CHAPTER 3

FUZZY SET THEORY AND FUZZY LOGIC

Fuzzy logic is introduced in this chapter starting with the history of how fuzzy logic was conceptualised. The basic aspects of fuzzy set theory is then discussed – taking into account how fuzzy sets, relations and inferencing combine together to create a fuzzy logic control using several fuzzy inference techniques.

3.1 Fuzzy Logic Through Time

Fuzzy logic is a problem-solving methodology often used to approximate reasoning in situations where information supplied is incomplete, imprecise, ambiguous or unreliable. Often times in real life, resulting answers to questions do not fall into the strict true or false categories but somewhere in between. Even the early philosophers such as Plato have contemplated the existence of a third region called the *uncertain* that lies between True and False. This was in contradiction with the "*Law of the Excluded Middle*" from "*The Laws of Thought*" by Aristotle in which states that each and every preposition must be either True or False (Matthews 1999).

Centuries later, in the 1900s, a tri-valued logic system was devised by mathematician Jan Lukasiewicz and was probably the first logical calculus of its type although it had only three axioms. Lukasiewicz later expanded his theories into four- and fivevalued logic systems and eventually concluded that an infinite-valued logic was no less plausible than a finite one. In 1965, Lofti A. Zadeh from the University of California, Berkeley introduced the first concepts of fuzzy logic in his work on *fuzzy set theory* to deal with the problem of partial truths – the truth values between "completely true" and "absolutely false". Zadeh applied Lukasiewicz's logic to all objects in a set and devised calculations to operate on the sets but even so, fuzzy sets were not implemented until the mid 1970s, when Ebrahim H. Mamdani of Queen Mary College in London successfully designed a fuzzy controller for a steam engine. Since then, the term "fuzzy logic" has come to mean any mathematical or computer system that reasons with fuzzy sets.

The importance of fuzzy logic is from the fact that most methods of human reasoning are approximate in nature and therefore the logic underlying the concepts of fuzzy logic should too be approximate instead of exact. The five basic principles of fuzzy logic as founded by Zadeh is as follows (Zadeh 1965):

1. In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
2. In fuzzy logic, everything is specified as a matter of degree.
3. Any logical system can be fuzzified
4. In fuzzy logic, knowledge is interpreted as a collection of elastic or, equivalently, fuzzy constraint on a collection of variables
5. Inference is viewed as a process of propagation of elastic constraints.

The understanding behind these principles are more clearly defined by explaining Zadeh's fuzzy set theory.

3.2 Fuzzy Set Theory

The now familiar concepts of fuzzy sets and fuzzy logic were introduced by Zadeh while working in the field of control engineering. The purpose of his work was as an approach to handling uncertainty and vagueness and, in particular, linguistic variables (John 1995) which do not conform well into the exacting black-and-white world of computing. Major advancements to his fuzzy set theory has now led to the implementation of fuzzy logic systems or more precisely fuzzy inferencing systems in many applications.

The fuzzy set theory can be divided into several sections which include fuzzy sets, membership functions, fuzzy operations and relations as well as fuzzy reasoning. The parts all combine to perform the various operations of a fuzzy logic control in a fuzzy inferencing system.

3.2.1 Fuzzy Sets and Notation

Conventional mathematical sets or bivalent sets are sets with objects that either clearly belong or do not belong to the set. There is no middle ground – the number three belongs fully to the set of odd numbers and not at all to the set of even numbers. In such sets, an object cannot belong to both a set and its complement set or to neither of the sets. This principle preserves the structure of Boolean logic and avoids the contradiction of an object that both is and is not a thing at the same time (Kosko & Isaka 2000). Sets like these are called *crisp sets* to differentiate them from the fuzzy sets as explained below.

Fuzzy sets are sets with objects that cannot clearly be defined as belonging or not belonging to the set. Items can belong only partially to a fuzzy set and they can also belong to more than one set (Kosko & Isaka 2000). One simple example to illustrate a fuzzy set is the set of tall people. Using conventional sets, a well-defined boundary is present as a threshold to separate the objects in the set into those that are tall and those that are not. However this strict division does not agree with our common sense because if a person being just a mere millimetre below this height threshold, he would be classified as "not tall" instead which is unreasonable.

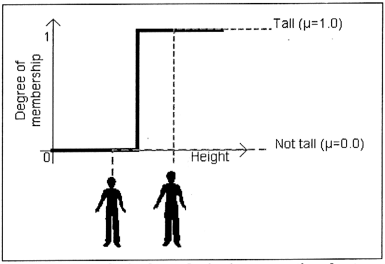

Figure 3.1 – Strict boundaries in conventional sets

One method of solving this problem is to assign a numerical value between 0 and 1 to represent the degree to which a person is evaluated as being tall. This is the fundamental idea of fuzzy sets, with the boundaries of a class being less defined and operate over a specific range of values as compared to the Boolean logic. The boundaries of fuzzy sets are curved or taper off, and this curvature creates partial contradictions. Using the same example, when plotted on the X-Y axis of a graph, a smooth varying curve is created that passes from "not so tall" to 'tall'. The Y output axis provides the membership value and thus the objects are classified into the same fuzzy set for 'tall' but to varying degrees. This process is often referred to as fuzzifying the inputs or *fuzzification*.

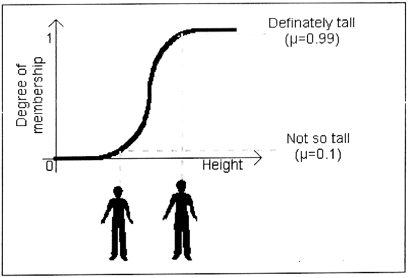

Figure 3.2 – Fuzzy sets

3.2.2 Membership functions

The degree to which an object is classified is determined by the curve and this curve is known as the *membership function* often with the given notation of either μ or m. In other words, membership functions are the functions that map each point of an input space to a point in the real interval [0.0, 1.0] thus giving the degree of membership. These membership functions are often represented graphically for ease of understanding such as those in the previous example to categorise tall people.

The input space to a membership function is sometimes referred to as the *universe of discourse* which is nothing but a fancy name. It should be understood that fuzzy degrees are not the same as probability percentages. Probabilities measure whether something will occur or not whilst fuzziness measures the degree to which something occurs or some condition exists. There is a one-to-one correspondence between fuzzy sets and their membership functions (Asai 1995) as defined by the following:

The fuzzy set A in the universe of discourse X is given by the membership function

$$\mu_A = X \rightarrow [0,1]$$

Similarly, since the fuzzy set can be viewed as a membership function, therefore

$$A : X \rightarrow [0,1]$$

Membership functions can be incredibly simple or extremely complex depending on the situation in which it is applied to. Common shapes for functions include triangular, trapezoidal, sinusoidal and bell curves but the shape is relatively less important than their number and placement to ensure adequate coverage of the input space and at times, as many as three to seven curves or functions are used.

The simplest membership functions use linear lines only and of these the most basic is the triangular membership function. A truncation of the triangular function results in the trapezoidal function. All straight-line membership functions have the advantage of simplicity in representation and calculation but the simplicity can also be a disadvantage as it cannot represent the fuzzy set accurately.

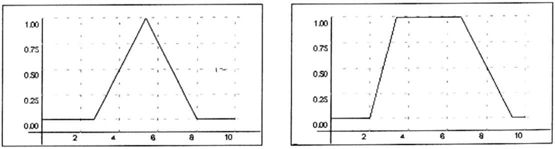

Figure 3.3 – Triangular and trapezoidal membership functions

Curved membership functions can be formed using gaussian, sigmoidal or polynomial functions depending on the type of curve needed. Gaussian functions usually form symmetrical bell shaped curves whilst sigmoidal functions can form asymmetrical bell curves. Polynomial functions create S and Z shaped curves with open ends. The advantage of curved functions is that it is nonzero at all points in the curve but the calculations are often much more complex.

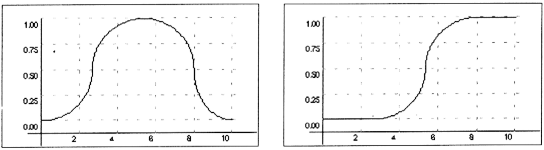

Figure 3.4 – Bell and sigmoidal membership functions

3.2.3 Operations on Fuzzy Sets

Fuzzy sets have the same operations defined as those for conventional sets. This includes operations of intersection, union and complement or more precisely the AND, OR and NOT operations. The common operations are described briefly:

<u>Union</u>

The membership function for the union of two fuzzy sets A and B with membership functions μ_A and μ_B respectively is defined as the maximum of the two individual membership functions. Union functions are also called the *maximum* criterion and are equivalent to the OR operation in Boolean algebra.

$$A \,\mathrm{Y}\, B \Leftrightarrow \mu_{A\mathrm{Y}B}(x) = \mu_A(x) \wedge \mu_B(x)$$

Intersection

The membership function for the intersection of two fuzzy sets A and B with membership functions μ_A and μ_B respectively is defined as the minimum of the two individual membership functions. Intersection functions are also called the *minimum* criterion and is equivalent to the AND operation in Boolean algebra.

$$A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \vee \mu_B(x)$$

Complement

The membership function for the complement of a fuzzy set A with the membership function μ_A is defined as the negation of the specified function. This is equivalent to the NOT operation in Boolean algebra.

$$\overline{A} \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

Other rules in conventional set theory such as De Morgan's Law and the laws of associativity, commutativity and distributivity also apply similarly to the fuzzy set theory. Although it is common to define intersection and union functions using min and max operations, it is also possible to use product operations and sum operations instead.

3.2.4 Fuzzy Relations

Computers cannot reason as human brains do as they can only manipulate precise facts that have been reduced to strings of zeros and ones and evaluate statements that are either true or false (Kosko 1995). In other words, computers handle only conventional relations that have Boolean results. Comparatively, the human brain often reasons with vague assertions or claims that involve uncertainties or value judgments such as "the temperature is *warm*" or "the car is *fast*" or "the hole is *too big*". To represent these ambiguous statements, fuzzy relations are required to show the strength of the relations between the elements.

The previously mentioned concepts of fuzzy sets using membership functions and fuzzy operations are combined to construct a fuzzy relationship. A fuzzy relation, *R*, can therefore be defined as a membership function μ_R from set X to Y. The value of

$$\mu_R (x,y) \in [0,1]$$

expresses the degree or strength of the relation between the elements x and y. In other words, a fuzzy relation generalizes a conventional relation into one that can allow partial memberships and describes a relationship that holds between two or more fuzzy objects. As the μ value approaches 1 the relation gets stronger and conversely the relation gets weaker when it approaches 0. An example of this is a fuzzy relation for "friend" which could describe the degree of friendship between two objects (people in this case) as opposed to the conventional relation which only affords the objects to either being a friend or not.

3.2.5 Fuzzy Reasoning (Fuzzy Inferencing)

Fuzzy reasoning is an inference method for deriving one fuzzy proposition from several fuzzy propositions (Asai 1995) and forms the basis of implementing fuzzy logic. Most inference methods have the following IF-THEN structure as fuzzy rules:

Rule	IF x is A THEN y is B
Fact	x is A'
Conclusion	y is B'

In the rule above A, A', B and B' are the fuzzy sets in the universe of discourse. The **IF** part is called the antecedent or premise and the **THEN** part is called the consequent or conclusion. Multiple antecedents can also be combined with the use of the fuzzy logic operators such as **AND** and **OR** to create more complex rules. For example:

IF service is poor **THEN** tip is small. (single antecedent and consequent)

IF skin is yellow **AND** flesh is soft **THEN** fruit is ripe

(multiple antecedents with single consequent)

The process of interpreting any rule is divided into two parts (Zadeh 1989). The first involves evaluating the antecedent to obtain the degree of membership of the fuzzy statements. This is known as fuzzifying the input or *fuzzification.* If there are multiple parts to the antecedents, the individual inputs are applied to their respective membership functions and the results are resolved to a single degree between 0 and 1 using the AND or OR operators.

The second step (*inferencing*) is to apply the implication method by applying the result to the consequent. The consequent of a fuzzy rule assigns an entire fuzzy set to the output. This fuzzy set is also represented by a membership function that is chosen to indicate the qualities of the consequent. Normally, a minimum of two and preferably more IF-THEN rules are used of which the output of each rule is mapped to a fuzzy set. These sets are then reduced into a single output fuzzy set in a process called *defuzzification* to obtain a crisp value. Defuzzification can be based on the result average (e.g. centroid of a graph) or through weighted calculations. Several inferencing methods are discussed on the following pages:

Mamdami's Inference Method (Min-Max)

The Mamdami inference method is one often found in fuzzy control and fuzzy expert systems. It is sometimes referred to as the min-max method simply because it uses both the min and max operations. The inferencing process is as follows:

1. Fuzzify the inputs and determine to what degree to which they belong to a fuzzy set using the corresponding membership functions. If more than one input is used (two or more antecedents present), then each input is matched to its corresponding membership function and its output obtained.
2. Once the inputs are fuzzified, apply the fuzzy operator (if more than one antecedent) to obtain one number that represents the result of the antecedent for that rule. This number will then be applied to the output function. In Mamdami's inference method the AND operator uses the min operation whilst the OR operator uses the max operation. The min operation simply selects the smaller antecedent value in the group and the max operator selects the largest.
3. The third step is to apply the implication method for the consequent. A consequent is also a fuzzy set represented by a membership function, which weighs appropriately with the linguistic characteristics that are attributed to it. The consequent is reshaped using a function associated with the antecedent
4. The next step involves the aggregation of the results of all the rules. In the example below, the results are aggregated using the max operator.
5. The final stage is the defuzzification stage of which the most popular is the centroid calculation which returns the centre of gravity under the curve as the result.

The diagram below illustrates the individual steps stated in the previous page. Two inputs, x0 and y0 are used and are fuzzified against their membership functions A1, A2, B1 and B2. The fuzzy operator is applied, which in this case is the AND operator and the smaller output value is taken. The result of the fuzzy operator is then applied to the consequent function C1 and C2. These results are finally aggregated and the centroid obtained as the result, z0.

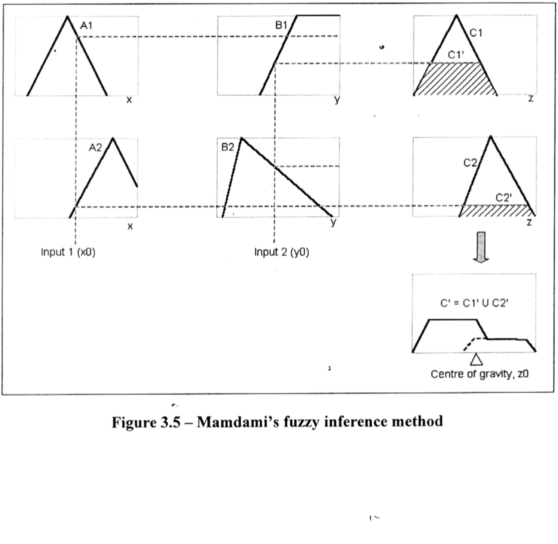

Figure 3.5 – Mamdami's fuzzy inference method

Mizumoto's Inference Method (Product-sum-gravity)

Mizomoto's product-sum-gravity method is quite similar to Mamdami's inference method but it replaces the min operation in the *Min-Max* method with the algebraic product (multiplication) and the max with the sum. The rest of the inferencing process remains the same and the centroid is used as the result of the inference and defuzzification process.

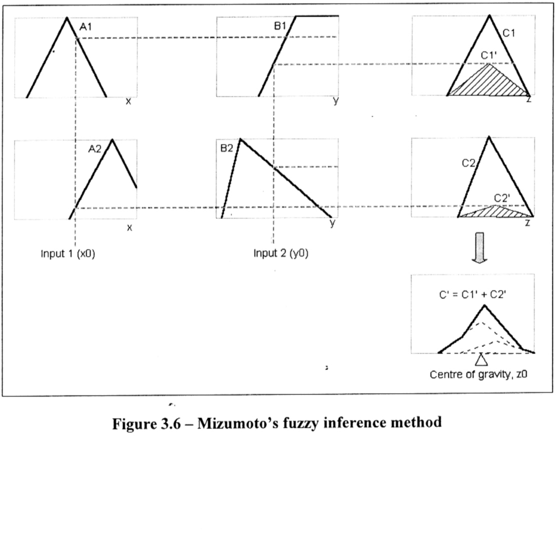

Figure 3.6 – Mizumoto's fuzzy inference method

Sugeno's Inference Method (Simple inference)

The Sugeno Inference method (or more precisely, the Takagi-Sugeno Kang inference method) is also very similar to the previous two other methods mentioned in terms of fuzzifying the inputs and applying the fuzzy operator. The major difference is that the outputs of the membership functions are linear values or constants that are often referred to as spikes or singletons. In this method, the output of the inference is calculated based on the fuzzy operator output and a weight associated with the rule. The immediate advantage is that the defuzzification process is greatly simplified but fine-tuning of the controller is much more difficult. There are variations of this simplified inference but they generally retain the same concept of linear outputs with weights.

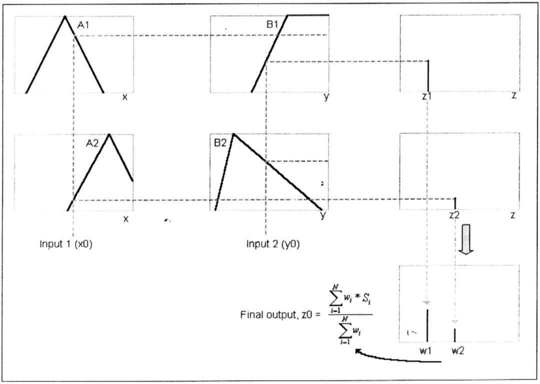

Figure 3.7 – Sugeno's fuzzy inference method

3.3 Applications of Fuzzy Logic

The concept of fuzzy logic has always and will probably remain controversial. Although widely accepted and used by engineering and computing communities, it is often rejected by mathematicians and statisticians. They argue that fuzzy logic cannot possibly be the superset of conventional (Boolean) logic since the membership functions are defined in the terms of conventional sets. Then there are those that argue that the fuzzy logic concept is unscientific since the membership values cannot be empirically verifiable.

Nevertheless, fuzzy logic has since been successfully used in many applications. Fuzzy logic can be used to control devices and appliances such as air-conditioners, refrigerators and washing machines giving them the ability to adjust their performance according to the situation presented. Vehicles also come with implemented additions using fuzzy logic such as ABS braking systems and cruise control features which are common in high-end automobiles.

One of the most common applications of fuzzy logic is in the form of artificial intelligence for the gaming industry. It is, for the most part, well-suited to simulate the actions of a 'virtual' player/opponent in any game whilst still retaining reasonable human-like faults and flaws to keep game-play as realistic as possible. One prominent implementation that stands out is IBM Corporation's Deep Blue system that analysed over 200 million chess moves per second to defeat Chess Grand Master Garry Kasparov in just six games with a 3.5-2.5 resulting victory for Deep Blue.

Another exciting field of fuzzy logic is its use in prediction and diagnosis systems such as in meteorological devices to predict weather changes including the possibility of dangerous weather occurring (e.g. hurricanes, typhoons and drought). Fuzzy logic is also being evaluated for use in calculating the probability of earthquakes and tsunamis occurring. It is hoped that using fuzzy logic, disasters like these can be accurately predicted and avoided to save thousands of lives that could be affected. Diagnostic systems benefit greatly from fuzzy logic inference because it closely matches how humans perform deductions and obtains results. Currently, there are working implementations of medical systems using fuzzy logic to diagnose multiple diseases and even one specialised to diagnose cancer.

3.5 Misconceptions of Fuzzy Logic

The term 'fuzzy' has often lead many to believe that the method used in fuzzy logic reasoning is imprecise or not serious. However, fuzzy logic is no less precise than any other form of logic but provides a mathematical equivalent to handle inherently uncertain concepts (i.e. the idea of 'fast' cannot be represented in conventional logic) (Entemann 2002). Another common misconception is that fuzzy logic is a method of measuring the probability of an occurrence. As previously stated above, the degrees of membership represent to what extent or degree something occurs and <u>not</u> how often it does or does not occur.

Fuzzy logic is also commonly referenced as fields of study in applications of artificial intelligence (AI). Although this is in part true since most applications of fuzzy logic tend to emulate human control (i.e. in fuzzy expert systems, control systems etc.), fuzzy logic is not restricted to AI type applications only and can certainly be applied to any situation requiring the handling of uncertainty.

Lastly, fuzzy logic is sometimes seen as an answer to situations in which conventional logic cannot handle. This is somewhat true since fuzzy logic can often be made suited for various situations but an effort must be made since no two situations can use the same fuzzy logic conditions (i.e. different membership functions, fuzzy sets and rules are required) and there is no one-size-fits-all solution available (Cox 1999). Finding the optimum conditions will probably take more time and effort but the end result will be more flexible and able to adapt to changing conditions it was applied to.

Once these misconceptions are clarified and understood, the strengths of fuzzy logic can then shine through. Based on implementation, the fuzzy logic methodology allows for simpler and faster design and development of systems controllers - instead of having to remodel, redesign and rewrite the control algorithm as in the conventional approach, often the only change required to fuzzy logic controls are to the definitions for the fuzzy relations/rules and fuzzy sets. This process of only modifying the rules for debugging and fine-tuning is considerably faster than redesigning the entire controller. In terms of simpler design, fuzzy logic lets complex control systems to be described in simple English-like clauses. This is in comparison with conventional systems which might require complex equations or math models to represent the same degree of control and relationship between the inputs and the result outputs.

Performance wise, fuzzy logic frequently outperforms conventional control design systems when facing the same situation. Many systems require non-linear control and conventional design controls have to use different methods to handle this non-linearity including using piecewise-linear and table-lookup approximations. This approximation comes at the cost of system performance, cost and complexity. Fuzzy logic provides the perfect alternative solution because it is closer to the real world in representation. The non-linearity is handled by fuzzy sets, membership functions, relations and the inference process which results in improved performance, simpler implementation, and reduced design costs.

Many fuzzy control implementations are smaller and require less memory to run than conventional controls thus making it easier to include as either software or hardware implementations. The inherent flexibility also gives fuzzy logic a natural capability to adapt to ever-changing situations more gracefully and with more stability. In uncontrolled situations like these, conventional control falters as it suffers an instability problem while trying to compensate for constant changes. On the whole, fuzzy logic can often be successfully used as a methodology for developing better control systems for all types of systems.