CHAPTER 2

Generalised Free Products of Residually p-Finite Groups

1. Introduction

In this chapter we study the residual p-finiteness of certain tree products. Higman [15] first proved that a free product of two finite p-groups amalgamating a cyclic subgroup is residually p-finite. Higman's result was generalised by Kim and McCarron [22] who proved that the free products of finitely many residually p-finite groups amalgamating a single cyclic subgroup are residually p-finite. More recently, Kim and McCarron [23] and Kim and Tang [26] gave characterisations for certain one-relator groups and the generalised free products of finitely generated torsion-free nilpotent groups and free groups amalgamating a cyclic subgroup to be residually p-finite.

We shall give, in this chapter, sufficient conditions for the tree products of finitely many groups amalgamating infinite cyclic subgroups to be residually p-finite. Our results are then applied to the tree products of finitely generated torsion-free nilpotent groups and free groups. Finally we give characterisations for some one-relator groups to be residually p-finite.

2. Preliminaries

We introduce two properties which will be used throughout our theorems and proofs.

Definition 2.1. *[24] Let H be a subgroup of a group G. Then G is said to be H-separable for p if for each $x \in G \backslash H$, there exists $N \lhd_p G$ such that $x \notin HN$.*

Definition 2.2. *A group G is said to be $\langle x \rangle$-potent for p ($\langle x \rangle$-pot for p in short) if for each positive integer n, there exists $N \lhd_p G$ such that xN has order exactly p^n in G/N.*

Note that if a group G is $\langle c \rangle$-pot for p and $\langle c \rangle$-separable for p for an element c in G then G is residually p-finite. On the other hand, if G is a residually p-finite group, then G is $\langle c \rangle$-pot for p for every element c of infinite order in G. (Kim &

McCarron [22]). Furthermore if G is a free group or a finitely generated (non-cyclic) torsion-free nilpotent group and $\langle h\rangle$ is a maximal cyclic subgroup of G, then G is $\langle c\rangle$-separable for all primes p (Kim & McCarron [22], Baumslag [6]).

3. The main results

In this section, we give sufficient conditions for the tree products of finitely many groups amalgamating infinite cyclic subgroups to be residually p-finite. We shall begin with the following result of Higman [15].

Theorem 2.3. *[15] Let $G = A \mathop{*}_H B$ where A, B are finite p-groups and H is cyclic. Then G is residually p-finite.*

Theorem 2.4. *Let $G = A \mathop{*}_H B$ where $H = \langle h\rangle$ is infinite cyclic. Suppose A, B are $\langle h\rangle$-separable for p and $\langle h\rangle$-pot for p. Then G is residually p-finite.*

Proof. Let $g \neq 1$ be a reduced element in G.

Case 1. $g \in \langle h\rangle$, say $g = h^r$. Let n be an integer such that $| r | < p^n$. Since A, B are $\langle h\rangle$-pot for p, there exist $N \triangleleft_p A$, $M \triangleleft_p B$ such that $N \cap \langle h\rangle = \langle h^{p^n}\rangle = M \cap \langle h\rangle$. Now we form $\bar{G} = \bar{A} \mathop{*}_{\bar{H}} \bar{B}$ where $\bar{A} = A/N, \bar{B} = B/M$ and $\bar{H} = \langle h\rangle N/N = \langle h\rangle M/M$. Clearly $\bar{G}$ is a homomorphic image of G. Let $\bar{x}$ denote the image of x in $\bar{G}$ for any element x in G. Then $\bar{g} = \bar{h}^r \neq \bar{1}$ in $\bar{G}$. By Theorem 2.3, $\bar{G}$ is residually p-finite and hence there exists $\bar{K} \triangleleft_p \bar{G}$ such that $\bar{g} \notin \bar{K}$. Let K be the preimage of $\bar{K}$ in G. Then $K \triangleleft_p G$ and $g \notin K$. Our result now follows.

Case 2. $g \notin \langle h\rangle$. WLOG, assume $g = a_1 b_1 \ldots a_r b_r$ where $a_i \in A\backslash\langle h\rangle$, $b_i \in B\backslash\langle h\rangle$ for all i. Since A, B are $\langle h\rangle$-separable for p, there exist $N_1 \triangleleft_p A, M_1 \triangleleft_p B$ such that $a_i \notin \langle h\rangle N_1$, $b_i \notin \langle h\rangle M_1$ for all i. Suppose $N_1 \cap \langle h\rangle = \langle h^{p^n}\rangle$ and $M_1 \cap \langle h\rangle = \langle h^{p^m}\rangle$ for some integers m, n. Since A, B are $\langle h\rangle$-pot for p, there exist $N_2 \triangleleft_p A$, $M_2 \triangleleft_p B$ such that $N_2 \cap \langle h\rangle = \langle h^{p^{m+n}}\rangle = M_2 \cap \langle h\rangle$. Let $N = N_1 \cap N_2$ and $M = M_1 \cap M_2$. Then $N \triangleleft_p A$, $M \triangleleft_p B$ and $N \cap \langle h\rangle = M \cap \langle h\rangle$. As in case 1, we form $\bar{G} = \bar{A} \mathop{*}_{\bar{H}} \bar{B}$. Then $\bar{g}$ is reduced and $\|\bar{g}\| = \|g\|$ in $\bar{G}$. This implies that $\bar{g} \neq \bar{1}$ in $\bar{G}$ and we are done as in case 1.

In order to extend Theorem 2.4 to a tree product, we need the next few lemmas.

Lemma 2.5. *Let $G = A \mathop{*}_H B$ where $H = \langle h\rangle$ is infinite cyclic. Suppose A is $\langle h\rangle$-pot for p. Let B be $\langle u\rangle$-pot for p where $u \in B$. Then G is $\langle u\rangle$-pot for p.*

Proof. Since B is $\langle u\rangle$-pot for p, then for each positive integer n, there exists $M \triangleleft_p B$ such that uM has order exactly p^n in B/M. Suppose $M \cap \langle h\rangle = \langle h^{p^k}\rangle$ for some integer k. Since A is $\langle h\rangle$-pot for p, there exists $N \lhd_p A$ such that $N \cap \langle h\rangle = \langle h^{p^k}\rangle$. Now we form $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$ where $\bar{A} = A/N, \bar{B} = B/M$ and $\bar{H} = \langle h\rangle N/N = \langle h\rangle M/M$. Clearly $\bar{G}$ is a homomorphic image of G. Then $\bar{u}$ has order exactly p^n in $\bar{G}$. By Theorem 2.3, $\bar{G}$ is residually p-finite and hence there exists $\bar{K} \lhd_p \bar{G}$ such that $\bar{u}, \bar{u}^2, \cdots, \bar{u}^{p^n-1} \notin \bar{K}$. Let K be the preimage of $\bar{K}$ in G. Then $K \lhd_p G$ and uK has order exactly p^n in G/K.

Lemma 2.6. *Let $G = \langle G_1, G_2, \ldots, G_n; a_{ij} = a_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}\rangle$ of G_i and $\langle a_{ji}\rangle$ of G_j. Suppose each G_i is $\langle a_{ij}\rangle$-pot for p. Let G_r be $\langle u\rangle$-pot for p where $u \in G_r$. Then G is $\langle u\rangle$-pot for p.*

Proof. We use induction on n. The case $n = 2$ follows from Lemma 2.5. Now, let $n > 2$. The tree product G has an extremal vertex, say G_n, which is joined to a unique vertex, say G_{n-1}. The subgroup of G generated by $G_1, G_2, \ldots, G_{n-1}$ is just their tree product. Let G' denote this subgroup. Then $G = \langle G', G_n; a_{(n-1)n} = a_{n(n-1)}\rangle$. By the inductive hypothesis, G' is $\langle a_{(n-1)n}\rangle$-pot for p and by assumption, G_n is $\langle a_{n(n-1)}\rangle$-pot for p.

Case 1. $u \in G'$. By the inductive hypothesis, G' is $\langle u\rangle$-pot for p and we are done by Lemma 2.5.

Case 2. $u \in G_n$. By assumption, G_n is $\langle u\rangle$-pot for p and we are done by Lemma 2.5.

Lemma 2.7. *Let $G = A \ast_H B$ where $H = \langle h\rangle$ is infinite cyclic. Suppose A, B are $\langle h\rangle$-separable for p and $\langle h\rangle$-pot for p. Let K be a subgroup of A and A is K-separable for p. Then G is K-separable for p.*

Proof. Let $g \in G\backslash K$.

Case 1. $g \in A\backslash K$. Since A is K-separable for p, there exists $N \lhd_p A$ such that $g \notin KN$. Suppose $N \cap \langle h\rangle = \langle h^{p^k}\rangle$ for some integer k. Since B is $\langle h\rangle$-pot for p, there exists $M \lhd_p B$ such that $M \cap \langle h\rangle = \langle h^{p^k}\rangle$. Let $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$ where $\bar{A} = A/N, \bar{B} = B/M$ and $\bar{H} = \langle h\rangle N/N = \langle h\rangle M/M$. Let $\bar{g}$, $\bar{K}$ denote the images of g, K in $\bar{G}$ respectively. Clearly $\bar{g} \notin \bar{K}$. Since $\bar{G}$ is residually p-finite and $\bar{K}$ is finite, there exists $\bar{L} \lhd_p \bar{G}$ such that $\bar{g} \notin \bar{K}\bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_p G$ and $g \notin KL$. The result now follows.

Case 2. $g \in B\backslash\langle h\rangle$. Since B is $\langle h\rangle$-separable for p, there exists $M \lhd_p B$ such that $g \notin \langle h\rangle M$. Since A is $\langle h\rangle$-pot for p, there exists $N \lhd_p A$ such that $N \cap \langle h\rangle = M \cap \langle h\rangle$. As in case 1, we form $\bar{G}$. Clearly $\bar{g} \notin \bar{K}$ and our result follows as above.

Case 3. $g \notin A \cup B$. WLOG, assume $g = a_1 b_1 \dots a_r b_r$ where $a_i \in A \backslash \langle h \rangle, b_i \in B \backslash \langle h \rangle$ for all i. As in case 2 of Theorem 2.4, we can find $N \lhd_p A$ and $M \lhd_p B$ such that $a_i \notin \langle h \rangle N, b_i \notin \langle h \rangle M$ for all i and $N \cap \langle h \rangle = M \cap \langle h \rangle$. Then we form $\bar{G}$ as above. Note that $\|\bar{g}\| = \|g\|$ and hence $\bar{g} \notin \bar{K}$ in $\bar{G}$. The result now follows as before.

Lemma 2.8. *Let $G = \langle G_1, G_2, \dots, G_n; a_{ij} = a_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij} \rangle$ of G_i and $\langle a_{ji} \rangle$ of G_j. Suppose each G_i is $\langle a_{ij} \rangle$-separable for p and $\langle a_{ij} \rangle$-pot for p. Let K be a subgroup of G_r and G_r is K-separable for p. Then G is K-separable for p.*

Proof. We use induction on n. The case $n = 2$ follows from Lemma 2.7. Now, let $n > 2$. As in Lemma 2.6, we write $G = \langle G', G_n; a_{(n-1)n} = a_{n(n-1)} \rangle$ where G' is the tree product generated by $G_1, G_2, \dots, G_{n-1}$. By the inductive hypothesis, G' is $\langle a_{(n-1)n} \rangle$-separable for p and by Lemma 2.6, G' is $\langle a_{(n-1)n} \rangle$-pot for p. Furthermore by assumption, G_n is $\langle a_{n(n-1)} \rangle$-separable for p and $\langle a_{n(n-1)} \rangle$-pot for p.

Case 1. $K \subseteq G'$. By the inductive hypothesis, G' is K-separable for p and we are done by Lemma 2.7.

Case 2. $K \subseteq G_n$. By assumption, G_n is K-separable for p and we are done by Lemma 2.7.

Now we extend Theorem 2.4 to a tree product.

Theorem 2.9. *Let $G = \langle G_1, G_2, \dots, G_n; a_{ij} = a_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij} \rangle$ of G_i and $\langle a_{ji} \rangle$ of G_j. Suppose each G_i is $\langle a_{ij} \rangle$-separable for p and $\langle a_{ij} \rangle$-pot for p. Then G is residually p-finite.*

Proof. We use induction on n. The case $n = 2$ follows from Theorem 2.4. Now, let $n > 2$. As in Lemma 2.6, we write $G = \langle G', G_n; a_{(n-1)n} = a_{n(n-1)} \rangle$ where G' is the tree product generated by $G_1, G_2, \dots, G_{n-1}$. By Lemma 2.8, G' is $\langle a_{(n-1)n} \rangle$-separable for p and by Lemma 2.6, G' is $\langle a_{(n-1)n} \rangle$-pot for p. Furthermore by assumption, G_n is $\langle a_{(n-1)n} \rangle$-separable for p and $\langle a_{(n-1)n} \rangle$-pot for p. Therefore G is residually p-finite by Theorem 2.4.

Next we shall strengthen Theorem 2.9 by using the following two lemmas:

Lemma 2.10. *Let A be a group and $a \in A$. Suppose A is $\langle a \rangle$-separable for p and $\langle a \rangle$-pot for p. Then A is $\langle a^{p^k} \rangle$-separable for p where k is any positive integer.*

Proof. Let $x \in A \backslash \langle a^{p^k} \rangle$.

Case 1. $x \notin \langle a \rangle$. Since A is $\langle a \rangle$-separable for p, there exists $N \lhd_p A$ such that $x \notin \langle a \rangle N$. Therefore $x \notin \langle a^{p^k} \rangle N$ and we are done.

Case 2. $x \in \langle a \rangle$. Since A is $\langle a \rangle$-pot for p, there exists $N \lhd_p A$ such that $N \cap \langle a \rangle = \langle a^{p^k} \rangle$. This implies that $x \notin \langle a^{p^k} \rangle N$ and we are done.

Lemma 2.11. *Let A be a group and $a \in A$. Suppose A is $\langle a \rangle$-pot for p. Then A is $\langle a^{p^k} \rangle$-pot for p where k is any positive integer.*

Proof. Let n be any positive integer. Since A is $\langle a \rangle$-pot for p, there exists $N \lhd_p G$ such that $N \cap \langle a \rangle = \langle a^{p^{k+n}} \rangle$. Clearly $N \cap \langle a^{p^k} \rangle = \langle a^{p^{k+n}} \rangle = \langle (a^{p^k})^{p^n} \rangle$ and we are done.

Theorem 2.12. *Let $G = \langle G_1, G_2, \dots, G_n; a_{ij}^{p^{n_{ij}}} = a_{ji}^{p^{n_{ji}}} \rangle$ be a tree product of $G_1, G_2 \dots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}^{p^{n_{ij}}} \rangle$ of G_i and $\langle a_{ji}^{p^{n_{ji}}} \rangle$ of G_j. Suppose each G_i is $\langle a_{ij} \rangle$-separable for p and $\langle a_{ij} \rangle$-pot for p. Then G is residually p-finite.*

Proof. By Lemmas 2.10 & 2.11, G_i is $\langle a_{ij}^{p^{n_{ij}}} \rangle$-separable for p and $\langle a_{ij}^{p^{n_{ij}}} \rangle$-pot for p. Therefore the result follows from Theorem 2.9.

Next we apply Theorems 2.9 & 2.12 to residually p-finite groups and then to free groups and finitely generated nilpotent groups. We shall need the following result of Kim & McCarron [22].

Lemma 2.13. *[22] Let G be a residually p-finite group and let $c \in G$ have infinite order. Then G is $\langle c \rangle$-pot for p.*

Corollary 2.14. *Let $G = \langle G_1, G_2, \dots, G_n; a_{ij}^{p^{n_{ij}}} = a_{ji}^{p^{n_{ji}}} \rangle$ be a tree product of $G_1, G_2 \dots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}^{p^{n_{ij}}} \rangle$ of G_i and $\langle a_{ji}^{p^{n_{ji}}} \rangle$ of G_j. Suppose each G_i is residually p-finite and G_i is $\langle a_{ij} \rangle$-separable for p. Then G is residually p-finite.*

Corollary 2.15. *Let $G_1, G_2, \dots, G_n$ be free groups or finitely generated torsion-free nilpotent groups. Let $G = \langle G_1, G_2, \dots, G_n; a_{ij}^{n_{ij}} = a_{ji}^{n_{ji}} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}^{n_{ij}} \rangle$ of G_i and $\langle a_{ji}^{n_{ji}} \rangle$ of G_j. Suppose each $\langle a_{ij} \rangle$ is a maximal cyclic subgroup of G_i.*

(i) If all the $| n_{ij} | = | n_{ji} | = 1$, then G is residually p-finite for all primes p.

(ii) If all the $| n_{ij} |$ and $| n_{ji} |$ are powers of a prime p, then G is residually p-finite.

Proof. Let p be a prime. It is well known that free groups and finitely generated torsion-free nilpotent groups are residually p-finite (Iwasawa [16], Gruenberg [13]).

Therefore G_i is $\langle a_{ij}\rangle$-pot for p by Lemma 2.13. Furthermore, maximal cyclic subgroups of a free group are separable for p (Theorem 3.9 of Kim & McCarron [22]). Also, maximal cyclic subgroups of a finitely generated (non-cyclic) torsion-free nilpotent group are isolated and hence are separable for p (see Theorem 2.5 of Baumslag [6]). Therefore (i) follows easily from Theorem 2.9 and (ii) follows from Theorem 2.12.

4. Some applications

In this section, we will give some applications of Theorem 2.12. First we give a characterisation for certain free products of groups with commuting subgroups to be residually p-finite.

Lemma 2.16. *The group $C = \langle x, y; [x, y]\rangle$ is $\langle x\rangle$-separable for p and $\langle y\rangle$-separable for all primes p.*

Proof. We shall only show that C is $\langle x\rangle$-separable for p. Let $u \in C\backslash\langle x\rangle$. Then $u = x^m y^n$ for some integers m, n where $n \neq 0$. Let ϕ be the projection from C onto $\langle y\rangle$. Then $u\phi \neq 1$ and since $\langle y\rangle$ is residually p-finite, we are done.

In [23], G. Kim & McCarron proved the following:

Theorem 2.17. *[23] The group $G = \langle u, v; [u^m, v^n]\rangle$ is residually p-finite if and only if m and n are powers of p.*

Now we prove the following:

Theorem 2.18. *Let $G = \langle A * B; [u^m, v^n]\rangle$ where $u \in A$ and $v \in B$.*
(i) If G is residually p-finite, then m and n are powers of p.
(ii) Suppose A is $\langle u\rangle$-separable for p, $\langle u\rangle$-pot for p and B is $\langle v\rangle$-separable for p, $\langle v\rangle$-pot for p. If m and n are powers of p, then G is residually p-finite.

Proof. (i) Suppose G is residually p-finite. Let H be the subgroup of G generated by u and v. Then $H = \langle u, v; [u^m, v^n]\rangle$ is residually p-finite and hence m and n are powers of p by Theorem 2.17.

(ii) Suppose m and n are powers of p. Note that G can be written as a generalized free product,

$$G = A \underset{u^m = x}{*} \langle x, y; [x, y]\rangle \underset{y = v^n}{*} B.$$

Let $C = \langle x, y; [x, y]\rangle$. Since C is free abelian, C is residually p-finite. Therefore by Lemma 2.13, C is $\langle x\rangle$-pot for p and $\langle y\rangle$-pot for p. By Lemma 2.16, C is $\langle x\rangle$-separable for p and $\langle y\rangle$-separable for p. Furthermore by assumption, A is $\langle u\rangle$-separable for p, $\langle u\rangle$-pot for p and B is $\langle v\rangle$-separable for p, $\langle v\rangle$-pot for p. Hence G is residually p-finite by Theorem 2.12.

Corollary 2.19. *Let $G = \langle A * B; [u^m, v^n]\rangle$ where A, B are free groups or finitely generated torsion-free nilpotent groups. Suppose that $\langle u\rangle$ and $\langle v\rangle$ are maximal cyclic subgroups of A and B respectively. If m and n are powers of p, then G is residually p-finite.*

Proof. As in Corollary 2.15, A is $\langle u\rangle$-separable for p, $\langle u\rangle$-pot for p and B is $\langle v\rangle$-separable for p, $\langle v\rangle$-pot for p. The result now follows from Theorem 2.18.

Next, we give a characterisation for the one-relator groups $\langle x, y; (x^l y^m)^t\rangle$ and the free products $G = \langle A * B; (u^l v^m)^t\rangle$ to be residually p-finite. First we note that the finite cyclic group $C = \langle c; c^t\rangle$ is residually p-finite if and only if t is a power of p.

Lemma 2.20. *The group $M = \langle u, v; (uv)^t\rangle$, $t > 1$, is residually p-finite if and only if t is a power of p.*

Proof. Let $z = uv$. Then by Tietze transformations, we have

$$\begin{aligned} M &= \langle u, v; (uv)^t\rangle = \langle u, v, z; (uv)^t, z = uv\rangle \\ &= \langle u, v, z; (uv)^t, v = u^{-1}z\rangle \\ &= \langle u, z; z^t\rangle \end{aligned}$$

Suppose M is residually p-finite. Then the cyclic subgroup $\langle z; z^t\rangle$ of M is residually p-finite and hence t is a power of p. Conversely, suppose t is a power of p. Then $M = \langle u\rangle * \langle z; z^t\rangle$ is residually p-finite since a free product of two residually p-finite groups is again residually p-finite.

Theorem 2.21. *The group $G = \langle x, y; (x^l y^m)^t\rangle$, $t > 1$, is residually p-finite if and only if t is a power of p.*

Proof. Suppose G is residually p-finite. Let M be the subgroup of G generated by x^l and y^m. Then $M = \langle x^l, y^m; (x^l y^m)^t\rangle$ is residually p-finite and hence t is a power of p by Lemma 2.20. Conversely, if t is a power of p, then G is residually p-finite by Lemma 1 in Baumslag [5].

Lemma 2.22. *Let G be a residually p-finite group and $x \in G$. If $C_G(x) = \langle x\rangle$, then G is $\langle x\rangle$-separable for p.*

Proof. Let $g \in G\backslash\langle x\rangle$. Since $C_G(x) = \langle x\rangle$, then $[g, x] \neq 1$. By residual p-finiteness of G, there exists $N \lhd_p G$ such that $[g, x] \notin N$. Clearly $g \notin \langle x\rangle N$ and we are done.

Lemma 2.23. *Let $G = \langle x, y; (x^l y^m)^t \rangle$, $t > 1$, be residually p-finite. Then G is $\langle x \rangle$-separable for p and $\langle y \rangle$-separable for p.*

Proof. We shall consider the following cases:

Case 1. $|l| \neq 1 \neq |m|$. Note that $G = \langle x \rangle \underset{x^l = c}{*} G_0$ where $G_0 = \langle c, y; (cy^m)^t \rangle$. Then $C_G(x) = \langle x \rangle$ and hence G is $\langle x \rangle$-separable for p by Lemma 2.22. Similarly, we can show that G is $\langle y \rangle$-separable for p.

Case 2. $|l| = 1 = |m|$. WLOG, we may assume that $G = \langle x, y; (xy)^t \rangle$. Let $z = xy$. As in Lemma 2.20, $G = \langle x, z; z^t \rangle = \langle x \rangle * \langle z; z^t \rangle$. Clearly $C_G(x) = \langle x \rangle$ and hence G is $\langle x \rangle$-separable for p by Lemma 2.22. Similarly, G is $\langle y \rangle$-separable for p.

Case 3. $|l| = 1, |m| \neq 1$. WLOG, we may assume that $G = \langle x, y; (xy^m)^t \rangle$. Let $z = xy^m$. Then $G = \langle y, z; z^t \rangle = \langle y \rangle * \langle z; z^t \rangle$. Now, $C_G(x) = C_G(zy^{-m}) = \langle zy^{-m} \rangle = \langle x \rangle$, $C_G(y) = \langle y \rangle$ and we are done by Lemma 2.22.

Case 4. $|l| \neq 1, |m| = 1$. This case is similar to case 3.

Now we prove the following:

Theorem 2.24. *Let $G = \langle A * B; (u^l v^m)^t \rangle, t > 1$, where $u \in A$ and $v \in B$.*
(i) If G is residually p-finite, then t is a power of p.
(ii) Suppose $p \mid l$ and $p \mid m$ and suppose A is $\langle u \rangle$-separable for p, $\langle u \rangle$-pot for p and B is $\langle v \rangle$-separable for p, $\langle v \rangle$-pot for p. If t is a power of p then G is residually p-finite.

Proof. (i) Suppose G is residually p-finite. Let M be the subgroup of G generated by u and v. Then $M = \langle u, v; (u^l v^m)^t \rangle$ is residually p-finite and hence t is a power of p by Theorem 2.21.

(ii) Suppose t is a power of p. Let $l = p^\alpha l_1, m = p^\beta m_1$ where $(p, l_1) = 1 = (p, m_1)$. We note that G can be written as a generalised free product

$$G = A \underset{u^{p^\alpha} = x}{*} \langle x, y; (x^{l_1} y^{m_1})^t \rangle \underset{y = v^{p^\beta}}{*} B$$

Let $M = \langle x, y; (x^{l_1} y^{m_1})^t \rangle$. By Theorem 2.21, M is residually p-finite and hence M is $\langle x \rangle$-pot for p and $\langle y \rangle$-pot for p. By Lemma 2.23, M is $\langle x \rangle$-separable for p and $\langle y \rangle$-separable for p. Furthermore by assumption, A is $\langle u \rangle$-separable for p, $\langle u \rangle$-pot for p and B is $\langle v \rangle$-separable for p, $\langle v \rangle$-pot for p. Therefore by Theorem 2.12, G is residually p-finite.

Applying Theorem 2.24 to free groups and finitely generated torsion-free nilpotent groups, we have the following:

Corollary 2.25. *Let* $G = \langle A * B; (u^l v^m)^t \rangle, t > 1$, *where* A, B *are free groups or finitely generated torsion-free nilpotent groups. Suppose that* $\langle u \rangle$ *and* $\langle v \rangle$ *are maximal cyclic subgroups of* A *and* B *respectively. Let* $p \mid l$ *and* $p \mid m$. *If* t *is a power of* p, *then* G *is residually* p*-finite.*