CHAPTER 2

TANGENT BUNDLES

2.1 Tangent bundles

Let M be a C^∞ n-manifold and T_pM be the tangent space of M at the point $p \in M$. The tangent bundle TM of M is defined to be union of all tangent spaces over M, that is,

$$TM = \bigcup_{p \in M} T_pM.$$

The projection mapping $\pi : TM \to M$ is defined by $\pi(X) = p$ for all $X \in T_pM$. We can always write a point in TM as (p, X), where p is in M and X is a tangent at p. As a result, $\pi(p, X) = p$ for any $X \in T_pM$.

If (ϕ, U) is a coordinate chart on M, we have $\phi : U \to \mathrm{R}^n$, with $x^i = u^i \circ \phi$. Let $\overline{U} = \pi^{-1}(U)$. For (p, X) in $\overline{U}$, $X = \sum_{i=1}^{n} a_i \left(\frac{\partial}{\partial x^i}\right)_p$, let $\overline{x}^i = x^i \circ \pi$ and $\overline{y}^i(p, X) = a_i$.

Define $\overline{\phi} : \overline{U} \to \mathrm{R}^{2n}$ so that $u^i \circ \overline{\phi} = \overline{x}^i$, $u^{n+i} \circ \overline{\phi} = \overline{y}^i$ for $i = 1, \ldots, n$. Then $(\overline{\phi}, \overline{U})$ is a $2n$ coordinate chart on TM with coordinate system $(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)$.

Now we would like to see whether these coordinate charts are C^∞-related to each other. Consider $(\overline{\phi}, \overline{U})$ and $(\overline{\varphi}, \overline{V})$. Let $\phi = (\phi_1, \ldots, \phi_n)$ and $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then

$$\begin{aligned}\overline{\phi} \circ \overline{\varphi}^{-1} &= (\overline{\phi}_1 \circ \overline{\varphi}^{-1}, \ldots, \overline{\phi}_{2n} \circ \overline{\varphi}^{-1}) \\ &= (\phi_1 \circ \pi \circ \overline{\varphi}^{-1}, \ldots, \phi_n \circ \pi \circ \overline{\varphi}^{-1}, \overline{\phi}_{n+1} \circ \varphi^{-1}, \ldots, \overline{\phi}_{2n} \circ \varphi^{-1}).\end{aligned}$$

Now for $i = 1, \ldots, n$,

$$\overline{\phi}_i \circ \overline{\varphi}^{-1}(p_1, \ldots, p_n, a_1, \ldots, a_n) = \phi_i \circ \pi(p, X)$$
$$= \phi_i(p) = \phi_i \circ \varphi^{-1} \circ \overline{P}_n(p_1, \ldots, p_n, a_1, \ldots, a_n),$$

where $\overline{\varphi}(p, X) = (p_1, \ldots, p_n, a_1, \ldots, a_n)$ and $\overline{P}_n(p_1, \ldots, p_n, a_1, \ldots, a_n) = (p_1, \ldots, p_n)$ is the projection function from R^{2n} to R^n. This shows that $\overline{\phi}_i \circ \overline{\varphi}^{-1}$ is C^∞ being a composition of C^∞ functions.

Next, if $X = \sum_{i=1}^{n} t_i \frac{\partial}{\partial \varphi_i} = \sum_{i,j=1}^{n} t_i \frac{\partial \phi_j}{\partial \varphi_i} \frac{\partial}{\partial \phi_j}$,

$$\overline{\phi}_{n+j} \circ \overline{\varphi}^{-1}(p_1, \ldots, p_n, a_1, \ldots, a_n) = \overline{\phi}_{n+j}(p, X)$$
$$= \sum_{i=1}^{n} t_i \frac{\partial \phi_j}{\partial \varphi_i}(p)$$
$$= \sum_{i=1}^{n} t_i \frac{\partial \phi_j}{\partial \varphi_i} \circ \varphi^{-1}(\varphi(p))$$
$$= \sum_{i=1}^{n} t_i \frac{\partial \phi_j}{\partial \varphi_i} \circ \varphi^{-1}(p_1, \ldots, p_n)$$
$$= \sum_{i=1}^{n} t_i \frac{\partial \phi_j}{\partial \varphi_i} \circ \varphi^{-1} \circ \overline{P}_n(p_1, \ldots, p_n, a_1, \ldots, a_n).$$

This shows that $\overline{\phi}_{n+j} \circ \overline{\varphi}^{-1}$ is C^∞. Hence the collection of all $(\overline{\phi}, \overline{U})$ form an atlas for *TM*.

The natural projection $\pi : TM \to M$ defines the natural bundle structure of *TM* over *M*, that is, for any chart $(\overline{\phi}, \overline{U})$, we can obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{U} \times \mathrm{R}^n & \xrightarrow{\tilde{\phi}} & TM|_U \\ & \pi_0 \searrow & \downarrow \pi \\ & & U \end{array}$$

where $\tilde{\phi} = \overline{\phi}^{-1} \circ (\phi, I)$, I the identity map from R^n to R^n, thus *TM* is a vector bundle over *M*.

We will denote by $\Im_s^r(M)$ the set of all tensor fields of type (r, s) on M and $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M)$. Obviously $\Im_0^0(M) = C^\infty(M)$ and $\Im_0^1(M) = \mathcal{X}(M)$.

2.2 Vertical lifts of functions

We define the vertical lift of a C^∞ function f in M as the composition of f with the projection function π, $f^V = f \circ \pi : TM \to R$, which is also C^∞ on TM.

$$\begin{array}{ccc} TM & & \\ \downarrow \pi & \searrow f^V & \\ M & \xrightarrow{\ f\ } & R \end{array}$$

If there is no confusion, we will write f^V as f.

Let $(x^1, \ldots, x^n)$ be the coordinates of the open set U in M and $(\bar{x}^1, \ldots, \bar{x}^n, \bar{x}^{\bar{1}}, \ldots, \bar{x}^{\bar{n}})$ or $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y}^1, \ldots, \bar{y}^n)$ the coordinates of $\pi^{-1}(U)$ in TM. The function f^V is constant on each fibre $T_p(M)$ and $f^V\left(T_p(M)\right) = \{f(p)\}$. We also have $(gf)^V = (g)^V (f)^V$.

A 1-form ω in M is regarded naturally as a function in TM, we denote it by $\iota\omega$. If we express ω locally as $\omega = \sum_{i=1}^{n} \omega_i dx^i$, then $\iota\omega = \sum_{i=1}^{n} \omega_i \bar{y}^i$ with respect to the induced coordinates $(\bar{x}^i, \bar{y}^i)$ in $\pi^{-1}(U)$. Thus for $f \in C^\infty(M)$, $\iota(df) = \sum_{i=1}^{n} (\partial_i f)\bar{y}^i$. A vector field $\bar{X} \in T_0^1(TM)$ is completely determined by its action on functions f in $C^\infty(TM)$. We can actually obtain the following proposition:

Proposition 2.1 [Ya1, page 5]

A vector field $\overline{X}$ in TM is completely determined by its action on $\iota(df)$, $f \in C^{\infty}(M)$. In other words, if $\overline{Y}$ is a vector field in TM such that $\overline{X}\iota df = \overline{Y}\iota df$ for all $f \in C^{\infty}(M)$, then $\overline{X} = \overline{Y}$.

Proof: It suffices to show that if $\overline{X}\iota df = 0$ for all $f \in C^{\infty}(M)$, then $\overline{X} = 0$. Let $\overline{X} = \sum_{A} \overline{X}^{A} \frac{\partial}{\partial \overline{x}^{A}} = \sum_{A} \overline{X}^{A} \partial_{A}$ with respect to the induced coordinate, where A takes value from 1, ..., n, $\overline{1}$, ..., $\overline{n}$. Then from $\overline{X}\iota df = 0$, $\sum_{A} \overline{X}^{A} \partial_{A}\left(\sum_{i=1}^{n} (\partial_{i} f)\overline{y}^{i}\right) = 0$, which implies that $\sum_{i,j=1}^{n}\left(\overline{X}^{j}\overline{y}^{i}\partial_{j}\partial_{i} f + \overline{X}^{\overline{j}}\delta_{ij}\partial_{i} f\right) = 0$ or $\sum_{i=1}^{n}\left(\sum_{j=1}^{n} \overline{X}^{j}\overline{y}^{i}\partial_{j}\partial_{i} f + \overline{X}^{\overline{i}}\partial_{i} f\right) = 0$. Since the equation holds for every $f \in C^{\infty}(M)$, we have $\overline{X}^{\overline{i}} = 0$ and $\overline{X}^{i}\overline{y}^{i} + \overline{X}^{i}\overline{y}^{i} = 0$ or $\overline{X}^{i}\overline{y}^{i} = 0$ for any i. Hence for any point $(\overline{x}^{1}, ..., \overline{x}^{n}, \overline{y}^{1}, ..., \overline{y}^{n})$, where $\overline{y}^{i} \neq 0$, $\overline{X}^{i}(\overline{x}^{1}, ..., \overline{x}^{n}, \overline{y}^{1}, ..., \overline{y}^{n}) = 0$. By continuity argument, $\overline{X}^{i} = 0$.

2.3 Vertical lifts of vector fields

Let $\bar{X} \in \Im_0^1(TM)$. Then $\bar{X}$ is called a *vertical vector field* if $\bar{X}f^V = 0$ for all $f \in C^\infty(M)$. If $\bar{X} = \begin{pmatrix} \bar{X}^h \\ \bar{X}^{\bar{h}} \end{pmatrix}$, then from $\bar{X}f^V = 0$, we have $\sum_{h=1}^{n} \bar{X}^h \partial_h f = 0$; which implies that $\bar{X}^h = 0$. So

$$\begin{pmatrix} \bar{X}^h \\ \bar{X}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{X}^{\bar{h}} \end{pmatrix}. \qquad (*)$$

Thus $\bar{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy (*).

The *vertical lift* of a vector field X in M is defined to be a vector field in TM, written as X^V, which satisfies $X^V(\iota\omega) = (\omega X)^V$ for all $\omega \in \Im_0^1(M)$. Let X^i and ω_i be the local components of X and ω in U. Let $\bar{X}^V = \begin{pmatrix} \bar{X}^h \\ \bar{X}^{\bar{h}} \end{pmatrix}$. Then from $X^V(\iota\omega) = (\omega X)^V$, we have

$$\sum_{i,h=1}^{n} (\partial_h \omega_i) \bar{X}^h y^i + \sum_{i=1}^{n} \omega_i \bar{X}^{\bar{i}} = \sum_{i=1}^{n} \omega_i X^i,$$

$$\sum_{i,h=1}^{n} (\partial_h \omega_i)(\bar{X}^h y^i) + \sum_{i=1}^{n} \omega_i (\bar{X}^{\bar{i}} - X^i) = 0$$

for arbitrary ω_i; which will give us $\bar{X}^{\bar{i}} = X^i$, and also $\bar{X}^i y^i = 0$ which implies that $\bar{X}^i = 0$ for $i = 1, \ldots, n$. Hence the vertical lift of a vector field with components $X^1, \ldots, X^n$ has components given by $X^V = \begin{pmatrix} \bar{X}^h \\ \bar{X}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ X^h \end{pmatrix}$.

2.4 Vertical lifts of 1-forms

Let $\overline{\omega}$ be a 1-form in *TM* with component $\overline{\omega}_A$ with respect to the coordinate in the open set $\pi^{-1}(U)$ in *TM*, where A takes value from $1, \ldots, n, \overline{1}, \ldots, \overline{n}$. We write $\omega = \sum_A \overline{\omega}_A d\overline{x}^A$. We say that $\overline{\omega}$ is a *vertical 1-form* if $\overline{\omega}(X^V) = 0$ for all $X \in \Im_0^1(M)$.

From $\overline{\omega}(X^V) = 0$, we obtain

$$\sum_A \overline{\omega}_A d\overline{x}^A \left(\sum_{i=1}^n X^i \partial_{\overline{i}} \right) = 0,$$

which implies that $\sum_{i=1}^n \overline{\omega}_{\overline{i}} X^i = 0$ for any $X^i \in \Im_0^0(M)$. It follows that $\overline{\omega}_{\overline{i}} = 0$ for $i = 1, \ldots, n$. Thus $\overline{\omega}$ have components $\overline{\omega} = (\overline{\omega}_i, 0)$.

Let $f, g \in \Im_0^0(M)$. We define the vertical lift $(df)^V$ and $(gdf)^V$ of 1-forms *df* and *gdf* respectively by $(df)^V = d(f^V)$ and $(gdf)^V = g^V(df)^V = g^V d(f^V)$. Suppose that $\omega \in \Im_0^1(M)$. We define the vertical lift ω^V of ω by $\omega^V = \sum_{i=1}^n \omega_i (dx^i)^V$, where $\omega = \sum_{i=1}^n \omega_i dx^i$. Obviously ω^V is a vertical 1-form and $\omega^V X^V = 0$.

We summarize some of the important properties of vertical lifts:

Proposition 2.2 [Ya1, page 7,9]

If $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$, $f \in \Im_0^0(M)$, then

(i) $X^V f^V = 0$,

(ii) $(X+Y)^V = X^V + Y^V$,

(iii) $(fX)^V = f^V X^V$,

(iv) $[X^V, Y^V] = 0$,

(v) $\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}$,

(vi) $(\omega + \theta)^V = \omega^V + \theta^V$,

(vii) $(f\omega)^V = f^V \omega^V$,

(viii) $(dx^h)^V = d\bar{x}^h$.

2.5 Vertical lifts of tensor fields

Having define the vertical lifts of vector fields and 1-forms, we now extend the definition of vertical lifts to arbitrary types of tensor fields in TM by requiring

$$(P \otimes Q)^V = P^V \otimes Q^V \text{ and } (P + R)^V = P^V + R^V$$

where Q is any tensor field and P, R any two tensor fields of the same type.

In the following, we consider the vertical lifts of some types of tensor fields.

Tensor of type (1, 1)

Let $F = \sum_{i,j=1}^{n} F_j^i \partial_i \otimes dx^j$, $F^V = \sum_{i,j=1}^{n} F_j^i (\partial_i)^V \otimes (dx^j)^V = \sum_{i,j=1}^{n} F_j^i \partial_{\bar{i}} \otimes d\bar{x}^j$. In terms of matrix, $F^V = \begin{pmatrix} 0 & 0 \\ F_j^i & 0 \end{pmatrix}$.

Tensor of type $(0, 2)$

Let $G = \sum_{i,j=1}^{n} G_{ji} dx^j \otimes dx^i$, $\quad G^V = \sum_{i,j=1}^{n} G_{ji} (dx^j)^V \otimes (dx^i)^V = \sum_{i,j=1}^{n} G_{ji} d\overline{x}^j \otimes d\overline{x}^i$,

$$G^V = \begin{pmatrix} G_{ji} & 0 \\ 0 & 0 \end{pmatrix}.$$

Tensor of type $(2, 0)$

$$H = \sum_{i,j=1}^{n} H^{ji} \partial_j \otimes \partial_i \,,\; H^V = \sum_{i,j=1}^{n} H^{ji} \partial_{\overline{j}} \otimes \partial_{\overline{i}} \,,\; H^V = \begin{pmatrix} 0 & 0 \\ 0 & H^{ji} \end{pmatrix}.$$

Tensor of type $(0, s)$

$$S = \sum_{i_s,\ldots,i_1=1}^{n} S_{i_s \cdots i_1} dx^{i_s} \otimes \cdots \otimes dx^{i_1},$$

$$S^V = \sum_{i_s,\ldots,i_1=1}^{n} S_{i_s \cdots i_1} (dx^{i_s})^V \otimes \cdots \otimes (dx^{i_1})^V = \sum_{i_s,\ldots,i_1=1}^{n} S_{i_s \cdots i_1} d\overline{x}^{i_s} \otimes \cdots \otimes d\overline{x}^{i_1}.$$

Tensor of type $(1, s)$

$$T = \sum_{j,i_s,\ldots,i_1=1}^{n} T^j_{i_s \cdots i_1} \partial_j \otimes dx^{i_s} \otimes \cdots \otimes dx^{i_1},$$

$$T^V = \sum_{j,i_s,\ldots,i_1=1}^{n} T^j_{i_s \cdots i_1} (\partial_j)^V \otimes (dx^{i_s})^V \otimes \cdots \otimes (dx^{i_1})^V = \sum_{j,i_s,\ldots,i_1=1}^{n} T^j_{i_s \cdots i_1} \partial_{\overline{j}} \otimes d\overline{x}^{i_s} \otimes \cdots \otimes d\overline{x}^{i_1}.$$

For $X, Y \in \Im_0^1(M)$, with the above expressions of X^V, Y^V, we have [Ya1, page 11]

(i) $F^V X^V = 0$ for $F \in \Im_1^1(M)$,

(ii) $G^V(X^V, Y^V) = 0$ for $G \in \Im_2^0(M)$.

From $\omega^V = (\omega^i, 0)$, where $\omega \in \Im_1^0(M)$, the differential of ω^V, $d(\omega^V)$ which is in $\Im_2^0(TM)$, has the following form:

$$\begin{aligned} d(\omega^V) &= \sum_{i=1}^{n} d\omega_i \wedge d\bar{x}^i \\ &= \sum_{A}\sum_{i=1}^{n} (\partial_A \omega_i) d\bar{x}^A \wedge d\bar{x}^i \\ &= \sum_{i,j=1}^{n} (\partial_j \omega_i) d\bar{x}^j \wedge d\bar{x}^i \\ &= \tfrac{1}{2}\sum_{i,j=1}^{n} (\partial_j \omega_i - \partial_i \omega_j) d\bar{x}^j \otimes d\bar{x}^i, \end{aligned}$$

$$\begin{aligned} (d\omega)^V &= \left(\sum_{i,j=1}^{n} \partial_j \omega_i dx^j \wedge dx^i \right)^V \\ &= \tfrac{1}{2}\sum_{i,j=1}^{n} (\partial_j \omega_i - \partial_i \omega_j) d\bar{x}^j \otimes d\bar{x}^i \\ &= d(\omega^V). \end{aligned}$$

This can be extended to any differential form $\theta \in \Im_s^0(M)$, namely $(d\theta)^V = d(\theta^V)$. The following is a simple observation.

Proposition 2.3 [Ya1, page 12]

For any differential forms ω and θ in *M*, $(\omega \wedge \theta)^V = \omega^V \wedge \theta^V$.

Now, we define an action, which is similar to contraction of tensor fields on M, on tensor fields on TM which will be used later. In the following, we will sometimes omit the ranges of the summations where they are too laborious to write, we will only use the symbol $\sum$ to indicate it, the summation for $i, j, k, l, m, n, p, q, s, t$ etc are from 1 to n, whereas $\alpha, \beta, \ldots, A, B, \ldots$ are from $1, \ldots; n, \bar{1}, \ldots, \bar{n}$.

Let $S \in \Im_{s+1}^{t}(M)$, write

$$S = \sum S_{l i_s \ldots i_1}^{j_t \ldots j_1} \frac{\partial}{\partial x^{j_t}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_1}} \otimes dx^l \otimes dx^{i_s} \otimes \cdots \otimes dx^{i_1} .$$

Suppose that X is a vector field. We define two tensor fields of type (t, s) in $\pi^{-1}(U)$, $\gamma_X S$ and γS by

$$\gamma_X S = \sum X^l S_{l i_s \ldots i_1}^{j_t \ldots j_1} \frac{\partial}{\partial \bar{y}^{j_t}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{j_1}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1} \text{ and}$$

$$\gamma S = \sum \bar{y}^l S_{l i_s \ldots i_1}^{j_t \ldots j_1} \frac{\partial}{\partial \bar{y}^{j_t}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{j_1}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

with respect to the induced coordinates in $\pi^{-1}(U)$. For $S \in \Im_0^0(M)$ we define $\gamma_X S = \gamma S = 0$.

For any $S \in \Im_{s+1}^{t}(M)$, if we define S_X by $S_X(X_s, \ldots, X_1) = S(X, X_s, \ldots, X_1)$ for any $X_s, \ldots, X_1 \in \Im_0^1(M)$, then $\gamma_X S = (S_X)^{V}$. In addition, we have for any $f \in \Im_0^0(M)$, $X, Y \in \Im_0^1(M)$, $F, G \in \Im_1^1(M)$, ω, $df \circ F \in \Im_1^0(M)$ [Ya1, page 13],

(1) $(\gamma_X F) f^V = 0, (\gamma F) f^V = 0$,

(2) $(\gamma_X F)(\iota df) = \gamma_X (df \circ F)$, $\gamma F(\iota df) = \gamma(df \circ F)$, where $(df \circ F)X = F(df, X)$,

(3) $\left[X^V, \gamma_Y F\right] = 0, \left[X^V, \gamma F\right] = \gamma_X F$,

(4) $\left[\gamma_X F, \gamma_Y G\right] = 0$,

(5) $\left[\gamma_X F, \gamma G\right] = \gamma_X (GF)$, where $GF = \sum_{i,j,k=1}^{n} G_j^i F_k^j \frac{\partial}{\partial x^i} \otimes dx^k$ and the local expression of G, F are $G = \sum_{i,j=1}^{n} G_j^i \frac{\partial}{\partial x^i} \otimes dx^j$, $F = \sum_{h,k=1}^{n} F_k^h \frac{\partial}{\partial x^h} \otimes dx^k$ respectively,

(6) $\left[\gamma F, \gamma G\right] = \gamma(GF - FG) = -\gamma\left[F, G\right]$,

(7) $\omega^V(\gamma_X F) = 0,\ \omega^V(\gamma F) = 0$.

2.6 Complete lifts of functions and vector fields

We work in the same way for complete lifts, that is, we define the liftings of functions, vector fields, 1-forms, and tensor fields.

Let f be a function in M. The complete lift of f, written as f^C, is defined to be $f^C = \iota(df) = \sum_{i=1}^{n} y^i \partial_i f$. We shall write ∂f for $\sum_{i=1}^{n} y^i \partial_i f$. We have $X^V f^C = (Xf)^V$ and $(gf)^C = g^C f^V + g^V f^C$. Thus, Proposition 2.1 can be written as

<u>Proposition 2.4 [Ya1, page 14]</u>

Let $\bar{X}, \bar{Y} \in \Im_0^1(TM)$. If $\bar{X} f^C = \bar{Y} f^C$ for any $f \in \Im_0^0(M)$, then $\bar{X} = \bar{Y}$.

Let $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$ be a vector field in *M*. We define X^C, the complete lift of *X* by

$X^C f^C = (Xf)^C$.

Now, we look at the components of X^C, $X^C = \begin{pmatrix} \overline{X}^h \\ \overline{X}^{\bar{h}} \end{pmatrix}$ or

$X^C = \sum_{h=1}^{n} \left(\overline{X}^h \frac{\partial}{\partial \overline{x}^h} + \overline{X}^{\bar{h}} \frac{\partial}{\partial \overline{y}^h} \right)$ with respect to the induced coordinates.

From the definition, $X^C f^C = (Xf)^C$, we have

$$\sum_{h,i=1}^{n} \overline{X}^h \overline{y}^i \frac{\partial}{\partial \overline{x}^h} (\partial_i f) + \sum_{h=1}^{n} \overline{X}^{\bar{h}} \partial_h f = \sum_{h,i=1}^{n} \overline{y}^i \partial_i (X^h \partial_h f)$$

$$= \sum_{h,i=1}^{n} \left(\overline{y}^i (\partial_i X^h) \partial_h f + \overline{y}^i X^h \partial_i \partial_h f \right)$$

for all f in $C^\infty(M)$ from which $\overline{X}^{\bar{h}} = \partial X^h$ and $\overline{X}^h = X^h$. Thus

$X^C = \sum_{h=1}^{n} \left(X^h \frac{\partial}{\partial \overline{x}^h} + \partial X^h \frac{\partial}{\partial \overline{y}^h} \right)$ or $X^C = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$.

We have a few properties of complete liftings of vector fields [Ya1, page 16]

(i) $(X+Y)^C = X^C + Y^C$,

(ii) $(fX)^C = f^C X^V + f^V X^C$,

(iii) $X^C f^V = X^V f^C = (Xf)^V$,

(iv) $\omega^V X^C = (\omega X)^V$,

(v) $X^C f^C = (Xf)^C$

for any $f \in \Im_0^0(M)$, $X, Y \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$.

Proposition 2.5 [Ya1, page 16]

For any $X, Y \in \Im_0^1(M)$,

(i) $[X^V, Y^V] = 0$,

(ii) $[X^V, Y^C] = [X, Y]^V$,

(iii) $[X^C, Y^C] = [X, Y]^C$.

Proof: We shall omit the proof of (i) since it can be easily verified.

Let $f \in \Im_0^0(M)$. Then

$$\begin{aligned}[X^V, Y^C]f^C &= X^V(Y^C f^C) - Y^C(X^V f^C) \\ &= X^V(Yf)^C - Y^C(Xf)^V \\ &= (XYf)^V - (YXf)^V \\ &= (XYf - YXf)^V \\ &= ([X, Y]f)^V \\ &= [X, Y]^V f^C\end{aligned}$$

and

$$\begin{aligned}[X^C, Y^C]f^C &= X^C Y^C f^C - Y^C X^V f^C \\ &= X^C(Yf)^C - Y^C(Xf)^C \\ &= (XYf)^C - (YXf)^C \\ &= (XYf - YXf)^C \\ &= ([X, Y]f)^C \\ &= [X, Y]^C f^C.\end{aligned}$$

Thus $[X^V, Y^C] = [X, Y]^V$ and $[X^C, Y^C] = [X, Y]^C$ by Proposition 2.4. ■

Proposition 2.6 [Ya1, page 17]

Let $\mathcal{L}_X$ denotes the Lie derivation with respect to X and $X, Y \in \Im_0^1(M)$, $F \in \Im_1^1(M)$, $\omega \in \Im_1^0(M)$. Then

(i) $\left[X^C, \gamma_Y F\right] = \gamma_Y(\mathcal{L}_X F) + \gamma_{[X,Y]}F$,

(ii) $\left[X^C, \gamma F\right] = \gamma(\mathcal{L}_X F)$,

(iii) $F^V X^C = (FX)^V$.

Proof: Let $X = \sum_{h=1}^{n} X^h \frac{\partial}{\partial x^h}$ in an open set U of M, where $(x^1, \ldots, x^n)$ is the coordinate system in U. Then we have $X^C = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$ with respect to the induced coordinates in $\pi^{-1}(U)$. Also, we have $\gamma_Y F = \sum_{i,l=1}^{n} Y^l F_l^i \frac{\partial}{\partial y^i}$, where $F = \sum_{i,l=1}^{n} F_l^i \frac{\partial}{\partial x^i} \otimes dx^l$, $Y = \sum_{l=1}^{n} Y^l \frac{\partial}{\partial x^l}$ in the open neighbourhood U in M. Now,

$$\begin{aligned}
\left[X^C, \gamma_Y F\right] = \sum_{h,i,l=1}^{n} \Bigg(& X^h \frac{\partial Y^l}{\partial x^h} F_l^i \frac{\partial}{\partial y^i} + X^h Y^l \frac{\partial F_l^i}{\partial x^h} \frac{\partial}{\partial y^i} + X^h Y^l F_l^i \frac{\partial^2}{\partial x^h \partial y^i} \\
& + \sum_{j=1}^{n} \frac{\partial X^j}{\partial x^i} y^i Y^l F_l^i \frac{\partial^2}{\partial y^j \partial y^h} - Y^l F_l^i \frac{\partial X^h}{\partial y^i} \frac{\partial}{\partial x^h} - Y^l F_l^i X^h \frac{\partial^2}{\partial x^h \partial y^i} \\
& - Y^l F_l^i \frac{\partial X^h}{\partial x^i} \frac{\partial}{\partial y^h} - \sum_{j=1}^{n} Y^l F_l^i \frac{\partial X^j}{\partial x^i} y^i \frac{\partial^2}{\partial y^j \partial y^i} \Bigg) \\
= \sum_{h,i,l=1}^{n} \Bigg(& X^h \frac{\partial Y^l}{\partial x^h} F_l^i \frac{\partial}{\partial y^i} + X^h Y^l \frac{\partial F_l^i}{\partial x^h} \frac{\partial}{\partial y^i} - Y^l F_l^i \frac{\partial X^h}{\partial x^i} \frac{\partial}{\partial y^h} \Bigg) \\
= \sum_{h,i,l=1}^{n} & \left(X^h F_l^i \partial_h Y^l + X^h Y^l \partial_h F_l^i - Y^l F_l^h \partial_h X^i \right) \partial_{\bar{i}} ,
\end{aligned}$$

$$\mathcal{L}_X F = \mathcal{L}_X \sum_{i,l=1}^{n}\left(F_l^i \frac{\partial}{\partial x^i} \otimes dx^l \right)$$

$$= \sum_{i,l=1}^{n}\left((XF_l^i)\frac{\partial}{\partial x^i} \otimes dx^l - \sum_{h=1}^{n} F_l^i \partial_i X^h \frac{\partial}{\partial x^h} \otimes dx^l + \sum_{h=1}^{n} F_l^i \frac{\partial}{\partial x^i} \otimes \frac{\partial X^l}{\partial x^h} dx^h \right)$$

$$= \sum_{h,i,l=1}^{n}\left(X^h \partial_h F_l^i - F_l^h \partial_h X^i + F_h^i \frac{\partial X^h}{\partial x^l} \right)\frac{\partial}{\partial x^i} \otimes dx^l \ ,$$

$$\gamma_Y \mathcal{L}_X F = \sum_{h,i,l=1}^{n} Y^l \left(X^h \partial_h F_l^i - F_l^h \partial_h X^i + F_h^i \partial_l X^h \right) \frac{\partial}{\partial y^i} \ ,$$

$$\gamma_{[X,Y]} F = \sum_{h,i,l=1}^{n} \left(X^h \partial_h Y^l - Y^h \partial_h X^l \right) F_l^i \frac{\partial}{\partial y^i} \ .$$

Therefore $\left[X^C, \gamma_Y F\right] = \gamma_Y(\mathcal{L}_Y F) + \gamma_{[X,Y]} F$. Similarly, we can prove (ii) while the others can be verified easily. ■

We state the following results without proof.

Proposition 2.7 [Ya1, page 18]

For $G \in \Im_2^0(M)$, $X, Y \in \Im_0^1(M)$

(i) $G^V(X^V, Y^V) = 0$,

(ii) $G^V(X^V, Y^C) = 0$,

(iii) $G^V(X^C, Y^V) = 0$,

(iv) $G^V(X^C, Y^C) = (G(X, Y))^V$.

2.7 Complete lifts of 1-forms

Suppose that $\omega \in \Im_1^0(M)$. We define the complete lift of ω, ω^C by $\omega^C(X^C) = (\omega X)^C$. Now, we would like to determine the components of ω^C with respect to the induced coordinate in $\pi^{-1}(U)$, where U is a coordinate neighbourhood in M. Let

$$\omega = \sum_{i=1}^{n} \omega_i dx^i \ , \ \omega^C = \sum_A \overline{\omega}_A d\overline{x}^A \ \text{and} \ X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \ .$$

From $\omega^C X^C = (\omega X)^C$,

$$\begin{aligned}\sum_{i=1}^{n}\left(\overline{\omega}_{\bar{i}} X^i + \overline{\omega}_i \partial X^i\right) &= \sum_{h,i=1}^{n} \bar{y}^h \partial_h(\omega_i X^i) \\ &= \sum_{h,i=1}^{n}\left(\bar{y}^h(\partial_h \omega_i) X^i + \bar{y}^h(\partial_h X^i)\omega_i\right)\end{aligned}$$

for arbitrary $X \in \Im_0^1(M)$. Thus we have $\overline{\omega}_{\bar{i}} = \sum_{h=1}^{n} \bar{y}^h \partial_h \omega_i = \partial\omega_i$, $\overline{\omega}_i = \omega_i$ or

$\omega^C = (\partial\omega_i, \omega_i)$.

We remark that there is a result similar to Proposition 2.4, that is, if $\overline{\omega}$ and $\overline{\theta}$ are 1-forms in *TM*, and if $\overline{\omega}$ and $\overline{\theta}$ agree on the complete lifts of arbitrary vector fields in *M*, then they are equal. The proof is also similar.

Now, we turn our attention to some simple consequences arising from the definition and the expression for components of the complete lift of a 1-form [Ya1, page 19]:

$$(\omega + \theta)^C = \omega^C + \theta^C,$$

$$(f\omega)^C = f^C \omega^V + f^V \omega^C,$$

$$\omega^C X^V = (\omega X)^V,$$

$$\omega^C X^C = (\omega X)^C,$$

for any $X \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$, $f \in \Im_0^0(M)$.

The following are some more results.

Proposition 2.8 [Ya1, page 20]

(i) $\omega^V(\gamma_X F) = 0$,

(ii) $\omega^V(\gamma F) = 0$,

(iii) $\omega^C(\gamma_X F) = (\omega(FX))^V$,

(iv) $\omega^C(\gamma F) = \gamma(\omega \circ F)$

for any $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$, $F \in \Im_1^1(M)$.

Proof: Let $\omega = \sum_{i=1}^{n} \omega_i dx^i$, $X = \sum_{i=1}^{n} X^i \partial_i$, $F = \sum_{i,l=1}^{n} F_l^i \partial_i \otimes dx^l$ in the coordinate neighbourhood U of M.

(i)

$$\begin{aligned}
\omega^V(\gamma_X F) &= \left(\sum_{h=1}^{n} \omega_h dx^h\right)^V \left(\sum_{i,j=1}^{n} F_j^i X^j \partial_{\bar{i}}\right) \\
&= \sum_{h=1}^{n} \omega_h d\bar{x}^h \left(\sum_{i,j=1}^{n} F_j^i X^j \partial_{\bar{i}}\right) \\
&= 0 .
\end{aligned}$$

(ii)

$$\begin{aligned}
\omega^V \gamma F &= \sum_{h=1}^{n} \omega_h d\bar{x}^h \left(\sum_{i,j=1}^{n} F_j^i \bar{y}^j \partial_{\bar{i}}\right) \\
&= 0 .
\end{aligned}$$

(iii)

$$\omega^C(\gamma_X F) = \sum_{h=1}^{n}(\partial\omega_h d\bar{x}^h + \omega_h d\bar{y}^h)\left(\gamma_X\left(\sum_{i,j=1}^{n} F_j^i \partial_i \otimes dx^j\right)\right)$$
$$= \sum_{h=1}^{n}(\partial\omega_h d\bar{x}^h + \omega_h d\bar{y}^h)\left(\sum_{i,j=1}^{n} F_j^i X^j \partial_{\bar{i}}\right)$$
$$= \sum_{i,j=1}^{n} \omega_i F_j^i X^j$$
$$= (\omega(FX))^V.$$

$$\omega^C(\gamma F) = \sum_{h=1}^{n}(\partial\omega_h d\bar{x}^h + \omega_h d\bar{y}^h)\left(\sum_{i,j=1}^{n} F_j^i \bar{y}^j \partial_{\bar{i}}\right)$$
$$= \sum_{i,j=1}^{n} \omega_i F_j^i \bar{y}^j$$
$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} \omega_i F_j^i\right)\bar{y}^j$$
$$= \gamma\left(\sum_{i,j=1}^{n} \omega_i F_j^i dx^j\right)$$
$$= \gamma(\omega \circ F),$$

where $\omega \circ F = F(\omega)$. (iv) can be proved similarly. ■

2.8 Complete lifts of tensor fields

By taking account of

$$(gf)^C = g^C f^V + g^V f^C,$$

$$(fX)^C = f^V X^C + f^C X^V,$$

$$(f\omega)^C = f^V \omega^C + f^C \omega^V,$$

where g, f are regarded as tensors of type $(0, 0)$, X tensor of type $(1, 0)$, ω tensor of type $(0, 1)$, we extend the definition of complete lift to arbitrary tensor field by imposing the following conditions:

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C \text{ and } (P + R)^C = P^C + R^C.$$

We now consider a few types of tensors and the components of their complete liftings.

Tensor of type $(1, 1)$

Let F_i^h be the local components of F, that is, $F = \sum_{h,i=1}^{n} F_i^h \frac{\partial}{\partial x^h} \otimes dx^i$. Then

$$F^C = \sum_{h,i=1}^{n} \left(F_i^h \left(\frac{\partial}{\partial x^h} \otimes \partial x^i \right)^C + (F_i^h)^C \left(\frac{\partial}{\partial x^h} \otimes \partial x^i \right)^V \right)$$

$$= \sum_{h,i=1}^{n} \left(F_i^h \left(\frac{\partial}{\partial x^h} \right)^V \otimes (\partial x^i)^C + F_i^h \left(\frac{\partial}{\partial x^h} \right)^C \otimes (\partial x^i)^V + \partial F_i^h \left(\frac{\partial}{\partial x^h} \right)^V \otimes (\partial x^i)^V \right)$$

$$= \sum_{h,i=1}^{n} \left(F_i^h \frac{\partial}{\partial y^h} \otimes \partial y^i + F_i^h \frac{\partial}{\partial x^h} \otimes \partial x^i + \partial F_i^h \frac{\partial}{\partial y^h} \otimes \partial x^i \right)$$

or in matrix form $F^C = \left(\widetilde{F}_B^A \right) = \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}$.

Tensor of type $(0, 2)$

Let $G = \sum_{i,j=1}^{n} G_{ji} dx^j \otimes dx^i$. Then

$$G^C = \sum_{i,j=1}^{n}\left(G_{ji}(dx^j \otimes dx^i)^C + (G_{ji})^C(dx^j \otimes dx^i)^V\right)$$
$$= \sum_{i,j=1}^{n}\left(G_{ji}d\bar{x}^j \otimes d\bar{y}^i + G_{ji}d\bar{y}^j \otimes d\bar{x}^i + \partial G_{ji}(d\bar{x}^j \otimes d\bar{x}^i)\right)$$

or in matrix form $G^C = \left(\widetilde{G}_{AB}\right) = \begin{pmatrix} \partial G_{ji} & G_{ji} \\ G_{ji} & 0 \end{pmatrix}$.

Tensor of type $(2, 0)$

Let $H = \sum_{i,j=1}^{n} H^{ji}\left(\frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^i}\right)$. Then

$$H^C = \sum_{i,j=1}^{n}\left(H^{ji}\left(\frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^i}\right)^C + (H^{ji})^C\left(\frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^i}\right)^V\right)$$
$$= \sum_{i,j=1}^{n}\left(H^{ji}\left(\frac{\partial}{\partial \bar{x}^j} \otimes \frac{\partial}{\partial \bar{y}^i} + \frac{\partial}{\partial \bar{y}^j} \otimes \frac{\partial}{\partial \bar{x}^i}\right) + \partial H^{ji}\,\frac{\partial}{\partial \bar{y}^j} \otimes \frac{\partial}{\partial \bar{y}^i}\right)$$

or in matrix form $H^C = \left(\widetilde{H}^{AB}\right) = \begin{pmatrix} 0 & H^{ji} \\ H^{ji} & \partial H^{ji} \end{pmatrix}$.

Tensor of type $(0, s)$ and $(1, s)$

Let $S = \sum_{i_s,\cdots,i_1=1}^{n} S_{i_s\cdots i_1} dx^{i_s} \otimes \cdots \otimes dx^{i_1} \in \Im_s^0(M)$. Then

$$S^C = \sum_{i_s,\cdots,i_1=1}^{n}(S_{i_s\cdots i_1})^C\left(dx^{i_s} \otimes \cdots \otimes dx^{i_1}\right)^V + \sum_{i_s,\cdots,i_1=1}^{n} S_{i_s\cdots i_1}\left(dx^{i_s} \otimes \cdots \otimes dx^{i_1}\right)^C$$
$$= \sum_{i_s,\cdots,i_1=1}^{n} \partial S_{i_s\cdots i_1} d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1} + \sum_{t=1}^{s}\sum_{i_s,\cdots,i_1=1}^{n} S_{i_s\cdots i_1} d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{\bar{i}_t} \otimes \cdots \otimes d\bar{x}^{i_1}.$$

Similarly,

Let $T=\sum_{h,i_s,\cdots,i_1=1}^{n} T^h_{i_s\cdots i_1}\frac{\partial}{\partial x^h}\otimes dx^{i_s}\otimes\cdots\otimes dx^{i_1}\in\Im^1_s(M)$. Then

$$T^C=\sum_{h,i_s,\cdots,i_1=1}^{n} T^h_{i_s\cdots i_1}\frac{\partial}{\partial\bar{x}^h}\otimes d\bar{x}^{i_s}\otimes\cdots\otimes d\bar{x}^{i_1}+\sum_{h,i_s,\cdots,i_1=1}^{n}\partial T^h_{i_s\cdots i_1}\frac{\partial}{\partial\bar{y}^h}\otimes d\bar{x}^{i_s}\otimes\cdots\otimes d\bar{x}^{i_1}$$

$$+\sum_{t=1}^{s}\sum_{h,i_s,\cdots,i_1=1}^{n} T^h_{i_s\cdots i_1}\frac{\partial}{\partial\bar{y}^h}\otimes d\bar{x}^{i_s}\otimes\cdots\otimes d\bar{x}^{i_{t+1}}\otimes d\bar{y}^{i_t}\otimes d\bar{x}^{i_{t-1}}\otimes\cdots\otimes d\bar{x}^{i_1}$$

all the others being zero, with respect to the induced coordinates in TM.

Now, let us have a look at some properties of complete liftings of tensor fields.

Proposition 2.9 [Ya1, page 23]

(i) $F^C X^V=(FX)^V$,

(ii) $F^V X^C=(FX)^V$,

(iii) $F^C X^C=(FX)^C$,

(iv) $F^V X^V=0$,

(v) $F^V(\gamma_X T)=0$,

(vi) $F^V(\gamma T)=0$,

(vii) $F^C(\gamma_X T)=\gamma_X(FT)$,

(viii) $F^C(\gamma T)=\gamma(FT)$,

where $X\in\Im^1_0(M)$, $F,\ T\in\Im^1_1(M)$ and $(FX)\omega=F(\omega,X)$ for any $\omega\in\Im^0_1(M)$.

Proof: We only consider case (vii) here.

$$F^C(\gamma_X T) = \sum_{i,j=1}^{n}\left(\partial F_i^j \frac{\partial}{\partial \bar{y}^j} \otimes d\bar{x}^i + F_i^j \frac{\partial}{\partial \bar{x}^j} \otimes d\bar{x}^i + F_i^j \frac{\partial}{\partial \bar{y}^j} \otimes d\bar{y}^i\right)\left(\sum_{h,l=1}^{n} X^h T_h^l \frac{\partial}{\partial \bar{y}^l}\right)$$

$$= \sum_{h,i,j=1}^{n} F_i^j X^h T_h^i \frac{\partial}{\partial \bar{y}^j},$$

$$\gamma_X(FT) = \gamma_X\left(\left(\sum_{i,j=1}^{n} F_i^j \frac{\partial}{\partial x^j} \otimes dx^i\right)\left(\sum_{h,l=1}^{n} T_h^l \frac{\partial}{\partial x^l} \otimes dx^h\right)\right)$$

$$= \gamma_X\left(\sum_{h,i,j=1}^{n} F_i^j T_h^i \frac{\partial}{\partial x^j} \otimes dx^h\right)$$

$$= \sum_{h,i,j=1}^{n} X^h F_i^j T_h^i \frac{\partial}{\partial \bar{y}^j}.$$

■

Similarly we have the following results:

Proposition 2.10 [Ya1, page 23]

(i) $G^C(X^V, Y^V) = 0,$

(ii) $G^C(X^V, Y^C) = (G(X, Y))^V,$

(iii) $G^C(X^C, Y^V) = (G(X, Y))^V,$

(iv) $G^C(X^C, Y^C) = (G(X, Y))^C$

for any $X, Y \in \Im_0^1(M)$, $G \in \Im_2^0(M)$.

We also have $(d\omega)^C = d(\omega^C)$ for any $\omega = \sum_{h=1}^{n} \omega_h dx^h \in \Im_1^0(M)$. To see this, we first consider $d(\omega^C)$. From $\omega^C = (\bar{\omega}_A) = (\partial\omega_h, \omega_h)$, we have

$$d(\omega^C) = \sum_{A,B} \partial_B \overline{\omega}_A d\overline{x}^B \Lambda d\overline{x}^A$$

$$= \sum_{h,k=1}^{n} \Big((\partial_k \partial\omega_h) d\overline{x}^k \Lambda d\overline{x}^h + (\partial_k \omega_h) d\overline{x}^k \Lambda d\overline{y}^h$$

$$+ \partial_{\bar{k}} (\partial\omega_h) d\overline{y}^k \Lambda d\overline{x}^h + \partial_{\bar{k}} \omega_h d\overline{y}^k \Lambda d\overline{y}^h \Big)$$

$$= \tfrac{1}{2} \sum_{h,k=1}^{n} \Big(\partial(\partial_k \omega_h)(d\overline{x}^k \otimes d\overline{x}^h - d\overline{x}^h \otimes d\overline{x}^k) + \partial_k \omega_h (d\overline{x}^k \otimes d\overline{y}^h - d\overline{y}^h \otimes d\overline{x}^k)$$

$$+ \partial_k \omega_h (d\overline{y}^k \otimes d\overline{x}^h - d\overline{x}^h \otimes d\overline{y}^k) + 0 \Big)$$

$$= \tfrac{1}{2} \sum_{h,k=1}^{n} \Big(\partial(\partial_k \omega_h - \partial_h \omega_k) d\overline{x}^k \otimes d\overline{x}^h + (\partial_k \omega_h - \partial_h \omega_k) d\overline{x}^k \otimes d\overline{y}^h$$

$$+ (\partial_k \omega_h - \partial_h \omega_k) d\overline{y}^k \otimes d\overline{x}^h \Big).$$

Next, we consider the complete lift of $d\omega$, where

$$d\omega = d\left(\sum_{i=1}^{n} \omega_i dx^i \right)$$

$$= \sum_{i,j=1}^{n} \partial_j \omega_i dx^j \Lambda dx^i$$

$$= \tfrac{1}{2} \sum_{i,j=1}^{n} (\partial_j \omega_i - \partial_i \omega_j) dx^j \otimes dx^i,$$

$$(d\omega)^C = \tfrac{1}{2} \sum_{i,j=1}^{n} \partial(\partial_j \omega_i - \partial_i \omega_j)(dx^j \otimes dx^i)^V + \tfrac{1}{2} \sum_{i,j=1}^{n} (\partial_j \omega_i - \partial_i \omega_j)(dx^j \otimes dx^i)^C$$

$$= \tfrac{1}{2} \sum_{i,j=1}^{n} \partial(\partial_j \omega_i - \partial_i \omega_j)((dx^j)^V \otimes (dx^i)^V)$$

$$+ \tfrac{1}{2} \sum_{i,j=1}^{n} (\partial_j \omega_i - \partial_i \omega_j)((dx^j)^V \otimes (dx^i)^C + (dx^j)^C \otimes (dx^i)^V)$$

$$= \tfrac{1}{2} \sum_{i,j=1}^{n} \Big(\partial(\partial_j \omega_i - \partial_i \omega_j) d\overline{x}^j \otimes d\overline{x}^i + (\partial_j \omega_i - \partial_i \omega_j) d\overline{x}^j \otimes d\overline{y}^i$$

$$(\partial_j \omega_i - \partial_i \omega_j) d\overline{y}^j \otimes d\overline{x}^i \quad \Big).$$

Thus we see that $(d\omega)^C = d(\omega^C)$ for any $\omega \in \Im_1^0(M)$.

Proposition 2.11 [Ya1, page 24]

Let S be an element of $\Im_s^0(M)$ or $\Im_s^1(M)$ and X an element of $\Im_0^1(M)$. Then

(i) $(S^V)_{X^V} = 0$,

(ii) $(S^C)_{X^V} = (S_X)^V = \gamma_X S$,

(iii) $(S^V)_{X^C} = (S_X)^V = \gamma_X S$,

(iv) $(S^C)_{X^C} = (S_X)^C$.

Proof: We will prove the case for tensor of type (1, s), the proof for tensor type (0, s) is similar. Let $S^h_{i_s \cdots i_1}$ be the components of S in the coordinate neighbour U of M. We know that the components of S^V and S^C, denoted by $\widetilde{S}^B_{A_s \cdots A_1}$ and $\widetilde{\widetilde{S}}^B_{A_s \cdots A_1}$ respectively, are given by

$$\widetilde{S}^{\bar{h}}_{i_s \cdots i_1} = S^h_{i_s \cdots i_1},$$

$$\widetilde{\widetilde{S}}^h_{i_s \cdots i_1} = S^h_{i_s \cdots i_1}, \quad \widetilde{\widetilde{S}}^{\bar{h}}_{i_s \cdots i_1} = \partial S^h_{i_s \cdots i_1}, \quad \widetilde{\widetilde{S}}^{\bar{h}}_{i_s \cdots i_{t+1} \bar{i}_t i_{t-1} \cdots i_1} = S^h_{i_s \cdots i_t \cdots i_1},$$

while the others being zero.

$$S^V = \sum_{B, A_s, \ldots, A_1} \widetilde{S}^B_{A_s \cdots A_1} \frac{\partial}{\partial \bar{x}^B} \otimes d\bar{x}^{A_s} \otimes \cdots \otimes d\bar{x}^{A_1}$$

$$= \sum_{h, i_s, \ldots, i_1 = 1}^{n} S^h_{i_s \cdots i_1} \frac{\partial}{\partial y^h} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1},$$

$$S^C = \sum_{B, A_s, \ldots, A_1} \widetilde{\widetilde{S}}^B_{A_s \cdots A_1} \frac{\partial}{\partial \bar{x}^B} \otimes d\bar{x}^{A_s} \otimes \cdots \otimes d\bar{x}^{A_1}$$

$$= \sum_{h, i_s, \ldots, i_1 = 1}^{n} S^h_{i_s \cdots i_1} \frac{\partial}{\partial \bar{x}^h} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1} + \sum_{h, i_s, \ldots, i_1 = 1}^{n} \partial S^h_{i_s \cdots i_1} \frac{\partial}{\partial y^h} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

$$+ \sum_{t=1}^{s} \sum_{h, i_s, \ldots, i_1 = 1}^{n} S^h_{i_s \cdots i_1} \frac{\partial}{\partial y^h} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_{t+1}} \otimes dy^{i_t} \otimes d\bar{x}^{i_{t-1}} \otimes \cdots \otimes d\bar{x}^{i_1}.$$

We shall write $(S^V(X^V))(\bar{X}^{s-1}, \ldots, \bar{X}^1) = S^V(X^V, \bar{X}^{s-1}, \ldots, \bar{X}^1)$, where $\bar{X}^{s-1}, \ldots, \bar{X}^1 \in \Im_0^1(TM)$. Then

$$(S^V)_{X^V} = S^V\left(\sum_{k=1}^n X^k \frac{\partial}{\partial \bar{y}^k}\right) = 0 ,$$

$$(S^C)_{X^V} = S^C\left(\sum_{k=1}^n X^k \frac{\partial}{\partial \bar{y}^k}\right) = \sum_{h,i_s,\ldots,i_1=1}^n X^{i_s} S^h_{i_s\cdots i_1} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} ,$$

$$\begin{aligned}(S^V)_{X^C} &= S^V \sum_{k=1}^n \left(X^k \frac{\partial}{\partial \bar{x}^k} + \partial X^k \frac{\partial}{\partial \bar{y}^k}\right) \\ &= \sum_{h,i_s,i_{s-1},\ldots,i_1=1}^n S^h_{i_s i_{s-1}\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} ,\end{aligned}$$

$$\begin{aligned}(S_X)^V &= \left(\sum_{h,i_s,i_{s-1},\ldots,i_1=1}^n S^h_{i_s i_{s-1}\cdots i_1} X^{i_s} \frac{\partial}{\partial x^h} \otimes dx^{i_{s-1}} \otimes \cdots \otimes dx^{i_1}\right)^V \\ &= \sum_{h,i_s,i_{s-1},\ldots,i_1=1}^n S^h_{i_s i_{s-1}\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} ,\end{aligned}$$

$$\gamma_X S = \sum_{h,i_s,i_{s-1},\ldots,i_1=1}^n S^h_{i_s i_{s-1}\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} .$$

So $(S^C)_{X^V} = (S^V)_{X^C} = (S_X)^V = \gamma_X S$.

$$\begin{aligned}(S^C)_{X^C} &= S^C \sum_{k=1}^n \left(X^k \frac{\partial}{\partial \bar{x}^k} + \partial X^k \frac{\partial}{\partial \bar{y}^k}\right) \\ &= \sum_{h,i_s,\ldots,i_1=1}^n S^h_{i_s\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{x}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} \\ &\quad + \sum_{h,i_s,\ldots,i=1}^n \partial S^h_{i_s\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} \\ &\quad + \sum_{h,i_s,\ldots,i_1=1}^n S^h_{i_s\cdots i_1} \partial X^{i_s} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} \\ &\quad + \sum_{t=1}^{s-1} \sum_{h,i_s,\ldots,i_1=1}^n S^h_{i_s\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_{t+1}} \otimes d\bar{y}^{i_t} \otimes d\bar{x}^{i_{t-1}} \otimes \cdots \otimes d\bar{x}^{i_1}\end{aligned}$$

$$\begin{aligned}&= \sum_{h,i_s,\ldots,i_1=1}^n S^h_{i_s\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{x}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} + \sum_{h,i_s,\ldots,i=1}^n \partial (S^h_{i_s\cdots i_1} X^{i_s}) \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_1} \\ &\quad + \sum_{t=1}^{s-1} \sum_{h,i_s,\ldots,i_1=1}^n S^h_{i_s\cdots i_1} X^{i_s} \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^{i_{s-1}} \otimes \cdots \otimes d\bar{x}^{i_{t+1}} \otimes d\bar{y}^{i_t} \otimes d\bar{x}^{i_{t-1}} \otimes \cdots \otimes d\bar{x}^{i_1}\end{aligned}$$

$$(S_X)^C = \left(\sum_{h,i_s,\ldots,i_1=1}^{n} S^h_{i_s \cdots i_1} X^{i_s} \frac{\partial}{\partial x^h} \otimes dx^{i_{s-1}} \otimes \cdots \otimes dx^{i_1} \right)^C$$

$$= \sum_{h,i_s,\ldots,i=1}^{n} \partial (S^h_{i_s \cdots i_1} X^{i_s}) \frac{\partial}{\partial \overline{y}^h} \otimes d\overline{x}^{i_{s-1}} \otimes \cdots \otimes d\overline{x}^{i_1}$$

$$+ \sum_{h,i_s,\ldots,i_1=1}^{n} S^h_{i_s \cdots i_1} X^{i_s} \frac{\partial}{\partial \overline{x}^h} \otimes d\overline{x}^{i_{s-1}} \otimes \cdots \otimes d\overline{x}^{i_1}$$

$$+ \sum_{t=1}^{s-1} \sum_{h,i_s,\ldots,i_1=1}^{n} S^h_{i_s \cdots i_1} X^{i_s} \frac{\partial}{\partial \overline{y}^h} \otimes d\overline{x}^{i_{s-1}} \otimes \cdots \otimes d\overline{x}^{i_{t+1}} \otimes d\overline{y}^{i_t} \otimes d\overline{x}^{i_{t-1}} \otimes \cdots \otimes d\overline{x}^{i_1}.$$

Thus $(S_X)^C = S^C{}_{X^C}$. ■

Proposition 2.12 [Ya1, page 24]

Let S be an element of $\Im^0_s(M)$ or $\Im^1_s(M)$. Then

(i) $S^V(\overline{X}_s, \ldots, \overline{X}_{i+1}, X_i^V, \overline{X}_{i-1}, \ldots, \overline{X}_1) = 0$,

(ii) $S^V(X_s^C, \ldots, X_1^C) = (S(X_s, \ldots, X_1))^V$,

(iii) $S^C(X_s^C, \ldots, X_1^C) = (S(X_s, \ldots, X_1))^C$

for any $X_s, \ldots, X_1 \in \Im^1_0(M)$ and $\overline{X}_s, \ldots, \overline{X}_1 \in \Im^1_0(TM)$.

Proof: We will prove the case for $S \in \Im^1_s(M)$.

$$S^V(\overline{X}_s, \ldots, \overline{X}_{i+1}, X_i^V, \overline{X}_{i-1}, \ldots, \overline{X}_1)$$

$$= \sum_{i_s,\ldots,i_1=1}^{n} \widetilde{S}^h_{i_s \cdots i_1} \frac{\partial}{\partial \overline{y}^h} \otimes d\overline{x}^{i_s} \otimes \cdots \otimes d\overline{x}^{i_1} (\overline{X}_s, \ldots, \overline{X}_{i+1}, X_i^V, \overline{X}_{i-1}, \ldots, \overline{X}_1)$$

$$= \sum_{i_s,\ldots,i_1=1}^{n} S^h_{i_s \cdots i_1} \frac{\partial}{\partial \overline{y}^h} \otimes d\overline{x}^{i_s} \otimes \cdots \otimes d\overline{x}^{i_1} (\overline{X}_s, \ldots, \overline{X}_{i+1}, \sum_{h=1}^{n} X_i^h \frac{\partial}{\partial \overline{y}^h}, \overline{X}_{i-1}, \ldots, \overline{X}_1)$$

$$= 0,$$

$$S^V(X_s^C, \ldots, X_1^C)$$

$$= \sum_{i_s,\ldots,i_1=1}^{n} S^h_{i_s\cdots i_1} \frac{\partial}{\partial y^h} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}\left(\sum_{h=1}^{n}(X_s^h\partial_h + \partial X_s^h \partial_{\bar{h}}), \ldots, \sum_{h=1}^{n}(X_1^h\partial_h + \partial X_1^h\partial_{\bar{h}})\right)$$

$$= \sum_{h,i_s,\ldots,i_1=1}^{n} S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h},$$

$$(S(X_s, \ldots, X_1))^V = \left(\sum_{h,i_s,\ldots,i_1=1}^{n} S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h}\right)^V$$

$$= \sum_{h,i_s,\ldots,i_1=1}^{n} S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h},$$

$$S^C(X_s^C, \ldots, X_1^C)$$

$$= \sum \tilde{\tilde{S}}^B_{A_s\cdots A_1} \frac{\partial}{\partial \bar{x}^B} \otimes d\bar{x}^{A_s} \otimes \cdots \otimes d\bar{x}^{A_1}\left(\sum_{h=1}^{n}(X_s^h\partial_h + \partial X_s^h\partial_{\bar{h}}), \ldots, \sum_{h=1}^{n}(X_1^h\partial_h + \partial X_1^h\partial_{\bar{h}})\right)$$

$$= \sum_{h,i_s,\ldots,i_1=1}^{n}\left(S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} + \partial S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h}\right.$$

$$\left. + \sum_{t=1}^{s} S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_{t+1}^{i_{t+1}} \partial X_t^{i_t} X_{t-1}^{i_{t-1}} \cdots X_1^{i_1} \frac{\partial}{\partial y^h}\right)$$

$$= \sum_{h,i_s,\ldots,i_1=1}^{n}\left(S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} + \partial(S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1})\frac{\partial}{\partial y^h}\right)$$

$$= \left(\sum_{h,i_s,\ldots,i_1=1}^{n} S^h_{i_s\cdots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h}\right)^C$$

$$= (S(X_s, \ldots, X_1))^C.$$

■

<u>Proposition 2.13 [Ya1, page 25]</u>

Let φ and ψ be any two differential forms. Then

(i) $(\varphi \wedge \psi)^C = \varphi^C \wedge \psi^V + \varphi^V \wedge \psi^C$,

(ii) $d(\varphi^C) = (d\varphi)^C$.

<u>Proof</u>: We have

$$
\begin{aligned}
(\varphi\Lambda\psi)^C &= \tfrac{1}{2}(\varphi\otimes\psi-\psi\otimes\varphi)^C \\
&= \tfrac{1}{2}(\varphi^C\otimes\psi^V+\varphi^V\otimes\psi^C-\psi^C\otimes\varphi^V-\psi^V\otimes\varphi^C) \\
&= \tfrac{1}{2}(\varphi^C\otimes\psi^V-\psi^V\otimes\varphi^C+\varphi^V\otimes\psi^C-\psi^C\otimes\varphi^V) \\
&= \varphi^C\Lambda\psi^V+\varphi^V\Lambda\psi^C .
\end{aligned}
$$

Let φ be any differential form with local expression

$$\varphi = f dx^{i_1}\Lambda\cdots\Lambda dx^{i_1},$$

$$d\varphi = df\Lambda dx^{i_1}\Lambda\cdots\Lambda dx^{i_1},\ f\in\Im_0^0(M).$$

Then

$$(d\varphi)^C = (df)^C\Lambda(dx^{i_1}\Lambda\ldots\Lambda dx^{i_1})^V+(df)^V\Lambda(dx^{i_1}\Lambda\ldots\Lambda dx^{i_1})^C .$$

Since

$$
\begin{aligned}
(df)^C &= \Big(\sum_{i=1}^{n}(\partial_i f)dx^i\Big)^C \\
&= \sum_{i=1}^{n}(\partial(\partial_i f)d\overline{x}^i+\partial_i f d\overline{y}^i),
\end{aligned}
$$

$$
\begin{aligned}
d(f^C) &= d(\partial f) \\
&= \sum_{A}\partial_A(\partial f)d\overline{x}^A \\
&= \sum_{i=1}^{n}(\partial(\partial_i f)d\overline{x}^i+\partial_i f d\overline{y}^i)
\end{aligned}
$$

and also

$$(df)^V = d(f)^V,$$

then

$$
\begin{aligned}
(d\varphi)^C &= (df)^C \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^V + (df)^V \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^C \\
&= d(f^C) \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^V + d(f^V) \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^C \\
&= d(f^C \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^V) + d(f^V \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^C) \\
&= d(f^C \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^V + f^V \wedge (dx^{i_s} \wedge \ldots \wedge dx^{i_1})^C) \\
&= d(f dx^{i_s} \wedge \ldots \wedge dx^{i_1})^C \\
&= d(\varphi^C).
\end{aligned}
$$

■

2.9 Lifts of derivations

Recall that a derivation D in M is a type preserving mapping $D : \Im(M) \to \Im(M)$ satisfying

(1) $D(S+T) = DS + DT$, for any $S, T \in \Im_s^r(M)$,

(2) $D(S \otimes T) = (DS) \otimes T + S \otimes (DT)$, for any $S, T \in \Im(M)$,

(3) $DI = 0$, where I is the identity tensor field of type (1, 1) in M. We write $I = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \otimes dx^i$ locally.

Furthermore, there exists a vector field P in M such that $Df = Pf$, for any $f \in \Im_0^0(M)$.

Let $\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^{n}$ be the coordinate vector fields in the coordinate neighbourhood U in M. If we put $D\left(\frac{\partial}{\partial x^i}\right) = \sum_{h=1}^{n} Q_i^h \frac{\partial}{\partial x^h}$, then for $dx^i\}_{i=1}^{n}$, the dual 1-form of $\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^{n}$, we have $D(dx^h) = \sum_{i=1}^{n} (-Q_i^h dx^i)$. For $X = \sum_{h=1}^{n} X^h \frac{\partial}{\partial x^h} \in \Im_0^1(M)$ and $\omega = \sum_{i=1}^{n} \omega_i dx^i \in \Im_1^0(M)$,

$$DX = \sum_{i=1}^{n}\left\{(DX^i)\frac{\partial}{\partial x^i} + X^i\left(D\frac{\partial}{\partial x^i}\right)\right\}$$
$$= \sum_{i,h=1}^{n}\left(P^h\frac{\partial X^i}{\partial x^h} + X^h Q_h^i\right)\frac{\partial}{\partial x^i},$$

$$D\omega = \sum_{h=1}^{n}((D\omega_h)dx^h + \omega_h(Ddx^h))$$
$$= \sum_{i,h=1}^{n}\left(P^i\frac{\partial \omega_h}{\partial x^i} + \omega_i(-Q_h^i)\right)dx^h .$$

(P^h, Q_i^h) are called the components of D in U. We define two vector fields in TM, denoted by D^V, D^C respectively, by

$$D^V f^C = (Df)^V,$$
$$D^C f^C = \gamma(Ddf)$$

for any $f \in \Im_0^0(M)$.

Let $\begin{pmatrix}\tilde{D}^h \\ \tilde{D}^{\bar{h}}\end{pmatrix}$ be the components of D^V with respect to the coordinates in $\pi^{-1}(U)$.

Then

$$D^V f^C = D^V \sum_{i=1}^{n} \bar{y}^i \partial_i f$$
$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} \bar{y}^i(\partial_j\partial_i f)\tilde{D}^j + (\partial_j f)\tilde{D}^{\bar{j}}\right),$$
$$(Df)^V = \sum_{h=1}^{n} P^h \partial_h f ,$$

we have $\sum_{j=1}^{n}\left(\sum_{i=1}^{n} \bar{y}^i(\partial_j\partial_i f)\tilde{D}^j + (\partial_j f)\tilde{D}^{\bar{j}}\right) = \sum_{j=1}^{n} P^j \partial_j f$ for arbitrary $f \in \Im_0^0(M)$. This implies that $\tilde{D}^{\bar{h}} = P^h$, $\tilde{D}^h = 0$. Thus $D^V : \begin{pmatrix} 0 \\ P^h \end{pmatrix}$.

If D^C has components $\begin{pmatrix} \widetilde{D}^h \\ \widetilde{D}^{\bar{h}} \end{pmatrix}$ in $\pi^{-1}(U)$, then

$$D^C f^C = \sum_{h=1}^{n} \{\widetilde{D}^h (\partial\, \partial_h f) + \widetilde{D}^{\bar{h}} (\partial_h f)\} , \tag{1}$$

$$\begin{aligned} \gamma(Ddf) &= \gamma\left\{ \sum_{j,h=1}^{n} \{P^h \partial_h \partial_j f + (-Q_j^h \partial_h f)\} dx^j \right\} \\ &= \sum_{j,h=1}^{n} (\bar{y}^j P^h \partial_h \partial_j f - \bar{y}^j Q_j^h \partial_h f) \qquad (2) \\ &= \sum_{h=1}^{n} (P^h \partial(\partial_h f) - \sum_{j=1}^{n} \bar{y}^j Q_j^h \partial_h f) . \end{aligned}$$

Since (1) and (2) are equal for arbitrary $f \in \Im_0^0(M)$, we have

$\widetilde{D}^{\bar{h}} = \sum_{j=1}^{n} (-\bar{y}^j Q_j^h)$, $\widetilde{D}^h = P^h$. Thus $D^C : \begin{pmatrix} P^h \\ -\sum_{j=1}^{n} \bar{y}^j Q_j^h \end{pmatrix}$.

Consequently, we have the following results [Ya1, page 28]:

(i) $(D_1 + D_2)^V = D_1^V + D_2^V$,

(ii) $(D_1 + D_2)^C = D_1^C + D_2^C$,

(iii) $(fD)^V = f^V D^V$,

(iv) $(fD)^C = f^V D^C$,

where D, D_1, D_2 are derivations in M, $f \in \Im_0^0(M)$.

Proposition 2.14 [Ya1, page28]

(i) $D^V f^V = 0$,

(ii) $D^V f^C = (Df)^V$,

(iii) $D^C f^V = (Df)^V$,

(iv) $D^C f^C = \gamma(D(df))$

for any $f \in \Im_0^0(M)$ and any derivation D in M.

Proposition 2.15 [Ya1, page 28]

Let D_1, D_2 be two derivations in M and (P_1^h, Q_{1i}^h), (P_2^h, Q_{2i}^h) be the components of D_1, D_2 respectively. Then

(i) $[D_1^V, D_2^V] = 0$,

(ii) $[D_1^V, D_2^C] = [D_1, D_2]^V$ if $Q_{2i}^h = -\partial_i P_2^h$ for all $i, h = 1, \ldots, n$,

(iii) $[D_1^C, D_2^C] = [D_1, D_2]^C$.

Proof:

(i) We have

$$
\begin{aligned}
[D_1^V, D_2^V]f^C &= D_1^V D_2^V f^C - D_2^V D_1^V f^C \\
&= D_1^V (D_2 f)^V - D_2^V (D_1 f)^V \\
&= 0 .
\end{aligned}
$$

(ii) From our assumption that $Q_{2i}^h = -\partial_i P_2^h$ for all $i,\ h = 1, \ldots, n$,

$$
\begin{aligned}
[D_1^V, D_2^C]f^C &= D_1^V D_2^C f^C - D_2^C D_1^V f^C \\
&= D_1^V (\gamma D_2 df) - D_2^C (D_1 f)^V \\
&= D_1^V \left(\gamma(D_2(\sum_{i=1}^{n} \partial_i f dx^i)) \right) - (D_2 D_1 f)^V \\
&= D_1^V \left(\gamma(\sum_{i,h=1}^{n} P_2^h (\partial_h \partial_i f) dx^i - \sum_{i,h=1}^{n} Q_{2i}^h (\partial_h f) dx^i) \right) - (D_2 D_1 f)^V \\
&= D_1^V \left(\sum_{i,h=1}^{n} \overline{y}^i \left(P_2^h (\partial_h \partial_i f) + (\partial_i P_2^h) \partial_h f \right) \right) - (D_2 D_1 f)^V \\
&= \sum_{i,h=1}^{n} \left(P_1^i (P_2^h \partial_i (\partial_h f) + (\partial_i P_2^h) \partial_h f) \right) - (D_2 D_1 f)^V \\
&= \sum_{i=1}^{n} \left(P_1^i \partial_i (\sum_{h=1}^{n} P_2^h \partial_h f) \right) - (D_2 D_1 f)^V \\
&= (D_1 D_2 f)^V - (D_2 D_1 f)^V \\
&= ([D_1, D_2] f)^V \\
&= [D_1, D_2]^V f^C .
\end{aligned}
$$

(iii)

$$
\begin{aligned}
[D_1^C, D_2^C]f^C &= D_1^C D_2^C f^C - D_2^C D_1^C f^C \\
&= D_1^C(\gamma D_2 df) - D_2^C(\gamma D_1 df) \\
&= D_1^C\left(\sum_{i,h=1}^{n} \bar{y}^i (P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f)\right) - D_2^C\left(\sum_{i,h=1}^{n} \bar{y}^i (P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f)\right) \\
&= \sum_{i,h,k=1}^{n} \{P_1^k \bar{y}^i \partial_k (P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f) - Q_{1k}^i \bar{y}^k (P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f) \\
&\qquad - P_2^k \bar{y}^i \partial_k (P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f) + Q_{2k}^i \bar{y}^k (P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f)\} \\
&= \sum_{i,h,k=1}^{n} \bar{y}^i \{P_1^k \partial_k (P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f) - Q_{1i}^k (P_2^h(\partial_h \partial_k f) - Q_{2k}^h \partial_h f) \\
&\qquad - P_2^k \partial_k (P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f) + Q_{2i}^k (P_1^h(\partial_h \partial_k f) - Q_{1k}^h \partial_h f)\} \\
&= \gamma \sum_{i,h,k=1}^{n} \{P_1^k \partial_k (P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f) - Q_{1i}^k (P_2^h(\partial_h \partial_k f) - Q_{2k}^h \partial_h f) \\
&\qquad - P_2^k \partial_k (P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f) + Q_{2i}^k (P_1^h(\partial_h \partial_k f) - Q_{1k}^h \partial_h f)\} dx^i \\
&= \gamma \sum_{i,h=1}^{n} \{[P_1(P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f)] dx^i + \sum_{k=1}^{n} (P_2^h(\partial_h \partial_k f) - Q_{2k}^h \partial_h f)(-Q_{1i}^k dx^i) \\
&\qquad - [P_2(P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f)] dx^i - \sum_{k=1}^{n} (P_1^h(\partial_h \partial_k f) - Q_{1k}^h \partial_h f)(-Q_{2i}^k dx^i)\} \\
&= \gamma \sum_{i,h=1}^{n} \{[D_1(P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f)] dx^i + (P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f)(D_1 dx^i) \\
&\qquad - [D_2(P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f)] dx^i - (P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f)(D_2 dx^i)\} \\
&= \gamma \sum_{i,h=1}^{n} \{D_1[(P_2^h(\partial_h \partial_i f) - Q_{2i}^h \partial_h f) dx^i] - D_2[(P_1^h(\partial_h \partial_i f) - Q_{1i}^h \partial_h f) dx^i]\} \\
&= \gamma \sum_{i=1}^{n} \{D_1[(P_2(\partial_i f)) dx^i + \sum_{h=1}^{n} (\partial_i f)(-Q_{2h}^i dx^h)] \\
&\qquad - D_2[(P_1(\partial_i f)) dx^i + \sum_{h=1}^{n} \partial_h f(-Q_{1i}^h dx^i)]\} \\
&= \gamma \sum_{i=1}^{n} \{D_1[(D_2(\partial_i f)) dx^i + (\partial_i f)(D_2 dx^i)] \\
&\qquad - D_2[(D_1(\partial_i f)) dx^i + \partial_i f(D_1 dx^i)]\} \\
&= \gamma(D_1 D_2 df - D_2 D_1 df) \\
&= \gamma(D_1 D_2 - D_2 D_1) df \\
&= [D_1, D_2]^C f^C.
\end{aligned}
$$

■

2.10 Lifts of Lie derivatives

Let $\mathcal{L}_X$ denotes the Lie derivative with respect to a vector field X in M. As a derivation, $\mathcal{L}_X$ has components $\mathcal{L}_X:(X^h, -\partial_i X^h)$ where X^h are the components of X. Since $\mathcal{L}_X f = Xf$ and $\mathcal{L}_X \frac{\partial}{\partial x^i} = \left[X, \frac{\partial}{\partial x^i}\right]$, we have $(\mathcal{L}_X)^V = \begin{pmatrix} 0 \\ X^h \end{pmatrix}$ and $(\mathcal{L}_X)^C = \begin{pmatrix} X^h \\ \sum_{i=1}^{n} \bar{y}^i \partial_i X^h \end{pmatrix} = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$. It is obvious that $(\mathcal{L}_X)^V = X^V$ and $(\mathcal{L}_X)^C = X^C$.

We note that if the derivations in Proposition 2.15 are Lie derivatives, then the condition in (ii) is automatically satisfied.

2.11 Lifts of covariant differentiations

Let ∇ be an affine connection in M. The covariant differentiation ∇_X with respect to an element X of $\Im_0^1(M)$ is a derivation in M. Since

$$\nabla_X f = Xf,\ \nabla_X \frac{\partial}{\partial x^i} = \sum_{j,h=1}^{n} X^j \Gamma_{ji}^h \frac{\partial}{\partial x^h} \text{ and } \nabla_X (dx^h) = -\sum_{j,i=1}^{n} X^j \Gamma_{ji}^h dx^i,$$

where X^h and Γ_{ji}^h are respectively the local components of X and ∇ in M and f is in $\Im_0^0(M)$, the covariant differentiation ∇_X has components

$$\nabla_X : (X^h, \sum_{j=1}^{n} X^j \Gamma_{ji}^h).$$

Thus, we have

$$(\nabla_X)^V : \begin{pmatrix} 0 \\ X^h \end{pmatrix} \text{ and } (\nabla_X)^C : \begin{pmatrix} X^h \\ -\sum_{i,j=1}^{n} X^j \bar{y}^i \Gamma_{ji}^h \end{pmatrix}$$

with respect to the induced coordinates in TM.

Proposition 2.16 [Ya1, page 30]

(i) $(\nabla_X)^V = X^V$,

(ii) $(\nabla_X)^C = X^C - \gamma(\hat{\nabla}X)$

for any $X \in \Im_0^1(M)$, where $\hat{\nabla}$ is an affine connection defined by $\hat{\nabla}_Y X = \nabla_X Y + [Y, X]$ and $\hat{\nabla}X$ is a (1, 1) tensor field defined by $(\hat{\nabla}X)Y = \hat{\nabla}_Y X$ for any $X, Y \in \Im_0^1(M)$.

Proof: The first equation is obvious. We will now derive the second equation. We have

$$(\nabla_X)^C = \sum_{h=1}^{n}(X^h \frac{\partial}{\partial x^h} - \sum_{i,j=1}^{n} X^j y^i \Gamma_{ji}^h \frac{\partial}{\partial y^h}).$$

Since

$$\begin{aligned}(\hat{\nabla}X)Y &= \hat{\nabla}_Y X \\ &= \nabla_X Y + [Y, X] \\ &= \sum_{i,h=1}^{n}(X^i \frac{\partial Y^h}{\partial x^i}\frac{\partial}{\partial x^h} + \sum_{j=1}^{n} X^i Y^j \Gamma_{ij}^h \frac{\partial}{\partial x^h} \\ &\quad + Y^i \frac{\partial X^h}{\partial x^i}\frac{\partial}{\partial x^h} - X^i \frac{\partial Y^h}{\partial x^i}\frac{\partial}{\partial x^h}) \\ &= \sum_{i,h=1}^{n} Y^i (\frac{\partial X^h}{\partial x^i} + \sum_{j=1}^{n} X^j \Gamma_{ij}^h)\frac{\partial}{\partial x^h},\end{aligned}$$

we have

$$\hat{\nabla}X = \sum_{i,h=1}^{n}(\frac{\partial X^h}{\partial x^i} + \sum_{j=1}^{n} X^j \Gamma_{ij}^h)\frac{\partial}{\partial x^h} \otimes dx^i .$$

Therefore

$$(X^C - \gamma(\hat{\nabla}X)) = \left(\sum_{h=1}^{n} (X^h \frac{\partial}{\partial x^h} + \partial X^h \frac{\partial}{\partial y^h}) - \sum_{i,h=1}^{n} \bar{y}^i (\frac{\partial X^h}{\partial x^i} + \sum_{j=1}^{n} X^j \Gamma_{ij}^h) \frac{\partial}{\partial y^h} \right)$$

$$= \left(\sum_{h=1}^{n} X^h \frac{\partial}{\partial x^h} - \sum_{i,j,h=1}^{n} X^j \bar{y}^i \Gamma_{ij}^h \frac{\partial}{\partial y^h} \right)$$

$$= (\nabla_X)^C.$$

■

Corollary 2.17 [Ya1, page 30]

For $X \in \Im_0^1(M)$, ∇ an affine connection in *M*, the derivation ∇_X has the properties $(\nabla_X)^C = X^C$ if and only if $\hat{\nabla}X = 0$. Moreover $(\nabla_X)^V = (\nabla_X)^C = 0$ if and only if $X = 0$ in *M*.

2.12 The lifts of a derivation determined by a tensor field of type (1,1)

If a derivation *D* in *M* satisfies the condition $Df = 0$ for any $f \in \Im_0^0(M)$, then *D* determines an elements *F* of $\Im_1^1(M)$ in such a way that $DX = FX$ for any $X \in \Im_0^1(M)$. In this case, *D* is denoted by D_F and is called the derivation determined by *F*.

If F_i^h are the local components of *F* in *M*, namely $F = \sum_{i,h=1}^{n} F_i^h \frac{\partial}{\partial x^h} \otimes dx^i$, then D_F has components D_F:(0, F_i^h).

Let $G, F \in \Im_1^1(M)$, and G_j^k, F_i^h be the local components of *G* and *F* respectively, i.e. $F = \sum_{i,h=1}^{n} F_i^h \frac{\partial}{\partial x^h} \otimes dx^i$, $G = \sum_{j,k=1}^{n} G_j^k \frac{\partial}{\partial x^k} \otimes dx^j$. We denote the tensor $\sum_{i,h,k=1}^{n} G_h^k F_i^h \frac{\partial}{\partial x^k} \otimes dx^i$ by *GF*. Then we have

Proposition 2.18 [Ya1, page 31]

$[D_G, D_F] = D_{[G,F]}$ for any $G, F \in \Im_1^1(M)$, where $[G, F] = GF - FG$.

Proof:

$$\begin{aligned}
[D_G, D_F]X &= D_G D_F X - D_F D_G X \\
&= D_G\left(\sum_{i,h=1}^{n} F_i^h X^i \frac{\partial}{\partial x^h}\right) - D_F\left(\sum_{j,k=1}^{n} G_j^k X^j \frac{\partial}{\partial x^k}\right) \\
&= \sum_{i,h,k=1}^{n} (G_h^k F_i^h X^i)\frac{\partial}{\partial x^k} - \sum_{i,j,h=1}^{n} (G_j^i F_i^h X^j)\frac{\partial}{\partial x^h} \\
&= \sum_{i,j,h=1}^{n} (G_i^h F_j^i - G_j^i F_i^h) X^j \frac{\partial}{\partial x^h},
\end{aligned}$$

$$\begin{aligned}
D_{[G,F]}X &= [G, F]X \\
&= (GF - FG)X \\
&= \sum_{i,j,h=1}^{n} (G_i^h F_j^i - G_j^i F_i^h) X^j \frac{\partial}{\partial x^h}.
\end{aligned}$$

■

Proposition 2.19 [Ya1, page 31]

Let $G, F \in \Im_1^1(M)$. Then

(i) $(D_F)^V = 0$,

(ii) $(D_F)^C = -\gamma F$,

(iii) $[(D_G)^C, (D_F)^C] = (D_{[G,F]})^C$.

Proof: Since we know that D_F has components $D_F : (0, F_i^h)$, where F_i^h are the local components of F in M, from the definition of lifting of derivation, we have

(i) $(D_F)^V : \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$,

(ii) $(D_F)^C : \begin{pmatrix} 0 \\ -\sum_{i=1}^{n} \bar{y}^i F_i^h \end{pmatrix} = -\sum_{i,h=1}^{n} \bar{y}^i F_i^h \frac{\partial}{\partial \bar{y}^h} = -(\gamma F)$,

(iii) from Proposition 2.15 and 2.18,

$$\begin{aligned}[(D_G)^C, (D_F)^C] &= [D_G, D_F]^C \\ &= (D_{[G, F]})^C.\end{aligned}$$

■

Next, we consider the curvature tensor R of the manifold M. Then there is a derivation $D_{R(X, Y)}$ determined by $R(X, Y)$, considered as a (1, 1) tensor. We have

$$D_{R(X, Y)} = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}. \qquad (*)$$

Thus by taking the vertical and complete lifts of both sides in (*), we have

$$0 = [\nabla_X, \nabla_Y]^V - (\nabla_{[X, Y]})^V \text{ and}$$

$$(D_{R(X, Y)})^C = [\nabla_X, \nabla_Y]^C - (\nabla_{[X, Y]})^C.$$

From $(D_F)^C = -\gamma F$ for $F \in \Im_1^1(M)$ we have

$$-\gamma R(X, Y) = [\nabla_X, \nabla_Y]^C - (\nabla_{[X, Y]})^C.$$

Now, from Proposition 2.15, we have the following proposition.

<u>Proposition 2.20 [Ya1, page 32]</u>

(i) $[(\nabla_X)^V, (\nabla_Y)^V] = 0$,

(ii) $[(\nabla_X)^V, (\nabla_Y)^C] = (\nabla_{[X, Y]})^V$ if and only if $\nabla_Z Y = 0$, for any $Z \in \Im_0^1(M)$, i.e. Y is a parallel vector field.

(iii) $[(\nabla_X)^C, (\nabla_Y)^C] = [\nabla_X, \nabla_Y]^C = (\nabla_{[X, Y]})^C - \gamma R(X, Y)$.

2.13 Complete lifts of tensor fields of type (1, 1)

We will first prove a useful proposition:

Proposition 2.21 [Ya1, page 33]

Let $\widetilde{S}, \widetilde{T}$ be elements of $\Im_s^0(TM)$ or $\Im_s^1(TM)$ where $s > 0$, such that $\widetilde{S}(X_s^C, \ldots, X_1^C) = \widetilde{T}(X_s^C, \ldots, X_1^C)$ for any $X_s, \ldots, X_1 \in \Im_0^1(M)$. Then $\widetilde{S} = \widetilde{T}$.

Proof: It is sufficient to prove that if $\widetilde{S}(X_s^C, \ldots, X_1^C) = 0$ for any $X_s, \ldots, X_1 \in \Im_0^1(M)$ then $\widetilde{S} = 0$. We shall prove this proposition by induction.

The case $s = 1$ has been mentioned in page 31. We assume the result for $s = n$. Let $\widetilde{S} \in \Im_{n+1}^1(TM)$ such that $\widetilde{S}(X_{n+1}^C, \ldots, X_1^C) = 0$ for any $X_{n+1}, X_n, \ldots, X_1 \in \Im_0^1(M)$. Take an arbitrary X_{n+1}, write $\widetilde{S}(X_{n+1}^C, \widetilde{X}_n, \ldots, \widetilde{X}_1) = \widetilde{S}_{X_{n+1}^C}(\widetilde{X}_n, \ldots, \widetilde{X}_1)$ for any $\widetilde{X}_n, \ldots, \widetilde{X}_1 \in \Im_0^1(TM)$. Then $\widetilde{S}_{X_{n+1}^C} \in \Im_n^1(TM)$. Since $\widetilde{S}_{X_{n+1}^C}(X_n^C, \ldots, X_1^C) = \widetilde{S}(X_{n+1}^C, X_n^C, \ldots, X_1^C) = 0$ for any $X_n, \ldots, X_1 \in \Im_0^1(M)$, we have $\widetilde{S}_{X_{n+1}^C} = 0$ by the induction hypothesis. Again, for arbitrary $\widetilde{X}_n, \ldots, \widetilde{X}_1 \in \Im_0^1(TM)$, we write $(\widetilde{S}(\widetilde{X}_n, \ldots, \widetilde{X}_1))\widetilde{X}_{n+1} = \widetilde{S}(\widetilde{X}_{n+1}, \widetilde{X}_n, \ldots, \widetilde{X}_1)$. Then $\widetilde{S}(\widetilde{X}_n, \ldots, \widetilde{X}_1) \in \Im_1^1(TM)$. Since $(\widetilde{S}(\widetilde{X}_n, \ldots, \widetilde{X}_1))X_{n+1}^C = \widetilde{S}_{X_{n+1}^C}(\widetilde{X}_n, \ldots, \widetilde{X}_1) = 0$ for any $X_{n+1} \in \Im_0^1(M)$, we have $\widetilde{S}(\widetilde{X}_n, \ldots, \widetilde{X}_1) = 0$ which implies that $\widetilde{S} = 0$. The case for $\widetilde{S} \in \Im_s^0(TM)$ is similar. ■

The complete lift of the identity tensor field I is the identity tensor field of the tangent bundle:

$$\begin{aligned} I^C &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\otimes dx^i\right)^C \\ &= \sum_{i=1}^{n}\left(\left(\frac{\partial}{\partial x^i}\right)^V \otimes (dx^i)^C + \left(\frac{\partial}{\partial x^i}\right)^C \otimes (dx^i)^V\right) \\ &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial \bar{y}^i}\otimes d\bar{y}^i + \frac{\partial}{\partial \bar{x}^i}\otimes d\bar{x}^i\right) \\ &= \sum_{A}\frac{\partial}{\partial \bar{x}^A}\otimes d\bar{x}^A. \end{aligned}$$

Moreover,

$$\begin{aligned} I^V &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\otimes dx^i\right)^V \\ &= \sum_{i=1}^{n}\frac{\partial}{\partial \bar{y}^i}\otimes d\bar{x}^i. \end{aligned}$$

Thus we have $I^V X^V = 0$, $I^V X^C = X^V$ for any $X \in \Im_0^1(M)$.

Proposition 2.22 [Ya1, page 34]

For any $F,\ G \in \Im_1^1(M)$, $(FG)^C = F^C G^C$.

Proof: For any $X \in \Im_0^1(M)$,

$$\begin{aligned} (FG)^C X^C &= (FGX)^C = (F(GX))^C = F^C (GX)^C \\ &= F^C G^C X^C. \ \blacksquare \end{aligned}$$

Corollary 2.23 [Ya1, page 35]

If $P(t)$ is a polynomial in one variable t, then $(P(F))^C = P(F^C)$ for any $F \in \Im_1^1(M)$.

2.14 Complete lifts of tensor fields of type (0,2)

Let $G \in \Im_2^0(M)$. If G_{ji} are local components of G in M, then G^C has components

$$G^C : \begin{pmatrix} \partial G_{ji} & G_{ji} \\ G_{ji} & 0 \end{pmatrix}$$

with respect to the induced coordinates in *TM*.

Let g be a Riemannian metric in M. Then $g = \sum_{j,i=1}^{n} g_{ji} dx^j \otimes dx^i \in \Im_2^0(M)$.

Proposition 2.24 [Ya1, page 38]

If we write $ds^2 = \sum_{j,i=1}^{n} g_{ji} dx^j \otimes dx^i = \sum_{j,i=1}^{n} g_{ji} dx^j dx^i$ for the Riemannian metric g in M, then the complete lift g^C of g is a pseudo-Riemannian metric in *TM*.

Moreover $g^C = \sum_{j,i=1}^{n} 2 g_{ji} \delta\overline{y}^j d\overline{x}^i$, where $\delta\overline{y}^j = d\overline{y}^j + \sum_{l,k=1}^{n} \overline{y}^k \Gamma_{lk}^i d\overline{x}^l$ and Γ_{lk}^i being the Christoffel symbols formed with g_{ji}.

Proof: We already know that

$$g^C = \sum_{j,i=1}^{n}(\partial g_{ji}d\bar{x}^j d\bar{x}^i + 2g_{ji}d\bar{y}^j d\bar{x}^i)$$

$$= \sum_{j,i=1}^{n}(\sum_{k=1}^{n}\bar{y}^k \partial_k g_{ji}d\bar{x}^j d\bar{x}^i + 2g_{ji}d\bar{y}^j d\bar{x}^i).$$

With $\partial_k g_{ji} = \sum_{h=1}^{n}(\Gamma_{kj}^h g_{hi} + \Gamma_{ki}^h g_{jh})$,

$$g^C = \sum_{j,i=1}^{n}(\partial g_{ji}d\bar{x}^j \otimes d\bar{x}^i + g_{ji}d\bar{y}^j \otimes d\bar{x}^i + g_{ji}d\bar{x}^j \otimes d\bar{y}^i)$$

$$= \sum_{j,i=1}^{n}(\sum_{h,k=1}^{n}\bar{y}^k(\Gamma_{kj}^h g_{hi} + \Gamma_{ki}^h g_{jh})d\bar{x}^j \otimes d\bar{x}^i + g_{ji}d\bar{y}^j \otimes d\bar{x}^i + g_{ji}d\bar{x}^j \otimes d\bar{y}^i)$$

$$= \sum_{j,i=1}^{n}\left(\sum_{h,k=1}^{n}(\bar{y}^k\Gamma_{kj}^h g_{hi}d\bar{x}^j + g_{ji}d\bar{y}^j)\otimes d\bar{x}^i + d\bar{x}^j \otimes(\sum_{k,h=1}^{n}\bar{y}^k\Gamma_{ki}^h g_{jh}d\bar{x}^i + g_{ji}d\bar{y}^i)\right)$$

$$= \sum_{j,i=1}^{n}g_{ji}\left(\left(\sum_{h,k=1}^{n}\bar{y}^k\Gamma_{kh}^j d\bar{x}^h + d\bar{y}^j\right)\otimes d\bar{x}^i + d\bar{x}^j \otimes\left(\sum_{k,h=1}^{n}\bar{y}^k\Gamma_{kh}^i d\bar{x}^h + d\bar{y}^i\right)\right)$$

$$= \sum_{j,i=1}^{n}g_{ji}(\delta\bar{y}^j \otimes d\bar{x}^i + d\bar{x}^j \otimes \delta\bar{y}^i)$$

$$= \sum_{j,i=1}^{n}2g_{ji}\delta\bar{y}^j d\bar{x}^i.$$

■

2.15 Complete lifts of affine connections

Let M be a manifold with an affine connection ∇. Then there exists a unique affine connection ∇^C in TM which satisfies

$$\nabla^C_{X^C}Y^C = (\nabla_X Y)^C.$$

We verify this by using the components of the connection. Let Γ_{ji}^h be the components of ∇ with respect to local coordinates $(x^1, \ldots, x^n)$ in M and denote $\overline{\Gamma}_{CB}^A$ as components of ∇^C with respect to the induced coordinates $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y}^1, \ldots, \bar{y}^n)$ in $T(M)$. Let X, Y be arbitrary vector fields with components X^h and Y^h respectively, with respect to the local coordinates $(x^1, \ldots, x^n)$ in M. Then X^C and Y^C have components $X^C\!:\begin{pmatrix} \overline{X}^h \\ \overline{X}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$ and $Y^C\!:\begin{pmatrix} \overline{Y}^h \\ \overline{Y}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} Y^h \\ \partial Y^h \end{pmatrix}$ respectively. Note that

$$\sum_{j=1}^{n}\{\overline{X}^j(\partial_j \overline{Y}^h + \sum_{i=1}^{n}(\overline{\Gamma}^h_{ji}\overline{Y}^i + \overline{\Gamma}^h_{j\bar{i}}\overline{Y}^{\bar{i}})) + \overline{X}^{\bar{j}}(\partial_{\bar{j}}\overline{Y}^h + \sum_{i=1}^{n}(\overline{\Gamma}^h_{\bar{j}i}\overline{Y}^i + \overline{\Gamma}^h_{\bar{j}\bar{i}}\overline{Y}^{\bar{i}}))\}$$
$$= \sum_{j=1}^{n} X^j(\partial_j Y^h + \sum_{i=1}^{n}\Gamma^h_{ji}Y^i),$$

from which we have

$$\overline{\Gamma}^h_{ji} = \Gamma^h_{ji}, \quad \overline{\Gamma}^h_{\bar{j}i} = \overline{\Gamma}^h_{j\bar{i}} = \overline{\Gamma}^h_{\bar{j}\bar{i}} = 0,$$

$$\overline{\Gamma}^{\bar{h}}_{ji} = \partial\Gamma^h_{ji}, \quad \overline{\Gamma}^{\bar{h}}_{\bar{j}i} = \overline{\Gamma}^{\bar{h}}_{j\bar{i}} = \Gamma^h_{ji}, \quad \overline{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}} = 0.$$

∇^C is called the *complete lift* of the affine connection ∇ to *TM*.

If $(z^1, z^2, \ldots, z^n)$ is another set of coordinates in M, we obtain another set of functions $\{\hat{\Gamma}^k_{ij}\}$ by

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \sum_{k=1}^{n} \hat{\Gamma}^k_{ij} \frac{\partial}{\partial z^k}$$

and they satisfy equation (*) on page 11, namely

$$\hat{\Gamma}^k_{ij} = \sum_{a,b,c=1}^{n} \frac{\partial x^a}{\partial z^i}\frac{\partial x^b}{\partial z^j}\frac{\partial z^k}{\partial x^c}\Gamma^c_{ab} + \sum_{a=1}^{n}\frac{\partial^2 x^a}{\partial z^i \partial z^j}\frac{\partial z^k}{\partial x^a}.$$

Let $(\bar{z}^1, \bar{z}^2, \ldots, \bar{z}^n, \bar{z}^{\bar{1}}, \bar{z}^{\bar{2}}, \ldots, \bar{z}^{\bar{n}})$ be the set of coordinates on *TM* with respect to $(z^1, z^2, \ldots, z^n)$. Then we have the set of functions $\{\hat{\Gamma}^k_{ij}\}$ defined by $\nabla^C_{\frac{\partial}{\partial \bar{z}^B}} \frac{\partial}{\partial \bar{z}^A} = \sum_C \hat{\overline{\Gamma}}^C_{BA} \frac{\partial}{\partial \bar{z}^C}$, where

$$\overline{\hat{\Gamma}}{}^{h}_{ji} = \hat{\Gamma}^{h}_{ji}\,, \quad \overline{\hat{\Gamma}}{}^{h}_{\bar{j}i} = \overline{\hat{\Gamma}}{}^{h}_{j\bar{i}} = \overline{\hat{\Gamma}}{}^{h}_{\bar{j}\bar{i}} = 0,$$

$$\overline{\hat{\Gamma}}{}^{\bar{h}}_{ji} = \partial\hat{\Gamma}^{h}_{ji}\,, \quad \overline{\hat{\Gamma}}{}^{\bar{h}}_{\bar{j}i} = \overline{\hat{\Gamma}}{}^{\bar{h}}_{j\bar{i}} = \hat{\Gamma}^{h}_{ji}\,, \quad \overline{\hat{\Gamma}}{}^{\bar{h}}_{\bar{j}\bar{i}} = 0.$$

We find that

$$\begin{aligned}
\sum_{A,B,C} \frac{\partial\bar{x}^A}{\partial\bar{z}^i}\frac{\partial\bar{x}^B}{\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^C}\overline{\Gamma}{}^{C}_{AB} + \sum_{A} \frac{\partial^2\bar{x}^A}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^A} &= \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^c}\overline{\Gamma}{}^{c}_{ab} + \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^{\bar{c}}}\overline{\Gamma}{}^{\bar{c}}_{ab} \\
&+ \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^{\bar{a}}}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^{\bar{c}}}\overline{\Gamma}{}^{\bar{c}}_{\bar{a}b} + \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^{\bar{b}}}{\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^{\bar{c}}}\overline{\Gamma}{}^{\bar{c}}_{a\bar{b}} \\
&+ \sum_{a=1}^{n} \frac{\partial^2\bar{x}^a}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^a} + \sum_{a=1}^{n} \frac{\partial^2\bar{x}^{\bar{a}}}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^{\bar{a}}} \\
&= \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^c}\overline{\Gamma}{}^{c}_{ab} + \sum_{a=1}^{n} \frac{\partial^2\bar{x}^a}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^a} \\
&= \sum_{a,b,c=1}^{n} \frac{\partial x^a}{\partial z^i}\frac{\partial x^b}{\partial z^j}\frac{\partial z^k}{\partial x^c}\Gamma^{c}_{ab} + \sum_{a=1}^{n} \frac{\partial^2 x^a}{\partial z^i\partial z^j}\frac{\partial z^k}{\partial x^a} \\
&= \hat{\Gamma}^{k}_{ij} = \overline{\hat{\Gamma}}{}^{k}_{ij},
\end{aligned}$$

$$\begin{aligned}
\sum_{A,B,C} \frac{\partial\bar{x}^A}{\partial\bar{z}^i}\frac{\partial\bar{x}^B}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^C}\overline{\Gamma}{}^{C}_{AB} + \sum_{A} \frac{\partial^2\bar{x}^A}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^A} &= \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^c}\overline{\Gamma}{}^{c}_{ab} + \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\overline{\Gamma}{}^{\bar{c}}_{ab} \\
&+ \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^{\bar{a}}}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\overline{\Gamma}{}^{\bar{c}}_{\bar{a}b} + \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^{\bar{b}}}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\overline{\Gamma}{}^{\bar{c}}_{a\bar{b}} \\
&+ \sum_{a=1}^{n} \frac{\partial^2\bar{x}^a}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^a} + \sum_{a=1}^{n} \frac{\partial^2\bar{x}^{\bar{a}}}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{a}}} \\
&= \sum_{a,b,c=1}^{n} \frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^c}\overline{\Gamma}{}^{c}_{ab} + \sum_{a=1}^{n} \frac{\partial^2\bar{x}^a}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^k}{\partial\bar{x}^a} \\
&= \sum_{a,b,c=1}^{n} \frac{\partial x^a}{\partial z^i}\frac{\partial x^b}{\partial z^j}\frac{\partial z^k}{\partial x^c}\Gamma^{c}_{ab} + \sum_{a=1}^{n} \frac{\partial^2 x^a}{\partial z^i\partial z^j}\frac{\partial z^k}{\partial x^a} \\
&= \hat{\Gamma}^{k}_{ij} = \overline{\hat{\Gamma}}{}^{k}_{ij},
\end{aligned}$$

$$\sum_{A,B,C}\frac{\partial\bar{x}^A}{\partial\bar{z}^i}\frac{\partial\bar{x}^B}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^C}\bar{\Gamma}^C_{AB}+\sum_A\frac{\partial^2\bar{x}^A}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^A}=\sum_{a,b,c=1}^n\frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^c}\bar{\Gamma}^c_{ab}+\sum_{a,b,c=1}^n\frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\bar{\Gamma}^{\bar{c}}_{ab}$$

$$+\sum_{a,b,c=1}^n\frac{\partial\bar{x}^{\bar{a}}}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\bar{\Gamma}^{\bar{c}}_{\bar{a}b}+\sum_{a,b,c=1}^n\frac{\partial\bar{x}^a}{\partial\bar{z}^i}\frac{\partial\bar{x}^{\bar{b}}}{\partial\bar{z}^j}\frac{\partial\bar{z}z^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\bar{\Gamma}^{\bar{c}}_{a\bar{b}}$$

$$+\sum_{a,b,c=1}^n\frac{\partial\bar{x}^{\bar{a}}}{\partial\bar{z}^i}\frac{\partial\bar{x}^{\bar{b}}}{\partial\bar{z}^j}\frac{\partial\bar{z}z^{\bar{k}}}{\partial\bar{x}^c}\bar{\Gamma}^c_{\bar{a}\bar{b}}+\sum_{a,b,c=1}^n\frac{\partial\bar{x}^{\bar{a}}}{\partial\bar{z}^i}\frac{\partial\bar{x}^{\bar{b}}}{\partial\bar{z}^j}\frac{\partial\bar{z}z^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\bar{\Gamma}^{\bar{c}}_{\bar{a}\bar{b}}$$

$$+\sum_{a=1}^n\frac{\partial^2\bar{x}^a}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^a}+\sum_{a=1}^n\frac{\partial^2\bar{x}^{\bar{a}}}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{a}}}$$

$$=\sum_{a,b,c=1}^n\frac{\partial\bar{x}^{\bar{a}}}{\partial\bar{z}^i}\frac{\partial\bar{x}^b}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{c}}}\bar{\Gamma}^{\bar{c}}_{\bar{a}b}+\sum_{a,b,c=1}^n\frac{\partial\bar{x}^{\bar{a}}}{\partial\bar{z}^i}\frac{\partial\bar{x}^{\bar{b}}}{\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^c}\bar{\Gamma}^c_{\bar{a}\bar{b}}$$

$$+\sum_{a=1}^n\frac{\partial^2\bar{x}^{\bar{a}}}{\partial\bar{z}^i\partial\bar{z}^j}\frac{\partial\bar{z}^{\bar{k}}}{\partial\bar{x}^{\bar{a}}}$$

$$=\sum_{a,b,c=1}^n\frac{\partial x^a}{\partial z^i}\frac{\partial x^b}{\partial z^j}\frac{\partial z^k}{\partial x^c}\Gamma^c_{ab}+\sum_{a=1}^n\frac{\partial^2x^a}{\partial z^i\partial z^j}\frac{\partial z^k}{\partial x^a}$$

$$=\hat{\Gamma}^k_{ij}=\bar{\hat{\Gamma}}^{\bar{k}}_{\bar{i}\bar{j}},$$

similarly for the rest of the $\{\bar{\hat{\Gamma}}^A_{BC}\}$ and $\{\bar{\Gamma}^A_{BC}\}$.

From the above computations, we can see that the families of functions $\{\bar{\hat{\Gamma}}^A_{BC}\}$ and $\{\bar{\Gamma}^A_{BC}\}$ satisfies equation (*) on page 10, hence they define the connection ∇^C.

Proposition 2.25 [Ya1, page 41]

If T and R are the torsion and curvature tensors of ∇ respectively, then the liftings T^C and R^C are the torsion and the curvature tensors of ∇^C respectively.

Proof: From Proposition 2.5, Proposition 2.12 and the definition of the connection ∇^C, we have

$$\begin{aligned}T^C(X^C,Y^C)&=(T(X,Y))^C\\&=(\nabla_XY-\nabla_YX-[X,Y])^C\\&=\nabla^C{}_{X^C}Y^C-\nabla^C{}_{Y^C}X^C-[X^C,Y^C],\end{aligned}$$

$$
\begin{aligned}
R^C(X^C, Y^C)Z^C &= (R(X, Y)Z)^C \\
&= (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z)^C \\
&= \nabla^C{}_{X^C} \nabla^C{}_{Y^C} Z^C - \nabla^C{}_{Y^C} \nabla^C{}_{X^C} Z^C - \nabla^C{}_{[X^C, Y^C]} Z^C
\end{aligned}
$$

for any $X, Y, Z \in \Im_0^1(M)$. Thus the proposition is proved. ■

The components of T and R are respectively given by

$$
T_{ji}^h = \Gamma_{ji}^h - \Gamma_{ij}^h ,
$$

$$
R_{kji}{}^h = \partial_k \Gamma_{ji}^h - \partial_j \Gamma_{ki}^h + \sum_{t=1}^{n} (\Gamma_{kt}^h \Gamma_{ji}^t - \Gamma_{jt}^h \Gamma_{ki}^t).
$$

Since $T \in \Im_2^1(M)$ and $R \in \Im_3^1(M)$, then from page 36, the components $\bar{T}_{CB}^A$ of T^C and $\bar{R}_{DCB}{}^A$ of R^C are given by

$$
\begin{aligned}
&\bar{T}_{ji}^h = T_{ji}^h , \\
&\bar{T}_{ji}^{\bar{h}} = \partial T_{ji}^h , \ \bar{T}_{j\bar{i}}^{\bar{h}} = T_{ji}^h , \ \bar{T}_{\bar{j}i}^{\bar{h}} = T_{ji}^h ,
\end{aligned}
$$

$$
\begin{aligned}
&\bar{R}_{kji}{}^h = R_{kji}{}^h , \ \bar{R}_{kji}{}^{\bar{h}} = \partial R_{kji}{}^h , \\
&\bar{R}_{kj\bar{i}}{}^{\bar{h}} = R_{kji}{}^h , \ \bar{R}_{k\bar{j}i}{}^{\bar{h}} = R_{kji}{}^h , \ \bar{R}_{\bar{k}ji}{}^{\bar{h}} = R_{kji}{}^h ,
\end{aligned}
$$

all the others being zero, with respect to the induced coordinates in *TM*.

We now consider the action of ∇^C on $\tilde{f} \in \Im_0^0(TM)$ and $\tilde{\omega} \in \Im_1^0(TM)$. Since ∇^C is an affine connection on *TM*, it follows that $\nabla^C{}_{\tilde{X}} \tilde{f} = \tilde{X}\tilde{f}$ for any $\tilde{X} \in \Im_0^1(TM)$. Thus we have [Ya1, page 42]

$$
\nabla^C{}_{X^V} f^V = X^V f^V = 0 ,
$$

$$
\begin{aligned}
\nabla^C{}_{X^V} f^C &= X^V f^C \\
&= (Xf)^V \\
&= (\nabla_X f)^V ,
\end{aligned}
$$

$$\begin{aligned}\nabla^C{}_{X^C} f^V &= X^C f^V \\ &= (Xf)^V \\ &= (\nabla_X f)^V,\end{aligned}$$

$$\begin{aligned}\nabla^C{}_{X^C} f^C &= X^C f^C \\ &= (Xf)^C \\ &= (\nabla_X f)^C,\end{aligned}$$

for any $X \in \Im_0^1(M)$ and $f \in \Im_0^0(M)$.

Furthermore, for any $X,\ Y \in \Im_0^1(M)$, let $X^h,\ Y^k$ be the local components of X, Y respectively. Then [Ya1, page 43]

$$\begin{aligned}\nabla^C{}_{X^V} Y^V &= \sum_{j,k=1}^{n}(X^j \partial_j Y^k \partial_k + \sum_{i=1}^{n}(X^j Y^i \overline{\Gamma}_{\bar{j}i}^{k} \partial_k + X^j(\partial Y^i)\overline{\Gamma}_{\bar{j}i}^{\bar{k}} \partial_{\bar{k}})) \\ &= 0,\end{aligned}$$

$$\begin{aligned}\nabla^C{}_{X^V} Y^C &= \sum_{j,k=1}^{n}\{X^j(\partial_{\bar{j}} Y^k)\partial_k + X^j(\partial_{\bar{j}} \partial Y^k)\partial_{\bar{k}} \\ &\quad + \sum_{i=1}^{n}(X^j Y^i(\overline{\Gamma}_{\bar{j}i}^{k}\partial_k + \overline{\Gamma}_{\bar{j}i}^{\bar{k}}\partial_{\bar{k}}) + X^j(\partial Y^i)(\overline{\Gamma}_{\bar{j}\bar{i}}^{k}\partial_k + \overline{\Gamma}_{\bar{j}\bar{i}}^{\bar{k}}\partial_{\bar{k}}))\} \\ &= \sum_{j,k=1}^{n}\{X^j(\partial_j Y^k)\partial_{\bar{k}} + \sum_{i=1}^{n}(X^j Y^i \overline{\Gamma}_{\bar{j}i}^{\bar{k}}\partial_{\bar{k}})\} \\ &= \sum_{j,k=1}^{n}\{X^j(\partial_j Y^k) + \sum_{i=1}^{n} X^j Y^i \Gamma_{ji}^{k}\}\partial_{\bar{k}} \\ &= (\nabla_X Y)^V,\end{aligned}$$

$$\begin{aligned}\nabla^C{}_{X^C} Y^V &= \sum_{j,k=1}^{n}\{X^j(\partial_j Y^k)\partial_{\bar{k}} + (\partial X^j)(\partial_{\bar{j}} Y^k)\partial_{\bar{k}} \\ &\quad + \sum_{i=1}^{n}(X^j Y^i(\overline{\Gamma}_{ji}^{k}\partial_k + \overline{\Gamma}_{ji}^{\bar{k}}\partial_{\bar{k}}) + (\partial X^j)Y^i(\overline{\Gamma}_{\bar{j}i}^{k}\partial_k + \overline{\Gamma}_{\bar{j}i}^{\bar{k}}\partial_{\bar{k}}))\} \\ &= \sum_{j,k=1}^{n}\{X^j(\partial_j Y^k) + \sum_{i=1}^{n} X^j Y^i \Gamma_{ji}^{k}\}\partial_{\bar{k}} \\ &= (\nabla_X Y)^V,\end{aligned}$$

$$\nabla^C{}_{X^C} Y^C = (\nabla_X Y)^C.$$

Since $\nabla^C{}_{X^V}$ and $\nabla^C{}_{X^C}$ are covariant differentiations with respect to X^V and X^C respectively, then for any $\omega \in \Im_1^0(M)$ and $X,\ Y \in \Im_0^1(M)$, [Ya1, page 44]

$$\begin{aligned}(\nabla^C{}_{X^V}\omega^V)Y^C &= \nabla^C{}_{X^V}(\omega^V Y^C) - \omega^V(\nabla^C{}_{X^V}Y^C)\\ &= \nabla^C{}_{X^V}(\omega Y)^V - \omega^V(\nabla_X Y)^V\\ &= 0,\end{aligned}$$

$$\begin{aligned}(\nabla^C{}_{X^V}\omega^C)Y^C &= \nabla^C{}_{X^V}(\omega^C Y^C) - \omega^C(\nabla^C{}_{X^V}Y^C)\\ &= \nabla^C{}_{X^V}(\omega Y)^C - \omega^C(\nabla_X Y)^V\\ &= (\nabla_X(\omega Y))^V - (\omega(\nabla_X Y))^V\\ &= ((\nabla_X\omega)Y)^V = (\nabla_X\omega)^V Y^C,\end{aligned}$$

$$\begin{aligned}(\nabla^C{}_{X^C}\omega^V)Y^C &= \nabla^C{}_{X^C}(\omega^V Y^C) - \omega^V(\nabla^C{}_{X^C}Y^C)\\ &= \nabla^C{}_{X^C}(\omega Y)^V - (\omega(\nabla_X Y))^V\\ &= ((\nabla_X\omega)Y)^V = (\nabla_X\omega)^V Y^C,\end{aligned}$$

$$\begin{aligned}(\nabla^C{}_{X^C}\omega^C)Y^C &= \nabla^C{}_{X^C}(\omega^C Y^C) - \omega^C(\nabla^C{}_{X^C}Y^C)\\ &= \nabla^C{}_{X^C}(\omega Y)^C - \omega^C(\nabla_X Y)^C\\ &= (\nabla_X(\omega Y) - \omega(\nabla_X Y))^C\\ &= ((\nabla_X\omega)Y)^C = (\nabla_X\omega)^C Y^C.\end{aligned}$$

Thus

$$\nabla^C{}_{X^V}\omega^V = 0, \qquad \nabla^C{}_{X^V}\omega^C = (\nabla_X\omega)^V,$$
$$\nabla^C{}_{X^C}\omega^V = (\nabla_X\omega)^V, \quad \nabla^C{}_{X^C}\omega^C = (\nabla_X\omega)^C.$$

As an extention of the above results, we have [Ya1, page 45]

$$\begin{aligned}\nabla^C{}_{X^V}K^V &= 0,\\ \nabla^C{}_{X^V}K^C &= (\nabla_X K)^V,\\ \nabla^C{}_{X^C}K^V &= (\nabla_X K)^V,\\ \nabla^C{}_{X^C}K^C &= (\nabla_X K)^C\end{aligned}$$

for any tensor field K in M. Furthermore

$$\left.\begin{aligned} \nabla^C K^V &= (\nabla K)^V, \\ \nabla^C K^C &= (\nabla K)^C. \end{aligned}\right\} \quad (**)$$

Proposition 2.26 [Ya1, page 45]

If ∇ is the Riemannian connection of a manifold M with respect to a Riemannian metric g, then ∇^C is the Riemannian connection of TM with respect to the pseudo-Riemannian metric g^C.

Proof: We already know that ∇^C defines an affine connection on TM which is torsion free, from (**), we have $\nabla^C g^C = (\nabla g)^C = 0$. This prove the result. ■

We observe that the above result is also true for a pseudo-Riemannian metric.

2.16 Horizontal lifts of vector fields

In the previous section, we have already defined two types of liftings of tensor fields on M to TM, the complete and vertical lifts. We shall now consider another important type of lifting, called *horizontal lifts* of tensor fields. In this case, an affine connection will be needed. Therefore we assume that the manifold M we deal with is an affine manifold, namely a differentiable manifold with an affine connection.

Let f be a function in M, γ the operation on tensor fields defined on page 25 and ∇ the affine connection of M. We write ∇f for the gradient of f in M. Then $\nabla_\gamma f = \gamma(\nabla f)$.

We now define the horizontal lift f^H of f in M to TM by $f^H = f^C - \nabla_\gamma f$. Then we have $f^H = 0$.

Let $X \in \Im_0^1(M)$, the horizontal lift X^H of X is defined by $X^H = X^C - \nabla_\gamma X$ in *TM*, where $\nabla_\gamma X = \gamma(\nabla X)$.

Suppose that X and ∇ have local components X^h and Γ_{ji}^h respectively in M. Then

$X^C = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$ and $\nabla_\gamma X = \begin{pmatrix} 0 \\ \sum_{j=1}^{n} \bar{y}^j \nabla_j X^h \end{pmatrix}$ with respect to the induced coordinates $(\bar{x}^h, \bar{y}^h)$ in *TM*, $\nabla_j X^h$ being the covariant derivative of X^h:

$$\nabla_j X^h = \partial_j X^h + \sum_{i=1}^{n} \Gamma_{ji}^h X^i .$$

We shall denote the horizontal lift of X in component form as follows:

$$X^H = \begin{pmatrix} X^h \\ -\sum_{i=1}^{n} \Gamma_i^h X^i \end{pmatrix}, \text{ where } \sum_{i=1}^{n} \Gamma_i^h X^i = \sum_{i,j=1}^{n} \Gamma_{ji}^h X^i \bar{y}^j ,$$

that is, $\Gamma_i^h = \sum_{i=1}^{n} \Gamma_{ji}^h \bar{y}^j$.

A vector field $\widetilde{X}$ is said to be projectable if there exists an element $X \in \Im_0^1(M)$ such that $\widetilde{X} - X^C$ is vertical, then X is called the projection of $\widetilde{X}$. Thus the horizontal lift X^H of X in M to *TM* is a projectable vector field with projection X.

Recall that $\hat{\nabla}_X Y = \nabla_Y X + [X, Y]$,

$$\hat{\nabla}_X Y = \sum_{i,h=1}^{n} \left(X^i \partial_i Y^h + \sum_{j=1}^{n} Y^j X^i \Gamma_{ji}^h \right) \partial_h . \qquad (*)$$

Since $\hat{\nabla}$ is an affine connection in M, then $\hat{\nabla}_X f = Xf$ and from (*), $\hat{\nabla}_X \partial_j = \sum_{i,h=1}^{n} X^i \Gamma_{ji}^h \partial_h$. Thus $\hat{\nabla}_X$ as a derivation in M has components $\hat{\nabla}_X : \left(X^h, \sum_{i=1}^{n} X^i \Gamma_{ji}^h \right)$. The complete lift of the derivation $\hat{\nabla}_X$, $(\hat{\nabla}_X)^C$ is a vector field in TM with components

$$(\hat{\nabla}_X)^C : \begin{pmatrix} X^h \\ -\sum_{i,j=1}^{n} X^i \bar{y}^j \Gamma_{ji}^h \end{pmatrix}$$

and this coincides with X^H. Thus $X^H = (\hat{\nabla}_X)^C$ for any $X \in \Im_0^1(M)$.

We can see that the components of X^H satisfy the following equations

$$\bar{X}^{\bar{h}} = -\sum_{i,j=1}^{n} \bar{X}^i \bar{y}^j \Gamma_{ji}^h, \qquad (**)$$

where we write $X^H : \begin{pmatrix} \bar{X}^h \\ \bar{X}^{\bar{h}} \end{pmatrix}$.

The *horizontal distribution* is the set consisting of all the vector fields that satisfy (**). Any vector field in this distribution is called a horizontal vector field. Thus a vector field $\bar{X}$ in TM with components $\bar{X} : \begin{pmatrix} \bar{X}^h \\ \bar{X}^{\bar{h}} \end{pmatrix}$ is horizontal if and only if

$$\bar{X}^{\bar{h}} + \sum_{i,j=1}^{n} \bar{X}^j \bar{y}^i \Gamma_{ij}^h = 0 \text{ for all } h = 1, \ldots, n.$$

Since $\nabla_\gamma X = \gamma(\nabla X) \in \Im_0^1(TM)$, then $(\nabla_\gamma X)f^V = 0$, $(\nabla_\gamma X)f^C = \gamma(df \circ \nabla X)$ for any $f \in \Im_0^0(M)$ and $X \in \Im_0^1(M)$ [Ya1, page 81]. Moreover, for any $X,\ Y \in \Im_0^1(M)$ [Ya1, page 89]

$$
\begin{aligned}
[\nabla_\gamma X,\ \nabla_\gamma Y] &= -\gamma(\nabla X \nabla Y - \nabla Y \nabla X)\,, \\
[X^V,\ \nabla_\gamma Y] &= \gamma_X(\nabla Y) = (\nabla_X Y)^V\,, \\
[X^C,\ \nabla_\gamma Y] &= \gamma(\mathcal{L}_X(\nabla Y))\,.
\end{aligned}
$$

We now look at some properties of the horizontal lifts of vector fields:

$$
\begin{aligned}
X^H f^V &= (X^C - \nabla_\gamma X) f^V \\
&= (Xf)^V - (\nabla_\gamma X) f^V \\
&= (Xf)^V,
\end{aligned}
$$

$$
\begin{aligned}
X^H f^C &= (X^C - \nabla_\gamma X) f^C \\
&= X^C f^C - (\nabla_\gamma X) f^C \\
&= (Xf)^C - \gamma(df \circ \nabla X)\,.
\end{aligned}
$$

For the Lie product, we have

<u>Proposition 2.27</u> [Ya1, page 89]

(i) $[\gamma_X F,\ Y^H] = -\gamma_X(\mathcal{L}_Y F + (\nabla Y)F) + \gamma_{[X,\,Y]}F\,,$

(ii) $[\gamma F,\ Y^H] = -\gamma(\mathcal{L}_Y F + (\nabla Y)F - F(\nabla Y))$

for any $X,\ Y \in \Im_0^1(M)$, $F \in \Im_1^1(M)$.

<u>Proof</u>: From Proposition 2.6 and the properties on page 25, we have

$$
\begin{aligned}
[\gamma_X F, Y^H] &= [\gamma_X F, Y^C - \gamma(\nabla Y)] \\
&= [\gamma_X F, Y^C] - [\gamma_X F, \gamma(\nabla Y)] \\
&= -\gamma_X(\mathcal{L}_Y F) - \gamma_{[X,Y]}F - \gamma_X((\nabla Y)F) \\
&= -\gamma_X((\mathcal{L}_Y F) + (\nabla Y)F) - \gamma_{[X,Y]}F,
\end{aligned}
$$

$$
\begin{aligned}
[\gamma F, Y^H] &= [\gamma F, Y^C - \gamma(\nabla Y)] \\
&= [\gamma F, Y^C] - [\gamma F, \gamma(\nabla Y)] \\
&= -\gamma(\mathcal{L}_Y F) + \gamma((\nabla Y)F - F(\nabla Y)) \\
&= -\gamma(\mathcal{L}_Y F + (\nabla Y)F - F(\nabla Y)),
\end{aligned}
$$

this yields the result. ■

Proposition 2.28 [Ya1, page 90]

(i) $[X^V, Y^H] = [X, Y]^V - (\nabla_X Y)^V = -(\hat{\nabla}_Y X)^V$,

(ii) $[X^C, Y^H] = [X, Y]^H - \gamma(L_X Y)$,

(iii) $[X^H, Y^H] = [X, Y]^H - \gamma\hat{R}(X, Y)$

for any $X, Y \in \mathfrak{I}^1_0(M)$, where $(L_X Y)Z$ is defined to be $(L_X Y)Z = (\mathcal{L}_X \hat{\nabla})(Y, Z) = (\mathcal{L}_X \nabla)(Z, Y)$ and $\hat{R}$ is the curvature tensor of the affine connection $\hat{\nabla}$.

Proof: We have

$$
\begin{aligned}
[X^V, Y^H] &= [X^V, Y^C - \gamma(\nabla Y)] \\
&= [X^V, Y^C] - [X^V, \gamma(\nabla Y)] \\
&= [X, Y]^V - (\nabla_X Y)^V,
\end{aligned}
$$

$$
\begin{aligned}
[X^C, Y^H] &= [X^C, Y^C] - [X^C, \gamma(\nabla Y)] \\
&= [X, Y]^C - \gamma(\mathcal{L}_X(\nabla Y)) \\
&= [X, Y]^C - \gamma(\nabla(\mathcal{L}_X Y)) - \gamma((\mathcal{L}_X \nabla) Y) \\
&= [X, Y]^C - \gamma(\nabla([X, Y]) - \gamma((\mathcal{L}_X \nabla) Y) \\
&= [X, Y]^H - \gamma(L_X Y).
\end{aligned}
$$

Also, we have $X^H = (\hat{\nabla}_X)^C$, $Y^H = (\hat{\nabla}_Y)^C$. It now follows from Proposition 2.20 that

$$
\begin{aligned}
[X^H, Y^H] &= [(\hat{\nabla}_X)^C, (\hat{\nabla}_Y)^C] \\
&= (\hat{\nabla}_{[X,Y]})^C - \gamma\hat{R}(X, Y) \\
&= [X, Y]^H - \gamma\hat{R}(X, Y).
\end{aligned}
$$

■

Proposition 2.29 [Ya1, page 91]

$$F^V X^H = (FX)^V, \; F^C X^H = (FX)^H + (\nabla_\gamma F) X^H$$

for any $X \in \Im_0^1(M)$, $F \in \Im_1^1(M)$.

Proof: Let F_i^h, X^j be local components of F and X respectively.

Then $F^V : \begin{pmatrix} 0 & 0 \\ F_i^h & 0 \end{pmatrix}$, $F^C : \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}$. Thus

$$
\begin{aligned}
F^V X^H &= \sum_{h,i=1}^{n} F_i^h X^i \partial_{\bar{h}} \\
&= (FX)^V,
\end{aligned}
$$

$$
\begin{aligned}
F^C X^H &= F^C (X^C - \gamma\nabla X) \\
&= F^C X^C - F^C (\gamma\nabla X) \\
&= (FX)^C - F^C (\gamma\nabla X),
\end{aligned}
$$

since $(FX)^H = (FX)^C - \gamma\nabla(FX)$, then

$$F^C X^H = (FX)^H + \gamma\nabla(FX) - F^C(\gamma\nabla X).$$

From

$$F^C(\gamma\nabla X) = \sum_{k,h,j=1}^{n} \bar{y}^j \left(\partial_j X^h + \sum_{i=1}^{n} \Gamma_{ji}^h X^i \right) F_h^k \partial_{\bar{k}} \text{ , and}$$

$$\gamma\nabla(FX) = \sum_{k,h,j=1}^{n} \bar{y}^j \left(\partial_j (F_h^k X^h) + \sum_{l=1}^{n} \Gamma_{ji}^k F_h^i X^h \right) \partial_{\bar{k}} \text{ ,}$$

$$\begin{aligned} \nabla_\gamma F &= \gamma(\nabla F) \\ &= \gamma \sum_{k,h,i=1}^{n} (\nabla_i F_h^k) \partial_k \otimes dx^i \otimes dx^h \\ &= \sum_{k,h,i=1}^{n} (\nabla_i F_h^k) \bar{y}^i \partial_{\bar{k}} \otimes dx^h, \end{aligned}$$

we have

$$\begin{aligned} (\nabla_\gamma F) X^H &= \sum_{k,h,i=1}^{n} (\nabla_i F_h^k) \bar{y}^i X^h \partial_{\bar{k}} \\ &= \sum_{k,h,i=1}^{n} (\partial_i F_h^k + \sum_{j=1}^{n} (\Gamma_{ih}^j F_j^k - \Gamma_{ij}^k F_h^j)) \bar{y}^i X^h \partial_{\bar{k}} \\ &= \gamma(\nabla FX) - F^C(\gamma\nabla X). \end{aligned}$$

Thus $F^C X^H = (FX)^H + (\nabla_\gamma F) X^H$. ■

<u>Proposition 2.30 [Ya1, page 91]</u>

(i) $\omega^V(X^H) = (\omega X)^V$,

(ii) $\omega^C(X^H) = (\omega X)^C - \gamma(\omega \circ (\nabla X))$

for any $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$.

<u>Proof</u>: $\omega^V(X^H) = \omega^V(X^C) - \omega^V(\nabla_\gamma X) = (\omega X)^V$

$$\omega^C(X^H) = \omega^C(X^C) - \omega^C(\nabla_\gamma X)$$
$$= (\omega X)^C - \gamma(\omega \circ (\nabla X)).$$

■

2.17 Horizontal lifts of 1-forms

Let ω be a 1-form in an affine manifold M with the afffine connection ∇. The horizontal lift of ω, denoted by ω^H, is defined by $\omega^H = \omega^C - \nabla_\gamma \omega$ in TM, where $\nabla_\gamma \omega = \gamma(\nabla \omega)$.

Let ω_i and Γ_{ji}^h be the local components of ω and ∇ in M respectively. We already know that ω^C has components $\omega^C:(\partial\omega_i, \omega_i)$. For $X, Y \in \Im_0^1(M)$, where X^i, Y^j, ω_h are local components of X, Y, ω respectively:

$$(\nabla_X \omega)Y = \nabla_X(\omega Y) - \omega(\nabla_X Y)$$
$$= \sum_{i,h=1}^{n} X^i \partial_i(\omega_h Y^h) - \sum_{i,h=1}^{n} \omega_h (X^i \partial_i Y^h + \sum_{j=1}^{n} X^i Y^j \Gamma_{ij}^h)$$
$$= \sum_{i,j=1}^{n} \{X^i(\partial_i \omega_j) Y^j - \sum_{h=1}^{n} \omega_h X^i Y^j \Gamma_{ij}^h\}$$
$$= \sum_{i,j=1}^{n} (\partial_i \omega_j - \sum_{h=1}^{n} \omega_h \Gamma_{ij}^h)(dx^i \otimes dx^j)(X, Y)$$

with $(\nabla\omega)(X,Y) = (\nabla_X \omega)Y$, we have

$$\nabla\omega = \sum_{i,j=1}^{n} (\partial_i \omega_j - \sum_{h=1}^{n} \omega_h \Gamma_{ij}^h)(dx^i \otimes dx^j).$$

We write $\nabla\omega = \sum_{i,j=1}^{n} (\nabla_i \omega_j) dx^i \otimes dx^j$, where $\nabla_i \omega_j = \partial_i \omega_j - \sum_{h=1}^{n} \omega_h \Gamma_{ij}^h$. Then

$$\gamma\nabla\omega = \sum_{i,j=1}^{n} \bar{y}^i(\partial_i\omega_j - \sum_{h=1}^{n}\omega_h\Gamma_{ij}^h)d\bar{x}^j$$
$$= \sum_{i,j=1}^{n}\bar{y}^i(\nabla_i\omega_j)d\bar{x}^j.$$

Thus

$$\omega^H = \sum_{h,i,j=1}^{n}\omega_h\Gamma_{ij}^h\bar{y}^i d\bar{x}^j + \sum_{j=1}^{n}\omega_j d\bar{y}^j \quad \text{or}$$

$$\omega^H : (\sum_{h=1}^{n}\omega_h\Gamma_j^h \, , \, \omega_j)$$

with respect to the induced coordinates in *TM*.

A 1-form $\tilde{\omega}$ in *TM* is called *horizontal* if it satisfies $\tilde{\omega}(X^H) = 0$ for any $X \in \Im_0^1(M)$. If $(\tilde{\omega}_i \, , \, \tilde{\omega}_{\bar{i}})$ are the components of $\tilde{\omega}$ with respect to the induced coordinates and X^h are local components of *X*, then

$$\tilde{\omega}(X^H) = \sum_{i=1}^{n}(\tilde{\omega}_i X^i - \sum_{h=1}^{n}\tilde{\omega}_{\bar{h}}\Gamma_i^h X^i) = 0$$

for any $X \in T_0^1(M)$. As a result, we have

$$\tilde{\omega}_i - \sum_{h=1}^{n}\tilde{\omega}_{\bar{h}}\Gamma_i^h = 0. \qquad (*)$$

Thus $\tilde{\omega}$ is a horizontal 1-form in *TM* if and only if (*) holds.

From the components of the horizontal lift ω^H of $\omega \in \Im_1^0(M)$, we can conclude that ω^H is horizontal. Furthermore, we have [Ya1, page 93]

$$\begin{aligned}
\omega^H(X^V) &= \sum_{i=1}^{n} \omega_i X^i \\
&= (\omega X)^V , \\
\omega^H(X^C) &= \sum_{h,i=1}^{n} \Gamma_i^h \omega_h X^i + \sum_{i=1}^{n} \omega_i \partial X^i \\
&= \sum_{h,i=1}^{n} \omega_h \bar{y}^i (\sum_{j=1}^{n} \Gamma_{ij}^h X^j + \partial_i X^h) \\
&= \sum_{h,i=1}^{n} \omega_h \bar{y}^i \nabla_i X^h \\
&= \sum_{j=1}^{n} \omega_j d\bar{y}^j \left(\sum_{i,h=1}^{n} \bar{y}^i (\nabla_i X^h) \frac{\partial}{\partial \bar{y}^h} \right) \\
&= \left(\sum_{j=1}^{n} \omega_j d\bar{y}^j \right) \gamma \left(\sum_{i,h=1}^{n} (\nabla_i X^h) \frac{\partial}{\partial x^h} \otimes dx^i \right) \\
&= \sum_{j=1}^{n} (\omega_j d\bar{y}^j - \sum_{k=1}^{n} \Gamma_j^k \omega_k d\bar{x}^j) \gamma (\sum_{i,h=1}^{n} (\nabla_i X^h) \frac{\partial}{\partial x^h} \otimes dx^i) \\
&= \omega^C \gamma(\nabla X) , \\
\omega^H(X^H) &= \sum_{h,j=1}^{n} (\Gamma_j^h \omega_h X^j - \Gamma_j^h X^j \omega_h) \\
&= 0 .
\end{aligned}$$

<u>Proposition 2.31 [Ya1, page 93]</u>

$$\omega^H(\gamma F) = \gamma(\omega \circ F) \text{ for any } \omega \in \Im_1^0(M),\ F \in \Im_1^1(M).$$

<u>Proof</u>: Let ω_h, F_j^i be the local components of ω and F respectively. Then

$$\gamma F = \sum_{i,j=1}^{n} F^i_j \bar{y}^j \frac{\partial}{\partial \bar{y}^i} ,$$

$$\omega^H(\gamma F) = \sum_{i,j=1}^{n} F^i_j \bar{y}^j \omega_i ,$$

$$\omega \circ F = \sum_{i,j=1}^{n} F^i_j \omega_i dx^j ,$$

$$\gamma(\omega \circ F) = \sum_{i,j=1}^{n} F^i_j \bar{y}^j \omega_i .$$

Thus $\omega^H(\gamma F) = \gamma(\omega \circ F)$. ■

2.18 Horizontal lifts of tensor fields

For any tensor field S in an affine manifold M, $S \in \Im^s_t(M)$, we let

$$S = \sum S^{i_s \ldots i_1}_{j_t \ldots j_1} \frac{\partial}{\partial x^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_1}} \otimes dx^{j_t} \otimes \cdots \otimes dx^{j_1} .$$

Then the action of the affine connection on S will send S to $\nabla S \in \Im^s_{t+1}(M)$,

$$\nabla S = \sum (\nabla_l S^{i_s \ldots i_1}_{j_t \ldots j_1}) \frac{\partial}{\partial x^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_1}} \otimes dx^l \otimes dx^{j_t} \otimes \cdots \otimes dx^{j_1} ,$$

where

$$\nabla_l S^{i_s \ldots i_1}_{j_t \ldots j_1} = \frac{\partial S^{i_s \ldots i_1}_{j_t \ldots j_1}}{\partial x^l} - \sum_{r=1}^{s} \sum_{i=1}^{n} S^{i_s \ldots i_{r+1} i i_{r-1} \ldots i_1}_{j_t \ldots j_1} \Gamma^{i_r}_{li} + \sum_{r=1}^{t} \sum_{j=1}^{n} S^{i_s \ldots i_1}_{j_t \ldots j_{r+1} j j_{r-1} \ldots j_1} \Gamma^{j}_{lj_r}$$

and $\quad \nabla_\gamma S = \gamma(\nabla S) = \sum \bar{y}^l (\nabla_l S^{i_s \ldots i_1}_{j_t \ldots j_1}) \frac{\partial}{\partial \bar{y}^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_1}} \otimes d\bar{x}^{j_t} \otimes \cdots \otimes d\bar{x}^{j_1} .$

The horizontal lift S^H of S is defined to be $S^H = S^C - \gamma(\nabla S)$. Thus $S^H = S^C$ if and only if $\nabla S = 0$ if and only if S is parallel with respect to the connection ∇. Since the metric g is parallel with respect to the Riemannian connection ∇, we have $g^H = g^C$.

For any tensor fields $P, Q \in \Im(M)$, if we write $P \otimes Q = S$, where $S^{i_s \ldots i_1}_{j_t \ldots j_1} = P^{i_s \ldots i_{p+1}}_{j_t \ldots j_{q+1}} Q^{i_p \ldots i_1}_{j_q \ldots j_1}$, $P \in \Im^{s-p}_{t-q}(M)$, $Q \in \Im^p_q(M)$, then

$$\nabla_\gamma (P \otimes Q) = \gamma \nabla (P \otimes Q)$$

$$= \gamma \Big(\sum \Big(\nabla_h (P^{i_s \ldots i_{p+1}}_{j_t \ldots j_{q+1}} Q^{i_p \ldots i_1}_{j_q \ldots j_1}) \Big) \frac{\partial}{\partial \bar{x}^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{x}^{i_{p+1}}} \otimes \frac{\partial}{\partial \bar{x}^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{x}^{i_1}}$$

$$\otimes d\bar{x}^h \otimes d\bar{x}^{j_t} \otimes \cdots \otimes d\bar{x}^{j_{q+1}} \otimes d\bar{x}^{j_q} \otimes \cdots \otimes d\bar{x}^{j_1} \Big)$$

$$= \sum \bar{y}^h \Big(\nabla_h (P^{i_s \ldots i_{p+1}}_{j_t \ldots j_{q+1}} Q^{i_p \ldots i_1}_{j_q \ldots j_1}) \Big) \frac{\partial}{\partial \bar{y}^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_{p+1}}} \otimes \frac{\partial}{\partial \bar{y}^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_1}}$$

$$\otimes d\bar{x}^{j_t} \otimes \cdots \otimes d\bar{x}^{j_{q+1}} \otimes d\bar{x}^{j_q} \otimes \cdots \otimes d\bar{x}^{j_1}$$

$$= \sum \bar{y}^h \Big((\nabla_h P^{i_s \ldots i_{p+1}}_{j_t \ldots j_{q+1}}) Q^{i_p \ldots i_1}_{j_q \ldots j_1} + P^{i_s \ldots i_{p+1}}_{j_t \ldots j_{q+1}} (\nabla_h Q^{i_p \ldots i_1}_{j_q \ldots j_1}) \Big) \frac{\partial}{\partial \bar{y}^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_{p+1}}}$$

$$\otimes \frac{\partial}{\partial \bar{y}^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_1}} \otimes d\bar{x}^{j_t} \otimes \cdots \otimes d\bar{x}^{j_{q+1}} \otimes d\bar{x}^{j_q} \otimes \cdots \otimes d\bar{x}^{j_1}$$

$$= \sum \bar{y}^h (\nabla_h P^{i_s \ldots i_{p+1}}_{j_t \ldots j_{q+1}}) \frac{\partial}{\partial \bar{y}^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_{p+1}}} \otimes d\bar{x}^{j_t} \otimes \cdots \otimes d\bar{x}^{j_{q+1}}$$

$$\otimes \sum Q^{i_p \ldots i_1}_{j_q \ldots j_1} \frac{\partial}{\partial \bar{y}^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_1}} \otimes d\bar{x}^{j_q} \otimes \cdots \otimes d\bar{x}^{j_1}$$

$$+ \sum P^{i_s \ldots i_{p+1}}_{j_t \ldots j_{q+1}} \frac{\partial}{\partial \bar{y}^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_{p+1}}} \otimes d\bar{x}^{j_t} \otimes \cdots \otimes d\bar{x}^{j_{q+1}}$$

$$\otimes \sum \bar{y}^h (\nabla_h Q^{i_p \ldots i_1}_{j_q \ldots j_1}) \frac{\partial}{\partial \bar{y}^{i_p}} \otimes \cdots \otimes \frac{\partial}{\partial \bar{y}^{i_1}} \otimes d\bar{x}^{j_q} \otimes \cdots \otimes d\bar{x}^{j_1}$$

$$= (\nabla_\gamma P) \otimes Q^V + P^V \otimes (\nabla_\gamma Q) .$$

Thus

$$\begin{aligned}
(P \otimes Q)^H &= (P \otimes Q)^C - \nabla_\gamma (P \otimes Q) \\
&= P^C \otimes Q^V + P^V \otimes Q^C - (\nabla_\gamma P) \otimes Q^V - P^V \otimes (\nabla_\gamma Q) \\
&= (P^C - \nabla_\gamma P) \otimes Q^V + P^V \otimes (Q^C - \nabla_\gamma Q) \\
&= P^H \otimes Q^V + P^V \otimes Q^H .
\end{aligned} \qquad (*)$$

We consider the horizontal lifts of some tensor fields. Let $F \in \Im_1^1(M)$, $G \in \Im_2^0(M)$, $H \in \Im_0^2(M)$ and F_i^h, G_{ji}, H^{ji} be the local components of F, G, H respectively. Then

$$F^H : \begin{pmatrix} F_i^h & 0 \\ \sum_{t=1}^{n}(\Gamma_i^t F_t^h - \Gamma_t^h F_i^t) & F_i^h \end{pmatrix},$$

$$G^H : \begin{pmatrix} \sum_{t=1}^{n}(\Gamma_j^t G_{ti} - \Gamma_i^t G_{jt}) & G_{ji} \\ G_{ji} & 0 \end{pmatrix},$$

$$H^H : \begin{pmatrix} 0 & H^{ji} \\ H^{ji} & \sum_{t=1}^{n}(-\Gamma_t^j H^{ti} - \Gamma_t^i H^{jt}) \end{pmatrix}.$$

Let $S \in \Im_s^0(M)$, $T \in \Im_s^1(M)$ and $S_{i_s \ldots i_1}$, $T^h_{i_s \ldots i_1}$ be the local components of S and T respectively. Recall that

$$\nabla_l S_{i_s \ldots i_1} = \partial_l S_{i_s \ldots i_1} - \sum_{t=1}^{s}\sum_{i=1}^{n} S_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^i_{l i_t},$$

$$\gamma\nabla S = \sum \bar{y}^l (\nabla_l S_{i_s \ldots i_1}) d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

$$= \sum \bar{y}^l (\partial_l S_{i_s \ldots i_1} - \sum_{t=1}^{s}\sum_{i=1}^{n} S_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^i_{l i_t}) d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

and

$$S^C = \sum (\partial S_{i_s \ldots i_1} d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1} + \sum_{t=1}^{s} S_{i_s \ldots i_1} d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_{t+1}} \otimes d\bar{y}^{i_t} \otimes d\bar{x}^{i_{t-1}} \otimes \cdots \otimes d\bar{x}^{i_1}).$$

Therefore

$$S^{H} = \sum \sum_{t=1}^{s} \bar{y}^{l} S_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^{i}_{l i_t} d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

$$+ \sum \sum_{t=1}^{s} S_{i_s \ldots i_1} d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_{t+1}} \otimes d\bar{y}^{i_t} \otimes d\bar{x}^{i_{t-1}} \otimes \cdots \otimes d\bar{x}^{i_1}.$$

If we write $S^{H} = \sum \widetilde{S}_{\alpha_s \ldots \alpha_1} d\bar{x}^{\alpha_s} \otimes \cdots \otimes d\bar{x}^{\alpha_1}$, where $\alpha_s, \ldots, \alpha_1$ vary from $1, \ldots, n, \bar{1}, \ldots, \bar{n}$, then $\widetilde{S}_{i_s \ldots i_1} = \sum_{t=1}^{s} \bar{y}^{l} S_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^{i}_{l i_t}$ and $\widetilde{S}_{i_s \ldots i_{t+1} \bar{i}_t i_{t-1} \ldots i_1} = S_{i_s \ldots i_1}$.

Similarly,

$$\nabla_l T^{h}_{i_s \ldots i} = \partial_l T^{h}_{i_s \ldots i_1} + \sum_{t=1}^{s} \sum_{i=1}^{n} T^{h}_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^{i}_{l i_t} - \sum_{i=1}^{n} T^{i}_{i_s \ldots i_1} \Gamma^{h}_{l i},$$

$$\nabla T = \sum \left(\partial_l T^{h}_{i_s \ldots i_1} + \sum_{t=1}^{s} \sum_{i=1}^{n} T^{h}_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^{i}_{l i_t} - \sum_{i=1}^{n} T^{i}_{i_s \ldots i_1} \Gamma^{h}_{l i} \right) \frac{\partial}{\partial x^{h}} \otimes dx^{l} \otimes dx^{i_s} \otimes \cdots \otimes dx^{i_1},$$

$$\gamma(\nabla T) = \sum \bar{y}^{l} \left(\partial_l T^{h}_{i_s \ldots i} + \sum_{t=1}^{s} \sum_{i=1}^{n} T^{h}_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^{i}_{l i_t} - \sum_{i=1}^{n} T^{i}_{i_s \ldots i_1} \Gamma^{h}_{l i} \right) \frac{\partial}{\partial \bar{y}^{h}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

and

$$T^{C} = \sum (\partial T^{h}_{i_s \ldots i} \frac{\partial}{\partial \bar{y}^{h}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1} + T^{h}_{i_s \ldots i} \frac{\partial}{\partial \bar{x}^{h}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

$$+ \sum_{t=1}^{s} T^{h}_{i_s \ldots i} \frac{\partial}{\partial \bar{y}^{h}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_{t+1}} \otimes d\bar{y}^{i_t} \otimes d\bar{x}^{i_{t-1}} \otimes \cdots \otimes d\bar{x}^{i_1}).$$

Therefore

$$T^{H} = \sum \bar{y}^{l} (T^{i}_{i_s \ldots i_1} \Gamma^{h}_{l i} - \sum_{t=1}^{s} T^{h}_{i_s \ldots i_{t+1} i i_{t-1} \ldots i_1} \Gamma^{i}_{l i_t}) \frac{\partial}{\partial \bar{y}^{h}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

$$+ \sum T^{h}_{i_s \ldots i_1} \frac{\partial}{\partial \bar{x}^{h}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1}$$

$$+ \sum \sum_{t=1}^{s} T^{h}_{i_s \ldots i_1} \frac{\partial}{\partial \bar{y}^{h}} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_{t+1}} \otimes d\bar{y}^{i_t} \otimes d\bar{x}^{i_{t-1}} \otimes \cdots \otimes d\bar{x}^{i_1}.$$

If we write $T^H = \sum \widetilde{T}^{\beta}_{\alpha_s \dots \alpha_1} \dfrac{\partial}{\partial \widetilde{x}^{\beta}} \otimes d\widetilde{x}^{\alpha_s} \otimes \cdots \otimes d\widetilde{x}^{\alpha_1}$, where $\beta, \alpha_s, \dots, \alpha_1$ vary from $1, \dots, n, \bar{1}, \dots, \bar{n}$, then the only non zero components of T^H are

$$\widetilde{T}^{h}_{i_s \dots i_1} = T^{h}_{i_s \dots i_1} ,$$

$$\widetilde{T}^{\bar{h}}_{i_s \dots i_1} = \sum_{i=1}^{n} \left(T^{i}_{i_s \dots i_1} \Gamma^{h}_{li} - \sum_{t=1}^{s} T^{h}_{i_s \dots i_{t+1} l i_{t-1} \dots i_1} \Gamma^{l}_{li_t} \right) ,$$

$$\widetilde{T}^{\bar{h}}_{i_s \dots i_{t+1} \bar{i}_t i_{t-1} \dots i_1} = T^{h}_{i_s \dots i_1} .$$

Propositon 2.32 [Ya1, page 96]

For any $X, Y \in \Im^1_0(M)$, $F \in \Im^1_1(M)$,

$$(\nabla_Y F) X^V = 0 ,$$

$$(\nabla_Y F) X^C = (\nabla_Y F) X^H = \gamma((\nabla F) X)$$
$$= \gamma\nabla(FX) - F^C(\nabla_Y X) = \nabla_Y (FX) - F^H(\nabla_Y X) ,$$

where $((\nabla F) X) Y = (\nabla_Y F) X$.

Proof: Let F^h_i, X^i, Y^k be the local components of F, X, Y respectively. Then

$$(\nabla_Y F) X^V = \sum_{i,h,l=1}^{n} (\bar{y}^l (\nabla_l F^h_i) \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^i) \sum_{j=1}^{n} X^j \frac{\partial}{\partial \bar{y}^j}$$
$$= 0 ,$$

$$(\nabla_Y F) X^C = \sum_{i,h,l=1}^{n} (\bar{y}^l (\nabla_l F^h_i) \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^i) \sum_{j=1}^{n} (X^j \frac{\partial}{\partial \bar{x}^j} + \partial X^j \frac{\partial}{\partial \bar{y}^j})$$
$$= \sum_{i,h,l=1}^{n} \bar{y}^l (\nabla_l F^h_i) X^i \frac{\partial}{\partial \bar{y}^h} ,$$

$$(\nabla_Y F) X^H = \sum_{i,h,l=1}^{n} (\bar{y}^l (\nabla_l F^h_i) \frac{\partial}{\partial \bar{y}^h} \otimes d\bar{x}^i) \sum_{j=1}^{n} (X^j \frac{\partial}{\partial \bar{x}^j} + \sum_{k=1}^{n} \Gamma^j_k X^k \frac{\partial}{\partial \bar{y}^j})$$
$$= \sum_{i,h,l=1}^{n} \bar{y}^l (\nabla_l F^h_i) X^i \frac{\partial}{\partial \bar{y}^h} .$$

Since

$$
\begin{aligned}
\gamma\nabla(FX) &= \gamma(\nabla\sum_{h,i=1}^{n} F_i^h X^i \frac{\partial}{\partial x^h}) \\
&= \gamma(\sum_{h,i,j=1}^{n} (\nabla_j(F_i^h X^i))\frac{\partial}{\partial x^h}\otimes dx^j) \\
&= \sum_{h,i,j=1}^{n} (\nabla_j(F_i^h X^i))\bar{y}^j \frac{\partial}{\partial \bar{y}^h}
\end{aligned}
$$

and

$$
\begin{aligned}
F^C(\nabla_\gamma X) &= F^C(\sum_{h,i=1}^{n}(\nabla_j X^i)\bar{y}^j \frac{\partial}{\partial \bar{y}^i}) \\
&= \sum_{h,i,j=1}^{n} (F_i^h(\nabla_j X^i)\bar{y}^j \frac{\partial}{\partial \bar{y}^h} \ , \\
F^H(\nabla_\gamma X) &= F^H(\sum_{i,j=1}^{n}(\nabla_j X^i)\bar{y}^j \frac{\partial}{\partial \bar{y}^i}) \\
&= \sum_{h,i,j=1}^{n} (F_i^h(\nabla_j X^i)\bar{y}^j \frac{\partial}{\partial \bar{y}^h} \\
&= F^C(\nabla_\gamma X) \ ,
\end{aligned}
$$

then

$$
\begin{aligned}
\gamma\nabla(FX) - F^C(\nabla_\gamma X) &= \sum_{h,i,j=1}^{n} (\partial_j(F_i^h X^i) + \sum_{k=1}^{n} F_i^k X^i \Gamma_{jk}^h - F_i^h \partial_j X^i - \sum_{k=1}^{n} F_i^h X^k \Gamma_{jk}^i)\bar{y}^j \frac{\partial}{\partial \bar{y}^h} \\
&= \sum_{h,i,j=1}^{n} X^i(\partial_j F_i^h + \sum_{k=1}^{n}(F_i^k \Gamma_{jk}^h - \Gamma_{ji}^k F_k^h))\bar{y}^j \frac{\partial}{\partial \bar{y}^h} \\
&= \sum_{h,i,j=1}^{n} X^i(\partial_j F_i^h - \sum_{k=1}^{n}\Gamma_{ji}^k F_k^h - \sum_{k=1}^{n}(-F_i^k \Gamma_{jk}^h))\bar{y}^j \frac{\partial}{\partial \bar{y}^h} \\
&= \sum_{h,i,j=1}^{n} (\nabla_j F_i^h) X^i \bar{y}^j \frac{\partial}{\partial \bar{y}^h} \ .
\end{aligned}
$$

Since

$$((\nabla F)X)Y = (\nabla_Y F)X \,,$$

$$(\nabla_Y F)X = \sum_{h,i,j=1}^{n} (\nabla_j F_i^h) X^i Y^j \frac{\partial}{\partial x^h} \,,$$

$$(\nabla F)X = \sum_{h,i,j=1}^{n} (\nabla_j F_i^h) X^i \frac{\partial}{\partial x^h} \otimes dx^j \,,$$

therefore

$$\gamma((\nabla F)X) = \sum_{h,i,j=1}^{n} \bar{y}^i (\nabla_j F_i^h) X^j \frac{\partial}{\partial \bar{y}^h} \,.$$

The result is proved. ■

Proposition 2.33 [Ya1 page 96]

(i) $F^H X^V = (FX)^V$,

(ii) $F^H X^C = (FX)^H + F^H(\nabla_\gamma X)$

$= (FX)^H + F^C(\nabla_\gamma X)$,

(iii) $F^H X^H = (FX)^H$

for any $X \in \Im_0^1(M)$, $F \in \Im_1^1(M)$.

Proof:

$$\begin{aligned} F^H X^V &= (F^C - \nabla_\gamma F)X^V \\ &= F^C X^V = (FX)^V \,, \\ F^H X^C &= (F^C - \nabla_\gamma F)X^C \\ &= (FX)^C - (\nabla_\gamma F)X^C \\ &= (FX)^C - \nabla_\gamma (FX) + F^H(\nabla_\gamma X) \\ &= (FX)^H + F^H(\nabla_\gamma X) \\ &= (FX)^H + F^C(\nabla_\gamma X) \,, \end{aligned}$$

$$\begin{aligned} F^H X^H &= (F^C - \nabla_\gamma F)X^H \\ &= (FX)^C - \nabla_\gamma (FX) \\ &= (FX)^H . \end{aligned}$$

■

Proposition 2.34 [Ya1, page 97]

(i) $G^H(X^V, Y^V) = 0$,

(ii) $G^H(X^V, Y^C) = G^H(X^C, Y^V) = (G(X, Y))^V$,

(iii) $G^H(X^V, Y^H) = G^H(X^H, Y^V) = (G(X, Y))^V$,

(iv) $G^H(X^C, Y^H) = \gamma(G(\nabla X, Y))$,

(v) $G^H(X^H, Y^C) = \gamma(G(X, \nabla Y))$,

(vi) $G^H(X^H, Y^H) = G(X, Y)^H$,

(vii) $G^H(X^C, Y^C) = (G(X, Y))^C - (\nabla_\gamma G)(X^C, Y^C)$

for any $X, Y \in \Im_0^1(M)$, $G \in \Im_2^0(M)$, where $G(\nabla X, Y)$, $G(X, \nabla Y)$ are 1-form such that $(G(\nabla X, Y))Z = G(\nabla_Z X, Y)$ and $(G(X, \nabla Y))Z = G(X, \nabla_Z Y)$ for arbitrary element Z of $\Im_0^1(M)$.

Proof: Let X^i, Y^j, G_{ji} be the local components of X, Y, G in M respectively. Then

$$X^V : \begin{pmatrix} 0 \\ X^i \end{pmatrix}, \quad X^C : \begin{pmatrix} X^i \\ \partial X^i \end{pmatrix}, \quad X^H : \begin{pmatrix} X^i \\ \sum_{j=1}^{n} \Gamma_j^i X^j \end{pmatrix} \text{ and}$$

$$G^H : \begin{pmatrix} \sum_{t=1}^{n} (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) & G_{ji} \\ G_{ji} & 0 \end{pmatrix},$$

where $\Gamma_j^t = \sum_{s=1}^n \Gamma_{sj}^t \bar{y}^s$. From

$$(G(X, Y))^V = \sum_{j,i=1}^n G_{ji} X^j Y^i ,$$

$$\begin{aligned} G^H(X^V, Y^V) &= \left(\sum_{i,j,t=1}^n (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) d\bar{x}^j \otimes d\bar{x}^i + \sum_{j,i=1}^n G_{ji} (d\bar{x}^j \otimes d\bar{y}^i + d\bar{y}^j \otimes d\bar{x}^i) \right) \\ &\quad \left(\sum_{h=1}^n X^h \frac{\partial}{\partial \bar{y}^h} , \sum_{k=1}^n Y^k \frac{\partial}{\partial \bar{y}^k} \right) \\ &= 0 , \end{aligned}$$

$$\begin{aligned} G^H(X^V, Y^C) &= \left(\sum_{i,j,t=1}^n (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) d\bar{x}^j \otimes d\bar{x}^i + \sum_{j,i=1}^n G_{ji} (d\bar{x}^j \otimes d\bar{y}^i + d\bar{y}^j \otimes d\bar{x}^i) \right) \\ &\quad \left(\sum_{h=1}^n X^h \frac{\partial}{\partial \bar{y}^h} , \sum_{k=1}^n \left(Y^k \frac{\partial}{\partial \bar{x}^k} + \partial Y^k \frac{\partial}{\partial \bar{y}^k} \right) \right) \\ &= \sum_{j,i=1}^n G_{ji} X^j Y^i \\ &= (G(X, Y))^V , \end{aligned}$$

$$\begin{aligned} G^H(X^C, Y^V) &= \left(\sum_{i,j,t=1}^n (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) d\bar{x}^j \otimes d\bar{x}^i + \sum_{j,i=1}^n G_{ji} (d\bar{x}^j \otimes d\bar{y}^i + d\bar{y}^j \otimes d\bar{x}^i) \right) \\ &\quad \left(\sum_{h=1}^n \left(X^h \frac{\partial}{\partial \bar{x}^h} + \partial X^h \frac{\partial}{\partial \bar{y}^h} \right) , \sum_{k=1}^n Y^k \frac{\partial}{\partial \bar{y}^k} \right) \\ &= \sum_{j,i=1}^n G_{ji} X^j Y^i \\ &= (G(X,Y))^V , \end{aligned}$$

$$\begin{aligned} G^H(X^V, Y^H) &= \left(\sum_{i,j,t=1}^n (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) d\bar{x}^j \otimes d\bar{x}^i + \sum_{j,i=1}^n G_{ji} (d\bar{x}^j \otimes d\bar{y}^i + d\bar{y}^j \otimes d\bar{x}^i) \right) \\ &\quad \left(\sum_{h=1}^n X^h \frac{\partial}{\partial \bar{y}^h} , \sum_{k=1}^n \left(Y^k \frac{\partial}{\partial \bar{x}^k} - \sum_{l=1}^n \Gamma_l^k Y^l \frac{\partial}{\partial \bar{y}^k} \right) \right) \\ &= \sum_{j,i=1}^n G_{ji} X^j Y^i = (G(X, Y))^V , \end{aligned}$$

$$G^H(X^H, Y^V) = \left(\sum_{i,j,t=1}^{n} (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) d\bar{x}^j \otimes d\bar{x}^i + \sum_{j,i=1}^{n} G_{ji} (d\bar{x}^j \otimes d\bar{y}^i + d\bar{y}^j \otimes d\bar{x}^i) \right)$$

$$\left(\sum_{h=1}^{n} \left(X^h \frac{\partial}{\partial \bar{x}^h} - \sum_{l=1}^{n} \Gamma_l^h X^l \frac{\partial}{\partial \bar{y}^h} \right), \sum_{k=1}^{n} Y^k \frac{\partial}{\partial \bar{y}^k} \right)$$

$$= \sum_{j,i=1}^{n} G_{ji} X^j Y^i = (G(X, Y))^V ,$$

therefore $G^H(X^V, Y^H) = G^H(X^H, Y^V) = (G(X, Y))^V$.

$$G^H(X^C, Y^H) = \left(\sum_{i,j,t=1}^{n} (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) d\bar{x}^j \otimes d\bar{x}^i + \sum_{j,i=1}^{n} G_{ji} (d\bar{x}^j \otimes d\bar{y}^i + d\bar{y}^j \otimes d\bar{x}^i) \right)$$

$$\left(\sum_{h=1}^{n} \left(X^h \frac{\partial}{\partial \bar{x}^h} + \partial X^h \frac{\partial}{\partial \bar{y}^h} \right), \sum_{k=1}^{n} \left(Y^k \frac{\partial}{\partial \bar{x}^k} - \sum_{l=1}^{n} \Gamma_l^k Y^l \frac{\partial}{\partial \bar{y}^k} \right) \right)$$

$$= \sum_{i,j,t=1}^{n} (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) X^j Y^i - \sum_{i,j,l=1}^{n} G_{ji} X^j \Gamma_l^i Y^l + \sum_{j,i=1}^{n} G_{ji} (\partial X^j) Y^i$$

$$= \sum_{i,j,t=1}^{n} \Gamma_j^t G_{ti} X^j Y^i + \sum_{j,i=1}^{n} G_{ji} (\partial X^j) Y^i$$

$$= \sum_{i,j,l=1}^{n} \bar{y}^l \left(\sum_{t=1}^{n} \Gamma_{lj}^t G_{tt} X^j Y^i + G_{ji} \left(\frac{\partial X^j}{\partial \bar{x}^l}\right) Y^i \right)$$

$$= \sum_{i,j,l=1}^{n} \bar{y}^l G_{ji} Y^i \left(\sum_{t=1}^{n} \Gamma_{lt}^j X^t + \frac{\partial X^j}{\partial \bar{x}^l} \right)$$

$$= \sum_{i,j,l=1}^{n} \bar{y}^l (\nabla_l X^j) G_{ji} Y^i .$$

Since $(G(\nabla X, Y))Z = G(\nabla_Z X, Y)$, so

$$(G(\nabla X, Y))Z = \sum_{i,j,l=1}^{n} G_{ji} Z^l (\nabla_l X^j) Y^i ,$$

$$G(\nabla X, Y) = \sum_{i,j,l=1}^{n} G_{ji} (\nabla_l X^j) Y^i dx^l ,$$

$$\gamma G(\nabla X, Y) = \sum_{i,j,l=1}^{n} \bar{y}^l G_{ji} (\nabla_l X^j) Y^i ,$$

$$= G^H(X^C, Y^H) .$$

Similarly, $\gamma G(X, \nabla Y) = G^H(X^H, Y^C)$.

$$G^H(X^H, Y^H) = \left(\sum_{i,j,t=1}^{n} (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) d\bar{x}^j \otimes d\bar{x}^i + \sum_{j,i=1}^{n} G_{ji} (d\bar{x}^j \otimes d\bar{y}^i + d\bar{y}^j \otimes d\bar{x}^i) \right)$$

$$\left(\sum_{h=1}^{n} \left(X^h \frac{\partial}{\partial \bar{x}^h} - \sum_{l=1}^{n} \Gamma_l^h X^l \frac{\partial}{\partial \bar{y}^h} \right), \sum_{k=1}^{n} \left(Y^k \frac{\partial}{\partial \bar{x}^k} - \sum_{l=1}^{n} \Gamma_l^k Y^l \frac{\partial}{\partial \bar{y}^k} \right) \right)$$

$$= \sum_{i,j,t=1}^{n} (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) X^j Y^i - \sum_{i,j,l=1}^{n} (\Gamma_l^j X^l Y^i G_{ji} + X^j \Gamma_l^i Y^l G_{ji})$$

$$= \sum_{i,j,t=1}^{n} (\Gamma_j^t G_{ti} + \Gamma_i^t G_{jt}) X^j Y^i - \sum_{i,j,t=1}^{n} (\Gamma_t^j X^t Y^i G_{ji} + X^j \Gamma_t^i Y^t G_{ji})$$

$$= 0 = (G(X, Y))^H,$$

also

$$G^H(X^C, Y^C) = (G^C(X^C, Y^C) - (\nabla_\gamma G)(X^C, Y^C)$$
$$= (G(X, Y))^C - (\nabla_\gamma G)(X^C, Y^C).$$

■

<u>Proposition 2.35 [Ya1, page 97]</u>

Let $S \in \Im_s^0(M)$ or $\Im_s^1(M)$, $X_1, \ldots, X_s \in \Im_0^1(M)$. Then

(i) $S^V(X_s^H, \ldots, X_1^H) = (S(X_s, \ldots, X_1))^V$,

(ii) $S^C(X_s^H, \ldots, X_1^H) = (S(X_s, \ldots, X_1))^H + (\nabla_\gamma S)(X_s^H, \ldots, X_1^H)$,

(iii) $S^H(X_s^V, \ldots, X_1^V) = 0$,

(iv) $S^H(X_s^H, \ldots, X_{t+1}^H, X_t^V, X_{t-1}^H, \ldots, X_1^H) = (S(X_s, \ldots, X_1))^V$,

(v) $S^H(X_s^H, \ldots, X_1^H) = (S(X_s, \ldots, X_1))^H$

for any $t = 1, \ldots, n$.

Proof: We only prove for the case $S \in \Im_s^1(M)$. We know that if $S_{i_s \dots i_1}^h$ are local components of S and if we denote $\tilde{S}_{B_s \dots B_1}^A$ and $\tilde{\tilde{S}}_{B_s \dots B_1}^A$ as local components of S^C and S^H respectively, then

$$\tilde{S}_{i_s \dots i_1}^h = S_{i_s \dots i_1}^h \,,\; \tilde{S}_{i_s \dots i_1}^{\bar{h}} = \partial S_{i_s \dots i_1}^h \,,\; \tilde{S}_{i_s \dots i_{t+1} \bar{i}_t i_{t-1} \dots i_1}^{\bar{h}} = S_{i_s \dots i_1}^h \,,$$

$$\tilde{\tilde{S}}_{i_s \dots i_1}^h = S_{i_s \dots i_1}^h \,,\; \tilde{\tilde{S}}_{i_s \dots i_1}^{\bar{h}} = \sum_{l=1}^{n} (-S_{i_s \dots i_1}^l \Gamma_l^h + \sum_{t=1}^{s} S_{i_s \dots i_{t+1} l i_{t-1} \dots i_1}^h \Gamma_{i_t}^l) \,,\; \tilde{\tilde{S}}_{i_s \dots i_{t+1} \bar{i}_t i_{t-1} \dots i_1}^{\bar{h}} = S_{i_s \dots i_1}^h \,,$$

all the others being zero,and

$$S^{V} = \sum S_{i_s \dots i_1}^h \frac{\partial}{\partial y^h} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1} \,.$$

Let $X_t^{i_t}$ be the local components of X_t in M. Then $X_t^H = \sum_{i_t=1}^{n} \left(X_t^{i_t} \frac{\partial}{\partial x^{i_t}} + \sum_{l=1}^{n} \Gamma_l^{i_t} X_t^l \frac{\partial}{\partial y^{i_t}} \right)$.

(i)

$$\begin{aligned}
& S^{V}(X_s^H, \dots, X_1^H) \\
&= S^{V}\left(\sum_{i_s=1}^{n} \left(X_s^{i_s} \frac{\partial}{\partial x^{i_s}} - \sum_{l=1}^{n} \Gamma_l^{i_s} X_s^l \frac{\partial}{\partial y^{i_s}} \right), \dots, \sum_{i_1=1}^{n} \left(X_1^{i_1} \frac{\partial}{\partial x^{i_1}} - \sum_{l=1}^{n} \Gamma_l^{i_1} X_1^l \frac{\partial}{\partial y^{i_1}} \right) \right) \\
&= \sum S_{i_s \dots i_1}^h X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h} \\
&= \left(\sum S_{i_s \dots i_1}^h X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} \right)^{V} \\
&= (S(X_1, \dots, X_s))^{V} \,.
\end{aligned}$$

(ii)

$$S^C(X_s{}^H, \dots, X_1{}^H) = \sum S^h_{i_s \dots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} + \sum (\partial S^h_{i_s \dots i_1}) X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h}$$

$$- \sum \sum_{t=1}^{s} S^h_{i_s \dots i_{t+1} i_t i_{t-1} \dots i_1} X_s^{i_s} \cdots X_{t+1}^{i_{t+1}} (-\Gamma_l^{i_t} X_t^l) X_{t-1}^{i_{t-1}} \cdots X_1^{i_1} \frac{\partial}{\partial y^h} ,$$

$$(S(X_s, \dots, X_1))^H = \sum S^h_{i_s \dots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} - \sum \Gamma_l^h S^l_{i_s \dots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h} ,$$

$$\nabla_\gamma S(X_s{}^H, \dots, X_1{}^H) = \left(\sum \bar{y}^l (\nabla_l S^h_{i_s \dots i_1}) \frac{\partial}{\partial y^h} \otimes d\bar{x}^{i_s} \otimes \cdots \otimes d\bar{x}^{i_1} \right)$$

$$\left(\sum_{i_s=1}^{n} \left(X_s^{i_s} \frac{\partial}{\partial x^{i_s}} - \sum_{l=1}^{n} \Gamma_l^{i_s} X_s^l \frac{\partial}{\partial y^{i_s}} \right), \dots , \sum_{i_1=1}^{n} \left(X_1^{i_1} \frac{\partial}{\partial x^{i_1}} - \sum_{l=1}^{n} \Gamma_l^{i_1} X_1^l \frac{\partial}{\partial y^{i_1}} \right) \right)$$

$$= \sum \bar{y}^l (\nabla_l S^h_{i_s \dots i_1}) X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h} .$$

therefore $(S(X_s, \dots, X_1))^H + \nabla_\gamma S(X_s{}^H, \dots, X_1{}^H) = S^C(X_s{}^H, \dots, X_1{}^H)$.

(iii)

$$S^H(X_1{}^V, \dots, X_s{}^V) = S^H\left(\sum_{i_s=1}^{n} X_s^{i_s} \frac{\partial}{\partial y^{i_s}}, \dots , \sum_{i_1=1}^{n} X_1^{i_1} \frac{\partial}{\partial y^{i_1}} \right)$$

$$= 0 .$$

(iv)

$$
\begin{aligned}
S^H(X_1{}^H, \ldots, X_{t+1}{}^H, X_t{}^V, X_{t-1}{}^H, \ldots, X_s{}^H) \\
&= \sum \tilde{\tilde{S}}^h_{i_s \ldots i_{t+1} i_t i_{t-1} \ldots i_1} X_s^{i_s} \cdots X_{t+1}^{i_{t+1}} X_t^{i_t} X_{t-1}^{i_{t-1}} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} \\
&= \sum S^h_{i_s \ldots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} \\
&= (S(X_1, \ldots, X_s))^V .
\end{aligned}
$$

(v)

$$
\begin{aligned}
S^H(X_s{}^H, \ldots, X_1{}^H) &= \sum \sum (\tilde{\tilde{S}}^h_{i_s \ldots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} + \tilde{\tilde{S}}^h_{i_s \ldots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h} \\
&\qquad + \tilde{\tilde{S}}^h_{i_s \ldots i_{t+1} i_t i_{t-1} \ldots i_1} X_s^{i_s} \cdots X_{t+1}^{i_{t+1}} (-\Gamma_l^{i_t} X_t^l) X_{t-1}^{i_{t-1}} \cdots X_1^{i_1} \frac{\partial}{\partial y^h}) \\
&= \sum \sum (S^h_{i_s \ldots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} + \{(\Gamma_{i_t}^l S^h_{i_s \ldots i_{t+1} l i_{t-1} \ldots i_1} - \Gamma_l^h S^l_{i_s \ldots i_1}) X_t^{i_t} \\
&\qquad - S^h_{i_s \ldots i_{t+1} i_t i_{t-1} \ldots i_1} \Gamma_l^{i_t} X_t^l \} X_s^{i_s} \cdots X_{t+1}^{i_{t+1}} X_{t-1}^{i_{t-1}} \cdots X_1^{i_1} \frac{\partial}{\partial y^h}) \\
&= \sum S^h_{i_s \ldots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial x^h} - \sum \Gamma_l^h S^l_{i_s \ldots i_1} X_s^{i_s} \cdots X_1^{i_1} \frac{\partial}{\partial y^h} \\
&= (S(X_s, \ldots, X_1))^H .
\end{aligned}
$$

■