CHAPTER 1

Introduction

1. General Introduction

The term residual finite groups was first introduced in 1955 by P. Hall. Then at P. Hall's suggestion, Gruenberg [13] began the first systematic study of these residual finite groups. Since then many mathematicians have investigated residual finiteness and its various extensions.

In this thesis we shall study some stronger residual finite properties in generalised free products, HNN extensions and certain one relator groups. These properties include residual p-finiteness, weak potency and cyclic subgroup separability. Before going into the details of these residual properties, we describe briefly the history and some uses of these properties.

Residually p-finite groups were among the main groups studied by Gruenberg in his pioneering paper [13] although it was already shown by others that free groups and finitely generated torsion-free nilpotent groups [13, 16] are residually p-finite for all primes p.

While investigating the residual finiteness of generalised free products, Stebe [39] in 1968, introduced the stronger concept of cyclic subgroup separability. He then used it to prove the residual finiteness of a class of knot groups.

In 1974, Evans [12] introduced the concept of weak potency (Evans named it as regular quotients) in order to prove the cyclic subgroup separability of certain generalised free products with cyclic amalgamation. In 1995, Tang [40] defined weak potency independently and he used it to determine the conjugacy separability of certain generalised free products of conjugacy separable groups. Tang's definition is derived from the stronger property of potency which was introduced by Allenby and Tang [3] in 1981 to prove the residual finiteness of a class of one-relator groups with torsion. Their investigation were motivated by the well known conjecture of G. Baumslag that all one-relator groups with torsion are residually finite.

We now give a brief description of the five chapters in this thesis.

In this chapter (Chapter 1), we give the general introduction and some basic facts about generalised free products, tree products and HNN extensions of groups. Definitions of the various strong residual finite properties which are studied in this thesis are also introduced.

In Chapter 2 we study the residual p-finiteness of certain tree products. We shall give sufficient conditions for the tree products of finitely many groups amalgamating infinite cyclic subgroups to be residually p-finite. Our results are then applied to tree products of polycyclic-by-finite groups and free-by-finite groups. We also give characterisations for some one-relator groups to be residually p-finite.

In Chapter 3, we study about the weak potency of generalised free products. We shall give sufficient conditions for the tree products of finitely many weakly potent groups to be weakly potent when the amalgamated subgroups are finite, infinite cyclic or finitely generated. Our results are then applied to tree products of polycyclic-by-finite groups and free-by-finite groups. Finally we shall show that certain one-relator groups are weakly potent.

We study the weak potency of HNN extensions in Chapter 4. We shall show that certain HNN extensions of weakly potent groups are again weakly potent. Our results are then applied to HNN extensions of polycyclic-by-finite groups and free-by-finite groups. We shall also give a characterisation for the one-relator groups $\langle h, t; t^{-1}h^{\gamma}t = h^{\delta}\rangle$ to be weakly potent. Finally we prove that one-relator groups with non-trivial centre are weakly potent.

Finally in Chapter 5 we study cyclic subgroup separability of HNN extensions. We shall show that certain HNN extensions of cyclic subgroup separable groups and subgroup separable groups are cyclic subgroup separable. Our results are then applied to HNN extensions of polycyclic-by-finite groups and free-by-finite groups with finitely generated associated subgroups. We also give a characterisation for the one-relator groups $\langle h, t; t^{-1}h^{\gamma}t = h^{\delta}\rangle$ to be cyclic subgroup separable. Finally we show that one-relator groups with non-trivial centre are cyclic subgroup separable.

2. Generalised Free Products

We give some facts about generalised free products. Let $G_1, G_2, \ldots, G_n$ be a collection of groups. Suppose that H is another group such that there exists a monomorphism φ_i from H into each of the group G_i. Then the generalised free product G of $G_1, G_2, \ldots, G_n$ amalgamating H is defined to be the group with presentation obtained by taking the union of the presentations of $G_1, G_2, \ldots, G_n$ together with the additional relations $\varphi_i(h) = \varphi_j(h)$, $h \in H$, $i \neq j, 1 \leq i, j \leq n$. We shall denote G by $G = \langle G_1, G_2, \ldots, G_n; H \rangle$ or by $G = G_1 \underset{H}{*} G_2 \underset{H}{*} \cdots \underset{H}{*} G_n$. If $n = 2$, this reduces to $G = \langle G_1, G_2; H \rangle$ or $G = G_1 \underset{H}{*} G_2$.

Suppose $G = G_1 \underset{H}{*} G_2$. Then every element g of G can be written as a reduced word $g_1 g_2 \cdots g_k$ where $g_i \in G_1 \cup G_2$, successive g_i, g_{i+1} belong strictly to different factors of G.

The length of the reduced element $g = g_1 g_2 \cdots g_k$ is denoted by $\|g\|$ and is defined as follows:

$$\|g\| = \begin{cases} 0, & if\ k = 1\ and\ g_1 \in H \\ 1, & if\ k = 1\ and\ g_1 \in G_1 \backslash H\ or\ g_1 \in G_2 \backslash H \\ k, & otherwise \end{cases}$$

The reduced element $g_1 g_2 \cdots g_k$ is called cyclically reduced if all of its cyclic permutations $g_i g_{i+1} \cdots g_k g_1 g_2 \cdots g_{i-1}$ are reduced. Note that every element of G is conjugate to a cyclically reduced element of G.

3. Tree Products

We give some facts about tree products. Let $G_1, G_2, \ldots, G_n$ be a collection of groups. Suppose that with certain pairs of groups G_i, G_j, there is associated an isomorphism φ_{ij} from a subgroup H_{ij} of G_i onto a subgroup H_{ji} of G_j and an isomorphism φ_{ji} from H_{ji} onto H_{ij} such that $\varphi_{ji} = \varphi_{ij}^{-1}$. Let G be the group with presentation obtained by presenting each of the generalised free products $\langle G_i, G_j; H_{ij} = H_{ji} \rangle$ with H_{ij} and H_{ji} amalgamated under φ_{ij}, and then taking the union of these presentations. With G we associated a linear graph, each of whose edges joins two vertices G_i and G_j if φ_{ij} exists. When this graph is a tree, G is called a tree product of $G_1, G_2, \ldots, G_n$ with the subgroups H_{ij} and H_{ji} amalgamated under φ_{ij}. We shall denote G by $\langle G_1, G_2, \ldots, G_n; H_{ij} = \varphi_{ij}(H_{ij}) \rangle$ or by $\langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ when the specific φ_{ij} are implicit.

4. HNN Extensions

Next, we collect some facts about HNN extensions. Let B be a group with pairs of isomorphic subgroups H_i and K_i and let φ_i be an isomorphism of H_i onto K_i, $1 \leq i \leq n$. Then the HNN extension of B with stable letters t_i and associated subgroups H_i and K_i is defined to be the group with presentation obtained by taking the presentation of B together with the additional generators t_i and the additional relations $t_i^{-1}ht_i = \varphi_i(h)$, $h \in H_i$, $1 \le i \le n$. We shall denote G by $G = \langle B, t_i; t_i^{-1}H_it_i = K_i, \varphi_i\rangle$ or by $G = \langle B, t_i; t_i^{-1}H_it_i = K_i\rangle$ when the specific φ_i are implicit. If $n = 1$, this reduces to $G = \langle B, t; t^{-1}Ht = K, \varphi\rangle$ or $G = \langle B, t; t^{-1}Ht = K\rangle$.

Suppose $G = \langle B, t; t^{-1}Ht = K\rangle$. Then every element g of G can be written as a reduced word $g_0t^{e_1}g_1t^{e_2}\dots t^{e_k}g_k$ where $g_i \in B, e_i = \pm 1$ and no subwords $t^{-1}ht, h \in H$ or $tkt^{-1}, k \in K$ occur. The length of the reduced element $g = g_0t^{e_1}g_1t^{e_2}\dots t^{e_k}g_k$, written $\|g\|$, is the number k of occurence of $t^{\pm 1}$ in g.

The reduced element $t^{e_1}g_1t^{e_2}\dots t^{e_k}g_k$ is called cyclically reduced if and only if all its cyclic permutations $t^{e_i}g_it^{e_{i+1}}\dots t^{e_k}g_kt^{e_1}g_1\dots t^{e_{i-1}}g_{i-1}$ are also reduced. Note that every element of G is conjugate to a cyclically reduced element.

5. Notations and definitions

The notations used in this thesis are standard. In addition the following will be used.

Let G be any group and p a prime.
$N \triangleleft_f G$ means N is a normal subgroup of finite index in the group G.
$N \triangleleft_p G$ means N is a normal subgroup of p-power index in the group G.

We give the definitions of those strong residual finite properties which will be discussed in this thesis.

Residual finiteness and residual p-finiteness

Definition 1.1. *A group G is said to be residually finite if, for each $1 \neq x \in G$, there exists $N \triangleleft_f G$ such that $x \notin N$.*

Definition 1.2. *A group G is said to be residually p-finite if, for each $1 \neq x \in G$, there exists $N \triangleleft_p G$ such that $x \notin N$.*

Subgroup separability and cyclic subgroup separability

Definition 1.3. *Let G be a group and H a subgroup of G. Then G is said to be H-separable if for each element $x \in G\backslash H$, there exists $N \triangleleft_f G$ such that $x \notin NH$.*

Definition 1.4. *A group G is said to be subgroup separable if G is H-separable for each finitely generated subgroup H of G.*

Definition 1.5. *A group G is cyclic subgroup separable (π_c) if G is H-separable for each cyclic subgroup H of G.*

Weak potency

Definition 1.6. *A group G is said to be $\langle x \rangle$-weakly potent for an element x of infinite order in G if we can find a positive integer r such that for every positive integer n, there exists $N \triangleleft_f G$ such that xN has order exactly rn in G/N.*

Definition 1.7. *A group G is said to be weakly potent if G is $\langle x \rangle$-weakly potent for every element x of infinite order in G.*