CHAPTER 3

NULLITY DISTRIBUTION

ON (TM, g^c, ∇^c)

3.1 Nullity distributions

The nullity distribution was first defined by Chern and Kuiper in their paper [Ch] on a Riemannian manifold (M, g). The distribution assigns to each point $m \in M$, the subspace

$$N(m) = N_m = \{X \in T_m M \mid R(X, Y) = 0, \text{ for all } Y \in T_m M\},$$

where ∇ is the Riemannian connection with respect to g in M and R is the curvature tensor. We denote the nullity distribution by N. The dimension of N_m is denoted by $\mu(m)$.

In the same way, the nullity distribution on an affine manifold (M, ∇), a manifold with a connection ∇ on M can be considered.

If U is an open set in M, then $\Gamma(U, N)$ denotes the set of all vector fields X such that $X(m) \in N_m$ for any $m \in U$. When $U = M$, we write $\Gamma(N)$ for short.

We say that a nullity distribution N on M is *involutive* if the presheaf of cross sections over open sets of M is involutive. Then we have

Theorem 3.1 [Tan, page 324]

The nullity distribution N on an affine manifold M is involutive.

Proof: Let U be an open set in M, $Y, Z \in \Gamma(U, N)$. It is sufficient to show that $R([Y, Z], X) = 0$ for arbitrary vector field X defined on U. Since

$$\nabla_X(R(Y, Z)) = \nabla_Y(R(Z, X)) = \nabla_Z(R(X, Y)) = 0,$$

we have

$$0 = S(\nabla_X(R(Y, Z))) = S\{(\nabla_X R)(Y, Z) + R(\nabla_X Y, Z) + R(Y, \nabla_X Z)\},$$

where S denotes the cyclic sum. Recall the following Bianchi Second Identity,

$$S((\nabla_X R)(Y, Z) + R(T(X, Y), Z)) = 0.$$

Therefore, when we substitute for $(\nabla_X R)(Y, Z)$, we have

$$0 = S(-R(T(X, Y), Z) + R(\nabla_X Y, Z) + R(Y, \nabla_X Z)).$$

By actually carrying out the cyclic sum, and noting that $Y, Z \in \Gamma(U, N)$, we obtain

$$0 = R(-T(Y, Z) + \nabla_Y Z - \nabla_Z Y, X).$$

Since $T(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z]$, we find that $R([Y, Z], X) = 0$. This proves the theorem. ■

The curvature tensor can also be considered as a bundle map $R: TM \to \mathrm{Hom}(TM, \mathrm{End}TM)$. The nullity distribution N of (M, ∇) can now be defined as the kernel of the map R, namely $N \equiv \ker R$.

Since R is a bundle map, the dimension at each fibre is always upper semicontinuous. Thus the function $\mu(m)$ is upper semicontinuous. Hence the set V where $\mu(m)$ assumes its minimum value is open in M.

The distribution N is nonsingular only on the set V. In general N is singular. The index μ will not take the value n-1 at any point, where n is the dimension of M. If so, then there exists linearly independent $X_1, \ldots, X_{n-1}$ in $T_m M$ such that $\{X_1, \ldots, X_{n-1}\}$ form a basis in N_m. Choose X_n such that $\{X_1, \ldots, X_{n-1}, X_n\}$ form a basis in $T_m M$. Since $R(X_i, X_n) = 0$ for all $i = 1, \ldots, n-1$, so $R(X_n, -) = 0$ implies that $\mu(m) = n$.

<u>Corollary 3.2 [Tan, page 325]</u>

If $\mu(m)$ is locally constant on an open submanifold M' of M, then N is integrable on M'.

Since the affine connection ∇ on M may not induce connections on the leaves, we cannot say that the leaves of N are flat.

<u>Proposition 3.3 [Tan, page 325]</u>

Let $f: M \to M$ be an affine transformation [Ko, page 226] of the affine manifold (M, ∇).Then $\mu(m) = \mu(f(m))$.

<u>Proof</u>: Since f is a transformation, for every $X \in \mathfrak{X}(M)$, X is f-related to $f_* X$. Also, since f is an affine transformation $f_* R(X, Y)Z = R(f_* X, f_* Y) f_* Z$ for any $X, Y, Z \in \mathfrak{X}(M)$, the result follows immediately because f_* is an isomorphism. ■

From now on we assume that the nullity distribution N of an affine manifold (M, ∇) is nonsingular.

Proposition 3.4 [Tan, page 326]

Let M be an affine manifold with curvature tensor parallel with respect to N. Then $\nabla_X Y \in \Gamma(N)$ for all $X,\ Y \in \Gamma(N)$.

Proof: We need to show that $R(\nabla_X Y, Z) = 0$ for all $Z \in \mathcal{H}(M)$. Since $\nabla_X R = 0$, then

$$(\nabla_X R)(Y, Z) = \nabla_X (R(Y, Z)) - R(\nabla_X Y, Z) - R(Y, \nabla_X Z) = 0$$

implies that $R(\nabla_X Y, Z) = 0$ since R is parallel with respect to N by our assumption. ■

Note that in a Riemannian manifold, the curvature tensor of its canonical connection is already parallel with respect to its nullity distribution N.

Let M' be a submanifold of M. We know that in a Riemannian manifold, ∇ will induce a connection on M' but this may not be true for an affine manifold. However, if M' is an autoparallel submanifold of the affine manifold (M, ∇), that is, for each $X \in TM'_m$ and every curve γ in M' with $\gamma(0) = m$, the parallel displacement of X along γ with respect to the affine connection of M, yields a vector tangent to M', then the affine connection on M induces a connection on M' in a natural manner.

Proposition 3.5 [Ko, volume II, page 55]

Let M' be a submanifold of the affine manifold (M,∇). Then M' is an autoparallel submanifold if and only if $\nabla_X Y$ is tangent to M' for all $X,\ Y \in \mathfrak{X}(M')$. When M' is an autoparallel submanifold, ∇ induces a connection on M' in a natural manner.

Proposition 3.6 [Tan, page 326]

If the curvature tensor is parallel with respect to ∇, then every leaf of N is autoparallel.

Proof: Let L be an integral submanifold of N. We need to show that $\nabla_X Y$ is tangent to L for $X,\ Y \in \Gamma(L)$. This follows from Proposition 3.4. ■

3.2 Characterization of nullity distributions

We have the following simple characterization of a nullity distribution which we will need in the sections to follow.

Proposition 3.7

$X \in \Gamma(N)$ if and only if $\sum_{k=1}^{n} X^k R_{kji}{}^h = 0$ for all $h,i,j = 1,\ 2,\ \ldots,\ n$ where $R_{kji}{}^h$, X^k are local components of X and R respectively.

Proof: If $X \in \Gamma(N)$, then $R(X,\ Y)Z = 0$ for all $Y,\ Z \in \mathfrak{X}(M)$. Let us express X, Y and Z in terms of local coordinates,

$$X=\sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i},\ Y=\sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i},\ Z=\sum_{i=1}^{n} Z^i \frac{\partial}{\partial x^i},$$

where X^i, Y^i, $Z^i \in C^\infty(M)$. Take $R_{kji}{}^h$ as components of the curvature tensor R with respect to the coordinate $(x^1, \ldots, x^n)$, namely $R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^i}=\sum_{h=1}^{n} R_{kji}{}^h \frac{\partial}{\partial x^h}$. Then

$$R(X, Y)Z=\sum_{i,j,k,h=1}^{n} X^k Y^j Z^i R_{kji}{}^h \frac{\partial}{\partial x^h}=0 \qquad \text{for all } Y^j,\ Z^i \in C^\infty(M).$$

Thus we have $\sum_{h,k=1}^{n} X^k R_{kji}{}^h \frac{\partial}{\partial x^h}=0$; which implies that $\sum_{k=1}^{n} X^k R_{kji}{}^h=0$ for all $h,\ i,j=1,\ \ldots,\ n$. Therefore, X is in $\Gamma(N)$ if and only if $\sum_{k=1}^{n} X^k R_{kji}{}^h=0$ for all $h,\ i,j$. ■

3.3 Nullity distribution on (TM, g^c, ∇^c)

We already know that for a Riemannian manifold (M, g, ∇), where R and T are the curvature and torsion tensors with respect to ∇, the tangent bundle TM with metric g^C, being the complete lift of g to TM, will form a pseudo-Riemannian manifold and the connection induced by g^C, will be ∇^C while the curvature and torsion tensors will be R^C and T^C respectively (see section 2.15). Let us consider the nullity distribution of the manifold (TM, g^c, ∇^c).

Recall that for the tangent bundle of (M, g, ∇), the curvature tensor R^C of ∇^C has components $\overline{R}_{CBA}{}^D$, $A,\ B,\ C,\ D=1,\ \ldots,\ n,\ \bar{1},\ \ldots,\ \bar{n}$, which can be expressed in terms of components of the curvature tensor R with respect to ∇,

$$\left.\begin{array}{l}\overline{R}_{kji}{}^{h} = R_{kji}{}^{h},\ \overline{R}_{kji}{}^{\bar{h}} = \partial R_{kji}{}^{h} = \sum_{l=1}^{n} \overline{y}^{l} \dfrac{\partial R_{kji}{}^{h}}{\partial \overline{x}^{l}}, \\ \overline{R}_{k\bar{j}i}{}^{\bar{h}} = R_{kji}{}^{h},\ \overline{R}_{kj\bar{i}}{}^{\bar{h}} = R_{kji}{}^{h},\ \overline{R}_{\bar{k}ji}{}^{\bar{h}} = R_{kji}{}^{h}, \\ \text{all the others being zero,}\end{array}\right\} \qquad (*)$$

with respect to the induced coordinates $(\overline{x}^1, \ldots, \overline{x}^n, \overline{x}^{\bar{1}}, \ldots, \overline{x}^{\bar{n}})$ or $(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)$ in TM.

If $\overline{X} = \sum_{A} \overline{X}^{A} \dfrac{\partial}{\partial \overline{x}^{A}} = \sum_{h=1}^{n} \left(\overline{X}^{h} \dfrac{\partial}{\partial \overline{x}^{h}} + \overline{X}^{\bar{h}} \dfrac{\partial}{\partial \overline{y}^{h}} \right)$ is in $\Gamma(\overline{N})$, where $\overline{N}$ is the nullity distribution in TM with respect to the curvature tensor R^C of ∇^C, then from Proposition 3.7, we have

$$\sum_{C} \overline{R}_{CBA}{}^{D} \overline{X}^{C} = 0 \qquad \text{for any } A, B, D. \qquad (**)$$

From (*), the above equations (**) reduce to the following two cases:

(i) $D = h,\ B = j,\ A = i$, we have

$$\sum_{C} \overline{R}_{Cji}{}^{h} \overline{X}^{C} = \sum_{k=1}^{n} R_{kji}{}^{h} \overline{X}^{k} = 0, \qquad (I)$$

(ii) $D = \bar{h},\ B = j,\ A = i$, we have $\sum_{C} \overline{R}_{Cji}{}^{\bar{h}} \overline{X}^{C} = 0$; which implies that

$$\sum_{k=1}^{n} (\overline{R}_{kji}{}^{\bar{h}} \overline{X}^{k} + \overline{R}_{\bar{k}ji}{}^{\bar{h}} \overline{X}^{\bar{k}}) = 0,$$

and from (*), we have

$$\sum_{k=1}^{n} (\partial R_{kji}{}^{h} \overline{X}^{k} + R_{kji}{}^{h} \overline{X}^{\bar{k}}) = 0. \qquad (II)$$

So the vector field $\overline{X}$ in TM is in $\Gamma(\overline{N})$ if and only if its components $\{\overline{X}^A\}$ satisfy

$$\left.\begin{aligned} \sum_{k=1}^{n} R_{kji}{}^{h}\overline{X}^{k} &= 0 \\ \sum_{k=1}^{n}(\partial R_{kji}{}^{h}\overline{X}^{k} + R_{kji}{}^{h}\overline{X}^{\bar{k}}) &= 0 \end{aligned}\right\} \qquad h,\, i, j = 1,\, \dots,\, n. \qquad (***)$$

Now we consider the vertical and complete lifts of vector fields in $\Gamma(N)$. If $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$ is in $\Gamma(N)$, then we know that the X^i's satisfy $\sum_{k=1}^{n} R_{kji}{}^{h} X^k = 0$, for all $h,\, i, j = 1,\, \dots,\, n$.

The vertical lift of X will be $X^V = \begin{pmatrix} \overline{X}^k \\ \overline{X}^{\bar{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ X^k \end{pmatrix} = \sum_{k=1}^{n} X^k \frac{\partial}{\partial y^k}$. Its components satisfy

$$\sum_{k=1}^{n} R_{kji}{}^{h}\overline{X}^{k} = 0 \qquad \text{and}$$

$$\begin{aligned} \sum_{k=1}^{n}(\partial R_{kji}{}^{h}\overline{X}^{k} + R_{kji}{}^{h}\overline{X}^{\bar{k}}) &= \sum_{k=1}^{n} R_{kji}{}^{h} X^k \\ &= 0 \end{aligned}$$

since $\overline{X}^k = 0$ for $k = 1,\, \dots,\, n$ and Proposition 3.7. Hence $X^V \in \Gamma(\overline{N})$.

The complete lift of $X \in \Gamma(N) \subseteq \mathcal{H}(M)$ will be $X^C = \begin{pmatrix} \overline{X}^k \\ \overline{X}^{\bar{k}} \end{pmatrix} = \begin{pmatrix} X^k \\ \partial X^k \end{pmatrix} = \sum_{k=1}^{n}(X^k \frac{\partial}{\partial x^k} + \partial X^k \frac{\partial}{\partial y^k})$. Its components satisfy

$$\sum_{k=1}^{n} R_{kji}{}^{h}\overline{X}^{k} = \sum_{k=1}^{n} R_{kji}{}^{h} X^k = 0 \qquad \text{and}$$

$$\sum_{k=1}^{n}(R_{kji}{}^{h}\bar{X}^{k}+\partial R_{kji}{}^{h}\bar{X}^{k})=\sum_{k=1}^{n}(R_{kji}{}^{h}\partial X^{k}+\partial R_{kji}{}^{h}X^{k})$$
$$=\sum_{k=1}^{n}\partial(R_{kji}{}^{h}X^{k})$$
$$=\partial\left(\sum_{k=1}^{n}R_{kji}{}^{h}X^{k}\right)$$
$$=0$$

for all $h,\ i,\ j=1,\ \ldots,\ n$. Hence X^{C} is in $\Gamma(\bar{N})$.

Proposition 3.8

For any $X\in\Gamma(N)$, we have $X^{V},\ X^{C}\in\Gamma(\bar{N})$.

Let π be the projection from *TM* onto *M*. The differential of π, π_{*} will map the tangent bundle of *T*(*TM*) to *TM*, $\pi_{*}: T(TM)\to TM$. If $\bar{Y}$ is in *T*(*TM*) at (p,X), then $\pi_{*}\bar{Y}$ will be a tangent at p with $(\pi_{*}\bar{Y})f=\bar{Y}(f\circ\pi)$ for all $f\in C^{\infty}(M)$.

The nullity distribution $\bar{N}$ of *TM* is a subspace of the tangent bundle of *TM*, we would like to see the effect of π_{*} over $\bar{N}$.

Consider the nullity subspace $\bar{N}(p,X)$ for $(p,X)\in TM$. For any $\bar{Y}(p,X)\in\bar{N}(p,X)$, we have

$$\bar{Y}(p,X)=\sum_{A}\bar{Y}^{A}(p,X)\frac{\partial}{\partial\bar{x}^{A}}(p,X),$$

$$\pi_{*}\bar{Y}(p,X)=\sum_{i=1}^{n}\bar{Y}^{i}(p,X)\frac{\partial}{\partial x^{i}}(p),$$

for all $h,\ i,j=1,\ \ldots,\ n$.

From (*) we have $\sum_{k=1}^{n} R_{kji}{}^{h}\overline{Y}^{k}(p,X) = 0$ and $\sum_{k=1}^{n}(\partial R_{kji}{}^{h}\overline{Y}^{k} + R_{kji}{}^{h}\overline{Y}^{\bar{k}})(p,X) = 0$.

Then $\pi_{*}\overline{Y}(p,X) = \sum_{i=1}^{n}\overline{Y}^{i}(p,X)\frac{\partial}{\partial x^{i}}(p)$ whose components clearly satisfy $\sum_{k=1}^{n} R_{kji}{}^{h}\overline{Y}^{k}(p,X) = 0$, for all $h, i, j = 1, \ldots, n$, which implies that $\pi_{*}\overline{Y}(p,X)$ is in $N(p)$. This is true for all $\overline{Y}(p,X)$ in $\overline{N}(p, X)$. Therefore, $\pi_{*}\overline{N}(p,X) \subseteq N(p)$. Hence we have $\pi_{*}: \overline{N} \to N$. Moreover from Proposition 3.8 $\pi_{*}: \overline{N} \to N$ is an onto map. This proves the following:

<u>Proposition 3.9</u>

$$N(p) = \pi_{*}(\overline{N}(p,X)).$$

3.4 The dimension of the nullity distribution on (TM, g^{c}, ∇^{c})

Suppose that the dimension of the nullity distribution N of M is d. Let $\{e_i\}_{i=1}^{d}$ be a local basis of $\Gamma(N)$. We already know that e_i^{V} and e_i^{C} are in $\Gamma(\overline{N})$ for all $i = 1, \ldots, d$. Let $e_i = \sum_{j=1}^{n} e_{ij}\partial_j$. Since the dimension of N is d, the rank of the matrix (e_{ij}) is also d. The set $\{e_i^{V}\}_{i=1}^{d}$ is linearly independent since $e_i^{V} = \sum_{j=1}^{n} e_{ij}\partial_{\bar{j}}$ and it has the same matrix (e_{ij}).

Consider $\sum_{i=1}^{d}(\gamma_i e_i^{V} + \beta_i e_i^{C}) = 0$. Then

$$\sum_{i=1}^{d}\sum_{j=1}^{n}(\gamma_i e_{ij}\partial_j + \beta_i \partial e_{ij}\partial_j + \beta_i e_{ij}\partial_j) = \sum_{i=1}^{d}\sum_{j=1}^{n}((\gamma_i e_{ij} + \beta_i \partial e_{ij})\partial_j + \beta_i e_{ij}\partial_j)$$
$$= 0.$$

Thus $\sum_{i=1}^{d}\sum_{j=1}^{n}\beta_i e_{ij}\partial_j = \sum_{j=1}^{d}\beta_i e_i = 0$ implies that $\beta_i = 0$ and $\sum_{i=1}^{d}\sum_{j=1}^{n}((\gamma_i e_{ij} + \beta_i \partial e_{ij})\partial_j) = \sum_{i=1}^{d}\sum_{j=1}^{n}\gamma_i e_{ij}\partial_j = \sum_{i=1}^{d}\gamma_i e_i^{V} = 0$ implies that $\gamma_i = 0$ since $\left\{e_i^{V}\right\}_{i=1}^{d}$ is linearly independent. Hence we know that $\left\{e_i^{V}, e_i^{C}\right\}$ is linearly independent. This shows that the dimension of $\overline{N}$ is greater or equal to $2d$.

We would like to show that under the <u>assumption</u> that both N and $\overline{N}$ are regular, the dimension of $\overline{N}$ is actually $2d$, where d is the dimension of N, i.e, at each point $(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)$, $\overline{N}_{(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)}$ is a $2d$ dimensional subspace of $T_{(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)}(TM)$.

For any $\overline{X} \in \Gamma(\overline{U}, \overline{N})$, let $\overline{X}^B$, $\overline{R}_{BCD}{}^A$ be the local components of $\overline{X}$ and R^C respectively with respect to the induced coordinates. Then from Proposition 3.7, we have

$$\sum_{B}\overline{R}_{BCD}{}^A\,\overline{X}^B = 0 \qquad \text{for any } A, C, D. \qquad (1)$$

We already know that

$$\overline{R}_{kji}{}^h(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n) = R_{kji}{}^h(\overline{x}^1, \ldots, \overline{x}^n),$$

$$\overline{R}_{kji}{}^{\overline{h}} = \partial R_{kji}{}^h = \sum_{l=1}^{n}\overline{y}^l\frac{\partial R_{kji}{}^h}{\partial \overline{x}^l}, \quad \overline{R}_{\overline{k}ji}{}^{\overline{h}} = \overline{R}_{k\overline{j}i}{}^{\overline{h}} = \overline{R}_{kj\overline{i}}{}^{\overline{h}} = R_{kji}{}^h \quad \text{for all } i, j, h, k = 1, \ldots, n$$

all the others being zero, where $R_{kji}{}^h$ are local components of R.

From (***) on page 96, we have

$$\left.\begin{aligned} &\sum_{k=1}^{n} R_{kji}{}^h \overline{X}^k = 0, \\ &\sum_{k=1}^{n} (R_{kji}{}^h \overline{X}^{\bar{k}} + \partial R_{kji}{}^h \overline{X}^k) = 0, \end{aligned}\right\} \quad \text{for all } i, j, h = 1, \ldots, n. \qquad (2)$$

The second equation of (2) can be further reduced:

$$\begin{aligned} 0 &= \sum_{k=1}^{n} (R_{kji}{}^h \overline{X}^{\bar{k}} + \partial R_{kji}{}^h \overline{X}^k) \\ &= \sum_{k=1}^{n} (R_{kji}{}^h \overline{X}^{\bar{k}} - R_{kji}{}^h \partial\overline{X}^k + R_{kji}{}^h \partial\overline{X}^k + \partial R_{kji}{}^h \overline{X}^k) \\ &= \sum_{k=1}^{n} R_{kji}{}^h (\overline{X}^{\bar{k}} - \partial\overline{X}^k) + \sum_{k=1}^{n} \partial(R_{kji}{}^h \overline{X}^k) \\ &= \sum_{k=1}^{n} R_{kji}{}^h (\overline{X}^{\bar{k}} - \partial\overline{X}^k) + \partial\sum_{k=1}^{n} R_{kji}{}^h \overline{X}^k = \sum_{k=1}^{n} R_{kji}{}^h (\overline{X}^{\bar{k}} - \partial\overline{X}^k), \end{aligned}$$

for all $h, i, j = 1, \ldots, n$, since $\sum_{k=1}^{n} R_{kji}{}^h \overline{X}^k = 0$. Thus we have $\overline{X} \in \Gamma(\overline{U}, \overline{N})$ if and only if its components $\overline{X}^B$ satisfy

$$\left.\begin{aligned} &\sum_{k=1}^{n} R_{kji}{}^h \overline{X}^k = 0, \\ &\sum_{k=1}^{n} R_{kji}{}^h (\overline{X}^{\bar{k}} - \sum_{l=1}^{n} \overline{y}^l \frac{\partial\overline{X}^k}{\partial\overline{x}^l}) = 0, \end{aligned}\right\} \quad \text{for all } i, j, h = 1, \ldots, n. \qquad (3)$$

Since $R_{kji}{}^h(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)$ are functions of $(\overline{x}^1, \ldots, \overline{x}^n)$ only, we can show that the dimension of $\overline{N}_{(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)}$ is $2d$.

The system (3) at $(\overline{x}^1, \ldots, \overline{x}^n, \overline{y}^1, \ldots, \overline{y}^n)$ will be

$$\left.\begin{aligned}&\left(\sum_{k=1}^{n} R_{kji}{}^{h}\bar{X}^{k}\right)(\bar{x}^{1}, \ldots, \bar{x}^{n}, \bar{y}^{1}, \ldots, \bar{y}^{n}) = 0\ , && 4(\mathrm{i})\\ &\left(\sum_{k=1}^{n} R_{kji}{}^{h}(\bar{X}^{\bar{k}} - \sum_{l=1}^{n}\bar{y}^{l}\frac{\partial \bar{X}^{k}}{\partial \bar{x}^{l}})\right)(\bar{x}^{1}, \ldots, \bar{x}^{n}, \bar{y}^{1}, \ldots, \bar{y}^{n}) = 0\ , && 4(\mathrm{ii})\end{aligned}\right\}\ \text{for all } i, j, h = 1, \ldots, n, \quad (4)$$

and its solutions will have the form $(\bar{X}^{k}, \bar{X}^{\bar{k}}) = \left(\bar{X}^{k}, \bar{Y}^{k} + \sum_{l=1}^{n}\bar{y}^{l}\frac{\partial \bar{X}^{k}}{\partial \bar{x}^{l}}\right)$, $k = 1, \ldots, n$

where $\{\bar{X}^{k}\}_{k=1}^{n}$, $\{\bar{Y}^{k}\}_{k=1}^{n}$ satisfy equation 4(i).

Since the degree of freedom of $\{\bar{X}^{k}\}_{k=1}^{n}$ is equal to the dimension of $N_{(x^{1}, \ldots, x^{n})}$ which is equal to d, we now have to show that the degree of freedom of $\left\{\bar{Y}^{k} + \sum_{l=1}^{n}\bar{y}^{l}\frac{\partial \bar{X}^{k}}{\partial \bar{x}^{l}}\right\}_{k=1}^{n}$ is d.

From the <u>assumption</u> that $\bar{N}$ is regular we have $dim\,\bar{N}_{(\bar{x}^{1}, \ldots, \bar{x}^{n}, \bar{y}^{1}, \ldots, \bar{y}^{n})} = dim\,\bar{N}_{(\bar{x}^{1}, \ldots, \bar{x}^{n}, 0, \ldots, 0)}$, thus, we only need to show that $\left\{\left(\bar{Y}^{k} + \sum_{l=1}^{n}\bar{y}^{l}\frac{\partial \bar{X}^{k}}{\partial \bar{x}^{l}}\right)(\bar{x}^{1}, \ldots, \bar{x}^{n}, 0, \ldots, 0)\right\}_{k=1}^{n}$ is of degree d. But $\left\{\left(\bar{Y}^{k} + \sum_{l=1}^{n}\bar{y}^{l}\frac{\partial \bar{X}^{k}}{\partial \bar{x}^{l}}\right)(\bar{x}^{1}, \ldots, \bar{x}^{n}, 0, \ldots, 0)\right\}_{k=1}^{n} = \{\bar{Y}^{k}(\bar{x}^{1}, \ldots, \bar{x}^{n}, 0, \ldots, 0) + 0\}_{k=1}^{n}$, hence the degree of freedom of $\{\bar{Y}^{k}(\bar{x}^{1}, \ldots, \bar{x}^{n}, 0, \ldots, 0)\}_{k=1}^{n}$ is clearly d since equation 4(ii) at $(\bar{x}^{1}, \ldots, \bar{x}^{n}, 0, \ldots, 0)$ reduces to $\left(\sum_{k=1}^{n} R_{kji}{}^{h}\bar{Y}^{k}\right)(\bar{x}^{1}, \ldots, \bar{x}^{n}, 0, \ldots, 0) = 0$.

We summarize the above result in the following theorem:

Theorem 3.10

If the nullity distribution $\overline{N}$ on TM is nonsingular, then the dimension of $\overline{N}$ is twice the dimension of N.

Remarks:-If we assume only that N is regular, $\overline{N}$ may still be singular. If we consider other liftings of g to TM, then their corresponding nullity distributions may also be studied.