colored in c_{t+1} for some $k \geq 1$. Further, we may assume that v_m and v_{2k+2} are not colored in c_{t+1}.

Then v_{2k+2} is colored with c_{2a+t+1}, $c_{b-a+t+1}$ or $c_{a-b+t+1}$. However, this means that, in the subgraph $v_1 v_2 \ldots v_{2k+1}$ of C, the number of vertices colored in c_{a+t+1} is one more than those colored in c_{t+1}. The same is true for other subgraph of C which contains vertices colored in c_{t+1}.

Thus there are more vertices colored in c_{a+t+1} than in c_{t+1}. This contradicts the fact that C contains an equal number of vertices of each color. □

In ([28]), Schwenk proposed also the problem of finding an open knight's tour on an $m \times n$ chessboard. We discuss this and give a complete solution to the open knight's tour problem in Section 4.2.

We observe, in passing, that other problems concerning knight's tours have also been discussed (see [12]). In [37], Watkins and Hoenigman consider knight's tour on the torus, that is a knight is allowed to leave the board from a side and reenter on the opposite side. It turns out unexpectedly that some of the closed knight's tours on the torus, when restricted to square chessboards, give rise to magic squares (see [2]).

Theorem 4.2 *([37]) Every rectangular chessboard has a closed knight's tour on a torus.*

Theorem 4.3 *([2]) The closed knight's tour produces a magic square for any $n \times n$ chessboard where n is not divisible by 2, 3 or 5.*

The knight's tour problem has also been considered on chessboards of other shapes, for example the triangular honeycomb ([15] and [36]). In the meantime, problems concerning the number of knight's tours on square chessboards has also been considered ([21]). Solving the Knight's tour problem by using neural network method has been discussed in [24] , [25] and [30].

Knight's moves are amenable to generalization. We consider the following one. Suppose the squares of the $m \times n$ chessboard are (i, j) where $1 \leq i \leq m$ and $1 \leq j \leq n$, counting from the upper left corner in matrix fashion. A move from square (i, j) to square (k, l) is termed an *(a, b)-knight's move* if $\{| k - i |, | l - j |\} = \{a, b\}$. In particular, a knight's move is a (1, 2)-knight's move. An *(a, b)-knight's tour* is a sequence of (a, b)-knight's moves that visits every square of an $m \times n$ chessboard exactly once. An (a, b)-knight's tour is *closed* if the knight can return to the starting square at the end of the tour, and *open* otherwise.

For a given (a, b)-knight's move on an $m \times n$ chessboard, there is associated with it a graph whose vertex set and edge set are $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\{(i, j)(k, l) \mid \{| k - i |, | l - j |\} = \{a, b\}\}$ respectively. Let $G((a, b), m, n)$ denote this graph, or just $G(m, n)$ for simplicity if the move (a, b) is understood or not to be emphasized. Then, a closed (a, b)-knight's tour in an $m \times n$ chessboard corresponds to a Hamilton cycle in the graph $G((a,b),m,n)$ and an open (a,b)-knight's tour corresponds to a Hamilton path.

The *generalized knight's tour problem* asks: which $m \times n$ chessboard admits a closed (a,b)-knight's tour? This amounts to asking: which graph $G((a,b),m,n)$ is hamiltonian? In Section 4.3, we show that certain rectangular chessboards do not admit a closed generalized knight's tour.

We shall make a few easy observations. First, if $a+b$ is even, then no closed (a,b)-knight's tour is possible because only cells of the same color (that is either all black or all white cells) are covered during the moves. Thus $a+b$ is assumed to be odd. Also, we shall assume that $a < b$ since (a,b)-knight's move and (b,a)-knight's move are the same.

Next, if m and n are both odd, then no closed (a,b)-knight's tour is possible because $G(m,n)$ is a bipartite graph with an odd number of vertices mn.

We may further assume that $m \leq n$ throughout this chapter. If $m \leq a+b-1$, then no closed (a,b)-knight's tour on the $m \times n$ chessboard is possible. This is because the vertex $(a,1)$ in $G(m,n)$ is of degree ≤ 1. Suppose $n < 2b$. Then the vertex $(b-1,b)$ could only be adjacent to the vertices $(2b-1,b+a)$ and $(2b-1,b-a)$. But this means that $m \geq 2b-1$ which implies that $n \geq 2b$, a contradiction.

We summarize the above observations in the following.

Theorem 4.4 *Suppose the $m \times n$ chessboard admits a closed (a,b)-knight's tour, where $a < b$ and $m \leq n$. Then*
(i) $a+b$ is odd;
(ii) m or n is even;
(iii) $m \geq a+b$; and
(iv) $n \geq 2b$.

Perhaps the simplest generalized knight's move is that of $(0,1)$-knight's move. In this case, the associated graph $G(m,n)$ is the horizontal grid whose hamiltonicity is easily decided. We shall henceforth assume that $1 \leq a < b$.

4.2 Open knight's tours on rectangular chessboards

In [28], Schwenk proposed the problem of finding an open knight's tour on an $m \times n$ chessboard. He mentioned that this problem can also be solved by the same method he has introduced in [28]. The solution was left as a challenge to the interested readers. In this section, we provide a complete solution to the open knight's tour problem. Earlier, Cull and De Curtins [10] has proved that every $m \times n$ chessboard with $5 \leq m \leq n$ has an open knight's tour.

Theorem 4.5 *([10]) Every $m \times n$ chessboard with $5 \leq m \leq n$ has an open knight's tour.*

The case $m = 3$ was considered in [26] where Van and Rees showed that the $3 \times n$ chessboard has an open knight's tour if and only if $n = 4$ or $n \geq 7$. Here we shall present the solution for the missing case $m = 4$ as well as some constructions for the open knight's tours on the $3 \times n$ chessboard.

We have a necessary condition for the existence of a Hamilton path in a graph. If G is a graph, we let $\omega(G)$ denote the number of components in G.

Theorem 4.6 *Let S be a proper subset of the vertex-set of a graph G and let $\omega(G - S)$ be the number of components of the graph $G - S$. If G contains a Hamilton path, then*

$$\omega(G - S) \leq |S| + 1.$$

Proof: Let P be a Hamilton path of G. Then for every proper subset S of the vertex-set of G, we have $\omega(P - S) \leq |S| + 1$.
But $P - S$ is a spanning subgraph of $G - S$, so

$$\omega(G - S) \leq w(P - S) \leq |S| + 1.$$

□

Theorem 4.7 *The $m \times n$ chessboard with $m \leq n$ admits an open knight's tour unless one or more of the following conditions holds:*
(i) m= 1 or 2;
(ii) m= 3 and n= 3, 5, 6; or
(iii) $m = 4$ and $n = 4$.

Proof: Both $G(3,3)$ and $G(m,n)$ for $m \leq 2$ are disconnected and hence do not have Hamilton paths.

For the remaining part on the non-existence of Hamilton paths, we shall use Theorem 4.6. Figure 4.1(a) shows that the removal of the five vertices $(1,2), (1,4), (2,3), (3,2)$ and $(3,4)$ of $G(3,5)$ leaves a disconnected graph with seven components. Figure 4.1(b) shows that removing the four vertices $(i,2)$ and $(i,3)$ for $i = 2,3$ from $G(4,4)$ results in a disconnected graph with six components. Figure 4.1(c) shows the resulting graph (which is disconnected) with eight components when the six vertices $(j,3)$ and $(j,4)$ for $j = 1,2,3$ are removed from $G(3,6)$. By Theorem 4.6, $G(3,5), G(3,6)$ and $G(4,4)$ do not contain Hamilton paths.

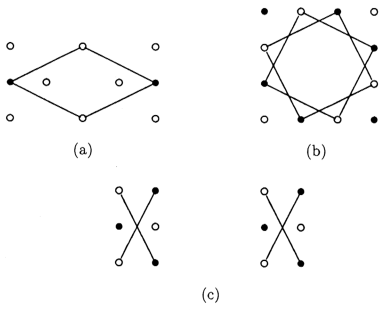


Figure 4.1

Next, we show that every other board admits an open knight's tour. Figure 4.2 depicts a Hamilton path in $G(3,n)$ for each $n \in \{4,7,8,9\}$ and $G(4,k)$ for each $k \in \{5,6,7\}$.

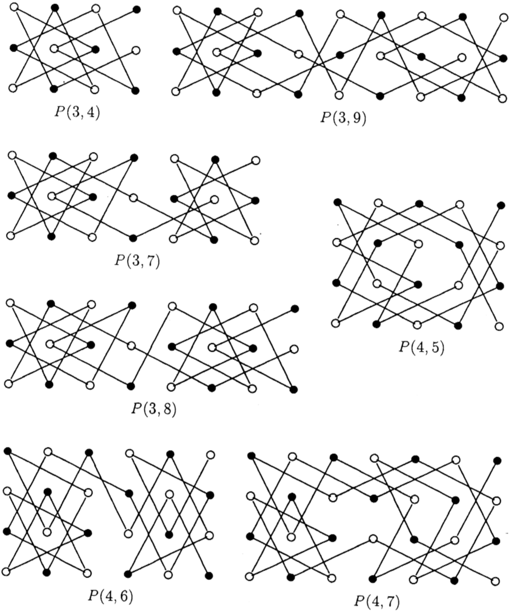
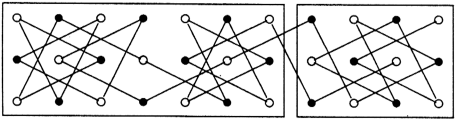


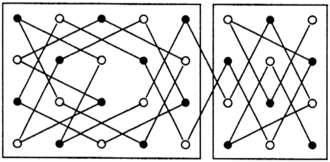
Figure 4.2: The Hamilton paths $P(3,4), P(3,7), P(3,8)$, $P(3,9)$, $P(4,5)$, $P(4,6)$ and $P(4,7)$

Let $P(m,n)$ denote a Hamilton path of $G(m,n)$. We shall show that each $P(3,n)$ for $n \in \{7,9\}$ in Figure 4.2 is extendable to a $P(3,n+4)$ and each $P(4,k)$ for $k \in \{5,6,7\}$ in Figure 4.2 is extendable to a $P(4,k+3)$. This can be done by placing the graphs $S(3,4)$ (a subgraph of $G(3,4)$) and $S(4,3)$ (a subgraph of $G(4,3)$) on the right-hand side of $P(3,n)$ and $P(4,k)$, respectively, and joining them by suitable edges as explained below. The graphs $S(3,4)$ and $S(4,3)$ are shown in Figure 4.3 and Figure 4.4, respectively.



$P(3,7)$ $S(3,4)$

Figure 4.3: Extension of $P(3,7)$ to $P(3,11)$



$P(4,5)$ $S(4,3)$

Figure 4.4: Extension of $P(4,5)$ to $P(4,8)$

For the case $m = 3$, note that each of the $P(3,n)$ for $n \in \{7,9\}$ has $(1,n)$ and $(2,n-1)$ as end vertices. Joining the vertices $(1,n)$ and $(2,n-1)$ of $P(3,n)$ to the vertices $(3,1)$ and $(1,1)$ of $S(3,4)$, respectively yields a Hamilton path in $G(3,n+4)$ with $(1,n+4)$ and $(2,n+3)$ as end vertices. The extension of a Hamilton path in $G(3,7)$ to a Hamilton path in $G(3,11)$ is shown in Figure 4.3. Repeating the process, we obtain a Hamilton path in $G(3,n)$ for every odd $n \geq 7$. For the case where $n \geq 10$ is even, Schwenk's result (Theorem 4.1) implies that $G(3,n)$ admits a Hamilton path.

For the case $m = 4$, note that each of the $P(4,k)$ for $k \in \{5,6,7\}$ has $(1,k)$ and $(4,k)$ as end vertices. Joining these two vertices to the vertices $(3,1)$ and $(2,1)$ of $S(4,3)$, respectively, produces a Hamilton path in $G(4,n+3)$ with $(1,k+3)$ and $(4,k+3)$ as end vertices. The extension of a Hamilton path in $G(4,5)$ to a Hamilton path in $G(4,8)$ is shown in Figure 4.4. Repeating the process, we obtain a Hamilton path in $G(4,n)$ for every $n \geq 5$.

By Theorem 4.5, $G(m,n)$ contains a Hamilton path for $m \geq 5$. This completes the proof. □

4.3 Forbidden rectangular boards

In this section, we show that certain rectangular chessboards do not admit a closed generalized knight's tour. The first two results generalize that of Pósa which states that the $4 \times n$ chessboard does not admit a closed $(1,2)$-knight's tour.

Theorem 4.8 *Suppose $m = a + b + 2t + 1$ where $0 \leq t \leq a - 1$. Then the $m \times n$ chessboard admits no closed (a,b)-knight's tour.*

Proof: As $a + b$ is odd, we may write $a + b = 2s + 1$. Then $a \leq s$ and $b > s$ because $a < b$.

Let $r = \frac{m}{2} = t + s + 1$ and let the vertices of the $m \times n$ chessboard B be colored using r distinct colors $c_1, c_2, \ldots, c_r$ in the following manner.

If $1 \leq i \leq s + t + 1$, then vertices in the i-th row of B are colored with c_i. If $s + t + 2 \leq i \leq m$, then vertices in the i-th row of B are colored with c_{m+1-i}.

Since the case $a + b = 3$ (where $a = 1$ and $b = 2$) has been settled by Pósa and discussed in [28], we may assume that $a + b \geq 5$ (so that $s \geq 2$).

Consider vertices in the $(t + 1)$-th row. They are all colored with c_{t+1}. Moreover these vertices are adjacent only to the vertices in the $(a + t + 1)$-th and $(b + t + 1)$-th rows because $0 \leq t \leq a - 1$.

Since $a + t + 1 \leq s + t + 1$ and $b + t + 1 > s + t + 1$, vertices in these two rows are colored with c_{a+t+1}.

Now look at those vertices in the $(a + t + 1)$-th row. They are adjacent only to the vertices in the $(t+1)$-th, $(2a+t+1)$-th and $(a+b+t+1)$-th rows and possibly to the vertices in the $(a + t + 1 - b)$-th row (if $b - a < t + 1$). Vertices in the $(a + b + t + 1)$-th row are colored with c_{t+1} while those in the $(2a + t + 1)$-th row are colored with c_{2a+t+1} if $2a \leq s$ and with $c_{b-a+t+1}$ if $2a > s$. Also, vertices in the $(a + t + 1 - b)$-th row are colored with $c_{a+t+1-b}$.

In other words, vertices in the $(a + t + 1)$-th row, which are colored in c_{a+t+1}, are adjacent only to vertices that are colored in c_{t+1}, c_{2a+t+1}, $c_{b-a+t+1}$ or $c_{a-b+t+1}$. By looking at those vertices in the $(b + t + 1)$-th row, which are colored in c_{a+t+1}, it is seen that they are adjacent only to vertices that are colored in c_{t+1}, $c_{a-b+t+1}$, $c_{b-a+t+1}$ or c_{2a+t+1}.

With these observations, we now show that it is not possible to have a closed (a, b)-knight's tour on B.

Suppose on the contrary that there is a closed (a, b)-knight's tour $C = v_1v_2 \ldots v_{mn}v_1$. Then any vertex in C colored with c_{t+1} must be sandwiched by two vertices colored in c_{a+t+1}.

We may assume without loss of generality that $v_1, v_3, \ldots, v_{2k+1}$ are colored in c_{a+t+1} and that $v_2, \ldots, v_{2k}$ are colored in c_{t+1} for some $k \geq 1$. Further, we may assume that v_m and v_{2k+2} are not colored in c_{t+1}.

Then v_{2k+2} is colored with c_{2a+t+1}, $c_{b-a+t+1}$ or $c_{a-b+t+1}$. However, this means that, in the subgraph $v_1v_2 \ldots v_{2k+1}$ of C, the number of vertices colored in c_{a+t+1} is one more than those colored in c_{t+1}. The same is true for other subgraph of C which contains vertices colored in c_{t+1}.

Thus there are more vertices colored in c_{a+t+1} than in c_{t+1}. This contradicts the fact that C contains an equal number of vertices of each color. □

Lemma 4.1 *Suppose the vertices of an $m \times n$ chessboard B are colored in equal number with two colors, red and blue. Suppose further that every red vertex is adjacent only to the blue vertices and that a blue vertex could be adjacent to a red vertex or a blue vertex. Then B admits no closed (a, b)-knight's tour.*

Proof: Suppose that there is a closed (a, b)-knight's tour $C = v_1 v_2 \dots v_{mn} v_1$ of B. Since B contains an equal number of vertices of each color and a red vertex must always be sandwiched by two blue vertices, the red and blue vertices must alternate around C. Let all the odd-labelled vertices v_{2r+1} be colored in red and all the even-labelled vertices v_{2r} be colored in blue. But from the original coloring of the chessboard B with black and white, we may conclude that all the vertices v_{2r+1} are also white. Thus all red vertices are white vertices, but this contradicts the different pattern chosen for the two colorings. We conclude that no closed (a, b)-knight's tour is possible. □

Pósa's theorem can also be generalized to the following.

Theorem 4.9 *Suppose $m = a(k + 2l)$ where $1 \leq l \leq \frac{k}{2}$. Then the $m \times n$ chessboard admits no closed (a, ak)-knight's tour.*

Proof: The proof is reminiscent of that of Pósa.

First note that, as $a + ak$ is odd, a is odd and k is even.

Next, let B be an $m \times n$ chessboard. For each $i = 1, 2, \dots, k + 2l$, let A_i denote the $a \times n$ chessboard which consists of the $((i-1)a+1)$-th, $((i-1)a+2)$-th, $\dots$, ia-th rows of B. In other words, B is partitioned into $k + 2l$ subchessboards $A_1, A_2, \dots, A_{k+2l}$ each of size $a \times n$.

Now, let the vertices of B be colored with two colors in the following manner.

For $1 \leq i \leq k$, let the vertices in A_i be colored with red if i is odd and with blue otherwise.

For $k + 1 \leq i \leq k + 2l$, let the vertices in A_i be colored with blue if i is odd and with red otherwise.

Consider the vertices in the j-th row. They are adjacent only to the vertices in the $(j \pm a)$-th and the $(j \pm ak)$-th rows. Note that not all the four rows are always possible. For example, if $j \leq a$, then the $(j - a)$-th and the $(j - ak)$-th rows do not exist.

Suppose the j-th row belongs to A_i. Then the $(j + a)$-th and the $(j - a)$-th rows belong to A_{i+1} and A_{i-1} respectively. Also, the $(j + ak)$-th and the $(j - ak)$-th rows belong to A_{i+k} and A_{i-k} respectively.

Suppose $1 \leq i \leq k$. Then a vertex in the j-th row is not adjacent to a vertex in the $(j - ak)$-th row (since there is no A_{i-k} chessboard).

If i is odd, then the vertices in A_i are colored with red whereas the vertices in A_{i+1} and A_{i-1} are colored with blue. Since $k + i$ is odd and $k + i \geq k + 1$, the vertices in A_{i+k} are colored with blue.

If i is even, then the vertices in A_i are colored with blue. Clearly, the vertices in A_{i-1} are colored with red. Since $k + i$ is even and $k + i \geq k + 1$, the vertices in A_{i+k} are colored with red. The vertices in A_{i+1} are colored with red when $i < k$, but they are colored with blue when $i = k$.

Suppose $k + 1 \leq i \leq k + 2l$. Then, a vertex in the j-th row is not adjacent to a vertex in the $(j + ak)$-th row (since there is no A_{i+k} chessboard).

If i is even and $i < 2k$, then the vertices in A_i are colored with red and the vertices in A_{i+1} and A_{i-1} are colored with blue. Since $i - k$ is even and $i - k \leq k$, the vertices in A_{i-k} are colored with blue. If $i = 2k$, then the vertices in A_{2k} are adjacent only to the vertices in A_{2k-1} and A_k which are both colored with blue.

If i is odd, then the vertices in A_i are colored with blue. Clearly, the vertices in A_{i+1} and A_{i-k} are colored with red. The vertices in A_{i-1} are colored with red when $i > k + 1$, but they are colored with blue when $i = k + 1$.

Thus, we may conclude that every red vertex in B is adjacent only to the blue vertices; however a blue vertex is adjacent to either a red vertex or a blue vertex. By Lemma 4.1, no closed (a, ak)-knight's tour is possible. □

Theorem 4.10 *Suppose* $m = 2(ak + l)$ *where* $1 \leq k \leq l \leq a$. *Then the* $m \times n$ *chessboard admits no closed* $(a, a + 1)$*-knight's tour.*

Theorem 4.10 *Suppose $m = 2(ak + l)$ where $1 \leq k \leq l \leq a$. Then the $m \times n$ chessboard admits no closed $(a, a+1)$-knight's tour.*

Proof: Let B be an $m \times n$ chessboard. As $k \leq l$, we have $m > k(2a+1)$. Partition the first $k(2a+1)$ rows of vertices into k subchessboards A_1, A_2, ...,A_k, each of size $(2a+1) \times n$. For each A_i, $i = 1, 2, ..., k$, we shall color the first a rows of vertices with red and the next $a+1$ rows of vertices that follow with blue. Note that in the chessboard B, we have ak rows of vertices colored with red, $k(a+1)$ rows of vertices colored with blue and $2l - k$ rows uncolored.

Let D denote the $(2l - k) \times n$ subboard that contains all the uncolored vertices of B. As $l \geq k$, we have $2l - k = k + s$ for some $s \geq 0$. Clearly, s is even. We shall color the first $k + \frac{s}{2}$ rows of vertices in D with red and the remaining $\frac{s}{2}$ rows of vertices with blue. The number of vertices colored with red in B is now equal to the number of vertices colored with blue.

Consider the vertices in the j-th row. They are adjacent only to the vertices in the $(j \pm a)$-th and the $(j \pm (a+1))$-th rows. Note that not all the four rows are always possible. For example, if $j \leq a$, then the $(j - a)$-th and the $(j - a - 1)$-th rows do not exist.

Suppose the j-th row belongs to A_i, for some $i = 1, 2, \ldots k$. If the j-th row is colored red, then the $(j \pm a)$-th and the $(j \pm (a+1))$-th rows are colored blue. So, every vertex colored with red in A_i is adjacent only to vertices colored with blue. However if the j-th row is blue, then a vertex in this row is adjacent to a red vertex or a blue vertex. For example, this is clearly the case for vertices in the $(a+1)$-th row.

Suppose the j-th row belongs to D. Since $k + \frac{s}{2} \leq a$, every vertex colored with red in D can only be adjacent to vertices colored in blue.

Thus, we may make the conclusion that every red vertex in B is adjacent only to the blue vertices; however a blue vertex could be adjacent to a red vertex or a blue vertex. By Lemma 4.1, B does not admit a closed $(a, a+1)$-knight's tour. □

The previous three results deal with forbidden boards of size $m \times n$ with m even. The next result considers a case where the move is $(a, a+1)$ and m is odd. However the result is not enjoyed by the $(1, 2)$-knight's move.

Theorem 4.11 *Suppose* $m = 2a + 2t + 1$ *where* $1 \leq t \leq a - 1$. *Then the* $m \times n$ *chessboard admits no closed* $(a, a + 1)$*-knight's tour.*

Proof: Let A_u (respectively A_l) denote the $a \times a$ subboard located at the upper (respectively lower) right corner of the $m \times n$ chessboard. It is easy to see that vertices in A_u or A_l are of degree 2 in $G(m, n)$.

Consider the vertex $(a+t+1, a+2)$. It is adjacent to the vertices $(t+1, 1)$, $(t, 2)$ and $(2a + t + 1, 1)$. Clearly, $(t + 1, 1)$ and $(t, 2)$ belong to A_u. Since $1 \leq t \leq a - 1$, it is easy to see that $(2a + t + 1, 1)$ belongs to A_l. Hence $(a + t + 1, a + 2)$ is adjacent to three vertices of degree 2 and thus $G(m, n)$ is non-hamiltonian. □