CHAPTER 4

$(TM,\tilde{g}_s)$ AND GEODESICS IN $(TM,\tilde{g}_s)$

4.1 Sasaki metric

Let (M,g) be a Riemannian manifold. The metric g as a $(0, 2)$ tensor on M can be written as $g=\sum_{j,i=1}^{n} g_{ji}dx^j\otimes dx^i$, with g_{ji} as its tensor components. Let Γ_{ji}^h be the Christoffel symbols formed with g_{ji}. In a neighbourhood $\pi^{-1}(U)$ of TM, where U is an open set in M, we write $\delta\bar{y}^h=d\bar{y}^h+\sum_{j,i=1}^{n}\Gamma_{ji}^h\bar{y}^j d\bar{x}^i$. Then $\sum_{j,i=1}^{n} g_{ji}\delta\bar{y}^j\otimes\delta\bar{y}^i$ will be a $(0, 2)$ tensor on TM. The *Sasaki metric* [Sa] is defined by

$$\tilde{g}_s=\sum_{j,i=1}^{n}\left(g_{ji}d\bar{x}^j\otimes d\bar{x}^i+g_{ji}\delta\bar{y}^j\otimes\delta\bar{y}^i\right).$$

Let us look at the components of the Sasaki metric,

$$\begin{aligned}
\tilde{g}_s &= \sum_{j,i=1}^{n}\left(g_{ji}d\bar{x}^j\otimes d\bar{x}^i+g_{ji}\delta\bar{y}^j\otimes\delta\bar{y}^i\right)\\
&=\sum_{j,i=1}^{n}\Bigg(g_{ji}d\bar{x}^j\otimes d\bar{x}^i+\sum_{h,k,t,s=1}^{n}\Gamma_{ki}^s\Gamma_{hj}^t g_{ts}\bar{y}^k\bar{y}^h d\bar{x}^j\otimes d\bar{x}^i+\sum_{k,s=1}^{n}\Gamma_{kj}^s\bar{y}^k g_{si}d\bar{x}^j\otimes d\bar{y}^i\\
&\quad+\sum_{s,k=1}^{n}\Gamma_{ki}^s\bar{y}^k g_{js}d\bar{y}^j\otimes d\bar{x}^i+g_{ji}d\bar{y}^j\otimes d\bar{x}^i\Bigg)\\
&=\sum_{j,i=1}^{n}\Bigg(\bigg(g_{ji}+\sum_{h,k,t,s=1}^{n}\Gamma_{ki}^s\Gamma_{hj}^t g_{ts}\bar{y}^k\bar{y}^h\bigg)d\bar{x}^j\otimes d\bar{x}^i+\sum_{s,k=1}^{n}\Gamma_{kj}^s\bar{y}^k g_{si}d\bar{x}^j\otimes d\bar{y}^i\\
&\quad+\sum_{s,k=1}^{n}\Gamma_{ki}^s\bar{y}^k g_{js}d\bar{y}^j\otimes d\bar{x}^i+g_{ji}d\bar{y}^j\otimes d\bar{y}^i\Bigg).
\end{aligned}$$

If we write $\Gamma_s^t=\sum_{h=1}^{n}\Gamma_{hs}^t\bar{y}^h$, then $\tilde{g}_s$ in matrix form will be:

$$\tilde{g}_s = (\tilde{g}_{AB}) = \begin{pmatrix} g_{ji} + \sum_{s,t=1}^{n} g_{ts}\Gamma_j^t\Gamma_i^s & \sum_{s=1}^{n}\Gamma_j^s g_{si} \\ \sum_{k,s=1}^{n}\Gamma_i^s g_{sj} & g_{ji} \end{pmatrix}.$$

We can see that for any $X, Y \in \mathfrak{I}_0^1(M)$,

$$\begin{aligned}
\tilde{g}_s(X^H, Y^H) &= \tilde{g}_s\left(\sum_{h=1}^{n}(X^h\partial_h - \sum_{m=1}^{n}\Gamma_m^h X^m \partial_{\bar{h}}),\ \sum_{h=1}^{n}(Y^h\partial_h - \sum_{m=1}^{n}\Gamma_m^h Y^m \partial_{\bar{h}})\right) \\
&= \sum_{i,j=1}^{n}(g_{ji} + \sum_{s,t=1}^{n} g_{ts}\Gamma_j^t\Gamma_i^s)X^jY^i - \sum_{i,j,m,s=1}^{n} g_{si}\Gamma_j^s X^j \Gamma_m^i Y^m \\
&\quad - \sum_{i,j,m,s=1}^{n} g_{sj}\Gamma_i^s Y^i \Gamma_m^j X^m + \sum_{i,j,m,s=1}^{n} g_{ji}\Gamma_m^j X^m \Gamma_s^i Y^s \\
&= \sum_{i,j=1}^{n} g_{ji} X^j Y^i ,
\end{aligned}$$

$$\begin{aligned}
\tilde{g}_s(X^V, Y^V) &= \tilde{g}_s\left(\sum_{h=1}^{n} X^h\partial_{\bar{h}},\ \sum_{h=1}^{n} Y^h\partial_{\bar{h}}\right) \\
&= \sum_{i,j=1}^{n} g_{ji} X^j Y^i ,
\end{aligned}$$

$$\begin{aligned}
\tilde{g}_s(X^H, Y^V) &= \tilde{g}_s\left(\sum_{h=1}^{n}(X^h\partial_h - \sum_{m=1}^{n}\Gamma_m^h X^m \partial_{\bar{h}}),\ \sum_{h=1}^{n} Y^h\partial_{\bar{h}}\right) \\
&= \sum_{i,j,s=1}^{n}\Gamma_j^s g_{si} X^j Y^i - \sum_{i,j,m=1}^{n}\Gamma_m^j X^m Y^i g_{ji} \\
&= 0 ,
\end{aligned}$$

thus we have

$$\tilde{g}_s(X^H, Y^H) = \tilde{g}_s(X^V, Y^V) = \left(g(X, Y)\right)^V ,$$

$$\tilde{g}_s(X^H, Y^V) = 0 .$$

For an alternative approach to the Sasaki metric, see [Kw] and [Do].

4.2 Adapted frames

In this section, we shall use some notation to simplify our structure: $\Gamma_s^l = \sum_{h=1}^{n}\Gamma_{hs}^l \bar{y}^h$

$\wedge_{ji} = \sum_{l=1}^{n}\Gamma_j^l g_{li} = \sum_{l,k=1}^{n}\Gamma_{kl}^l \bar{y}^k g_{li}$, from $\sum_{l=1}^{n}\Gamma_{ij}^l g_{lk} = \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ii}}{\partial x^k}$, it is clear that $\wedge_{ji} = \wedge_{ij}$.

So we can write $\tilde{g}_s = (\bar{g}_{AB}) = \begin{pmatrix} g_{ji} + \sum_{s,t=1}^{n} g_{ts}\Gamma_j^t\Gamma_i^s & \wedge_{ji} \\ \wedge_{ji} & g_{ji} \end{pmatrix}$.

Let U be an open set in M with local coordinate $(x^1, \ldots, x^n)$. Take $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ as its local basis on $\pi^{-1}(U)$. Consider $\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}$ and $\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial \bar{x}^i} - \sum_{h,j=1}^{n}\Gamma_{ji}^h \bar{y}^j \frac{\partial}{\partial \bar{y}^h} = \frac{\partial}{\partial \bar{x}^i} - \sum_{h=1}^{n}\Gamma_i^h \frac{\partial}{\partial \bar{y}^h}$. We put $D_i = \left(\frac{\partial}{\partial x^i}\right)^H$ and $D_{\bar{i}} = \left(\frac{\partial}{\partial x^i}\right)^V$. Then the set $\{D_i, D_{\bar{i}}\}$ is a local basis for $\pi^{-1}(\pi^{-1}(U)) \subseteq T(TM)$. We call this the *adapted frame*.

If we write

$$\left(A_B{}^A\right) = \begin{pmatrix} \delta_i^h & -\Gamma_i^h \\ 0 & \delta_i^h \end{pmatrix} \text{ and } \left(A_B{}^A\right)^{-1} = \left(A^A{}_B\right) = \begin{pmatrix} \delta_i^h & \Gamma_i^h \\ 0 & \delta_i^h \end{pmatrix},$$

then we will have

$$D_\alpha = \sum_A A_\alpha{}^A \frac{\partial}{\partial \bar{x}^A},$$

and if we write the coframe of $\{D_i, D_{\bar{i}}\}$ as $\{\theta^i, \theta^{\bar{i}}\}$, where

$$\theta^\beta = \sum_B A^\beta{}_B d\bar{x}^B, \ \alpha,\ \beta,\ A,\ B = 1, \ldots, n, \ \bar{1}, \ldots, \bar{n},$$

we can see clearly that $\theta^i = d\bar{x}^i$, $\theta^{\bar{i}} = \delta\bar{y}^i$.

Since $\left(A_B{}^A\right)^{-1} = \left(A^A{}_B\right)$, we have $d\bar{x}^A = \sum_\alpha A_\alpha{}^A \theta^\alpha$. Therefore

$$\begin{aligned}
\tilde{g}_s &= \sum_{A,B} \bar{g}_{AB} d\bar{x}^A \otimes d\bar{x}^B \\
&= \sum_{A,B} \bar{g}_{AB} \left(\sum_\gamma A_\gamma{}^A \theta^\gamma\right) \otimes \left(\sum_\beta A_\beta{}^B \theta^\beta\right) \\
&= \sum_{A,B,\gamma,\beta} \bar{g}_{AB} A_\gamma{}^A A_\beta{}^B \theta^\gamma \otimes \theta^\beta.
\end{aligned}$$

If we let $\tilde{g}_{\gamma\beta}$ be the components of $\tilde{g}_s$ with respect to the coframe $\{\theta^\alpha\}$, that is, $\tilde{g}_s = \sum_{\gamma,\beta} \tilde{g}_{\gamma\beta} \theta^\gamma \otimes \theta^\beta$, then we have

$$\tilde{g}_{\gamma\beta} = \sum_{A,B} \bar{g}_{AB} A_\gamma{}^A A_\beta{}^B \text{, for all } \gamma \text{ and } \beta.$$

From $\begin{pmatrix} D_h \\ D_{\bar{h}} \end{pmatrix} = \begin{pmatrix} \delta_i^h & -\Gamma_i^h \\ 0 & \delta_i^h \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial\bar{x}^h} \\ \dfrac{\partial}{\partial\bar{y}^h} \end{pmatrix}$, $\begin{pmatrix} \theta^h & \theta^{\bar{h}} \end{pmatrix} = \begin{pmatrix} d\bar{x}^h & d\bar{y}^h \end{pmatrix} \begin{pmatrix} \delta_i^h & \Gamma_i^h \\ 0 & \delta_i^h \end{pmatrix}$, we have

$$\begin{pmatrix} \dfrac{\partial}{\partial\bar{x}^h} \\ \dfrac{\partial}{\partial\bar{y}^h} \end{pmatrix} = \begin{pmatrix} \delta_i^h & \Gamma_i^h \\ 0 & \delta_i^h \end{pmatrix} \begin{pmatrix} D_h \\ D_{\bar{h}} \end{pmatrix}, \quad \begin{pmatrix} d\bar{x}^h & d\bar{y}^h \end{pmatrix} = \begin{pmatrix} \theta^h & \theta^{\bar{h}} \end{pmatrix} \begin{pmatrix} \delta_i^h & -\Gamma_i^h \\ 0 & \delta_i^h \end{pmatrix},$$

which implies that $\left(\tilde{g}_{\gamma\beta}\right) = \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{pmatrix}$. Using the same argument, $\left(\tilde{g}^{\gamma\beta}\right) = \begin{pmatrix} g^{ji} & 0 \\ 0 & g^{ji} \end{pmatrix}$.

Define $\Omega_{\gamma\beta}{}^\alpha = \sum_A \left(D_\gamma A_\beta{}^A - D_\beta A_\gamma{}^A\right) A^\alpha{}_A$, where $\alpha, \beta, \gamma = 1, \ldots, n, \bar{1}, \ldots, \bar{n}$.

The only non zero term are $\Omega_{ji}{}^{\bar h} = -\Omega_{ij}{}^{\bar h} = -\sum_{k=1}^{n} R_{jik}{}^{h}\bar{y}^{k}$ and $\Omega_{\bar{j}i}{}^{\bar h} = -\Omega_{i\bar{j}}{}^{\bar h} = \Gamma_{ji}^{h}$,

where $R_{jik}{}^{h}$ are the local components of the curvature tensor with respect to ∇.

$$\begin{aligned}
\Omega_{ji}{}^{\bar h} &= \sum_{A}\left(D_j A_i{}^{A} - D_i A_j{}^{A}\right)A^{\bar h}{}_{A} \\
&= \sum_{k=1}^{n}\left(D_j(-\Gamma_i^k)\delta_k^h - D_i(-\Gamma_j^k)\delta_k^h\right) \\
&= \sum_{l=1}^{n}\bar{y}^l\left(-\partial_j\Gamma_{li}^k + \sum_{m=1}^{n}\Gamma_{mi}^h\Gamma_{lj}^m\right) - \sum_{l=1}^{n}\bar{y}^l\left(-\partial_i\Gamma_{lj}^k + \sum_{m=1}^{n}\Gamma_{mj}^h\Gamma_{li}^m\right) \\
&= \sum_{l=1}^{n}\bar{y}^l\left(\partial_i\Gamma_{lj}^k - \partial_j\Gamma_{li}^k + \sum_{m=1}^{n}\Gamma_{mi}^h\Gamma_{lj}^m - \sum_{m=1}^{n}\Gamma_{mj}^h\Gamma_{li}^m\right) \\
&= -\sum_{k=1}^{n} R_{jik}{}^{h}\bar{y}^k = \sum_{k=1}^{n} R_{ijk}{}^{h}\bar{y}^k = -\Omega_{ij}{}^{\bar h}.
\end{aligned}$$

$\Omega_{\bar{j}i}{}^{h} = -\Omega_{i\bar{j}}{}^{h} = \Gamma_{ij}^{h}$ can be proved similarly.

Let us evaluate the Lie Bracket $[D_\alpha, D_\beta]$,

$$\begin{aligned}
[D_\alpha, D_\beta] &= \left[\sum_B A_\alpha{}^B\partial_B, \sum_C A_\beta{}^C\partial_C\right] \\
&= \sum_{B,C}\left(A_\alpha{}^B\partial_B(A_\beta{}^C)\partial_C - A_\beta{}^C\partial_C(A_\alpha{}^B)\partial_B\right) \\
&= \sum_{B,C}\left(A_\alpha{}^B\partial_B(A_\beta{}^C) - A_\beta{}^B\partial_B(A_\alpha{}^C)\right)\partial_C \\
&= \sum_{C}\left(D_\alpha A_\beta{}^C - D_\beta A_\alpha{}^C\right)\partial_C \\
&= \sum_{B,C,\gamma}\left(D_\alpha A_\beta{}^C - D_\beta A_\alpha{}^C\right)A^\gamma{}_C A_\gamma{}^B\partial_B \\
&= \sum_{\gamma}\Omega_{\alpha\beta}{}^{\gamma}D_\gamma .
\end{aligned}$$

Now let $\tilde{\nabla}$ be the metric connection in *TM* with respect to the Sasaki metric $\tilde{g}_s$.

Let $\tilde{g}_{\gamma\beta}$ be the tensor components of $\tilde{g}_s$ with respect to the adapted frame $\{D_\alpha\}$, $\tilde{\Gamma}_{\gamma\beta}^{\alpha}$ be the Christoffel symbols with respect to the adapted frame $\{D_\alpha\}$ and $\overline{\Gamma}^A_{CB}$ be the Christoffel symbols with respect to the coordinate vector fields $\{\partial_B\}$. We have

$$\begin{aligned}\widetilde{\nabla}_{D_\gamma} D_\beta &= \sum_\alpha \widetilde{\Gamma}^\alpha_{\gamma\beta} D_\alpha = \sum_A D_\gamma A_\beta{}^A \partial_A + \sum_{B,C} A_\gamma{}^C A_\beta{}^B (\widetilde{\nabla}_{\partial_C} \partial_B) \\ &= \sum_{A,\alpha} (D_\gamma A_\beta{}^A) A^\alpha{}_A D_\alpha + \sum_{A,B,C} A_\gamma{}^C A_\beta{}^B \overline{\Gamma}^A_{CB} \partial_A \\ &= \sum_{A,\alpha} (D_\gamma A_\beta{}^A) A^\alpha{}_A D_\alpha + \sum_{A,B,C,\alpha} A_\gamma{}^C A_\beta{}^B \overline{\Gamma}^A_{CB} A^\alpha{}_A D_\alpha \\ &= \sum_{A,\alpha} (D_\gamma A_\beta{}^A + \sum_{B,C} A_\gamma{}^C A_\beta{}^B \overline{\Gamma}^A_{CB}) A^\alpha{}_A D_\alpha .\end{aligned}$$

Thus, we have $\widetilde{\Gamma}^\alpha_{\gamma\beta} = \sum_A (D_\gamma A_\beta{}^A + \sum_{B,C} A_\gamma{}^C A_\beta{}^B \overline{\Gamma}^A_{CB}) A^\alpha{}_A$.

Since $\widetilde{\nabla}$ is a metric connection, that is, $\widetilde{\nabla}_\delta \widetilde{g}_s = 0$, $\delta = 1, \ldots, n, \bar{1}, \ldots, \bar{n}$.

$$\begin{aligned}(\widetilde{\nabla}_\delta \widetilde{g}_s)(D_\gamma, D_\beta) &= \widetilde{\nabla}_\delta \big(\widetilde{g}_s(D_\gamma, D_\beta)\big) - \widetilde{g}_s(\widetilde{\nabla}_\delta D_\gamma, D_\beta) - \widetilde{g}_s(D_\gamma, \widetilde{\nabla}_\delta D_\beta) \\ &= D_\delta \widetilde{g}_{\gamma\beta} - \widetilde{g}_s(\sum_\varepsilon \widetilde{\Gamma}^\varepsilon_{\delta\gamma} D_\varepsilon, D_\beta) - \widetilde{g}_s(D_\gamma, \sum_\varepsilon \widetilde{\Gamma}^\varepsilon_{\delta\beta} D_\varepsilon) \\ &= D_\delta \widetilde{g}_{\gamma\beta} - \sum_\varepsilon \big(\widetilde{\Gamma}^\varepsilon_{\delta\gamma} \widetilde{g}_{\varepsilon\beta} - \widetilde{\Gamma}^\varepsilon_{\delta\beta} \widetilde{g}_{\gamma\varepsilon}\big) = 0 ,\end{aligned}$$

we have $D_\delta \widetilde{g}_{\gamma\beta} = \sum_\varepsilon \big(\widetilde{\Gamma}^\varepsilon_{\delta\gamma} \widetilde{g}_{\varepsilon\beta} - \widetilde{\Gamma}^\varepsilon_{\delta\beta} \widetilde{g}_{\gamma\varepsilon}\big)$. Thus we have

Proposition 4.1 [Ya1, page 160]

$$\widetilde{\Gamma}^\alpha_{\gamma\beta} = \tfrac{1}{2} \sum_\varepsilon (\widetilde{g}^{\alpha\varepsilon})(D_\gamma \widetilde{g}_{\varepsilon\beta} + D_\beta \widetilde{g}_{\gamma\varepsilon} - D_\varepsilon \widetilde{g}_{\gamma\beta}) + \tfrac{1}{2}(\Omega_{\gamma\beta}{}^\alpha + \Omega^\alpha{}_{\gamma\beta} + \Omega^\alpha{}_{\beta\gamma})$$

where $\Omega^\alpha{}_{\gamma\beta} = \sum_{\varepsilon,\delta} \widetilde{g}^{\alpha\varepsilon} \widetilde{g}_{\delta\beta} \Omega_{\varepsilon\gamma}{}^\delta$.

Proof:

$$\tfrac{1}{2}\sum_{\varepsilon}(\widetilde{g}^{\alpha\varepsilon})(D_{\gamma}\widetilde{g}_{\varepsilon\beta}+D_{\beta}\widetilde{g}_{\gamma\varepsilon}-D_{\varepsilon}\widetilde{g}_{\gamma\beta})+\tfrac{1}{2}(\Omega_{\gamma\beta}{}^{\alpha}+\Omega^{\alpha}{}_{\gamma\beta}+\Omega^{\alpha}{}_{\beta\gamma})$$

$$=\tfrac{1}{2}\sum_{\varepsilon}\widetilde{g}^{\alpha\varepsilon}(D_{\gamma}\widetilde{g}_{\varepsilon\beta}+D_{\beta}\widetilde{g}_{\gamma\varepsilon}-D_{\varepsilon}\widetilde{g}_{\gamma\beta})+\tfrac{1}{2}\Omega_{\gamma\beta}{}^{\alpha}+\tfrac{1}{2}\sum_{\varepsilon,\delta}(\widetilde{g}^{\alpha\varepsilon}\widetilde{g}_{\delta\beta}\Omega_{\varepsilon\gamma}{}^{\delta}+\widetilde{g}^{\alpha\varepsilon}\widetilde{g}_{\delta\gamma}\Omega_{\varepsilon\beta}{}^{\delta})$$

$$=\tfrac{1}{2}\sum_{\varepsilon,\theta}(\widetilde{\Gamma}^{\varepsilon}_{\gamma\theta}\widetilde{g}_{\varepsilon\beta}-\widetilde{\Gamma}^{\varepsilon}_{\gamma\beta}\widetilde{g}_{\theta\varepsilon}+\widetilde{\Gamma}^{\varepsilon}_{\beta\gamma}\widetilde{g}_{\varepsilon\theta}-\widetilde{\Gamma}^{\varepsilon}_{\beta\theta}\widetilde{g}_{\gamma\varepsilon}-\widetilde{\Gamma}^{\varepsilon}_{\theta\gamma}\widetilde{g}_{\varepsilon\beta}+\widetilde{\Gamma}^{\varepsilon}_{\theta\beta}\widetilde{g}_{\gamma\varepsilon})\widetilde{g}^{\alpha\theta}$$

$$+\tfrac{1}{2}\left(\widetilde{\Gamma}^{\alpha}_{\gamma\beta}-\widetilde{\Gamma}^{\alpha}_{\beta\gamma}+\sum_{\varepsilon,\theta}\widetilde{g}^{\alpha\theta}\widetilde{g}_{\varepsilon\beta}(\widetilde{\Gamma}^{\varepsilon}_{\theta\gamma}-\widetilde{\Gamma}^{\varepsilon}_{\gamma\theta})+\sum_{\varepsilon,\theta}\widetilde{g}^{\alpha\theta}\widetilde{g}_{\varepsilon\gamma}(\widetilde{\Gamma}^{\varepsilon}_{\theta\beta}-\widetilde{\Gamma}^{\varepsilon}_{\beta\theta})\right)$$

$$=\tfrac{1}{2}\left(\widetilde{\Gamma}^{\alpha}_{\gamma\beta}-\widetilde{\Gamma}^{\alpha}_{\beta\gamma}+\sum_{\varepsilon,\theta}\widetilde{\Gamma}^{\varepsilon}_{\gamma\beta}\widetilde{g}_{\theta\varepsilon}\widetilde{g}^{\alpha\theta}+\sum_{\varepsilon,\theta}\widetilde{\Gamma}^{\varepsilon}_{\beta\gamma}\widetilde{g}_{\varepsilon\theta}\widetilde{g}^{\alpha\theta}\right)$$

$$=\widetilde{\Gamma}^{\alpha}_{\gamma\beta}\,.$$

■

Now we find all the Christoffel symbols of $\widetilde{\nabla}$, let $R^{h}{}_{jki}=\sum_{s,t=1}^{n}g^{ht}g_{is}R_{tjk}{}^{s}$:

$$\widetilde{\Gamma}^{h}_{ji}=\tfrac{1}{2}\sum_{\varepsilon}\widetilde{g}^{h\varepsilon}(D_{j}\widetilde{g}_{\varepsilon i}+D_{i}\widetilde{g}_{j\varepsilon}-D_{\varepsilon}\widetilde{g}_{ji})+\tfrac{1}{2}(\Omega_{ji}{}^{h}+\Omega^{h}{}_{ji}+\Omega^{h}{}_{ij})$$

$$=\tfrac{1}{2}\sum_{k=1}^{n}(\widetilde{g}^{hk}\partial_{j}\widetilde{g}_{ki}+\widetilde{g}^{hk}\partial_{i}\widetilde{g}_{jk}-\widetilde{g}^{hk}\partial_{k}\widetilde{g}_{ji})+\tfrac{1}{2}\sum_{\varepsilon,\delta}(\widetilde{g}^{h\varepsilon}\widetilde{g}_{\delta i}\Omega_{\varepsilon j}{}^{\delta}+\widetilde{g}^{h\varepsilon}\widetilde{g}_{\delta j}\Omega_{\varepsilon i}{}^{\delta})$$

$$=\tfrac{1}{2}\sum_{k=1}^{n}(\widetilde{g}^{hk}\partial_{j}\widetilde{g}_{ki}+\widetilde{g}^{hk}\partial_{i}\widetilde{g}_{jk}-\widetilde{g}^{hk}\partial_{k}\widetilde{g}_{ji})$$

$$=\tfrac{1}{2}\sum_{k=1}^{n}(g^{hk}\partial_{j}g_{ki}+g^{hk}\partial_{i}g_{jk}-g^{hk}\partial_{k}g_{ji})$$

$$=\Gamma^{h}_{ji}\,,$$

$$
\begin{aligned}
\widetilde{\Gamma}_{ji}^{h} &= \tfrac{1}{2}\sum_{\varepsilon}\widetilde{g}^{h\varepsilon}(D_j\widetilde{g}_{\varepsilon i} + D_i\widetilde{g}_{j\varepsilon} - D_\varepsilon\widetilde{g}_{ji}) \\
&\quad + \tfrac{1}{2}\left(\Omega_{ji}{}^{h} + \sum_{\varepsilon,\delta}\left(\widetilde{g}^{h\varepsilon}\widetilde{g}_{\delta i}\Omega_{\varepsilon j}{}^{\delta} + \widetilde{g}^{h\varepsilon}\widetilde{g}_{\delta j}\Omega_{\varepsilon i}{}^{\delta}\right)\right) \\
&= \tfrac{1}{2}\sum_{s,t=1}^{n}\widetilde{g}^{ht}\widetilde{g}_{\bar{s}\bar{i}}\Omega_{tj}{}^{\bar{s}} \\
&= \tfrac{1}{2}\sum_{s,t,k=1}^{n}\widetilde{g}^{ht}\widetilde{g}_{\bar{s}i}R_{jtk}{}^{s}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{s,t,k=1}^{n}g^{ht}g_{si}R_{jtk}{}^{s}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{s,t,k=1}^{n}g^{ht}g_{si}R_{tjk}{}^{s}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{k=1}^{n}R^{h}{}_{jki}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{s,t,k=1}^{n}g^{ht}g_{is}R_{tjk}{}^{s}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{s,t,k=1}^{n}g^{ht}(g_{is}R_{tjk}{}^{s})\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{t,k=1}^{n}g^{ht}R_{tjki}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{t,k=1}^{n}g^{ht}R_{kitj}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{t,k=1}^{n}g^{ht}R_{kijt}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{s,t,k=1}^{n}g^{ht}R_{kij}{}^{s}g_{st}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{s,k=1}^{n}\delta_{s}^{h}R_{kij}{}^{s}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{k=1}^{n}R_{kij}{}^{h}\bar{y}^{k}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Gamma}^h_{ji} &= \tfrac{1}{2}\sum_{\varepsilon} \widetilde{g}^{h\varepsilon}(D_j \widetilde{g}_{\varepsilon i} + D_i \widetilde{g}_{j\varepsilon} - D_{\varepsilon}\widetilde{g}_{ji}) \\
&\quad + \tfrac{1}{2}\left(\Omega_{ji}{}^{h} + \sum_{\delta,\varepsilon}\left(\widetilde{g}^{h\varepsilon}\widetilde{g}_{\delta i}\Omega_{\varepsilon j}{}^{\delta} + \widetilde{g}^{h\varepsilon}\widetilde{g}_{\delta j}\Omega_{\varepsilon i}{}^{\delta}\right)\right) \\
&= \sum_{s,t=1}^{n} \tfrac{1}{2}\widetilde{g}^{ht}\widetilde{g}_{\bar{s}j}\Omega_{ti}{}^{\bar{s}} \\
&= -\tfrac{1}{2}\sum_{s,t,k=1}^{n} \widetilde{g}^{ht}\widetilde{g}_{\bar{s}j}R_{tik}{}^{s}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{s,t,k=1}^{n} g^{ht}g_{sj}R_{tik}{}^{s}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{k=1}^{n} R^{h}{}_{ikj}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{s,t,k=1}^{n} R_{tik}{}^{s}g^{ht}g_{js}\bar{y}^{k} \\
&= -\tfrac{1}{2}\sum_{t,k=1}^{n} R_{tikj}g^{ht}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{t,k=1}^{n} R_{kjit}g^{ht}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{s,t,k=1}^{n} R_{kji}{}^{s}g_{ts}g^{ht}\bar{y}^{k} \\
&= \tfrac{1}{2}\sum_{k=1}^{n} R_{kji}{}^{h}\bar{y}^{k} \,.
\end{aligned}
$$

With the same argument, we will have

$$
\widetilde{\Gamma}^h_{ji} = \Gamma^h_{ji}\,,\quad \widetilde{\Gamma}^h_{j\bar{i}} = -\tfrac{1}{2}\sum_{k=1}^{n} R^{h}{}_{jki}\bar{y}^{k} = \tfrac{1}{2}\sum_{k=1}^{n} R_{kij}{}^{h}\bar{y}^{k}\,,\quad \widetilde{\Gamma}^h_{\bar{j}i} = -\tfrac{1}{2}\sum_{k=1}^{n} R^{h}{}_{ikj}\bar{y}^{k} = \tfrac{1}{2}\sum_{k=1}^{n} R_{kji}{}^{h}\bar{y}^{k}\,,\quad \widetilde{\Gamma}^h_{\bar{j}\bar{i}} = 0\,,
$$

$$
\widetilde{\Gamma}^{\bar{h}}_{ji} = \tfrac{1}{2}\sum_{k=1}^{n} R_{jik}{}^{h}\bar{y}^{k}\,,\quad \widetilde{\Gamma}^{\bar{h}}_{j\bar{i}} = \Gamma^h_{ji}\,,\quad \widetilde{\Gamma}^{\bar{h}}_{\bar{j}i} = 0\,,\quad \widetilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}} = 0\,.
$$

4.3 Geodesics in $(TM, \widetilde{g}_s)$

For a curve $\gamma : I \to M$, $\gamma : t \to \gamma(t)$, we define along γ the notion $\dfrac{\delta}{dt}$ by

$\nabla_{\frac{d}{dt}} f = \frac{df}{dt} = \frac{\delta f}{dt}$, $\nabla_{\frac{d}{dt}} X = \sum_{i=1}^{n} (\nabla_{\frac{d}{dt}} X^i) \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} \frac{\delta X^i}{dt} \frac{\partial}{\partial x^i}$ for f a function in M, X a vector field in M, where X can be written locally as $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$.

We have $\frac{\delta X^h}{dt} = \nabla_{\frac{d}{dt}} X^h = \frac{dX^h}{dt} + \sum_{i,j=1}^{n} \Gamma_{ji}^h \frac{d\gamma^j}{dt} X^i$.

Let $\bar{\gamma} : I \to TM$ be a curve in TM defined by

$$\bar{\gamma}(t) = (\gamma(t), V(t)) = (\gamma^1(t), \ldots, \gamma^n(t), V^1(t), \ldots, V^n(t)).$$

Obviously for a parallel vector field $V(t)$ along $\gamma(t)$, where $V(t) = (V^1(t), \ldots, V^n(t))$, $\frac{\delta V^h(t)}{dt} = 0$ for all $h = 1, \ldots, n$. Now we write

$$\begin{aligned} \bar{\gamma}(t) &= (\gamma^1(t), \ldots, \gamma^n(t), V^1(t), \ldots, V^n(t)) \\ &= (\bar{\gamma}^1(t), \ldots, \bar{\gamma}^n(t), \bar{\gamma}^{\bar{1}}(t), \ldots, \bar{\gamma}^{\bar{n}}(t)) \\ &= (\bar{\gamma}^A(t)). \end{aligned}$$

If $\bar{\gamma} : I \to TM$ is a geodesic in TM with respect to the Sasaki metric $\tilde{g}_s$, then $\tilde{\nabla}_{\dot{\bar{\gamma}}(t)} \dot{\bar{\gamma}}(t) = 0$, which implies that $\tilde{\nabla}_{\frac{d}{dt}} \sum_A \frac{d\bar{\gamma}^A}{dt} \frac{\partial}{\partial x^A} = 0$, thus

$$\sum_A \left(\frac{d^2 \bar{\gamma}^A}{dt^2} + \sum_{B,C} \bar{\Gamma}_{BC}^A \frac{d\bar{\gamma}^B}{dt} \frac{d\bar{\gamma}^C}{dt} \right) \frac{\partial}{\partial x^A} = 0,$$

hence

$$\frac{d^2 \bar{\gamma}^A}{dt^2} + \sum_{B,C} \bar{\Gamma}_{BC}^A \frac{d\bar{\gamma}^B}{dt} \frac{d\bar{\gamma}^C}{dt} = 0 \text{ for all } A = 1, \ldots, n, \bar{1}, \ldots, \bar{n},$$

It will be much more convenient to consider the geodesic equation with respect to the adapted frame $\{D_i, D_{\bar{i}}\}$.

We write

$$\frac{\theta^h}{dt}=\sum_B A^h{}_B\frac{d\bar{\gamma}^B}{dt}=\frac{d\bar{\gamma}^h}{dt},$$

$$\begin{aligned}\frac{\theta^{\bar{h}}}{dt}&=\sum_B A^{\bar{h}}{}_B\frac{d\bar{\gamma}^B}{dt}\\&=\frac{d\bar{\gamma}^{\bar{h}}}{dt}+\sum_{i=1}^{n}\Gamma_i^h\frac{d\bar{\gamma}^i}{dt}\\&=\frac{d\bar{\gamma}^{\bar{h}}}{dt}+\sum_{j,i=1}^{n}\Gamma_{ji}^h\frac{d\bar{\gamma}^i}{dt}\bar{y}^j\\&=\frac{d\bar{\gamma}^{\bar{h}}}{dt}+\sum_{j,i=1}^{n}\Gamma_{ji}^h\frac{d\bar{\gamma}^i}{dt}\bar{\gamma}^j\\&=\frac{\delta\bar{\gamma}^{\bar{h}}}{dt}\\&=\frac{\delta V^h}{dt}.\end{aligned}$$

Now, from $\frac{\theta^\beta}{dt}=\sum_A A^\beta{}_A\frac{d\gamma^A}{dt}$, $\frac{d\bar{\gamma}^A}{dt}=\sum_\beta A_\beta{}^A\frac{\theta^\beta}{dt}$ and

$\tilde{\Gamma}^\alpha_{\gamma\beta}=\sum_A(D_\gamma A_\beta{}^A+\sum_{B,C}\bar{\Gamma}^A_{CB}A_\gamma{}^C A_\beta{}^B)A^\alpha{}_A$, where $\bar{\Gamma}^A_{CB}$ and $\tilde{\Gamma}^\alpha_{\gamma\beta}$ are the Christoffel symbols of $\tilde{\nabla}$ with respect to the coordinate vector fields and adapted forms $\{\theta^\alpha\}_\alpha$ respectively, we have $\bar{\Gamma}^A_{CB}=\sum_{\gamma,\beta}(\sum_\alpha\tilde{\Gamma}^\alpha_{\gamma\beta}A_\alpha{}^A-D_\gamma A_\beta{}^A)A^\beta{}_B A^\gamma{}_C$.

Thus from $\frac{d^2\gamma^A}{dt^2}+\sum_{B,C}\bar{\Gamma}^A_{CB}\frac{d\gamma^C}{dt}\frac{d\gamma^B}{dt}=0$, we have

$$\frac{d}{dt}\left(\sum_\beta A_\beta{}^A\frac{\theta^\beta}{dt}\right)+\sum_{B,C,\gamma,\beta}\left(\sum_\alpha\tilde{\Gamma}^\alpha_{\gamma\beta}A_\alpha{}^A-D_\gamma A_\beta{}^A\right)A^\beta{}_B A^\gamma{}_C\left(\sum_{\xi,\eta}A_\xi{}^C\frac{\theta^\xi}{dt}A_\eta{}^B\frac{\theta^\eta}{dt}\right)=0,$$

$$\frac{d}{dt}\left(\sum_{\beta} A_{\beta}{}^{A}\frac{\theta^{\beta}}{dt}\right)+\sum_{\beta,\gamma}\left(\sum_{\alpha}\tilde{\Gamma}^{\alpha}_{\gamma\beta}A_{\alpha}{}^{A}-D_{\gamma}A_{\beta}{}^{A}\right)\frac{\theta^{\beta}}{dt}\frac{\theta^{\gamma}}{dt}=0\ .$$

If $A=h$, then

$$\frac{d}{dt}\left(\sum_{\beta} A_{\beta}{}^{h}\frac{\theta^{\beta}}{dt}\right)+\sum_{\beta,\gamma}\left(\sum_{\alpha}\tilde{\Gamma}^{\alpha}_{\gamma\beta}A_{\alpha}{}^{h}-D_{\gamma}A_{\beta}{}^{h}\right)\frac{\theta^{\beta}}{dt}\frac{\theta^{\gamma}}{dt}=0,$$

$$\frac{d}{dt}\left(\frac{\theta^{h}}{dt}\right)+\sum_{\beta,\gamma}\tilde{\Gamma}^{h}_{\gamma\beta}\frac{\theta^{\beta}}{dt}\frac{\theta^{\gamma}}{dt}=0\ . \qquad (*)$$

If $A=\bar{h}$, then

$$\frac{d}{dt}\left(\sum_{\beta} A_{\beta}{}^{\bar{h}}\frac{\theta^{\beta}}{dt}\right)+\sum_{\beta,\gamma}\left(\sum_{\alpha}\tilde{\Gamma}^{\alpha}_{\gamma\beta}A_{\alpha}{}^{\bar{h}}-D_{\gamma}A_{\beta}{}^{\bar{h}}\right)\frac{\theta^{\beta}}{dt}\frac{\theta^{\gamma}}{dt}=0. \qquad (**)$$

Since

$$\begin{aligned}\sum_{\beta,\gamma}D_{\gamma}A_{\beta}{}^{\bar{h}}\frac{\theta^{\beta}}{dt}\frac{\theta^{\gamma}}{dt}&=\sum_{\gamma}\sum_{i=1}^{n}D_{\gamma}(-\Gamma^{h}_{i})\frac{\theta^{i}}{dt}\frac{\theta^{\gamma}}{dt}\\&=\sum_{i,k=1}^{n}\left(\partial_{k}-\sum_{l=1}^{n}\Gamma^{l}_{k}\partial_{\bar{l}}\right)\left(-\sum_{j=1}^{n}\Gamma^{h}_{ji}\bar{y}^{j}\right)\frac{\theta^{i}}{dt}\frac{\theta^{k}}{dt}+\sum_{i,k=1}^{n}\partial_{\bar{k}}\left(-\sum_{j=1}^{n}\Gamma^{h}_{ji}\bar{y}^{j}\right)\frac{\theta^{i}}{dt}\frac{\theta^{\bar{k}}}{dt}\\&=\sum_{i,k=1}^{n}\left(\sum_{l,j=1}^{n}\bar{y}^{j}\Gamma^{l}_{jk}\Gamma^{h}_{li}-\sum_{j=1}^{n}\bar{y}^{j}\partial_{k}\Gamma^{h}_{ji}\right)\frac{\theta^{i}}{dt}\frac{\theta^{k}}{dt}-\sum_{i,k=1}^{n}\Gamma^{h}_{ki}\frac{\theta^{i}}{dt}\frac{\theta^{\bar{k}}}{dt},\end{aligned}$$

(**) becomes

$$\frac{d}{dt}\left(\sum_{i=1}^{n}(-\Gamma^{h}_{i}\frac{\theta^{i}}{dt})+\frac{\theta^{\bar{h}}}{dt}\right)+\sum_{\gamma,\beta}\left(\tilde{\Gamma}^{\bar{h}}_{\gamma\beta}-\sum_{i=1}^{n}\tilde{\Gamma}^{i}_{\gamma\beta}\Gamma^{h}_{i}\right)\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}$$

$$-\sum_{i,k=1}^{n}\left(\sum_{j,l=1}^{n}\bar{y}^{j}\Gamma^{l}_{jk}\Gamma^{h}_{li}-\bar{y}^{j}\partial_{k}\Gamma^{h}_{ji}\right)\frac{\theta^{i}}{dt}\frac{\theta^{k}}{dt}+\sum_{i,k=1}^{n}\Gamma^{h}_{ki}\frac{\theta^{i}}{dt}\frac{\theta^{\bar{k}}}{dt}=0,$$

$$\frac{d}{dt}\left(-\sum_{i,l=1}^{n}\Gamma_{li}^{h}\bar{y}^{l}\right)\frac{\theta^{i}}{dt}-\sum_{i=1}^{n}\Gamma_{i}^{h}\frac{d}{dt}\left(\frac{\theta^{i}}{dt}\right)+\frac{d}{dt}\left(\frac{\theta^{\bar{h}}}{dt}\right)+\sum_{\beta,\gamma}\widetilde{\Gamma}_{\gamma\beta}^{\bar{h}}\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}-\sum_{\beta,\gamma}\sum_{i=1}^{n}\Gamma_{i}^{h}\widetilde{\Gamma}_{\gamma\beta}^{i}\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}$$

$$-\sum_{i,k=1}^{n}\left(\sum_{j,l=1}^{n}\bar{y}^{j}\Gamma_{jk}^{l}\Gamma_{li}^{h}-\sum_{j=1}^{n}\bar{y}^{j}\partial_{k}\Gamma_{ji}^{h}\right)\frac{\theta^{i}}{dt}\frac{\theta^{k}}{dt}+\sum_{i,k=1}^{n}\Gamma_{ki}^{h}\frac{\theta^{i}}{dt}\frac{\theta^{k}}{dt}=0\,,$$

$$\left(\frac{d}{dt}\left(\frac{\theta^{\bar{h}}}{dt}\right)+\sum_{\beta,\gamma}\widetilde{\Gamma}_{\gamma\beta}^{\bar{h}}\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}\right)-\sum_{i=1}^{n}\Gamma_{i}^{h}\left(\frac{d}{dt}\left(\frac{\theta^{i}}{dt}\right)+\sum_{\beta,\gamma}\widetilde{\Gamma}_{\gamma\beta}^{i}\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}\right)$$

$$-\sum_{i,j,k=1}^{n}(\partial_{k}\Gamma_{ji}^{h})\bar{y}^{j}\frac{d\bar{y}^{k}}{dt}\frac{\theta^{i}}{dt}-\sum_{i,j=1}^{n}\Gamma_{ji}^{h}\frac{d\bar{y}^{j}}{dt}\frac{\theta^{i}}{dt}-\sum_{i,j,k,l=1}^{n}\bar{y}^{j}\Gamma_{jk}^{l}\Gamma_{li}^{h}\frac{\theta^{i}}{dt}\frac{\theta^{k}}{dt}\qquad(***)$$

$$+\sum_{i,j,k=1}^{n}\bar{y}^{j}\partial_{k}\Gamma_{ji}^{h}\frac{\theta^{i}}{dt}\frac{\theta^{k}}{dt}+\sum_{i,k=1}^{n}\Gamma_{ki}^{h}\frac{\theta^{i}}{dt}\left(\frac{d\bar{y}^{k}}{dt}+\sum_{l,m=1}^{n}\Gamma_{lm}^{k}\frac{d\bar{y}^{l}}{dt}\bar{y}^{m}\right)=0\,.$$

From (*) and $\frac{\theta^{k}}{dt}=\frac{d\bar{y}^{k}}{dt}$, equation (***) can be reduced to $\frac{d}{dt}\frac{\theta^{\bar{h}}}{dt}+\sum_{\beta,\gamma}\widetilde{\Gamma}_{\gamma\beta}^{\bar{h}}\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}=0$.

Hence a curve $\bar{\gamma}$ in TM is a geodesic if with respect to the adapted forms $\{\theta^{\alpha}\}_{\alpha}$, we have $\frac{d}{dt}\frac{\theta^{\alpha}}{dt}+\sum_{\beta,\gamma}\widetilde{\Gamma}_{\gamma\beta}^{\alpha}\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}=0$, $\alpha=1,\ldots,n,\bar{1},\ldots,\bar{n}$.

Then we have for $\alpha=h$,

$$\frac{d}{dt}\left(\frac{d\gamma^{h}}{dt}\right)+\sum_{\beta,\gamma}\widetilde{\Gamma}_{\gamma\beta}^{h}\frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt}=0.$$

$$\frac{d}{dt}\left(\frac{d\gamma^{h}}{dt}\right)+\sum_{i,j=1}^{n}\Gamma_{ji}^{h}\frac{\theta^{j}}{dt}\frac{\theta^{i}}{dt}+\tfrac{1}{2}\sum_{i,j,k=1}^{n}R_{kij}{}^{h}\bar{y}^{k}\frac{\theta^{j}}{dt}\frac{\theta^{\bar{i}}}{dt}+\tfrac{1}{2}\sum_{i,j,k=1}^{n}R_{kji}{}^{h}\bar{y}^{k}\frac{\theta^{\bar{j}}}{dt}\frac{\theta^{i}}{dt}=0\,,$$

$$\frac{d}{dt}\left(\frac{d\gamma^{h}}{dt}\right)+\sum_{i,j=1}^{n}\Gamma_{ji}^{h}\frac{d\gamma^{j}}{dt}\frac{d\gamma^{i}}{dt}+\sum_{i,j,k=1}^{n}R_{kji}{}^{h}\bar{y}^{k}\frac{\delta V^{j}}{dt}\frac{d\gamma^{i}}{dt}=0\,,$$

$$\frac{\delta}{dt}\left(\frac{d\gamma^{h}}{dt}\right)+\sum_{i,j,k=1}^{n}R_{kji}{}^{h}\bar{y}^{k}\frac{\delta V^{j}}{dt}\frac{d\gamma^{i}}{dt}=0\,,$$

$$\frac{\delta^2 \gamma^h}{dt^2} + \sum_{i,j,k=1}^{n} R_{kji}{}^{h} \bar{y}^k \frac{\delta V^j}{dt} \frac{d\gamma^i}{dt} = 0.$$

For $\alpha = \bar{h}$,

$$\frac{d}{dt}\left(\frac{\delta V^h}{dt}\right) + \sum_{\beta,\gamma} \widetilde{\Gamma}^{\bar{h}}_{\gamma\beta} \frac{\theta^\gamma}{dt} \frac{\theta^\beta}{dt} = 0 ,$$

$$\frac{d}{dt}\left(\frac{\delta V^h}{dt}\right) + \sum_{i,j=1}^{n} \Gamma^h_{ji} \frac{d\gamma^j}{dt} \frac{\delta V^i}{dt} - \tfrac{1}{2} \sum_{i,j,k=1}^{n} R_{kji}{}^{h} \bar{y}^k \frac{d\gamma^j}{dt} \frac{d\gamma^i}{dt} = 0 ,$$

$$\frac{\delta^2 V^h}{dt^2} - \tfrac{1}{2} \sum_{i,j,k=1}^{n} R_{kji}{}^{h} \bar{y}^k \frac{d\gamma^j}{dt} \frac{d\gamma^i}{dt} = 0 ,$$

but $\sum_{i,j,k=1}^{n} R_{kji}{}^{h} \bar{y}^k \frac{d\gamma^i}{dt} \frac{d\gamma^i}{dt} = \sum_{i,j,k=1}^{n} \left(\partial_j \Gamma^h_{ik} - \partial_i \Gamma^h_{jk} + \sum_{m=1}^{n} \Gamma^h_{jm} \Gamma^m_{ik} - \sum_{m=1}^{n} \Gamma^h_{im} \Gamma^m_{jk} \right) \bar{y}^k \frac{d\gamma^j}{dt} \frac{d\gamma^i}{dt} = 0$,

therefore the equations for a geodesic $\bar{\gamma}(t) = (\gamma(t), V(t)) = (\gamma^1(t), \ldots, \gamma^n(t), V^1(t), \ldots, V^n(t))$ in $(TM, \widetilde{g}_s)$ are

(a) $$\frac{\delta^2 \gamma^h}{dt^2} + \sum_{i,j,k=1}^{n} R_{kji}{}^{h} \bar{y}^k \frac{\delta V^j}{dt} \frac{d\gamma^i}{dt} = 0 ,$$

(b) $$\frac{\delta^2 V^h}{dt^2} = 0$$

for all $h = 1, \ldots, n$.

If we put t as arc length in TM, then we have $\widetilde{g}_s(\dot{\bar{\gamma}}(t), \dot{\bar{\gamma}}(t)) = \sum_{\beta,\gamma} \widetilde{g}_{\beta\gamma} \frac{\theta^\beta}{dt} \frac{\theta^\gamma}{dt} = 1$ or

in full $\sum_{i,j=1}^{n} \left(g_{ji} \frac{d\gamma^j}{dt} \frac{d\gamma^i}{dt} + g_{ji} \frac{\delta V^j}{dt} \frac{\delta V^i}{dt} \right) = 1$.

Since $\widetilde{\nabla}$ is a metric connection, $(\widetilde{\nabla}_{\dot{\bar{\gamma}}(t)} \widetilde{g}_s)\left(\sum_{h=1}^{n} \frac{\theta^{\bar{h}}}{dt} D_{\bar{h}} , \sum_{h=1}^{n} \frac{\theta^{\bar{h}}}{dt} D_{\bar{h}} \right) = 0$. Hence

$$(\widetilde{\nabla}_{\dot{\gamma}(t)}\widetilde{g}_s)\left(\sum_{h=1}^{n}\frac{\theta^{h}}{dt}D_{h}\ ,\ \sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\right)$$

$$=\widetilde{\nabla}_{\dot{\gamma}(t)}\left(\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\right)\right)-\widetilde{g}_s\left(\widetilde{\nabla}_{\dot{\gamma}(t)}\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\right)$$

$$-\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \widetilde{\nabla}_{\dot{\gamma}(t)}\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\right)$$

$$=\widetilde{\nabla}_{\dot{\gamma}(t)}\left(\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\right)\right)$$

$$-2\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{d}{dt}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}+\sum_{i,j,h=1}^{n}\widetilde{\Gamma}_{i\bar{h}}^{j}\theta^{i}(\dot{\bar{\gamma}}(t))\frac{\theta^{\bar{h}}}{dt}D_{j}+\sum_{i,j,h=1}^{n}\widetilde{\Gamma}_{i\bar{h}}^{j}\theta^{i}(\dot{\bar{\gamma}}(t))\frac{\theta^{\bar{h}}}{dt}D_{j}\right)$$

$$=\widetilde{\nabla}_{\dot{\gamma}(t)}\left(\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\right)\right)$$

$$-2\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\left(\frac{d}{dt}\frac{\theta^{\bar{h}}}{dt}+\sum_{i,j=1}^{n}\widetilde{\Gamma}_{ij}^{\bar{h}}\theta^{i}(\dot{\bar{\gamma}}(t))\frac{\theta^{j}}{dt}\right)D_{\bar{h}}+\sum_{i,j,h=1}^{n}\widetilde{\Gamma}_{i\bar{h}}^{j}\theta^{i}(\dot{\bar{\gamma}}(t))\frac{\theta^{\bar{h}}}{dt}D_{j}\right)$$

$$=\widetilde{\nabla}_{\dot{\gamma}(t)}\left(\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\right)\right)$$

$$-2\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\left(\frac{d}{dt}\left(\frac{\theta^{\bar{h}}}{dt}\right)+\sum_{i,j=1}^{n}\Gamma_{ij}^{h}\frac{d\gamma^{i}}{dt}\frac{\theta^{j}}{dt}\right)D_{\bar{h}}\right)$$

$$=\widetilde{\nabla}_{\dot{\gamma}(t)}\left(\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\delta V^{h}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\delta V^{h}}{dt}D_{\bar{h}}\right)\right)$$

$$-2\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\left(\frac{d}{dt}\left(\frac{\delta V^{h}}{dt}\right)+\sum_{i,j=1}^{n}\Gamma_{ij}^{h}\frac{d\gamma^{i}}{dt}\frac{\delta V^{j}}{dt}\right)D_{\bar{h}}\right)$$

$$=\widetilde{\nabla}_{\dot{\gamma}(t)}\left(\sum_{i,j=1}^{n}g_{ji}\frac{\delta V^{j}}{dt}\frac{\delta V^{i}}{dt}\right)-2\widetilde{g}_s\left(\sum_{h=1}^{n}\frac{\theta^{\bar{h}}}{dt}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\delta^{2}V^{h}}{dt^{2}}D_{\bar{h}}\right)$$

$$=0$$

implies that

$$\frac{\delta}{\delta t}\left(\sum_{i,j=1}^{n}g_{ji}\frac{\delta V^{j}}{dt}\frac{\delta V^{i}}{dt}\right)-2\widetilde{g}_s\left(\sum_{h=1}^{n}\theta^{\bar{h}}D_{\bar{h}}\ ,\ \sum_{h=1}^{n}\frac{\delta^{2}V^{h}}{dt^{2}}D_{\bar{h}}\right)=0.$$

Since $\dfrac{\delta^2 V^h}{dt^2} = 0$ thus $\dfrac{\delta}{\delta t}\left(\sum_{i,j=1}^{n} g_{ji} \dfrac{\delta V^j}{dt}\dfrac{\delta V^i}{dt}\right) = 0$. Therefore

$$\left(\frac{ds}{dt}\right)^2 = \sum_{i,j=1}^{n} g_{ji} \frac{d\gamma^j}{dt}\frac{d\gamma^i}{dt} = \text{constant},$$

s being the arc length in M so that s and t are related linearly and may conveniently be taken to be identical. Thus the arc length for a geodesic in TM is linearly related with the arc length parameter of its projection on M.

4.4 Geodesics on a fibre

Take a curve on a fibre, that is, $\gamma^h = \text{constant}$ for $h = 1, \ldots, n$. Then (b) on page 116 can be reduced to

$$\frac{\delta^2 V^h}{dt^2} = \frac{d}{dt}\left(\frac{\delta V^h}{dt}\right) + \sum_{i,j=1}^{n} \Gamma_{ji}^{h} \frac{d\gamma^j}{dt}\frac{\delta V^i}{dt} = \frac{d^2 V^h}{dt^2} + \frac{d}{dt}\left(\sum_{i,j=1}^{n} \Gamma_{ji}^{h} \frac{d\gamma^j}{dt} V^i\right) = \frac{d^2 V^h}{dt^2} = 0,$$

$h = 1, \ldots, n$, these imply that $V^h = a^h t + b^h$, a^h, b^h are constants. Thus a curve on a fibre is a geodesic if and only if it is of the form $\bar{\gamma}(t) = (\gamma(t), V(t)) = (\gamma^1, \ldots, \gamma^n, a^1 t + b^1, \ldots, a^n t + b^n)$, where γ^h, a^h, b^h are constants for $h = 1, \ldots, n$.

4.5 Natural and horizontal lifts

Let $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ be a geodesic in M. The horizontal lift of $\gamma(t)$ will be:

$$\bar{\gamma}(t) = \gamma^H(t) = (\gamma^1(t), \ldots, \gamma^n(t), V^1(t), \ldots, V^n(t)),$$

where $\frac{\delta V^h}{dt}=\frac{dV^h}{dt}+\sum_{i,j=1}^{n}\Gamma_{ji}^{h}\frac{d\gamma^j}{dt}V^i=0$ and $\frac{\delta^2\gamma^h}{dt^2}=0$.

We note that a proposition in [Ya1, proposition 7.2, page 174], stated that the horizontal lift of a geodesic is a geodesic in *TM*, the proof was not given. In this thesis we show that the converse is also true.

Proposition 4.2

The horizontal lift of a curve γ in M is a geodesic in $(TM,\tilde{g}_s)$ if and only if γ is a geodesic in (M,g).

Proof: Let $\gamma : I \to M$ be a curve in (M,g) defined by $\gamma(t)=(\gamma^1(t),\ \dots,\ \gamma^n(t))$. The horizontal lift of γ at a point (p,V), where $p=\gamma(0)$ and $V\in T_pM$, is a unique curve $\bar{\gamma} : I \to TM$ given by

$$\bar{\gamma}(t)=(\gamma(t),V(t))=(\gamma^1(t),\ \dots,\ \gamma^n(t),\ V^1(t),\ \dots,\ V^n(t)),$$

where $\bar{\gamma}$ projects onto γ, the vector field $V(t)=(V^1(t),\ \dots,\ V^n(t))$ has these properties: $V(0)=V$ and is parallel along $\gamma(t)$, that is, $\nabla_{\dot{\gamma}(t)}V(t)=0$. Therefore,

$\frac{\delta^2\gamma^h}{dt^2}+\sum_{i,j,k=1}^{n}R_{kji}{}^{h}\bar{y}^k\frac{\delta V^j}{dt}\frac{d\gamma^i}{dt}=\frac{\delta^2\gamma^h}{dt^2}=0$ and $\frac{\delta^2 V^h}{dt^2}=0$ if and only if γ^h satisfy $\frac{\delta^2\gamma^h}{dt^2}=0$, that is, $\gamma(t)$ is a geodesic in (M,g). ■

Consider now the natural lift γ^N of a curve γ in *M*. Let $\bar{\gamma}^N : I\to TM$, $\bar{\gamma}^N(t)=(\bar{\gamma}^1(t),\ \dots,\ \bar{\gamma}^n(t),\ \bar{\gamma}^{\bar{1}}(t),\ \dots,\ \bar{\gamma}^{\bar{n}}(t))$, $\bar{\gamma}^{\bar{h}}$ satisfy

$$\bar{\gamma}^h(t) = \frac{d\bar{\gamma}^h(t)}{dt} = \frac{d\gamma^h(t)}{dt} \text{ for } \gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t)).$$

It is clear that

(a) $$\frac{\delta^2\gamma^h}{dt^2} + \sum_{i,j,k=1}^{n} R_{kji}{}^h \frac{d\gamma^k}{dt}\frac{\delta}{dt}\left(\frac{d\gamma^j}{dt}\right)\frac{d\gamma^i}{dt} = \frac{\delta^2\gamma^h}{dt^2} + \sum_{i,j,k=1}^{n} R_{kji}{}^h \frac{d\gamma^k}{dt}\frac{\delta^2\gamma^j}{dt^2}\frac{d\gamma^i}{dt} = 0,$$

(b) $$\frac{\delta}{dt}\left(\frac{\delta}{dt}\frac{d\gamma^h}{dt}\right) = \frac{\delta^3\gamma^h}{dt^3} = 0.$$

Thus we have

Proposition 4.3 [Ya1, proposition 7.3, page 174]

If γ is a geodesic in (M,g), then γ^N is a geodesic in $(TM,\tilde{g}_s)$.

On the other hand, if a natural lift of a curve is a geodesic in $(TM,\tilde{g}_s)$, then, it satisfies the equations

(a) $$\frac{\delta^2\gamma^h}{dt^2} + \sum_{i,j,k=1}^{n} R_{kji}{}^h \frac{d\gamma^k}{dt}\frac{\delta^2\gamma^j}{dt^2}\frac{d\gamma^i}{dt} = 0,$$

(b) $$\frac{\delta}{dt}\left(\frac{\delta}{dt}\frac{d\gamma^h}{dt}\right) = \frac{\delta^3\gamma^h}{dt^3} = 0.$$

Thus from (b), we can see that $(\tilde{\nabla}_{\dot{\gamma}(t)}\tilde{g}_s)\left(\sum_{h=1}^{n}\frac{\delta^2\gamma^h}{dt^2}D_{\bar{h}}, \sum_{h=1}^{n}\frac{\delta^2\gamma^h}{dt^2}D_{\bar{h}}\right) = 0$ which implies $\frac{d}{dt}\left(\sum_{i,j=1}^{n} g_{ji}\frac{\delta^2\gamma^j}{dt^2}\frac{\delta^2\gamma^i}{dt^2}\right) = 0$, thus $\frac{\delta^2\gamma^h}{dt^2} = \rho Y^h$, where ρ is a constant, Y^h the unit vector in the direction of the vector $\frac{\delta^2 \gamma^h}{dt^2}$. If we write $X^h = \frac{d\gamma^h}{dt}$, the unit vector in the direction of $\dot{\gamma}(t)$ and if $\rho \neq 0$, then we can write equation (a) on page 120 as follow

$$Y^h + \sum_{i,j,k=1}^{n} R_{kji}{}^h X^k Y^j X^i = 0.$$

Transvecting this with Y^h, we will have

$$\sum_{i,j,k=1}^{n} R_{kjih} X^k Y^j X^i Y^h = -1,$$

hence we can conclude that the Riemannian sectional curvature with respect to the section determined by the osculating plane of the curve in M is constant. Thus if the natural lift γ^N of a curve γ is a geodesic, then either γ is a geodesic or the first curvature of γ is a constant and the Riemannian sectional curvature with respect to the section determined by the osculating plane of γ at any point is a constant [Ya1, proposition 7.4, page 175].

4.6 Tangent vector fields of the liftings of the geodesics

Let $\gamma : I \to M$ be a geodesic in (M,g), $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. We already know that the natural and horizontal lifts of γ are both geodesics in $(TM, \tilde{g}_s)$. Now we consider the tangent to each γ^N and γ^H.

The natural lift is given by

$$\gamma^N(t) = \left(\gamma^1(t), \ldots, \gamma^n(t), \frac{d\gamma^1(t)}{dt}, \ldots, \frac{d\gamma^n(t)}{dt}\right) = \left(\gamma^h(t), \frac{d\gamma^h(t)}{dt}\right).$$

The horizontal lift of γ at a point (p,V) is given by

$$\gamma^H(t) = (\gamma^1(t)\ ,\ \ldots,\ \gamma^n(t)\ ,\ V^1(t)\ ,\ \ldots,\ V^n(t)) = (\gamma^h(t)\ ,V^h(t))$$

with $\gamma(0) = p$, $V(0) = V$, $\nabla_{\dot\gamma(t)}V(t) = 0$.

We already know that $\nabla_{\dot\gamma(t)}V(t) = 0$, and since γ is a geodesic in M, we also have $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$, thus $(V^1(t)\ ,\ \ldots,\ V^n(t))$ and $\left(\frac{d\gamma^1(t)}{dt},\ \ldots,\ \frac{d\gamma^n(t)}{dt}\right)$ are two parallel vector fields along γ. It is clear that the natural lift of a geodesic is actually a horizontal lift with initial properties $\gamma(0) = p$, $V(0) = \dot\gamma(0)$ and $V(t) = \dot\gamma(t)$.

The tangent vector fields along γ^H and γ^N denoted by $\frac{d\gamma^H(t)}{dt}$ and $\frac{d\gamma^N(t)}{dt}$ respectively, are

$$\frac{d\gamma^H(t)}{dt} = \left(\frac{d\gamma^h(t)}{dt}\ ,\frac{dV^h(t)}{dt}\right) = \left(\frac{d\gamma^h(t)}{dt}\ ,\ -\sum_{i,j=1}^{n}\Gamma^h_{ji}\frac{d\gamma^j(t)}{dt}V^i\right) \text{ and}$$

$$\frac{d\gamma^N(t)}{dt} = \left(\frac{d\gamma^1(t)}{dt}\ ,\ \ldots,\ \frac{d\gamma^n(t)}{dt}\ ,\ \frac{d^2\gamma^1(t)}{dt^2}\ ,\ \ldots,\ \frac{d^2\gamma^n(t)}{dt^2}\right) = \left(\frac{d\gamma^h(t)}{dt}\ ,\ \frac{d^2\gamma^h(t)}{dt^2}\right).$$

The tangent of $\gamma(t)$ is

$$\dot\gamma(t) = \frac{d\gamma(t)}{dt} = \left(\frac{d\gamma^1(t)}{dt}\ ,\ \ldots,\ \frac{d\gamma^n(t)}{dt}\right) = \left(\frac{d\gamma^h(t)}{dt}\right),$$

and the horizontal lift of $\frac{d\gamma(t)}{dt}$ as a vector field along $\gamma(t)$ will be

$$\left(\frac{d\gamma(t)}{dt}\right)^H = \left(\frac{d\gamma^1(t)}{dt}\ ,\ \ldots,\ \frac{d\gamma^n(t)}{dt}\right)^H = \left(\frac{d\gamma^h(t)}{dt}\ ,\ -\sum_{i,j=1}^{n}\Gamma^h_{ji}\frac{d\gamma^j(t)}{dt}V^i\right).$$

Along the curve γ^H ,we have

$$\left(\frac{d\gamma(t)}{dt}\right)^H \circ \gamma^H = \left(\frac{d\gamma^h(t)}{dt}\ ,\ -\sum_{i,j=1}^{n}\Gamma_{ji}^h \frac{d\gamma^j}{dt}V^i\right).$$

Along the curve γ^N, we have

$$\left(\frac{d\gamma(t)}{dt}\right)^H \circ \gamma^N = \left(\frac{d\gamma^h(t)}{dt}\ ,\ -\sum_{i,j=1}^{n}\Gamma_{ji}^h \frac{d\gamma^j}{dt}\frac{d\gamma^i}{dt}\right),$$

but since γ is a geodesic, $\dfrac{d^2\gamma^h(t)}{dt^2} + \displaystyle\sum_{i,j=1}^{n}\Gamma_{ji}^h \frac{d\gamma^j}{dt}\frac{d\gamma^i}{dt} = 0$, then

$$\left(\frac{d\gamma(t)}{dt}\right)^H \circ \gamma^N = \left(\frac{d\gamma^h(t)}{dt}\ ,\ \frac{d^2\gamma^h(t)}{dt^2}\right).$$

Thus we have

Proposition 4.4

If γ is a geodesic in M, then for any horizontal lift γ^H of γ, we have $\left(\dfrac{d\gamma(t)}{dt}\right)^H \circ \gamma^H = \dfrac{d\gamma^H(t)}{dt}$. In particular $\left(\dfrac{d\gamma(t)}{dt}\right)^H \circ \gamma^N = \dfrac{d\gamma^N(t)}{dt}$.