APPENDIX A

DIRAC ALGEBRA AND LOOP MOMENTUM

INTEGRATION IN \( n \) DIMENSION

Dirac Matrices In \( n \) Dimensions

We shall adopt the following convention:

\[
Tr(1) = 4
\]

\[
\{\gamma_{\mu}, \gamma_{5}\} = 0.
\]  \hspace{1em} (A.1)

The basic algebra is given by

\[
\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}.
\]  \hspace{1em} (A.2)

The metric tensor \( g_{\mu\nu} \) satisfies

\[
g_{\mu\nu}g^{\mu\nu} = n.
\]  \hspace{1em} (A.3)

Combining (A.2) and (A.3) we obtain

\[
\gamma_{\nu} = n
\]

\[
\gamma_{\nu} \gamma_{\mu} \gamma_{\nu} = (2-n)\gamma_{\mu}
\]

\[
\gamma_{\nu} \gamma_{\mu} \gamma_{5} \gamma_{\nu} = 4g_{\mu\nu} + (n-4)\gamma_{\mu} \gamma_{\nu}
\]

\[
\gamma_{\nu} \gamma_{\mu} \gamma_{\nu} \gamma_{\sigma} = -2\gamma_{\alpha} \gamma_{\nu} \gamma_{\mu} + (4-n)\gamma_{\mu} \gamma_{\nu} \gamma_{\sigma}.
\]  \hspace{1em} (A.4)
By using (A.1) we find that

\[
Tr(\gamma_\mu \gamma_\nu) = 4g_{\mu\nu}
\]

\[
Tr(\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho) = 4\left[ g_{\mu\nu} g_{\lambda\rho} + g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} \right]
\]  \hspace{1cm} (A.5)

\[
Tr(\gamma_{\mu1} \cdots \gamma_{\mu m}) = 0 \quad \text{for } m \text{ odd.}
\]

There is no trouble concerning the \(\gamma_3\) matrix in this case since Glashow-Weinberg-Salam model is an anomaly-free theory.

The Feynman parameterization is given by:

\[
\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 U_1^{n-2} dU_1 \int_0^1 U_2^{n-3} dU_2 \cdots \int_0^1 dU_{n-1} \times \left[ (a_1 - a_2) U_1 \cdots U_{n-1} + (a_2 - a_3) U_1 \cdots U_{n-2} + \cdots + a_n \right]^n
\]  \hspace{1cm} (A.6)

which is very convenient as it allows possible cancellations of the terms of two different propagators and also the advantage of finite bounds of integration.

Special cases of (A.6) used in our calculations are:

\[
\frac{1}{ab} = \int_0^1 dx \left[ b + (a - b)x \right]^{-2}
\]  \hspace{1cm} (A.7)

\[
\frac{1}{abc} = 2 \int_0^1 dx \int_0^{1-x} dy \left[ a + (b - a)x + (c - a)y \right]^{-3}
\]  \hspace{1cm} (A.8)
APPENDIX B

Momentum Integrals For One-loop Diagram

Generally

\[
I(m,r) = \int \frac{d^n\vec{q}}{(2\pi)^n} \frac{(\vec{q})^2}{[\vec{q}^2 - R^2]^m} = \frac{i}{(16\pi^2)^{n/4}} (-1)^{r-m}(R^2)^{r-m+n/2} \frac{\Gamma\left(r + \frac{n}{2}\right)\Gamma\left(m - r - \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(m)}.
\] (B.1)

By symmetrical integration, we can show that

\[
\int \frac{d^n\vec{q}}{(2\pi)^n} \frac{\vec{q}_\mu \vec{q}_\nu}{[\vec{q}^2 - R^2]^m} = \frac{1}{n} g_{\mu\nu} \int \frac{d^n\vec{q}}{(2\pi)^n} \frac{\vec{q}^2}{[\vec{q}^2 - R^2]^m}
\] (B.2)

and

\[
\int \frac{d^n\vec{q}}{(2\pi)^n} q_{\mu_1}q_{\mu_2}\cdots q_{\mu_k} = 0 \quad \text{for } k \text{ odd.}
\] (B.3)

The Gamma function \( \Gamma(x) \) has the following properties:

\[
\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon)
\] (B.4)

\[
\Gamma(\varepsilon - 1) = \frac{1}{\varepsilon} - (1 - \gamma) + O(\varepsilon)
\] (B.5)

where \( \gamma = 0.5772 \) is the Euler constant.
APPENDIX C

Romberg Integration

For the composite trapezoid rule approximation

\[ T_N = T_N(f) = h \sum_{i=1}^{N-1} f_i + \frac{h(f_0 + f_N)}{2} \]  \hspace{1cm} (C.1)

to the number

\[ I = I(f) = \int_a^b f(x)dx \]  \hspace{1cm} (C.2)

where \( N \) is a positive integer related to \( h \) by

\[ h = \frac{b-a}{N} \]  \hspace{1cm} (C.3)

and \( f_i = f_{i,N} = f(a + ih) \) where \( i = 0, \ldots, N \).

If \( f(x) \) is four times continuously differentiable

\[ I(f) = T_N(f) + c_1 h^2 + \theta(h^4) \]  \hspace{1cm} (C.5)

does not contain \( h \). Therefore the extrapolation to the limit is applicable. Here

\[ T_{N,q}(f) = T_N + \frac{T_N(f) - T_{N/q}(f)}{q^2 - 1} \]  \hspace{1cm} (C.6)

is an \( \theta(h^4) \) approximation to \( I(f) \), while in general \( T_N(f) \) has only an error of \( \theta(h^2) \). The choice of \( q \) or \( N \) is restricted by the condition that \( N, q \) be an integer. We choose \( q = 2 \) so that \( N \) must be even. For even \( N \),

\[ T_N(f) = \frac{T_{N/2}(f)}{2} + \frac{h}{2} \sum_{i=1}^{N/2} f(a + (2i - 1)h). \]  \hspace{1cm} (C.7)
From (B.1),

$$T_N(f) = h \sum_{i=1}^{N-1} f(a + ih) + \frac{h(f(a) + f(b))}{2} \tag{C.8}$$

$$= h \sum_{i=1}^{\frac{N}{2}} f(a + (2i - 1)h) + h \sum_{i=1}^{\frac{N}{2} - 1} f(a + 2ih) + \frac{h(f(a) + f(b))}{2}. \tag{C.8}$$

The first sum extends over the 'odd' points and the second sum over the 'even' points. The last two terms can be written as

$$\left[ 2h \sum_{j=1}^{\frac{N}{2} - 1} f(a + ij(2h)) + \frac{2h(f(a) + f(b))}{2} \right] / 2. \tag{C.9}$$

But $2h = \frac{2(b - a)}{N} = \frac{b - a}{N/2}$, therefore the last two terms add up to $T_{N/2}(f) / 2$. Finally (B.7) can be written as

$$T_N(f) = T_{N/2}(f) + \frac{M_{N/2}(f)}{2} \tag{C.10}$$

with $M$ denoting the composite midpoint.

If the integrand has $2k + 2$ continuous derivatives,

$$I(f) = T_N(f) + C_1 h^2 + C_2 h^4 + C_3 h^6 + \ldots + C_k h^{2k} + \theta(h^{2k+2}) \tag{C.11}$$

where $C_1, \ldots, C_k$ do not depend on $h$. Therefore

$$I(f) = T_N'(f) + C_1' h^2 + C_2' h^4 + \ldots + C_k' h^{2k} + \theta(h^{2k+2}) \tag{C.12}$$

where $T_N'(f) = T_N(f) + \frac{T_N(f) - T_{N/2}(f)}{3}$

with the constants $C_1', \ldots, C_k'$ independent of $h$. By taking

$$T_N^2(f) = T_N'(f) + \frac{T_N'(f) - T_{N/2}'(f)}{15} \tag{C.13}$$
where

\[ I(f) = T_N^2(f) + C^2 \theta h^2 + \ldots + C^2 h^2 \theta + \theta(h^{2k+2}). \]  

(C.14)

Generally for \( m = 1, \ldots, k \)

\[ T_N^m(f) = T_N^{m-1}(f) + \frac{T_N^{m-1}(f) - T_{N/2}^{m-1}(f)}{4^m - 1} \]  

(C.15)

is an \( \theta(h^{2m+2}) \) approximation to \( I(f) \).

The calculation of \( T_N^m \) involves \( T_{N/2}^{m-1} \) and \( T_N^{m-1} \) therefore \( T_{N/4}^{m-2} \), \( T_{N/2}^{m-2} \) and \( T_N^{m-2} \), \ldots, and finally, \( T_{N/2^m}^{m}, \ldots, T_{N/2} \) and \( T_N \). Therefore \( N/2^m \) must be an integer where

\[ \frac{N}{2^m} = M \]  

(C.16)

for \( T_N^m \) to be defined.

It will be more convenient to visualize these various approximations to \( I(f) \) as entries of a triangular array, the so-called \( T \) table:

\[
\begin{array}{cccccc}
T_m^0 & & & & & \\
T_{2m}^0 & T_{2m}^1 & & & & \\
T_{4m}^0 & T_{4m}^1 & T_{4m}^2 & & & \\
& & & & & \\
T_{2^{m}N}^0 & T_{2^{m}N}^1 & T_{2^{m}N}^2 & \ldots & T_{2^{m}M}^m & \\
\end{array}
\]  

(C.17)
We present below an algorithm for Romberg integration.

Given a function \( f(x) \) which is defined on \([a,b]\) and a positive integer \( M \) (usually, \( M = 1 \))

\[
h = \frac{(b-a)}{M} \tag{C.18}
\]

calculate \( T^0_M = \sum_{i=1}^{M-1} f(a + ih) + \frac{h(f(a) + f(b))}{2} \). \tag{C.19}

For \( k = 1,2,3,\ldots \), do:

\[
h = h / 2
\]

calculate \( T^0_{2^k M} = \frac{1}{2} T^0_{2^{k-1} M} + h \sum_{i=1}^{2^{k-1} M} f(a + (2i-1)h). \) \tag{C.20}

For \( m = 1,\ldots,k \), do:

calculate \( T^m_{2^k M} = T^m_{2^{k-1} M} + \left( T^{m-1}_{2^k M} - T^{m-1}_{2^{k-1} M} \right)/(4^m - 1) \). \tag{C.21}

If \( f(x) \) has \( 2m+2 \) continuous derivatives, then

\[
I(f) = \int_a^b f(x)dx = T^m_{2^k M} + \theta \left( \frac{b-a}{2^k M} \right)^{2m+2} \tag{C.22}
\]

where \( k = m, m+1, \ldots \).

If \( k \) is sufficiently large, then

\[
|I(f) - T^m_{2^k M}| < |T^m_{2^k M} - T^{m-1}_{2^k M}|. \tag{C.23}
\]

There is a need to check that

\[
R^{m-1}_k = \frac{T^{m-1}_{2^{k-1} M} - T^{m-1}_{2^{k-2} M}}{T^{m-1}_{2^k M} - T^{m-1}_{2^{k-1} M}} \approx 4^m. \tag{C.24}
\]

A sample of the FORTRAN program to execute the numerical integration using Romberg's method is given in APPENDIX D.
APPENDIX D

PROGRAM CALC
REAL *8, T(1000)
REAL *8, k,J,W,KK, JJ, WW, b, k0, KSTEP, s
REAL *8, TOL, CL, CR
COMMON k, J, W, JJ, WW, KK, a, b
OPEN (222, FILE='300C(TOL6)', ACCESS='SEQUENTIAL', FORM='FORMATTED',
& STATUS='UNKNOWN')
OPEN (123, FILE='PRN')

TOL = 1.0D-6
PRINT *, 'TOL =', TOL

WRITE(222,111) TOL
111 FORMAT(' TOL = ', D9.3)

C WRITE(6,*)'k(GeV)    CL    CR'
WRITE(222,*)'k(GeV)    CL    CR'

k0=300.0D0
KSTEP=20.0D0

C PRINT *, 'k0 =', k0
C WRITE(122,*)'k0 =', k0

J = 1.3D0
W=80.22D0
b = (J*J)/(W*W)
s=1.0D0-b
WW= W*W
JJ= J*J
C --------------- analytical functions ( k as the variable ) ----- 
DO 368 K=K0,3.0D0*KSTEP+K0,KSTEP

    KK= K*K
    IF (K.EQ.0.0D0) THEN
        a=0.0D0
    ELSE
        a = (W*W-J*J)/(K*K)
    END IF

C ------------------ PART I : CALCULATION OF CL ---------------------

C  CALL RMBERG(3,0.0D0,1.0D0,2,T,30,TOL,CL)

    CALL RMBERG(5,0.0D0,1.0D0,2,T,30,TOL,CR)

C  WRITE(6,888)k,CL,CR
C  WRITE(222,888)k,CL,CR

    WRITE(6,888)k,CR
    WRITE(123,888)k,CR
    WRITE(222,888)k,CR

368 CONTINUE
888 FORMAT(F15.6,3X,D15.5)
C 888 FORMAT(F8.3,3X,D15.5,3X,D15.5)

    CLOSE (222)
    END
SUBROUTINE RMBERG(GNum, AL, BU,MROM, T, NROW, TOL,INTE)
INTEGER MROM,NROW,I,KLOOP,M, GNum
REAL*8,K
REAL *8, AL,BU, T(NROW,NROW), Gfunc, H,SUM,TOL,RR,INTE
COMMON K
M=MROM
H=(BU-AL)/M

SUM=(GFunc(GNum,AL)+GFunc(GNum,BU))/(2.0D0)
IF (M .GT. 1) THEN
   DO 10 I=1,M-1
       SUM=SUM + GFunc(GNum,AL+DBLE(I)*H)
   10    CONTINUE
END IF
T(1,1)=SUM*H

KLOOP=1

DO WHILE (KLOOP .LE. NROW)
   KLOOP=KLOOP+1
   H=H/(2.0D0)
   M=M*(2.0D0)
   SUM=0.0D0
   DO 11 I=1,M,2
       SUM=SUM + Gfunc(GNum,AL+DBLE(I)*H)
   11 CONTINUE
   T(KLOOP,1)=T(KLOOP-1,1)/(2.0D0) + SUM*H
   DO 12 J=1,KLOOP-1
   T(KLOOP,J+1) = T(KLOOP,J)
   & + (T(KLOOP,J) - T(KLOOP-1,J))/(4.0D0**J - 1.0D0)
   12 CONTINUE
PRINT*,'(T(KLOOP,KLOOP) = ', T(KLOOP,KLOOP)
PRINT*,'(T(KLOOP-1,KLOOP-1) = ', T(KLOOP-1,KLOOP-1)
PRINT*,'KLOOP = ',KLOOP,'GNUM = ',GNUM

RR= (T(KLOOP,KLOOP)-T(KLOOP-1,KLOOP-1))/T(KLOOP,KLOOP)
PRINT*, ' RR =',RR ', K = ',K
IF (DABS(RR) .LE. TOL) THEN
   INTE=T(KLOOP,KLOOP)

   KLOOP=NROW+1
END IF
IF (KLOOP .EQ. NROW) THEN
   INTE=T(KLOOP,KLOOP)
END IF
END DO

RETURN
END

C --------------- GFunc functions to be integrated numerically ---------------

FUNCTION GFunc(GNum,Y)
REAL *8, GFunc, Y
INTEGER GNum
REAL *8, k, j, w, a, b, KK, WW, JJ
REAL *8, V1, V2, V3, V4, V5, V6
COMMON k, j, w, j, WW, KK, a, b

IF (GNum .EQ. 3) THEN

   V1=(DLOG(DABS((WW-JJ)*Y+JJ)/DABS((KK)*(Y*Y)-(KK)*Y+WW))
   V2=(DLOG(DABS((JJ-WW)*Y+WW)/DABS((KK)*(Y*Y)-(KK)*Y+JJ))
   V3=DLOG((1.0D0-(JJ/(WW)))*Y+(JJ/(WW))
   V4=DLOG(DABS((KK*Y*Y)/(WW)-(KK*Y)/(WW)+1.0D0))
   V5=DLOG(DABS((JJ/(WW)-1.0D0)*Y+1.0D0))
   V6=DLOG(DABS((KK/(WW))*(Y*Y)-(KK/(WW))*Y+(JJ)/(WW)))

   GFunc=
& (3.0D0*Y-(Y*Y)-2.0D0)*(KK)/(Y*(KK)+(JJ)-(WW))
&-(((KK)*(Y*Y*Y)-3.0D0*(KK)*(Y*Y)
&+(2.0D0*(KK)+(WW))*(Y-2.0D0*(WW))*(KK))
&/(Y*(KK)+(JJ)-(WW))*(Y*(KK)+(JJ)-(WW)))

   V1
&+=(Y*Y)*(KK)+Y*(-4.0D0*(WW)-3.0D0*(KK)-2.0D0*(JJ)
&-(JJ/(WW))*(KK)+2.0D0*(KK)+(JJ/(WW))*(KK))
&(Y*(KK)+(JJ)-(WW)))
&+1.0D0
&+(Y*Y)*(J/J(WW))*K)/((K*Y+WW-JJ)

&-((Y*Y)*(KK*KK)+(J/J(WW))
&-(Y*Y)*(KK*KK)*J/(WW))+Y*(J/J(WW))*J/(KK)*V2
&*(1.0D0/((K*Y+WW-JJ)/(K*Y+WW-JJ)))

&+(Y*Y)*(J/J(WW))*(KK)
&+Y*(4.0D0*(JJ-J/J(WW))*(KK)+2.0D0*(J/J(WW)-2.0D0*(JJ))*V2
&*(1.0D0/(K*K+Y+WW-JJ))

&+2.0D0*(JJ+WW-JJ)*Y/(JJ+WW+KK*Y)))*V3
&-((KK*Y+Y-KK*Y+WW)/(KK*Y+JJ-JJ)))*V4
&-0.25D0

&+2.0D0*(JJ/(WW))*(((JJ-JW)*Y+WW)/(Y*(KK)+WW-JJ)))*V5
&-((KK*Y+Y-KK*Y+JJ)/(Y*(KK)+WW-JJ)))*V6-0.50D0

&+1.5D0*(J/J(WW)+((J/J(WW)))/(J/J(WW)))/(1.0D0-JJ/(WW))

ELSE IF (Gnum .EQ. 5) THEN

V1=(DLOG(DABS((WW-JJ)*Y+JJ)/DABS((KK)*(Y*Y)-(KK)*Y+WW)))
V2=(DLOG(DABS((JJ-JW)*Y+WW))/DABS((KK)*(Y*Y)-(KK)*Y+JJ)))
V3=DLOG((1.0D0-((J/J(WW)))*Y+(J/J(WW))
V4=DLOG(DABS(K*K*Y*/Y)/(WW)-(K*Y)/(WW)+1.0D0))
V5=DLOG(DABS((J/J(WW)-1.0D0)*Y+1.0D0))
V6=DLOG(DABS((K*/K*/(WW)))/K*K*/Y*-(K*/K*/(WW)))/(Y*(J/J(WW)))

Gfunc=

&-(Y*Y)*(KK)+Y*(KK-4.0D0*(WW)
&-2.0D0*(JJ)-(J/J(WW))*(KK))
&+4.0D0*(WW)+2.0D0*(JJ)+(KK)*(J/J(WW)))/(Y*(KK)+JJ-JW)

&-(K*K*KK)*(Y*Y*Y)
&+(Y*Y)*(4.0D0*(WW)*(KK)-(KK*KK)
&+2.0D0*(JJ-*(KK)+(J/J(WW))*(KK*KK))
&+(Y*Y*(-3.0D0*(WW)*(KK)-2.0D0*(JJ)*(KK)-(J/J(WW))*(KK*KK))
&+4.0D0*(WW*WW)+2.0D0*(JJ)*(WW)+(J/J*(KK)))*V1
&*(1.0D0/(((KK)*Y+JJ-JW))*((K*K)*Y+JJ-WW)))

&+(K*K*(Y*Y)+Y*(4.0D0*(WW)+KK)
&-2.0D0*(JJ)+(J/J(WW))*(KK))
&-4.0D0*(WW-(JJ))*V1*(1.0D0/((K*K)*Y+(JJ)-(WW)))
&+1.0D0}
\&+(-\langle JJ/(WW)\rangle \langle KK\rangle \langle YY \rangle)
\&+\langle YY \rangle \langle 4.0 \text{D}0 \rangle \langle JJ \rangle + \langle JJ/(WW) \rangle \langle KK \rangle + 2.0 \text{D}0 \langle \langle JJ*JJ \rangle/(WW) \rangle)
\&-4.0 \text{D}0 \langle JJ \rangle - 2.0 \text{D}0 \langle \langle JJ*JJ \rangle/(WW) \rangle \langle KK \rangle \langle YY + WW-JJ \rangle)

\&+((\langle KK \rangle \langle JJ/(WW) \rangle \langle YY \rangle)
\&+\langle YY \rangle \langle -4.0 \text{D}0 \rangle \langle JJ \rangle - \langle JJ/(WW) \rangle \langle KK \rangle - 2.0 \text{D}0 \langle \langle JJ*JJ \rangle/(WW) \rangle)
\&+2.0 \text{D}0 \langle JJ \rangle + 2.0 \text{D}0 \langle \langle JJ*JJ \rangle/(WW) \rangle \langle V2 \rangle
\&*(1.0 \text{D}0 \langle \langle KK \rangle \langle YY + WW-JJ \rangle - \langle JJ \rangle \rangle)

\&-((\langle JJ/(WW) \rangle \langle KK*KK \rangle \langle YY \rangle))
\&+\langle YY \rangle \langle -4.0 \text{D}0 \rangle \langle KK \rangle \langle JJ/(WW) \rangle \langle KK \rangle \langle KK \rangle
\&-2.0 \text{D}0 \langle \langle JJ*JJ \rangle/(WW) \rangle \langle KK \rangle \langle KK \rangle
\&+\langle YY \rangle \langle 4.0 \text{D}0 \rangle \langle JJ \rangle \langle KK \rangle + 3.0 \text{D}0 \langle \langle JJ*JJ \rangle/(WW) \rangle \langle KK \rangle \langle KK \rangle - 4.0 \text{D}0 \langle JJ*JJ \rangle
\&-2.0 \text{D}0 \langle \langle JJ*JJ \rangle/(WW) \rangle \langle V2 \rangle
\&*(1.0 \text{D}0 \langle \langle KK \rangle \langle YY + WW-JJ \rangle \langle KK \rangle \langle YY + WW-JJ \rangle \rangle)

\&+2.0 \text{D}0 \langle \langle JJ+(WW-JJ) \rangle \langle YY \rangle \rangle \langle JJ-WW+KK*YY \rangle \langle V3 \rangle
\&-((\langle KK*YY-KK*YY+WW \rangle) \langle KK*YY+JJ-WW \rangle \langle V4 \rangle
\&-0.25 \text{D}0 \langle JJ/(WW) \rangle

\&+2.0 \text{D}0 \langle JJ/(WW) \rangle \langle ((JJ-WW)*YY+WW)/(YY*KK+WW-JJ) \rangle \langle V5 \rangle
\&-((\langle KK*YY-KK*YY+JJ \rangle/(YY*KK)+WW-JJ \rangle \langle V6 \rangle
\&-0.50 \text{D}0 \langle JJ/(WW) \rangle

\&+1.50 \text{D}0 \langle JJ/(WW) \rangle \langle (JJ/(WW)) \rangle \langle JJ/(WW) \rangle
\&*\text{DLOG}(JJ/(WW)))/(1.0 \text{D}0 \langle JJ/(WW) \rangle)

\text{ELSE}

\text{STOP}

\text{END IF}

\text{END}
APPENDIX E

THE KM MIXING MATRIX

In the Standard Model with $SU(2) \times U(1)$ as the gauge group of electroweak interactions, both the quarks and the leptons are assigned to be left-handed doublets and right-handed singlets. The quark mass eigenstates are not the same as the weak eigenstates, and the matrix relating these bases was defined for six quarks and given an explicit parameterization by Kobayashi and Maskawa [51].

By convention, the three quarks ($u$, $c$, $t$) with charge $\frac{2}{3}$ are unmixed, and all the mixing is expressed in terms of a $3 \times 3$ unitary matrix $U$ operating on the quarks $(d, s, b)$ with charge $-\frac{1}{3}$

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} U_{ud} & U_{us} & U_{ub} \\ U_{cd} & U_{cs} & U_{cb} \\ U_{td} & U_{ts} & U_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}.$$  \hspace{1cm} (E.1)

Kobayashi and Maskawa [51] originally chose a parameterization involving the four angles $\theta_1$, $\theta_2$, $\theta_3$ and $\delta$ where

$$U = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix}.$$  \hspace{1cm} (E.2)

and $c_i = \cos \theta_i$, $s_i = \sin \theta_i$ for $i = 1, 2, 3$. In the limit $\theta_2 = \theta_3 = 0$, this reduces to the usual Cabibbo mixing with $\theta_1$ identified as the Cabibbo angle [49].
There are several parameterization of the Cabibbo-Kobayashi-Maskawa matrix but no physics can depend on which of the above parameterization is used as long as a single one is used consistently and care is taken to be sure that no other choice of phases is in conflict.

CP-violating processes will involve the phase in $K_M$ matrix assuming that the observed CP violation is solely related to the non-zero value of the phase, $\delta$.

The phase factor, $\delta$ can actually be determined from [55] where

$$J = \text{Im}(U_{cb} U_{ub}^* U_{cs}^* U_{ub}^*) = c_1 s_1^2 c_2 s_2 c_3 s_3 \sin \delta = 4.2 \times 10^{-5}$$

$$+2.7 -1.4$$

(E.3)

which gives

$$\sin \delta = \frac{J}{U_{ud}^* U_{us}^* U_{cd}^* s_2 s_3} = 1.2 \times 10^{-2} \quad \text{and} \quad \delta = 0.67^\circ.$$  

(E.4)

The angle $\theta_1$ is very well determined from the results of nuclear beta decay, by comparing the muon decay [58],

$$|U_{us}| = 0.9744 \pm 0.010.$$  

(E.5)

The magnitude of $|U_{cd}|$ is deduced from neutrino and antineutrino production of valence quarks. The experimental value of the dimuon production cross section [59], supplemented with the measurements of the semileptonic branching fractions of charm mesons [60] gives

$$|U_{cs}| = 0.204 \pm 0.017.$$  

(E.6)

Analysis of $K$ decay [61] and of hyperon decays [62] gives

$$|U_{ub}| = 0.2205 \pm 0.008.$$  

(E.7)