CHAPTER 3

Generalized Free Products of Weakly Potent Groups

1. Introduction

In this chapter, we shall study the weak potency of tree products of finitely many weakly potent groups. The concept of weak potency was first introduced by Evans [12] (with the name regular quotients) and he showed that free groups and finitely generated torsion-free nilpotent groups are weakly potent. Later, Tang [40] defined weak potency independently and he proved that finite extensions of free groups and finitely generated torsion-free nilpotent groups are weakly potent. Evan [12] used weak potency to show the cyclic subgroup separability of certain generalised free products while Kim and Tang [25] and Tang [40] used it to determine the conjugacy separability of certain generalised free products of conjugacy separable groups. Despite its usefulness, few groups are known to be weakly potent. Hence in this chapter, we extend the weak potency property to tree products.

We shall show in this chapter that the tree products of finitely many weakly potent groups amalgamating finite subgroups are again weakly potent. Then we give sufficient conditions for the tree products of finitely many weakly potent groups to be weakly potent when the amalgamated subgroups are infinite cyclic subgroups or finitely generated subgroups. Our results are then applied to tree products of polycyclic-by-finite groups and free-by-finite groups. Finally we show that certain one-relator groups are weakly potent.

2. Tree products amalgamating finite subgroups

In this section we show that tree products of finitely many weakly potent groups amalgamating finite subgroups are weakly potent. First we have the following lemmas.

Lemma 3.1. *[40] Let G be a free-by-finite group. Then G is weakly potent.*

Lemma 3.2. *Let $G = G_1 \underset{H}{*} G_2$ where G_1, G_2 are finite. Then G is weakly potent.*

Proof. Since G is free-by-finite ([17]), the result follows from Lemma 3.1.

Theorem 3.3. *Let $G = G_1 \underset{H}{*} G_2$ where G_1, G_2 are weakly potent and H is finite. Then G is weakly potent.*

Proof. Let g be an element of infinite order in G.

Case 1. $g \in G_1 \cup G_2$. WLOG, assume $g \in G_1$. Since G_1, G_2 are residually finite and H is finite, there exist $N_1 \lhd_f G_1$, $M \lhd_f G_2$ such that $N_1 \cap H = 1 = M \cap H$. Suppose $N_1 \cap \langle g \rangle = \langle g^s \rangle$ for some positive integer s. By the weak potency of G_1, we can find a positive integer r such that for each positive integer n, there exists $N_2 \lhd_f G_1$ such that $N_2 \cap \langle g \rangle = \langle g^{rsn} \rangle$. Let $N = N_1 \cap N_2$. Then $N \lhd_f G_1$, $N \cap H = 1$ and $N \cap \langle g \rangle = \langle g^{rsn} \rangle$. Now we form $\bar{G} = \bar{G}_1 \underset{\bar{H}}{*} \bar{G}_2$ where $\bar{G}_1 = G_1/N$, $\bar{G}_2 = G_2/M$ and $\bar{H} = HN/N = HM/M$. Clearly $\bar{G}$ is a homomorphic image of G. Let $\bar{g}$ denote the image of g in $\bar{G}$. Since $\bar{G}$ is residually finite, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{g}, \ldots, \bar{g}^{rsn-1} \notin \bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ and gP has order exactly rsn in G/P. The result now follows.

Case 2. $g \notin G_1 \cup G_2$. WLOG, assume $g = a_1 b_1 \ldots a_n b_n$ where $a_i \in G_1 \backslash H$ and $b_i \in G_2 \backslash H$ for all i. Since G_1, G_2 are residually finite and H is finite, there exist $N \lhd_f G_1$, $M \lhd_f G_2$ such that $a_i \notin HN$, $b_i \notin HM$ for all i and $N \cap H = 1 = M \cap H$. As in case 1, we form $\bar{G}$. Then $\|\bar{g}\| = \|g\|$ and hence $\bar{g}$ has infinite order in $\bar{G}$. By Lemma 3.2, $\bar{G}$ is weakly potent and the result follows.

Theorem 3.3 can be easily extended as follows:

Theorem 3.4. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the finite subgroups H_{ij} of G_i and H_{ji} of G_j. Suppose each G_i is weakly potent. Then G is weakly potent.*

Proof. We use induction on n. The case $n = 2$ follows from Theorem 3.3. Now, let $n > 2$. The tree product G has an extremal vertex, say G_n, which is joined to a unique vertex, say G_{n-1}. The subgroup of G generated by $G_1, G_2, \ldots, G_{n-1}$ is just their tree product. Let G' denote this subgroup. Then $G = \langle G', G_n; H_{(n-1)n} = H_{n(n-1)} \rangle$ where $H_{(n-1)n}$ and $H_{n(n-1)}$ are finite. By inductive hypothesis, G' is weakly potent and by assumption, G_n is weakly potent. Therefore G is weakly potent by Theorem 3.3.

3. Tree products amalgamating infinite cyclic subgroups

In this section, we shall give sufficient conditions for the tree products of finitely many weakly potent groups amalgamating infinite cyclic subgroups to be weakly potent.

Theorem 3.5. *Let $G = G_1 \overset{*}{_H} G_2$ where G_1, G_2 are weakly potent and $H = \langle h \rangle$ is infinite cyclic. Suppose G_1, G_2 are $\langle h \rangle$-separable. Then G is weakly potent.*

Proof. First we note that since G_1, G_2 are weakly potent, we can find positive integers r_1, r_2 such that for each positive integer n, there exist $P \triangleleft_f G_1, Q \triangleleft_f G_2$ such that $P \cap \langle h \rangle = \langle h^{r_1 n} \rangle, Q \cap \langle h \rangle = \langle h^{r_2 n} \rangle$. Let g be an element of infinite order in G.
Case 1. $g \in G_1 \cup G_2$. WLOG, assume $g \in G_1$. Let $N_1 \triangleleft_f G_1$ be such that $N_1 \cap \langle h \rangle = \langle h^{r_1 r_2} \rangle$. Suppose $N_1 \cap \langle g \rangle = \langle g^s \rangle$ for some positive integer s. By the weak potency of G_1, we can find a positive integer r such that for each positive integer n, there exists $N_2 \triangleleft_f G_1$ such that $N_2 \cap \langle g \rangle = \langle g^{rsn} \rangle$. Let $N = N_1 \cap N_2$. Then $N \triangleleft_f G_1$, $N \cap \langle g \rangle = \langle g^{rsn} \rangle$ and $N \cap \langle h \rangle = \langle h^{r_1 r_2 t} \rangle$ for some positive integer t. Let $M \triangleleft_f G_2$ be such that $M \cap \langle h \rangle = \langle h^{r_1 r_2 t} \rangle$. Now we form $\bar{G} = \bar{G}_1 \overset{*}{_{\bar{H}}} \bar{G}_2$ where $\bar{G}_1 = G_1/N$, $\bar{G}_2 = G_2/M$ and $\bar{H} = \langle h \rangle N/N = \langle h \rangle M/M$. Clearly $\bar{G}$ is a homomorphic image of G. Then $\bar{g}$ has order exactly rsn in $\bar{G}$. Our result now follows as in case 1 of Theorem 3.3.

Case 2. $g \notin G_1 \cup G_2$. WLOG, assume $g = a_1 b_1 \ldots a_n b_n$ where $a_i \in G_1 \backslash \langle h \rangle$ and $b_i \in G_2 \backslash \langle h \rangle$ for all i. Since G_1, G_2 are $\langle h \rangle$-separable, there exist $N_1 \triangleleft_f G_1$, $M_1 \triangleleft_f G_2$ such that $a_i \notin \langle h \rangle N_1$ and $b_i \notin \langle h \rangle M_1$ for all i. Suppose $N_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some positive integers s_1 and s_2. Let $N_2 \triangleleft_f G_1$, $M_2 \triangleleft_f G_2$ be such that $N_2 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2} \rangle = M_2 \cap \langle h \rangle$. Let $N = N_1 \cap N_2$ and $M = M_1 \cap M_2$. Then $N \triangleleft_f G_1, M \triangleleft_f G_2$ and $N \cap \langle h \rangle = M \cap \langle h \rangle$. As in case 1, we form $\bar{G}$. Then $\|\bar{g}\| = \|g\|$ and hence $\bar{g}$ has infinite order in $\bar{G}$. By Lemma 3.2, $\bar{G}$ is weakly potent and the result follows.

To extend Theorem 3.5 to a tree product, we need the next few lemmas.

Lemma 3.6. *[19] Let $G = G_1 \overset{*}{_H} G_2$. Suppose that*

(a) *G_1, G_2 are H-separable;*

(b) *for each $R \triangleleft_f H$, there exist $N \triangleleft_f G_1, M \triangleleft_f G_2$ such that $N \cap H = M \cap H \subseteq R$.*

Let K be a subgroup of G_1 and G_1 is K-separable. Then G is K-separable.

Lemma 3.7. *Let $G = G_1 \overset{*}{_H} G_2$ where $H = \langle h \rangle$ is infinite cyclic. Suppose that G_1, G_2 are $\langle h \rangle$-weakly potent and $\langle h \rangle$-separable. Let K be a subgroup of G_1 and G_1 is K-separable. Then G is K-separable.*

Proof. Let $R \triangleleft_f \langle h \rangle$ be given. Then $R = \langle h^s \rangle$ for some positive integer s. Since G_1, G_2 are $\langle h \rangle$-weakly potent, there exist $N \triangleleft_f G_1, M \triangleleft_f G_2$ such that $N \cap \langle h \rangle = M \cap \langle h \rangle = \langle h^{r_1 r_2 s} \rangle \subseteq R$ for some positive integers r_1 and r_2. This proves Lemma 3.6(b). Therefore G is K-separable.

Lemma 3.8. *Let $G = \langle G_1, G_2, \ldots, G_n; a_{ij} = a_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}\rangle$ of G_i and $\langle a_{ji}\rangle$ of G_j. Suppose G is weakly potent and each G_i is $\langle a_{ij}\rangle$-separable. Let K be a subgroup of G_r and G_r is K-separable. Then G is K-separable.*

Proof. We use induction on n. The case $n = 2$ follows from Lemma 3.7. Now, let $n > 2$. As in Theorem 3.4, we write $G = \langle G', G_n; a_{(n-1)n} = a_{n(n-1)}\rangle$ where G' is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$. By inductive hypothesis, G' is $\langle a_{(n-1)n}\rangle$-separable and by assumption, G_n is $\langle a_{n(n-1)}\rangle$-separable. Furthermore, G' is $\langle a_{(n-1)n}\rangle$-weakly potent and G_n is $\langle a_{n(n-1)}\rangle$-weakly potent since G is weakly potent.

Case 1. $K \subseteq G'$. By inductive hypothesis, G' is K-separable and we are done by Lemma 3.7.

Case 2. $K \subseteq G_n$. By assumption, G_n is K-separable and we are done by Lemma 3.7.

Now, Theorem 3.5 can be extended to a tree product as follows:

Theorem 3.9. *Let $G = \langle G_1, G_2, \ldots, G_n; a_{ij} = a_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}\rangle$ of G_i and $\langle a_{ji}\rangle$ of G_j. Suppose each G_i is weakly potent and $\langle a_{ij}\rangle$-separable. Then G is weakly potent.*

Proof. We use induction on n. The case $n = 2$ follows from Theorem 3.5. Now, let $n > 2$. As in Theorem 3.4, we write $G = \langle G', G_n; a_{(n-1)n} = a_{n(n-1)}\rangle$ where G' is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$. By inductive hypothesis, G' is weakly potent. Hence G' is $\langle a_{(n-1)n}\rangle$-separable by Lemma 3.8. Furthermore by assumption, G_n is weakly potent and G_n is $\langle a_{n(n-1)}\rangle$-separable. Therefore G is weakly potent by Theorem 3.5.

Next we show that Theorem 3.9 can be strengthened by using the following result:

Lemma 3.10. *Let A be a group and $a \in A$. Suppose A is $\langle a\rangle$-weakly potent and $\langle a\rangle$-separable. Then A is $\langle a^k\rangle$-separable for each positive integer k.*

Proof. Let $x \in A \backslash \langle a^k\rangle$.

Case 1. $x \notin \langle a\rangle$. Since A is $\langle a\rangle$-separable, there exists $N \triangleleft_f A$ such that $x \notin \langle a\rangle N$. Therefore $x \notin \langle a^k\rangle N$ and we are done.

Case 2. $x \in \langle a\rangle$. Since A is $\langle a\rangle$-weakly potent, there exists $N \triangleleft_f A$ such that $N \cap \langle a\rangle = \langle a^{rk}\rangle \subseteq \langle a^k\rangle$ for some positive integer r. This implies $x \notin \langle a^k\rangle N$ and we are done.

Theorem 3.11. *Let $G = \langle G_1, G_2, \dots, G_n; a_{ij}^{n_{ij}} = a_{ji}^{n_{ji}} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}^{n_{ij}} \rangle$ of G_i and $\langle a_{ji}^{n_{ji}} \rangle$ of G_j. Suppose each G_i is weakly potent and $\langle a_{ij} \rangle$-separable. Then G is weakly potent.*

Proof. By Lemma 3.10, each G_i is $\langle a_{ij}^{n_{ij}} \rangle$-separable and we are done by Theorem 3.9.

It is well known that polycylic groups and free groups are subgroup separable (Mal'cev [30], M. Hall [14]) and finite extensions of subgroup separable groups are again subgroup separable (Romanovski [37], Scott [38]). Hence polycyclic-by-finite groups and free-by-finite groups are subgroup separable. Furthermore, polycyclic-by-finite groups and free-by-finite groups are weakly potent (Wehrfritz [41], Tang [40]). Hence from Theorem 3.11, we have the following:

Corollary 3.12. *Let $G_1, G_2, \dots, G_n$ be polycyclic-by-finite groups or free-by-finite groups. Let $G = \langle G_1, G_2, \dots, G_n; a_{ij}^{n_{ij}} = a_{ji}^{n_{ji}} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij}^{n_{ij}} \rangle$ of G_i and $\langle a_{ji}^{n_{ji}} \rangle$ of G_j. Then G is weakly potent.*

Corollary 3.13. *The group $G = \langle a_1, a_2, \dots, a_m; a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$ is weakly potent.*

We note that Campbell [10] has shown that the infinite tree product $G = \langle \dots, a_{-1}, a_0, a_1, \dots; \dots, a_i^p = a_{i+1}^q, \dots \rangle$, where p and q are mutually coprime, is not residually finite. On the other hand, many of the groups in Corollary 3.13 are one-relator groups (see Meskin, Pietrowski and Steinberg [34], Collins [11] and McCool [32]).

4. Some applications

We apply Theorem 3.9 & 3.11 to certain free products with commuting subgroups and some one-relator groups with torsion.

Theorem 3.14. *The group $G = \langle a, b; [a^m, b^n] \rangle$ is weakly potent.*

Proof. We note that G can be written as a generalised free product,

$$G = \langle a \rangle \underset{a^m = u}{*} \langle u, v; [u, v] \rangle \underset{v = b^n}{*} \langle b \rangle.$$

Let $C = \langle u, v; [u, v] \rangle$. Since C is free abelian, C is weakly potent and subgroup separable. Therefore G is weakly potent by Theorem 3.9.

Theorem 3.15. *Let $G = \langle A * B; [a^m, b^n] \rangle$ where $a \in A$ and $b \in B$. Suppose that A, B are weakly potent and A is $\langle a \rangle$-separable, B is $\langle b \rangle$-separable. Then G is weakly potent.*

Proof. Since G can be written as

$$G = A \underset{a^m = u}{*} \langle u, v; [u, v] \rangle \underset{v = b^n}{*} B.$$

and by assumption, A, B are weakly potent and A is $\langle a \rangle$-separable, B is $\langle b \rangle$-separable. Therefore G is weakly potent by Theorem 3.11.

Lemma 3.16. *The group $M = \langle u, v; (uv)^t \rangle, t > 1$, is weakly potent.*

Proof. Let $z = uv$. Then by Tietze transformations, we obtain

$$\begin{aligned} M &= \langle u, v; (uv)^t \rangle = \langle u, v, z; (uv)^t, z = uv \rangle \\ &= \langle u, v, z; (uv)^t, v = u^{-1} z \rangle \\ &= \langle u, z; z^t \rangle \end{aligned}$$

Since a free product of a weakly potent group and a finite group is weakly potent, it follows that M is weakly potent.

Theorem 3.17. *The group $G = \langle a, b; (a^l b^m)^t \rangle, t > 1$, G is weakly potent.*

Proof. We note that G can be written as a generalised free product,

$$G = \langle a \rangle \underset{a^l = u}{*} \langle u, v; (uv)^t \rangle \underset{v = b^m}{*} \langle b \rangle.$$

Let $M = \langle u, v; (uv)^t \rangle$. By Lemma 3.16, M is weakly potent and by Lemma 1 of [1], M is $\langle u \rangle$-separable and $\langle v \rangle$-separable. Therefore G is weakly potent by Theorem 3.9.

Theorem 3.18. *Let $G = \langle A * B; (a^l b^m)^t \rangle, t > 1$, where $a \in A$ and $b \in B$. Suppose that A, B are weakly potent and A is $\langle a \rangle$-separable, B is $\langle b \rangle$-separable. Then G is weakly potent.*

Proof. The proof is similar to Theorem 3.15.

Corollary 3.19. *Let A, B be polycyclic-by-finite groups or free-by-finite groups and $a \in A$, $b \in B$. Then $G_1 = \langle A * B; [a^m, b^n] \rangle$ and $G_2 = \langle A * B; (a^l b^m)^t \rangle, t > 1$, are weakly potent.*

5. Tree products amalgamating finitely generated subgroups

We give sufficient conditions for the tree products of finitely many weakly potent groups amalgamating finitely generated subgroups to be weakly potent. We shall begin with the following criterion:

Theorem 3.20. *Let $G = G_1 \ast_H G_2$. Suppose that*

(a) *G_1, G_2 are weakly potent and H-separable;*

(b) *for each $R \triangleleft_f H$, there exist $N \triangleleft_f G_1, M \triangleleft_f G_2$ such that $N \cap H = R = M \cap H$.*

Then G is weakly potent.

Proof. Let g be an element of infinite order in G.

Case 1. $g \in G_1 \cup G_2$. WLOG, assume $g \in G_1$. Since G_1 is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $N \triangleleft_f G_1$ such that $N \cap \langle g \rangle = \langle g^{rn} \rangle$. Let $R = N \cap H$. Then $R \triangleleft_f H$. Hence by (b), there exists $M \triangleleft_f G_2$ such that $M \cap H = R$. Now we form $\bar{G} = \bar{G}_1 \ast_{\bar{H}} \bar{G}_2$ where $\bar{G}_1 = G_1/N, \bar{G}_2 = G_2/M$ and $\bar{H} = HN/N = HM/M$. Clearly $\bar{G}$ is a homomorphic image of G. Then $\bar{g}$ has order exactly rsn in $\bar{G}$. Our result now follows as in case 1 of Theorem 3.3.

Case 2. $g \notin G_1 \cup G_2$. WLOG, assume $g = a_1 b_1 \dots a_n b_n$ where $a_i \in G_1 \backslash H$ and $b_i \in G_2 \backslash H$ for all i. Since G_1, G_2 are H-separable, there exist $N_1 \triangleleft_f G_1$, $M_1 \triangleleft_f G_2$ such that $a_i \notin HN_1$ and $b_i \notin HM_1$ for all i. Let $R = N_1 \cap M_1$. Then $R \triangleleft_f H$. Hence by (b), there exist $N_2 \triangleleft_f G_1$, $M_2 \triangleleft_f G_2$ such that $N_2 \cap H = R = M_2 \cap H$. Let $N = N_1 \cap N_2$ and $M = M_1 \cap M_2$. Then $N \triangleleft_f G_1, M \triangleleft_f G_2$ and $N \cap H = M \cap H$. As in case 1, we form $\bar{G}$. Then $\|\bar{g}\| = \|g\|$ and hence $\bar{g}$ has infinite order in $\bar{G}$. By Lemma 3.2, $\bar{G}$ is weakly potent and the result follows.

In order to extend Theorem 3.20 to a tree product, we need the next few lemmas.

Lemma 3.21. *Let $G = G_1 \ast_H G_2$. Let K be a subgroup of G_2. Suppose that*

(a) *for each $R \triangleleft_f H$, there exists $N \triangleleft_f G_1$ such that $N \cap H = R$;*

(b) *for each $S \triangleleft_f K$, there exists $M \triangleleft_f G_2$ such that $M \cap K = S$.*

Then for each $S \triangleleft_f K$, there exists $P \triangleleft_f G$ such that $P \cap K = S$.

Proof. Let $S \triangleleft_f K$ be given. Then by (b), there exists $M \triangleleft_f G_2$ such that $M \cap K = S$. Let $R = M \cap H$. Then $R \triangleleft_f H$. Hence by (a), there exists $N \triangleleft_f G_1$ such that $N \cap H = R$. Now we form $\bar{G} = \bar{G}_1 \ast_{\bar{H}} \bar{G}_2$ where $\bar{G}_1 = G_1/N, \bar{G}_2 = G_2/M$ and $\bar{H} = HN/N = HM/M$. Clearly $\bar{G}$ is a homomorphic image of G. Since $\bar{G}$ is residually finite and $\bar{K}$ is finite in $\bar{G}$, there exists $\bar{P} \triangleleft_f \bar{G}$ such that $\bar{P} \cap \bar{K} = \bar{1}$. Let P be the preimage of $\bar{P}$ in G. Then $P \triangleleft_f G$ and $P \cap K = S$.

Lemma 3.22. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Let K be a subgroup of G_r. Suppose that*

(a) *for each $R_i \triangleleft_f H_{ij}$, there exists $N_i \triangleleft_f G_i$ such that $N_i \cap H_{ij} = R_i$;*

(b) *for each $S \triangleleft_f K$, there exists $M \triangleleft_f G_r$ such that $M \cap K = S$.*

Then for each $S \triangleleft_f K$, there exists $P \triangleleft_f G$ such that $P \cap K = S$.

Proof. We use induction on n. The case $n = 2$ follows from Lemma 3.21. Now, let $n > 2$. As in Theorem 3.4, we write $G = \langle G', G_n; H_{(n-1)n} = H_{n(n-1)}\rangle$ where G' is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$.

Case 1. $K \subseteq G'$. By assumption, for each $R_n \triangleleft_f H_{n(n-1)}$, there exists $N_n \triangleleft_f G_n$ such that $N_n \cap H_{n(n-1)} = R_n$. By inductive hypothesis, for each $S \triangleleft_f K$, there exists $M' \triangleleft_f G'$ such that $M' \cap K = S$. Hence the result follows from Lemma 3.21.

Case 2. $K \subseteq G_n$. By inductive hypothesis, for each $R \triangleleft_f H_{(n-1)n}$, there exists $N' \triangleleft_f G'$ such that $N' \cap H_{(n-1)n} = R$. Furthermore by assumption, for each $S \triangleleft_f K$, there exists $M_n \triangleleft_f G_n$ such that $M_n \cap K = S$. Therefore, the result follows from Lemma 3.21.

Lemma 3.23. *Let $G = \langle G_1, G_2, \ldots, G_n; H_{ij} = H_{ji}\rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Suppose that*

(a) *G_i is H_{ij}-separable;*

(b) *for each $R_i \triangleleft_f H_{ij}$, there exists $N_i \triangleleft_f G_i$ such that $N_i \cap H_{ij} = R_i$.*

Let K be a subgroup of G_r and G_r is K-separable. Then G is K-separable.

Proof. We use induction on n. The case $n = 2$ follows from Lemma 3.6. Now, let $n > 2$. As in Theorem 3.4, we write $G = \langle G', G_n; H_{(n-1)n} = H_{n(n-1)}\rangle$ where G' is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$. By inductive hypothesis, G' is $H_{(n-1)n}$-separable and by Lemma 3.22, for each $R \triangleleft_f H_{(n-1)n}$, there exists $N' \triangleleft_f G'$ such that $N' \cap H_{(n-1)n} = R$. Furthermore by assumption, G_n is $H_{n(n-1)}$-separable and for each $R_n \triangleleft_f H_{n(n-1)}$, there exists $N_n \triangleleft_f G_n$ such that $N_n \cap H_{n(n-1)} = R_n$.

Case 1. $K \subseteq G'$. By inductive hypothesis, G' is K-separable and we are done by Lemma 3.6.

Case 2. $K \subseteq G_n$. By assumption, G_n is K-separable and we are done by Lemma 3.6.

Now we extend Theorem 3.20 to a tree product.

Theorem 3.24. *Let $G = \langle G_1, G_2, \dots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Suppose that*

(a) *G_i is weakly potent and H_{ij}-separable;*

(b) *for each $R_i \triangleleft_f H_{ij}$, there exists $N_i \triangleleft_f G_i$ such that $N_i \cap H_{ij} = R_i$.*

Then G is weakly potent.

Proof. We use induction on n. The case $n = 2$ follows from Theorem 3.20. Now, let $n > 2$. As in Theorem 3.4, we write $G = \langle G', G_n; H_{(n-1)n} = H_{n(n-1)} \rangle$ where G' is the tree product generated by $G_1, G_2, \dots, G_{n-1}$. By inductive hypothesis, G' is weakly potent. Furthermore, by Lemma 3.23, G' is $H_{(n-1)n}$-separable and by Lemma 3.22, for each $R \triangleleft_f H_{(n-1)n}$, there exists $N' \triangleleft_f G'$ such that $N' \cap H_{(n-1)n} = R$. By assumption, G_n is weakly potent, $H_{n(n-1)}$-separable and for each $R_n \triangleleft_f H_{n(n-1)}$, there exists $N_n \triangleleft_f G_n$ such that $N_n \cap H_{n(n-1)} = R_n$. Hence the result follows from Theorem 3.20.

We now give some applications of Theorem 3.24. We shall prove that tree products of finitely many weakly potent groups are again weakly potent if the amalgamated subgroups are in the centre of their respective groups or the amalgamated subgroups are retracts of their respective groups.

Lemma 3.25. *Let G be a subgroup separable group and H be a finitely generated subgroup of $Z(G)$. Then for each $R \triangleleft_f H$, there exists $N \triangleleft_f G$ such that $N \cap H = R$.*

Proof. Let $R \triangleleft_f H$ be given. Since $R \subseteq Z(G)$, we can form $\bar{G} = G/R$. Then $\bar{G}$ is residually finite since G is subgroup separable and R is finitely generated. Furthermore, the subgroup $\bar{H}$ is finite in $\bar{G}$. Hence there exists $\bar{N} \triangleleft_f \bar{G}$ such that $\bar{N} \cap \bar{H} = \bar{1}$. Let N be the preimage of $\bar{N}$ in G. Then $N \triangleleft_f G$ and $N \cap H = R$.

Corollary 3.26. *Let $G = \langle G_1, G_2, \dots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the finitely generated subgroups H_{ij} of $Z(G_i)$ and H_{ji} of $Z(G_j)$. Suppose each G_i is weakly potent and subgroup separable. Then G is weakly potent.*

Proof. Follows from Theorem 3.24 and Lemma 3.25.

Lemma 3.27. *[20] Let H be a retract of a group G. If G is residually finite, then G is H-separable.*

Lemma 3.28. *Let H be a retract of a group G. Then for each $R \triangleleft_f H$, there exists $N \triangleleft_f G$ such that $N \cap H = R$.*

Proof. Let $R \triangleleft_f H$ be given. Since H is a retract of G, then G has a normal subgroup L such that $G = LH$ and $L \cap H = 1$. Let $N = LR$. First we show $N \triangleleft G$. Let $n = l_1 r \in N$ and $g = l_2 h \in G$ where $l_1, l_2 \in L$ and $r \in R, h \in H$. Since $L \triangleleft G$, we have $r l_2 r^{-1} = l_2' \in L$. Therefore $g^{-1} n g = h^{-1} l_2^{-1} l_1 r l_2 h = h^{-1} l_2^{-1} l_1 l_2' r h = l_3 r_1 \in LR$ where $l_3 = h^{-1} l_2^{-1} l_1 l_2' h \in L$ and $r_1 = h^{-1} r h \in R$. Hence $N \triangleleft G$. Now clearly $G/N \cong H/R$ and hence $N \triangleleft_f G$. Furthermore, $N \cap H = LR \cap H = (L \cap H)R = R$ and we are done.

Corollary 3.29. *Let $G = \langle G_1, G_2, \dots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Suppose each G_i is weakly potent and H_{ij} is a retract of G_i. Then G is weakly potent.*

Proof. Follows from Theorem 3.24, Lemma 3.27 and Lemma 3.28.

Since polycyclic-by-finite groups and free-by-finite groups are weakly potent and subgroup separable, from Corollary 3.26 & 3.29, we have the following:

Corollary 3.30. *Let $G_1, G_2, \dots, G_n$ be polycyclic-by-finite groups or free-by-finite groups. Let $G = \langle G_1, G_2, \dots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Suppose that*

(a) *each H_{ij} is a finitely generated subgroup of $Z(G_i)$ or*

(b) *each H_{ij} is a retract of G_i.*

Then G is weakly potent.

Corollary 3.31. *Let $G_1, G_2, \dots, G_n$ be finitely generated abelian groups. Let $G = \langle G_1, G_2, \dots, G_n; H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$, amalgamating the subgroups H_{ij} of G_i and H_{ji} of G_j. Then G is weakly potent.*