CHAPTER 5

Cyclic Subgroup Separability of Certain HNN Extensions

1. Introduction

In this chapter we shall study the cyclic subgroup separability of certain HNN extensions of cyclic subgroup separable groups and subgroup separable groups. The concept of cyclic subgroup separability was introduced by Stebe [39] in 1968 who used it to prove the residual finiteness of a class of knot groups. Later, in 1974, Evans [12] proved the cyclic subgroup separability of certain generalised free products with cyclic amalgamation. Kim [19] in 1992 proved that certain polygonal products of finitely generated abelian groups are cyclic subgroup separable. More recently, Kim [21] and Kim and Tang [27] gave characterisations for HNN extensions of cyclic subgroup separable groups with cyclic associated subgroups to be again cyclic subgroup separable. They applied their results to HNN extensions of abelian and nilpotent groups with cyclic associated subgroups.

We shall show in this chapter that certain HNN extensions of cyclic subgroup separable groups and subgroup separable groups are cyclic subgroup separable. Our results are then applied to HNN extensions of polycyclic-by-finite groups and free-by-finite groups with finitely generated associated subgroups. We also give a characterisation for the one-relator groups $\langle h, t; t^{-1}h^{\gamma}t = h^{\delta}\rangle$ to be cyclic subgroup separable. Finally we show that one-relator groups with non-trivial centre are cyclic subgroup separable.

For ease of exposition, we shall use the term π_c from now on.

2. Preliminaries

We begin with the following lemmas which will be used in the proof of the main theorem.

Lemma 5.1. *[43] Let $G = \langle B, t; t^{-1}Ht = K, \psi\rangle$ be an HNN extension. Let g_1 be a reduced element of G and g_2 be a cyclically reduced element of G. Then $g_1 \in \langle g_2\rangle$ only under the following conditions:*

(a) *If $\|g_2\| = 0$, say $g_2 = b \in B$, then $g_1 = b^k$ for some integer k.*

(b) *Suppose* $\|g_2\| > 0$.

If $g_2 = y_0 t^{\epsilon_1} y_1 t^{\epsilon_2} \dots y_{m-1} t^{\epsilon_m}, m \geq 0$, *and* $g_1 = x_0 t^{e_1} x_1 t^{e_2} \dots x_{n-1} t^{e_n} x_n, n \geq 0$, *are both reduced, then* $n = km$ *for some positive integer* k. *So, if* $g_1 = g_2^k$ *then* $z = \pm k$.

If $g_1 = g_2^k$, *then* $e_j = e_{j+m} = \cdots = e_{j+(k-1)m} = \epsilon_j, j = 1, 2, \dots, m$, *and there exists a finite sequence of elements* $c_1, \dots, c_{km}, d_0, d_1, \dots, d_{km}$ *in* $H \cup K$ *for which* $t^{-e_j} d_{j-1} t^{e_j} = c_j, j = 1, 2, \dots, km$, *and*

$$\begin{aligned} y_0^{-1} x_0 &= d_0 \\ y_1^{-1} c_1 x_1 &= d_1 \\ &\vdots \\ y_j^{-1} c_j x_j &= d_j \\ &\vdots \\ y_{km-1}^{-1} c_{km-1} x_{km-1} &= d_{km-1} \\ c_{km} x_{km} &= d_{km} = 1 \end{aligned}$$

where $y_j = y_{j+m} = \cdots = y_{j+(k-1)m}, j = 0, 1, \dots, m-1$.

If $g_1 = g_2^{-k}$, *then* $e_j = e_{j+m} = \cdots = e_{j+(k-1)m} = -\epsilon_{m-j+1}, j = 1, 2, \dots, m$, *and there exist a finite sequence of elements* $\hat{c}_1, \dots, \hat{c}_{km}, \hat{d}_0, \hat{d}_1, \dots, \hat{d}_{km}$ *in* $H \cup K$ *for which* $t^{-e_j} \hat{d}_{j-1} t^{e_j} = \hat{c}_j, j = 1, 2, \dots, km$, *and*

$$\begin{aligned} y_0^{-1} x_0 &= \hat{d}_0 \\ y_{km-1}^{-1} \hat{c}_1 x_1 &= \hat{d}_1 \\ &\vdots \\ y_{km-j}^{-1} \hat{c}_j x_j &= \hat{d}_j \\ &\vdots \\ y_1^{-1} \hat{c}_{km-1} x_{km-1} &= \hat{d}_{km-1} \\ y_0 \hat{c}_{km} x_{km} &= \hat{d}_{km} = 1 \end{aligned}$$

where again $y_j = y_{j+m} = \cdots = y_{j+(k-1)m}, j = 0, 1, \dots, m-1$.

Let $G = \langle B, t; t^{-1} H t = K, \psi \rangle$ as in Lemma 5.1. Let $g_1 = x_0 t^{e_1} x_1 t^{e_2} \dots t^{e_n} x_n$, and $g_2 = y_0 t^{\epsilon_1} y_1 t^{\epsilon_2} \dots t^{\epsilon_m} y_m$, $n, m \geq 1$, be reduced elements of G. Then g_1 is said to be circumparallel with g_2 if $n = m$ and $e_i = \epsilon_i$ for all i.

Lemma 5.2. *[43] Let $G = \langle B, t; t^{-1}Ht = K, \psi\rangle$ as in Lemma 5.1. Let $g_1 = x_0t^{e_1}x_1t^{e_2}\dots t^{e_{km}}x_{km}, k \geq 1$, be a reduced element of G and $g_2 = y_0t^{\epsilon_1}y_1t^{\epsilon_2}\dots y_{m-1}t^{\epsilon_m}$, $m \geq 1$, be a cyclically reduced element of G. Then $g_1 \notin \langle g_2\rangle$ only if one of the following conditions hold.*

(1) *g_1 is circumparallel to neither g_2^k nor g_2^{-k}*

(2) *g_1 is circumparallel with g_2^k only (g_2^{-k} only) and at least one equation in the set of equations of Lemma 5.1(b) is violated for the case $g_1 = g_2^k$ ($g_1 = g_2^{-k}$)*

(3) *g_1 is circumparallel with both g_2^k and g_2^{-k} and at least one equation in the set of equations of Lemma 5.1(b) is violated for both the cases $g_1 = g_2^k$ and $g_1 = g_2^{-k}$.*

Lemma 5.3. *[43] Let $G = \langle B, t; t^{-1}Ht = K, \psi\rangle$ be an HNN extension where B is finite. Then G is π_c.*

3. A criterion

First we prove a criterion (i.e. Theorem 5.4) for an HNN extension of a π_c group to be again π_c. Then we apply this criterion to certain HNN extensions of subgroup separable groups and to one-relator groups with non-trivial centre.

Theorem 5.4. *Let $G = \langle B, t; t^{-1}Ht = K, \psi\rangle$ be an HNN extension where B is a π_c group. Suppose that*
(a) B is H-separable and K-separable;
(b) for each $M \triangleleft_f B$, there exists $N \triangleleft_f B$ such that $N \subseteq M$ and $\psi(N \cap H) = N \cap K$.
Then G is π_c.

Proof. Let g_1, g_2 be elements of G such that $g_1 \notin \langle g_2\rangle$. WLOG, we may assume that g_1 is reduced and g_2 is cyclically reduced in G. We shall consider the various cases:
Case 1. Suppose $g_1 \notin \langle g_2\rangle$ is implied by the lengths of g_1 and g_2, that is,

(1) if $\|g_2\| = 0$, then $\|g_1\| > 0$ or
(2) if $\|g_2\| = 1$, then $\|g_1\| = 0$ or
(3) if $\|g_2\| = m$, then $\|g_1\|$ is not divisible by m or $\|g_1\| = 0$.

Suppose $g_1 = x_0t^{e_1}x_1t^{e_2}\dots x_{n-1}t^{e_n}x_n, n \geq 0$, and $g_2 = y_0t^{\epsilon_1}y_1t^{\epsilon_2}\dots y_{m-1}t^{\epsilon_m}, m \geq 0$. Let u_i denote those x_i, y_i in $B\backslash H$, v_i denote those x_i, y_i in $B\backslash K$ and w_i denote those x_i, y_i in $H \cap K\backslash\{1\}$. Since B is residually finite and H-separable, K-separable, there exists $M \triangleleft_f B$ such that $u_i \notin HM, v_i \notin KM$ and $w_i \notin M$. By (b), there exists $N \triangleleft_f B$ such that $N \subseteq M$ and $\psi(N \cap H) = N \cap K$. Clearly $u_i \notin HN$, $v_i \notin KN$ and $w_i \notin N$. Now we form $\bar{G} = \langle \bar{B}, t; t^{-1}\bar{H}t = \bar{K}, \bar{\psi}\rangle$ where $\bar{B} = B/N, \bar{H} = HN/N, \bar{K} =$

KN/N and $\bar{\psi}$ is the isomorphism from $\bar{H}$ onto $\bar{K}$ induced by ψ. Clearly $\bar{G}$ is a homomorphic image of G. Let $\bar{g}$ denote the image of any element g in $\bar{G}$. Then $\bar{g}_1$ is reduced, $\bar{g}_2$ is cyclically reduced and $\|\bar{g}_1\| = \|g_1\|$, $\|\bar{g}_2\| = \|g_2\|$ in $\bar{G}$. This implies $\bar{g}_1 \notin \langle \bar{g}_2 \rangle$ in $\bar{G}$. Since $\bar{B}$ is finite, then $\bar{G}$ is π_c by Lemma 5.3. The result now follows.

Case 2. Suppose $\|g_1\| = \|g_2\| = 0$, that is, $g_1, g_2 \in B$. Since B is π_c, there exists $M \lhd_f B$ such that $g_1 \notin \langle g_2 \rangle M$. By (b), we can find $N \lhd_f B$ such that $N \subseteq M$ and $\psi(N \cap H) = N \cap K$. As in case 1, we form $\bar{G} = \langle \bar{B}, t; t^{-1}\bar{H}t = \bar{K}, \bar{\psi} \rangle$. Then $\bar{g}_1 \notin \langle \bar{g}_2 \rangle$ in $\bar{G}$ and we are done.

Case 3. Suppose $\|g_1\| = km$ and $\|g_2\| = m$ where $k \geq 1$ and $m \geq 1$. By Lemma 5.1, if $g_1 = g_2^z$, then $z = \pm k$. So we may assume $g_1 \neq g_2^{\pm k}$. Now $g_1 \neq g_2^{\pm k}$ implies that one of the conditions of Lemma 5.2 occurs.

Suppose g_1 is circumparallel to neither g_2^k nor g_2^{-k}. Then we construct $\bar{G}$ as in case 1. Clearly $\bar{g}_1$ is circumparallel to neither $\bar{g}_2^k$ nor $\bar{g}_2^{-k}$ in $\bar{G}$. We can now proceed as in case 1 to obtain our result.

Suppose now g_1 is circumparallel with g_2^k only. Let $g_1 = x_0 t^{e_1} x_1 t^{e_2} \ldots t^{e_{km}} x_{km}$, and $g_2 = y_0 t^{\epsilon_1} y_1 t^{\epsilon_2} \ldots y_{m-1} t^{\epsilon_m}$. As in case 1, let u_i denote those x_i, y_i in $B \backslash H$, v_i denote those x_i, y_i in $B \backslash K$ and w_i denote those x_i, y_i in $H \cap K \backslash \{1\}$. Since B is residually finite and H-separable, K-separable, there exists $M_1 \lhd_f B$ such that $u_i \notin HM_1, v_i \notin KM_1$ and $w_i \notin M_1$. Since g_1 is circumparallel to g_2^k, at least one equation in the set of equations of Lemma 5.1(b) is violated for the case $g_1 = g_2^k$. Suppose $y_i^{-1} c_i x_i = d_i$ is the first such equation, that is, $d_i, i \neq km$, is not in the appropriate associated subgroup or $d_{km} \neq 1$. Since B is residually finite and H-separable, K-separable, there exists $M_2 \lhd_f B$ such that $d_i \notin HM_2$ (if $d_i \in B \backslash H$) or $d_i \notin KM_2$ (if $d_i \in B \backslash K$) or $d_{km} \notin M_2$. By (b), there exists $N \lhd_f B$ such that $N \subseteq M_1 \cap M_2$ and $\psi(N \cap H) = N \cap K$. Now we form $\bar{G} = \langle \bar{B}, t; t^{-1}\bar{H}t = \bar{K}, \bar{\psi} \rangle$ as before. Then $\bar{g}_1$ is reduced, $\bar{g}_2$ is cyclically reduced and $\|\bar{g}_1\| = \|g_1\|$, $\|\bar{g}_2\| = \|g_2\|$ in $\bar{G}$. Hence $\bar{g}_1 \neq \bar{g}_2^z$ for all integers $z \neq \pm k$. Since $N \subseteq M_2$, we have $\bar{g}_1 \neq \bar{g}_2^k$. Furthermore, $\bar{g}_1$ is not circumparallel with $\bar{g}_2^{-k}$ and hence $\bar{g}_1 \neq \bar{g}_2^{-k}$. This implies that $\bar{g}_1 \notin \langle \bar{g}_2 \rangle$ and we are done as before.

The proof for the remaining cases when g_1 is circumparallel with g_2^{-k} only or with both g_2^k and g_2^{-k} are similar. This completes the proof of the theorem.

4. HNN extensions of subgroup separable groups

Now we use Theorem 5.4 to show that certain HNN extensions of subgroup separable groups are π_c.

Lemma 5.5. *Let B be a subgroup separable group and H, K be finitely generated subgroups of $Z(B)$ such that $H \cap K = 1$. Then for $R \triangleleft_f H, S \triangleleft_f K$, there exists $N \triangleleft_f B$ such that $N \cap H = R$ and $N \cap K = S$.*

Proof. Let $R \triangleleft_f H, S \triangleleft_f K$ be given. Since $R, S \subseteq Z(B)$, we can form $\bar{B} = B/RS$. Then $\bar{B}$ is residually finite since B is subgroup separable and R and S are finitely generated. Furthermore, the subgroups $\bar{H}$ and $\bar{K}$ are finite in $\bar{B}$. Hence there exists $\bar{N} \triangleleft_f \bar{B}$ such that $\bar{N} \cap \bar{H} = \bar{1}$ and $\bar{N} \cap \bar{K} = \bar{1}$. Let N be the preimage of $\bar{N}$ in B. Then $N \triangleleft_f B$ is such that $N \cap H = R$ and $N \cap K = S$.

Theorem 5.6. *Let $G = \langle B, t; t^{-1}Ht = K, \psi \rangle$ be an HNN extension where B is a subgroup separable group. Suppose that H, K are finitely generated subgroups of $Z(B)$ such that $H \cap K = 1$. Then G is π_c.*

Proof. Since B is a subgroup separable group and H, K are finitely generated, B is H-separable and K-separable. This proves Theorem 5.4(a). Let $M \triangleleft_f B$ be given. Now let $R = M \cap H$ and $S = M \cap K$. Then $R \triangleleft_f H$ and $S \triangleleft_f K$. Let $R_1 = R \cap \psi^{-1}(S)$ and $S_1 = S \cap \psi(R)$. Then $R_1 \triangleleft_f H, S_1 \triangleleft_f K$ and $\psi(R_1) = S_1$. By Lemma 5.5, there exists $P \triangleleft_f B$ such that $P \cap H = R_1$ and $P \cap K = S_1$. Let $N = M \cap P$. Then $N \triangleleft_f B$ and $N \cap H = R_1, N \cap K = S_1$. Since $\psi(R_1) = S_1$, it follows that $\psi(N \cap H) = N \cap K$. This proves Theorem 5.4(b). Therefore G is π_c.

Lemma 5.7. *Let B be a subgroup separable group and H be a finitely generated subgroup of B. Let R be a normal subgroup of B and R is of finite index in H. Then there exists $N \triangleleft_f B$ such that $N \cap H = R$.*

Proof. Let $R \triangleleft B$ where R is of finite index in H be given. Since R is normal in B, we can form $\bar{B} = B/R$. Then $\bar{B}$ is residually finite since B is subgroup separable and R is finitely generated. Furthermore, the subgroup $\bar{H}$ is finite in $\bar{B}$. Hence there exists $\bar{N} \triangleleft_f \bar{B}$ such that $\bar{N} \cap \bar{H} = \bar{1}$. Let N be the preimage of $\bar{N}$ in B. Then $N \cap H = R$.

Theorem 5.8. *Let $G = \langle B, t; t^{-1}Ht = K, \psi \rangle$ be an HNN extension where B is a subgroup separable group and $\psi(H \cap K) = H \cap K$. Suppose that H, K are finitely generated normal subgroups of B such that $H \cap K \triangleleft_f H$ and $H \cap K \triangleleft_f K$. Then G is π_c.*

Proof. Since B is a subgroup separable group and H, K are finitely generated, B is H-separable and K-separable. This proves Theorem 5.4(a). Let $M \triangleleft_f B$ be given. Now let $R = M \cap H \cap K$. Then $R \triangleleft_f H \cap K$. Suppose R is of index r in $H \cap K$. Since $H \cap K$ is finitely generated, there exist only a finite number of subgroups of index r in $H \cap K$. Let R_1 be the intersection of all these subgroups of index r in $H \cap K$. Then $R_1 \subset R$ and R_1 is characteristic and of finite index in $H \cap K$. Since $\psi(H \cap K) = H \cap K$, it follows that $\psi(R_1) = R_1$. Since $R_1 \lhd_f H$ and $R_1 \lhd_f K$, then by Lemma 5.7, there exist $P_1 \lhd_f B, P_2 \lhd_f B$ such that $P_1 \cap H = R_1 = P_2 \cap K$. Let $N = M \cap P_1 \cap P_2$. Then $N \lhd_f B$ and $N \cap H = R_1 = N \cap K$. Since $\psi(R_1) = R_1$, it follows that $\psi(N \cap H) = N \cap K$. This proves Theorem 5.4(b). Therefore G is π_c.

An easy consequence of Theorem 5.8 is as follows:

Corollary 5.9. *Let $G = \langle B, t; t^{-1}Ht = H, \psi \rangle$ be an HNN extension where B is a subgroup separable group. Suppose that H is a finitely generated normal subgroup of B. Then G is π_c.*

Proof. Let $H = K$ in Theorem 5.8 and we are done.

It is well known that polycyclic groups and free groups are subgroup separable (Mal'cev [31], Hall [14]) and finite extensions of subgroup separable groups are again subgroup separable (Romanovski [37], Scott [38]). Hence polycyclic-by-finite groups and free-by-finite groups are subgroup separable. Therefore from Theorems 5.6, 5.8 and Corollary 5.9, we have the following:

Corollary 5.10. *Let $G = \langle B, t; t^{-1}Ht = K, \psi \rangle$ be an HNN extension where B is a polycyclic-by-finite group or free-by-finite group. Suppose that H, K are finitely generated subgroups of B and*

(a) *$H, K \subseteq Z(B)$ such that $H \cap K = 1$ or*

(b) *$H \lhd B$, $K \lhd B$ such that $H \cap K \lhd_f H$, $H \cap K \lhd_f K$ and $\psi(H \cap K) = H \cap K$ or*

(c) *$H \lhd B$ and $H = K$.*

Then G is π_c.

Corollary 5.11. *Let $G = \langle B, t; t^{-1}Ht = K, \psi \rangle$ be an HNN extension where B is a finitely generated abelian group. Suppose that*

(a) *$H \cap K = 1$ or $H = K$ or*

(b) *$H \cap K \lhd_f H$, $H \cap K \lhd_f K$ and $\psi(H \cap K) = H \cap K$.*

Then G is π_c.

5. HNN extensions of π_c groups

In this section, we show that certain HNN extensions of π_c groups with identical associated subgroups are again π_c. For completeness we restate Lemma 3.27 and 3.28 as follows:

Lemma 5.12. *Let H be a retract of a group B. If B is residually finite, then B is H-separable.*

Proof. Lemma 3.27.

Lemma 5.13. *Let H be a retract of a group B. Then for each $R \lhd_f H$, there exists $N \lhd_f B$ such that $N \cap H = R$.*

Proof. Lemma 3.28.

Theorem 5.14. *Let $G = \langle B, t; t^{-1}Ht = H, \psi \rangle$ be an HNN extension where B is a π_c group. Suppose that H is a finitely generated subgroup of B and H is a retract of B. Then G is π_c.*

Proof. By Lemma 5.12, B is H-separable. This proves Theorem 5.4(a). Let $M \lhd_f B$ be given. Let $R = M \cap H$. Then $R \lhd_f H$. As in the proof of Theorem 5.8, we can find $R_1 \subset R$ such that R_1 is characteristic and of finite index in H. Since $\psi(H) = H$, it follows that $\psi(R_1) = R_1$. By Lemma 5.13, we can find $P \lhd_f B$ such that $P \cap H = R_1$. Let $N = M \cap P$. Then $N \lhd_f B$ and $N \cap H = R_1$. Since $\psi(R_1) = R_1$, it follows that $\psi(N \cap H) = N \cap H$. This proves Theorem 5.4(b). Therefore G is π_c.

Theorem 5.15. *Let $G = \langle B, t; t^{-1}Ht = H, \psi \rangle$ be an HNN extension where B is a π_c group. Suppose that H is a finitely generated normal subgroup of finite index in B. Then G is π_c.*

Proof. Since $H \lhd_f B$, B is H-separable. This proves Theorem 5.4(a). Let $M \lhd_f B$ be given. Let $R = M \cap H$. Then $R \lhd_f H$. As in the proof of Theorem 5.8, we can find $R_1 \subset R$ such that R_1 is characteristic and of finite index in H. Since $\psi(H) = H$, it follows that $\psi(R_1) = R_1$. Let $N = R_1$. Then $N \lhd_f B$ and $N \cap H = R_1$. Since $\psi(R_1) = R_1$, it follows that $\psi(N \cap H) = N \cap H$. This proves Theorem 5.4(b). Therefore G is π_c.

Apply Theorem 5.14 & 5.15 to polycyclic-by-finite groups and free-by-finite groups, we have the following:

Corollary 5.16. *Let $G = \langle B, t; t^{-1}Ht = H, \psi \rangle$ be an HNN extension where B is a polycyclic-by-finite group or free-by-finite group. Suppose that H is a finitely generated subgroup of B and*

(a) *$H \lhd_f B$ or*
(b) *H is a retract of B.*

Then G is π_c.

Corollary 5.17. *Let $G = \langle B, t; t^{-1}Ht = H, \psi\rangle$ be an HNN extension where B is a finitely generated abelian group. Suppose that H is a subgroup of finite index in B. Then G is π_c.*

Theorem 5.18. *Let $G = \langle B, t; t^{-1}Ht = H, \psi\rangle$ be an HNN extension where B is a π_c group. Suppose that $\psi(a) = a$ for all $a \in H$ (or $\psi(a) = a^{-1}$ for all $a \in H$ if H is abelian). Then G is π_c if and only if B is H-separable.*

Proof. We show the case when $\psi(a) = a^{-1}$ for all $a \in H$ where H is abelian. The other case is similar. Suppose G is π_c but B is not H-separable. Choose $a \in B\backslash H$ such that $a \in RH$ whenever $R \triangleleft_f B$. Then $g = t^{-1}ata \neq 1$ in G since $a \in B\backslash H$. Since G is residually finite, there exists $M \triangleleft_f G$ such that $g \notin M$. Let $N = M \cap B$. Then $N \triangleleft_f B$ and hence $a \in NH$. We form $\bar{G} = \langle \bar{B}, \bar{t}; \bar{t}^{-1}\bar{H}\bar{t} = \bar{H}, \bar{\psi}\rangle$ where $\bar{B} = B/N$, $\bar{H} = HN/N$ and $\bar{\psi}$ is the automorphism of $\bar{H}$ induced by ψ. Clearly $\bar{G}$ is a homomorphic image of G. Since $a \in NH$, we have $\bar{a} = \bar{h}$ for some $\bar{h} \in \bar{H}$. Thus $\bar{g} = \bar{t}^{-1}\bar{h}\bar{t}\bar{h} = \bar{1}$ in $\bar{G}$ and hence $g \in \langle N\rangle^G \leq M$, a contradiction. Therefore B is H-separable.

Conversely, if B is H-separable, then G is π_c by Theorem 5.4.

Next we give a characterisation for the one-relator groups $\langle h, t; t^{-1}h^{\gamma}t = h^{\delta}\rangle$ to be π_c. We shall need the following lemma.

Lemma 5.19. *The group $G = \langle h, t; t^{-1}ht = h^{\delta}\rangle$, $\mid \delta \mid \neq 1$, is not π_c.*

Proof. Clearly $h \notin \langle h^{\delta}\rangle$ in G. Let $\bar{G}$ denote a homomorphic image of G of order n. Then $\bar{h} = \bar{t}^{-n}\bar{h}\bar{t}^{n} = \bar{h}^{\delta^n} \in \langle \bar{h}^{\delta}\rangle$. So G is not π_c.

Corollary 5.20. *The group $G = \langle h, t; t^{-1}h^{\gamma}t = h^{\delta}\rangle$ is π_c if and only if $\gamma = \pm\delta$.*

Proof. First suppose that G is π_c. Since G is residually finite, then by Meskin [33] $\mid \gamma \mid = 1$ or $\mid \delta \mid = 1$ or $\mid \gamma \mid = \mid \delta \mid$. Hence by Lemma 5.19, $\mid \gamma \mid = \mid \delta \mid$.

Conversely, suppose that $\gamma = \pm\delta$. Then G is π_c by Theorem 5.18.

6. One-relator groups with non-trivial centre

Next we apply Theorem 5.4 to show that one-relator groups with non-trivial centre are π_c. First we have the following theorem.

Theorem 5.21. *Let $G = \langle B, t; t^{-1}ht = k\rangle$ be an HNN extension where B is a π_c and weakly potent group and $\langle h\rangle$, $\langle k\rangle$ are infinite cyclic subgroups such that $\langle h\rangle \cap \langle k\rangle \neq 1$. Then G is π_c if and only if $h^{\delta} = k^{\pm\delta}$ for some $\delta > 0$.*

Proof. Suppose $h^\delta = k^{\pm\delta}$ for some $\delta > 0$. Since B is a π_c group and $\langle h \rangle$, $\langle k \rangle$ are cyclic subgroups, B is $\langle h \rangle$-separable and $\langle k \rangle$-separable. This proves Theorem 5.4(a). Let $M \triangleleft_f B$ be given. Suppose $M \cap \langle h \rangle = \langle h^s \rangle$ and $M \cap \langle k \rangle = \langle k^t \rangle$ for some integers s, t. Since B is weakly potent, we can find $N_1 \triangleleft_f B, N_2 \triangleleft_f B$ such that $N_1 \cap \langle h \rangle = \langle h^{r_1 r_2 st\delta} \rangle, N_2 \cap \langle k \rangle = \langle k^{r_1 r_2 st\delta} \rangle$ for some positive integers r_1 and r_2. Let $N = M \cap N_1 \cap N_2$. Then $N \triangleleft_f B$ and $N \cap \langle h \rangle = \langle h^{r_1 r_2 st\delta} \rangle, N \cap \langle k \rangle = \langle k^{r_1 r_2 st\delta} \rangle$. Since $h^\delta = k^{\pm\delta}$, it follows that $\psi(N \cap \langle h \rangle) = N \cap \langle k \rangle$. This proves Theorem 5.4(b). Therefore G is π_c.

Conversely, suppose that G is π_c. Since $\langle h \rangle \cap \langle k \rangle \neq 1$, then $h^\delta = k^\gamma$ for some integers δ, γ. Since G is π_c, the subgroup $\langle h, t; t^{-1} h^\gamma t = h^\delta \rangle$ is π_c. Hence by Corollary 5.20, we have $\gamma = \pm\delta$. The result now follows.

Since polycyclic-by-finite groups and free-by-finite groups are subgroup separable (see above) and weakly potent (Wehrfritz [41], Kim [19]) we obtain, from Theorem 5.21, the following:

Corollary 5.22. *Let $G = \langle B, t; t^{-1} h t = k \rangle$ be an HNN extension where B is a polycyclic-by-finite group or free-by-finite group and $\langle h \rangle$, $\langle k \rangle$ are infinite cyclic subgroups such that $\langle h \rangle \cap \langle k \rangle \neq 1$. Then G is π_c if and only if $h^\delta = k^{\pm\delta}$ for some $\delta > 0$.*

Now we apply Theorem 5.21 to show that one-relator groups with non-trivial centre are π_c.

Theorem 5.23. *Let G be a one-relator group with non-trivial centre. Then G is π_c.*

Proof. First suppose that the abelianisation of G is not free abelian of rank two. Then by Pietrowski [35, Theorem 1], G has a presentation of the form

$$\langle a_1, a_2, \ldots, a_m; a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \ldots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$$

where $m, p_i, q_i \geq 2$ and $(p_i, q_j) = 1$ for $i > j$. Clearly G is a tree product of infinite cyclic groups and hence G is π_c by Corollary 3 of [44].

Now suppose that the abelianisation of G is free abelian of rank two. Again by Pietrowski [35, Theorem 3], G has a presentation of the form

$$\langle t, a_1, a_2, \ldots, a_m; t^{-1} a_1 t = a_m, a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \ldots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$$

where $m, p_i, q_i \geq 2$ and $(p_i, q_j) = 1$ for $i > j$ such that $p_1 p_2 \cdots p_{m-1} = q_1 q_2 \cdots q_{m-1}$. Then $G = \langle t, B; t^{-1} a_1 t = a_m \rangle$ is an HNN extension where $B = \langle a_1, a_2, \cdots, a_m; a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \cdots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$ and $a_1^{\delta} = a_m^{\delta}$ where $\delta = p_1 p_2 \cdots p_{m-1} = q_1 q_2 \cdots q_{m-1}$. Now B is weakly potent by Corollary 3.13 of Chapter 3 and B is π_c by Corollary 3 of [44]. Therefore G is π_c by Theorem 5.21.