CHAPTER 1

RIEMANNIAN MANIFOLD

1.1 Introduction

We shall give some basic notations before we proceed.

Let R be the set of real numbers. If n is a positive integer, let R^n be the standard n-dimensional Euclidean space of ordered n-tuples of real numbers, $R^n = \{(a_1, \ldots, a_n) : a_i \in R,\ i = 1, \ldots, n\}$.

Let u^i, $i = 1, \ldots, n$, be the natural coordinate functions for R^n, $u^i : R^n \to R$, where $u^i(a_1, \ldots, a_n) = a_i$.

Let U be an open set of R^m. Then a function $f: U \to R^n$ is said to be C^∞ if the components are C^∞, that is, it has continuous partial derivatives of all orders and types.

Let M be a topological space, p a point in M. A *coordinate chart* at p is a function $\phi : U \to R^n$, where U is an open set of M containing p and ϕ a homeomorphism onto an open subset of R^n.

The coordinate functions x^i, $i = 1, \ldots, n$ of the coordinate chart are the real valued functions on U, given by the entries of values of ϕ, that is, $x^i = u^i \circ \phi : U \to R$. Thus for each $p \in U$, $\phi(p) = (x^1(p), \ldots, x^n(p))$, so we shall write $\phi = (x^1, \ldots, x^n)$. We shall call ϕ a coordinate map, U the coordinate neighbourhood and $(x^1, \ldots, x^n)$ a coordinate system at p.

An n-dimensional manifold is a paracompact, second countable topological space M such that M is a Hausdorff space and there exists a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$, where (U_α, ϕ_α) is a homeomorphism of the connected open set ${}_{\alpha} \subset M$ onto an open subset of R^n satisfying the following:

(i) $M = \bigcup U_\alpha$,

(ii) for any α, β , if $U_\alpha \cap U_\beta \neq \varnothing$, then the mapping

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a diffeomorphism, and

(iii) the collection of charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$ is maximal with respect to (ii).

If furthermore the mapping $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is C^∞ for any α, β, then M is said to be a C^∞ manifold.

In this thesis, we shall only consider C^∞ manifolds.

Let M, N be two manifolds. If f is a function from M to N, $f: M \to N$ such that for any coordinate charts (U, ϕ), (V, φ), of M, N respectively, with $f(U) \subset V$, $\varphi \circ f \circ \phi^{-1}$ is C^∞, then f is said to be a C^∞ function from M to N.

Let M be an n-dimensional manifold and denote by $C^\infty(m)$ to be the set consisting of all C^∞ real valued functions in some open neighbourhood of m in M. A *tangent* or a *tangent vector* X at $m \in M$ is a function (operator) $X : C^\infty(m) \to \mathrm{R}$ such that for every $f, g \in C^\infty(m)$, $a, b \in \mathrm{R}$,

(i) $X(af + bg) = a(Xf) + b(Xg)$,

(ii) $X(fg) = (Xf)g(m) + f(m)(Xg)$.

The set of all tangents at m will be called the *tangent space* at m and will be denoted by M_m. It is easy to show that the tangent space M_m is an n-dimensonal vector space with the usual operations.

The set of all tangents on M, that is, the union of M_m for all m in M, $\bigcup_{m\in M} M_m$ is denoted by TM and called the *tangent bundle* of M. We will sometimes write M_m as $T_m M$. We denote the dual of M_m or $T_m M$ by M_m^* or $T_m^* M$.

Let (U, ϕ) be a chart at $m \in M$ and $\phi = (x^1, \ldots, x^n)$ a coordinate system about m. We define for each i, a *coordinate vector* at m, denoted $\left(\frac{\partial}{\partial x^i}\right)_m$, by

$$\left(\frac{\partial}{\partial x^i}\right)_m f = \frac{\partial(f \circ \phi^{-1})}{\partial u^i}(\phi(m)),$$

where $x^i = u^i \circ \phi$ and the differentiation on the right side of the equation is as usual on R^n for any $f \in C^\infty(m)$.

A *vector field* X on a set $U \subset M$ is a mapping that assigns to each point $p \in U$ a vector X_p in M_p. A vector field X is said to be C^∞ on an open set U if for every $f \in C^\infty(m)$, where $m \in U$, the function define by $(Xf)(p) = X_p f$ is C^∞ on $U \cap V$, where V is the domain of f. We denote by $\mathfrak{X}(M)$ the set of all C^∞ vector fields on M.

The *Lie Bracket* of two vector fields X, Y, denoted $[X, Y]$, is defined by $[X, Y]f = X(Yf) - Y(Xf)$ for any $f \in C^\infty(M)$. This bracket operation satisfies the following identity which is called the *Jacobi Identity*:

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0,$$

where X, Y, Z are C^∞ vector fields in M.

It is obvious that in a coordinate system $(x^1, \ldots, x^n)$, the Lie Bracket of its coordinate vector fields vanishes, that is, $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]=0$ for all i, j.

We define a curve in M as a mapping of an interval of the real line into M such that there is an extension to an open interval which is a C^∞ map. A curve γ is said to be *closed* if γ is defined on a closed interval $[a, b]$ and for which $\gamma(a)=\gamma(b)$.

If γ is a C^∞ curve in M such that $\gamma(c)=p$, we define $\dot{\gamma}(c) \in M_p$ the tangent to γ at c by requiring for every $f \in C^\infty(p)$, $\dot{\gamma}(c)f=\frac{d(f\circ\gamma)}{dt}(c)$, where t is the parameter along γ.

We would like to have a look at the coordinate expression of a curve γ in a coordinate neighbourhood U with coordinate system $(x^1, \ldots, x^n)$. We write the i-th component of γ as $x^i \circ \gamma = \gamma^i$. Thus

$$\begin{aligned}\gamma(t) &= (x^1\circ\gamma(t), \ldots, x^n\circ\gamma(t)) \\ &= (\gamma^1(t), \ldots, \gamma^n(t)).\end{aligned}$$

As a consequence,

$$\begin{aligned}\dot{\gamma}(t) &= \sum_{i=1}^{n}\frac{d\gamma^i(t)}{dt}\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} \\ &= \sum_{i=1}^{n}\dot{\gamma}^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}.\end{aligned}$$

A curve $\gamma : I \to M$ is called an *integral curve* of $X \in \mathcal{H}(M)$ if $\dot{\gamma} = X \circ \gamma$, that is $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in I$.

We can express the equation $\dot{\gamma} = X \circ \gamma$ in terms of a coordinate system $(x^1, \ldots, x^n)$, this will give us a system of first-order ordinary differential equations. Thus

Proposition 1.1 [Bi, page 122]

A curve γ is an integral curve of a vector field X if and only if for every coordinate chart, the coordinate expressions γ^i of γ and X^i of X satisfy the system of differential equations $\dfrac{d\gamma^i}{dt} = X^i(\gamma(t))$ for $1 \le i \le n$.

As a consequence, we have

Proposition 1.2 [Hi, page 12]

If $X \in \mathcal{H}(M)$, then for each $p \in M$, there is an interval I around 0 and an integral curve $\gamma : I \to M$ of X through p.

1.2 Submanifolds

Let M and N be two m-dimensional and n-dimensional C^∞ manifolds respectively. Let $\varphi: M \to N$ be a C^∞ map from M to N. The *differential* of φ, denoted by $d\varphi$ or φ_*, is defined by $\varphi_*(m) = d\varphi_m : M_m \to N_{\varphi(m)}$ where $(d\varphi_m(X))f = X(f \circ \varphi)$ for any $X \in M_m$, and $f \in C^\infty(N)$. Observe that $d\varphi$ maps TM to TN.

A C^{∞} mapping $\varphi : M \to N$ is said to be an *immersion* if $d\varphi_m : M_m \to N_{\varphi(m)}$ is injective for all m in M. We said that M is immersed in N or that M is an *immersed submanifold* of N. If, in addition, φ is injective, we say that φ is an *imbedding* and M is called an *imbedded submanifold* or just a *submanifold* of N.

1.3 Tensors

Let U and V be finite dimensional vector spaces and $F(U, V)$ be the free vector space over R whose generators are the points of $U \times V$. Hence $F(U, V)$ consists of all finite linear combinations of pairs (u, v) with $u \in U$ and $v \in V$. Let $W(U, V)$ be the subspace of $F(U, V)$ generated by the set of all elements of $F(U, V)$ of the form $(u_1 + u_2, v) - (u_1, v) - (u_2, v)$, $(u, v_1 + v_2) - (u, v_1) - (u, v_2)$, $(ru, v) - r(u, v)$ and $(u, rv) - r(u, v)$, where $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $r \in \mathrm{R}$. The tensor product of U and V is defined as the quotient space $F(U, V) / W(U, V)$ and is denoted by $U \otimes V$.

The *contravariant tensor space of degree r* for a vector space V, is defined as $V \otimes \cdots \otimes V$ (r times tensor product), whereas the *covariant tensor space of degree s* for a vector space V, is defined as $V^* \otimes \cdots \otimes V^*$ (s times tensor product), where V^* is the dual vector space of V. We note that $T_0^1(V) = V$ and $T_1^0(V) = V^*$. The *tensor space of type* (r, s) of a vector space V, $T_s^r(V)$ is defined as $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ (r times tensor product of V and s times tensor product of V^*). It can be shown that

$T_s^r(M) = \bigcup_{p \in M} T_s^r(T_p M)$ is a vector bundle (see page 14 for definition) over M with fibre $\pi^{-1}(p) = T_s^r(T_p M)$. A tensor field of type (r, s) is just a cross section of $T_s^r(M)$.

It is well known that a tensor can be viewed as a multilinear function.

A tensor is symmetric if and only if its value remains the same for all possible permutations of its arguments (thus only T_0^s or T_s^0 tensors can be symmetric). A tensor is skew symmetric if and only if its value after any permutation of its arguments is the product of its value before the permutation and the sign of the permutation.

A skew symmetric covariant tensor of degree s in $T_s^0(M)$ is also called an *s-form*.

As part of the vector space structure, we have that tensors of the same type can be added and multiplied by scalars. Now we shall define the tensor product of tensors of possibly different types. The tensor product of a tensor S of type (r, s) and a tensor T of type (t, u) is a tensor $S \otimes T$ of type $(r+t, s+u)$ defined as a function on $(V^*)^{r+t} \times V^{s+u}$ by

$$(S \otimes T)(w^1, \ldots, w^{r+t}, v_1, \ldots, v_{s+u}) = S(w^1, \ldots, w^r, v_1, \ldots, v_s)\, T(w^{r+1}, \ldots, w^{r+t}, v_{s+1}, \ldots, v_{s+u})$$

The associative and distributive laws of tensor product are true and easily verified:

$$(S \otimes T) \otimes W = S \otimes (T \otimes W)\ ,$$
$$S \otimes (T + W) = S \otimes T + S \otimes W\ ,$$
$$(S + T) \otimes W = S \otimes W + T \otimes W$$

whenever the types of S, T, W are such that these formulas make sense.

Let ω and θ be forms of degree p and q respectively. The *exterior product* $\omega\Lambda\theta$ is defined to be the $(p+q)$ form

$$\omega\Lambda\theta = \frac{1}{(p+q)!}\sum_{\pi}\operatorname{sgn}(\pi)(\omega\otimes\theta)\circ\pi ,$$

where the sum is taken over all permutations π of the set $\{1, 2, \ldots, p+q\}$.

We also have

$$\omega\Lambda\theta = (-1)^{pq}\,\theta\Lambda\omega ,$$
$$\omega\Lambda(\theta+\beta) = \omega\Lambda\theta + \omega\Lambda\beta ,$$
$$(\omega\Lambda\theta)\Lambda\beta = \omega\Lambda(\theta\Lambda\beta) .$$

Let X be a vector field on M. An operator $\mathcal{L}_X$ called the *Lie derivative* via X which maps $T_s^r(M)$ into itself, is defined as follows:

(a) $\mathcal{L}_X f = Xf$ for $f \in C^\infty(M)$,

(b) $\mathcal{L}_X Y = [X, Y]$ for $Y \in T_0^1(M)$,

(c) $(\mathcal{L}_X\omega)Y = X(\omega Y) - \omega([X, Y])$ for $Y \in T_0^1(M),\ \omega \in T_1^0(M)$, and

(d) $(\mathcal{L}_X\theta)(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s)$

$$\begin{aligned}&= \mathcal{L}_X(\theta(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s))\\ &\quad - \theta(\mathcal{L}_X\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s) - \cdots - \theta(\omega_1, \ldots, \mathcal{L}_X\omega_r, Y_1, \ldots, Y_s)\\ &\quad - \theta(\omega_1, \ldots, \omega_r, \mathcal{L}_X Y_1, \ldots, Y_s) - \cdots - \theta(\omega_1, \ldots, \omega_r, Y_1, \ldots, \mathcal{L}_X Y_s)\end{aligned}$$

for $\theta \in T_s^r(M),\ Y_1, \ldots, Y_s \in T_0^1(M),\ \omega_1, \ldots, \omega_r \in T_1^0(M)$.

We list down a few important properties of $\mathcal{L}_X$: [Hi, page 92]

(1) $\mathcal{L}_X$ preserves forms,

(2) $\mathcal{L}_X(S+T)=\mathcal{L}_X S+\mathcal{L}_X T$,

(3) $\mathcal{L}_X(S\otimes P)=(\mathcal{L}_X S)\otimes P+S\otimes(\mathcal{L}_X P)$,

(4) $\mathcal{L}_X(\omega\Lambda\alpha)=(\mathcal{L}_X\omega)\Lambda\alpha+\omega\Lambda(\mathcal{L}_X\alpha)$

for S, T tensors of the same type, P any tensor and ω, α any two forms.

1.4 Affine connections on manifolds

We begin this section with the definition of an affine connection on a manifold M. An *affine connection* on M is a mapping

$$\nabla: \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M), \qquad (X, Y)\to\nabla_X Y$$

which satisfies the following conditions:

(a1) $\nabla_{fX+gY}Z=f\nabla_X Z+g\nabla_Y Z$,

(a2) $\nabla_X(fY+Z)=f\nabla_X Y+(Xf)Y+\nabla_X Z$

for any $f,\ g\in C^\infty(M)$ and $X,\ Y,\ Z\in\mathfrak{X}(M)$. The operator ∇_X is called the *covariant derivative* with respect to X.

We define the covariant derivative of a function f with respect to X by $\nabla_X f=Xf$.

We define here the covariant derivative $\nabla_X K$ with respect to X of a tensor field K of type $(0,\ k)$ or $(1,\ k)$ as follow,

$$(\nabla_X K)(X_1,\ \ldots,\ X_k)=(\nabla_X)(K(X_1,\ \ldots,\ X_k))-\sum_{i=1}^{k}K(X_1,\ \ldots,\ \nabla_X X_i,\ \ldots,\ X_k)$$

for any $X_i\in\mathfrak{X}(M)$, $i=1,\ \ldots,\ k$.

The tensor field K is said to be *parallel* with respect to the affine connection if $\nabla_X K=0$ for any $X\in\mathfrak{X}(M)$.

For any $p \in M$, choose a system of coordinates about p and write $X = \sum_{i=1}^{n} X^i \partial_i$,

$Y = \sum_{j=1}^{n} Y^j \partial_j$, where $\partial_i = \dfrac{\partial}{\partial x^i}$ for $i = 1, \ldots, n$, we have

$$\begin{aligned}\nabla_X Y &= \sum_{i=1}^{n} X^i \nabla_{\partial_i} (\sum_{j=1}^{n} Y^j \partial_j) \\ &= \sum_{i,j=1}^{n} X^i (\partial_i Y^j) \partial_j + \sum_{i,j=1}^{n} X^i Y^j \nabla_{\partial_i} \partial_j .\end{aligned}$$

Setting $\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k$, we see that the set of functions $\{\Gamma_{ij}^k\}$ called the Christoffel symbols, satisfying the following property (*), will completely determine the connection.

If $(z^1, z^2, \ldots, z^n)$ is another coordinate system at p, we obtain another set of functions $\{\hat{\Gamma}_{ij}^k\}$:

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \sum_{k=1}^{n} \hat{\Gamma}_{ij}^k \frac{\partial}{\partial z^k} ,$$

by using the axioms (a1), (a2) of connections, we find that

$$\hat{\Gamma}_{ij}^k = \sum_{a,b,c=1}^{n} \frac{\partial x^a}{\partial z^i} \frac{\partial x^b}{\partial z^j} \frac{\partial z^k}{\partial x^c} \Gamma_{ab}^c + \sum_{a=1}^{n} \frac{\partial^2 x^a}{\partial z^i \partial z^j} \frac{\partial z^k}{\partial x^a} . \qquad (*)$$

Conversely, suppose that for any coordinate system about p, there is a set of functions $\{\Gamma_{ij}^k\}$ such that (*) holds. Then we can define an affine connection on M by using the equation $\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k$.

We define the *torsion tensor* T of type (1, 2) as $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ for any $X, Y \in \mathfrak{X}(M)$. An affine connection ∇ with vanishing torsion tensor field is called a *torsion-free connection.*

We define the *curvature tensor* R of type (1, 3) as $R(X, Y)Z = \nabla_X Z - \nabla_Y Z - \nabla_{[X, Y]} Z$ for any $X, Y, Z \in \mathfrak{X}(M)$.

A vector field Y along a curve γ is said to be parallel along γ if and only if $\nabla_{\dot\gamma} Y = 0$.

The curve is said to be a *geodesic* if and only if $\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0$. Thus a curve is a geodesic if and only if its tangent vector field is parallel along the curve.

<u>Theorem 1.3 [Hi, page 57]</u>

Let $\gamma : [a, b] \to M$ be a curve. For each vector Y in $M_{\gamma(a)}$ there is a unique C^∞ field $Y(t)$ on γ such that $Y(a) = Y$ and the field $Y(t)$ is parallel along γ. The mapping $P_{a,t}: M_{\gamma(a)} \to M_{\gamma(t)}$ defined by $P_{a,t}(Y) = Y(t)$ is a linear isomorphism which is called *parallel translation* along γ from $\gamma(a)$ to $\gamma(t)$.

<u>Theorem 1.4 [Hi, page 58]</u>

Let $m \in M$, $X \in M_m$. Then for any real number b, there exists a real number $r > 0$ and a unique curve γ, defined on $(b - r, b + r)$ such that $\gamma(b) = m$, $\dot\gamma(b) = X$ and γ a geodesic.

1.5 Riemannian manifolds

A Riemannian metric on M is a tensor field g of type $(0, 2)$ which satisfies the following:

(i) $g(X, Y) = g(Y, X)$ for any $X, Y \in M_p$, $p \in M$, that is, g is symmetric,

(ii) $g(X,X) \geq 0$ for any $X \in M_p$ and $g(X,X) = 0$ if and only if $X = 0$, that is, g is positive definite.

The manifold M endowed with a Riemannian metric g is called a *Riemannian manifold.* The length of a vector X is denoted by $\|X\|$ and is defined by $\|X\|^2 = g(X,X)$. A tensor field g of type $(0, 2)$ which satisfies (i) above and (ii)$'$ below is called a pseudo-Riemannian metric on M:

(ii)$'$ $g(X, Y) = 0$ for all X implies $Y = 0$.

The following is a well known theorem which can be found in [Ya2, page 29].

Theorem 1.5

There exists one and only one affine connection on a Riemannian manifold that satisfies the following conditions:

(i) the torsion tensor T vanishes, that is,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

(ii) g is parallel, that is, $\nabla_X g = 0$. Therefore, we have

$$X(g(Y,Z)) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z)$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

The affine connection stated in the theorem above is called the *Riemannian connection* or the *Levi-Civita connection*. It is characterized by

$$2g(\nabla_X Y,Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)$$

for any $X, Y, Z \in \mathfrak{X}(M)$ [Ko, page 160].

Next, we shall define the *Riemannian curvature tensor R* of type $(0,4)$ by

$$R(X,Y,U,V) = g(R(X,Y)U,V)$$

for any $X, Y, U, V \in \mathfrak{X}(M)$.

The *Ricci tensor field* is defined by

$$S(X,Y) = \sum_{i=1}^{n} g(R(E_i,X)Y,E_i),$$

where $(E_1, \ldots, E_n)$ is a local field of orthonormal frames and $X, Y \in \mathfrak{X}(M)$.

1.6 Distributions on manifolds

An *r-dimensional distribution* on a manifold M is a mapping D defined on M which assigns to each point p of M an r-dimensional linear subspace D_p of T_pM. A vector field X is said to *belong to D* if we have $X_p \in D_p$ for every $p \in M$. We denote this by writing $X \in \Gamma(D)$. The distribution D is said to be *differentiable* if for any $p \in M$, there exists r

differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of p. In this dissertation, we only consider differentiable distributions of class C^∞.

A submanifold N of M is said to be an *integral manifold* of D if for every $p \in N$, $f_*(T_p N) = D_{f(p)}$, where f is the imbedding of N into M. The distribution is said to be *integrable* if for every $q \in M$, there exists an integral manifold of D through q. If there exists no integral manifold of D that properly contains N, then N is called the *maximal integral manifold* or *leaf* of D.

The distribution D is said to be *involutive* if for all $X, Y \in \Gamma(D)$, we have $[X, Y] \in \Gamma(D)$. The following is the classical Frobenius theorem [Be, page 8].

Theorem 1.6

Let D be an involutive distribution on a manifold M. Then D is integrable and through every point $p \in M$, there passes a unique maximal integral manifold of D. Any integral manifold through p is an open submanifold of the maximal one.

1.7 Vector bundles

Let E and M be any arbitrary manifolds and π be a differentiable mapping of E onto M. The manifold E is called the *vector bundle* over M under the projection π if the following conditions are satisfied:

(i) $\pi^{-1}(p)$ is a real vector space called the *fibre* above p and each $\pi^{-1}(p)$ is isomorphic to R^k, for some fixed k, and

(ii) for each $p \in M$, there exist an open neighborhood U of p such that the mapping $\phi : \mathrm{U} \times \mathrm{R}^k \to \pi^{-1}(U)$ is a diffeomorphism satisfying the commutative diagram below:

$$\begin{array}{ccc} \mathrm{U} \times \mathrm{R}^k & \xrightarrow{\phi} & \pi^{-1}(U) \\ & \pi_0 \searrow & \downarrow \pi \\ & & U \end{array}$$

where $\pi \circ \phi = \pi_0$.

A *cross section* is a differentiable mapping $\psi : M \to E$ such that $\pi(\psi(p)) = p$ for all $p \in M$ .The set of all cross sections is denoted by $\Gamma(E)$. We note that

(i) for $\psi_1, \psi_2 \in \Gamma(E)$ and $p \in M$,

$$(\psi_1 + \psi_2)(p) = \psi_1(p) + \psi_2(p),$$

(ii) for $\psi \in \Gamma(E)$ and a function $f \in C^\infty(M)$,

$$(f\psi)(p) = f(p)\psi(p).$$

Therefore, $\Gamma(E)$ forms a module over the ring $C^\infty(M)$. We note that in the case $E = TM$, a cross section X defines a vector field on M.