CHAPTER 2

Fuzzy Set Theory and Fuzzy Logic Control

2.1 Fuzzy Logic

Conventional logic states that a statement is either true or false, with nothing in between. This principle of true or false was formulated by Aristotle some 2000 years ago and has since been widely used today. It is also known as bivalent logic. However, if examined closely, saying that a statement is entirely true or entirely false, is in many cases does not represent the real world. For instance, saying that a person is either young or old. A small child might consider a person in his thirties old. On the other hand, a sixty-year-old might consider that person young. Another example is when one may ask is she beautiful? The answer may be yes or no or neither. This idea of gradation of truth is very much familiar to everyone.

When Prof. Lofti Zadeh of the University of California in Berkeley published a paper on fuzzy set theory in 1965, it was breakthrough in the area fuzzy mathematics. Fuzzy logic offers a better way of representing reality. In fuzzy logic, a statement is true to various degrees, ranging from completely true through half-truth to completely false. Fuzzy logic is a departure from classical two-valued sets and logic. It is basically a multi-valued logic that allows intermediate values to be defined between conventional evaluations like yes/no, true/false, etc.

Fuzzy logic is the logic underlying modes of reasoning which are approximate rather than exact [13]. Its importance lies to the fact that most modes of human reasoning are approximate in nature. Some essential characteristics of fuzzy logic relates to the following [13]:

a) In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning
b) In fuzzy logic, everything is a matter of degree
c) Any logical system can be fuzzified
d) In fuzzy logic, knowledge is interpreted a collection of elastic, or equivalently, fuzzy constraint on a collection of variables
e) Inference is viewed as a process of propagation of elastic constraints.

Fuzzy logic uses linguistic variables such as small, large or cold and a continuous range of truth values in the interval [0,1] rather than strict binary (True or False) decisions and assignments. Some of the principal differences between fuzzy logic and conventional logic are [13]:

i) **Truth:** In bivalent logical systems, truth can only have two values: true or false. In a multi-valued system, the truth value of a proposition may be an element of a finite set, an interval such as [0,1], or a Boolean algebra.
ii) **Predicates:** In bivalent systems, the predicates are crisp, e.g. larger than, even. In fuzzy logic, the predicates are fuzzy, e.g. much larger than, short, young.
iii) **Predicates modifiers:** In classical systems, the only widely used modifier is the negation, not. In fuzzy logic, there are a variety of modifiers such as very, quite, rather, extremely. These modifiers play an important role in generation of values of the linguistic variable, e.g. very short, rather short, not very short.
iv) **Quantifiers:** In classical systems, there are only 2 quantifiers, universal and existential. Fuzzy logic adds a wide variety of quantifiers exemplified by few, several, most, above 2, etc. A fuzzy quantifier is interpreted as a fuzzy number/proposition.
v) **Probabilities:** In classical systems, probability is numerical or interval-valued. In fuzzy logic, one can employ linguistic or fuzzy probabilities such as likely, very unlikely, around 0.1, etc. These probabilities can be interpreted as fuzzy numbers.
vi) **Possibilities:** In classical systems, concept of possibility is bivalent as opposed to fuzzy logic whereby possibilities are graded. Also, possibilities can be treated as linguistic variables with values such as maybe, impossible, very sure, etc.

2.2 Fuzzy Set Theory

A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$ which can be finite, countable, or overcountable. Each single element can either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belongs to A" is true, whereas in the latter case, this statement is false. A classical set can be described in many ways. For example, one can list out all elements that belong to a set, or one can describe the set analytically by stating conditions for membership ($A = \{x \mid x \leq 2\}$). Another way is to use characteristic function to define the member elements, where 1 indicates membership and 0 for non-membership.

In 1965, Prof. Lofti Zadeh proposed the idea of a fuzzy set. A fuzzy set is an extension of a crisp set [14]. Crisp sets only allow full membership or no membership at all. In the case of fuzzy sets, elements can belong to a fuzzy set with different degrees of membership called grades of membership. This is to say that an element may partially belong to a set.

To illustrate this, in a crisp set, the membership or non-membership of an element x in set A is described by a characteristic function $\mu_A(x)$:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Fuzzy set theory extends this concept by defining partial memberships which can take values ranging from 0 to 1:

$$\mu_A(x) : X \rightarrow [0, 1]$$

where X refers to the universal set defined in a specific problem.

For example, a fuzzy number 2 as shown in Figure 2.1 may be defined to have the following membership function:

$$\mu_{fuzzy\,2}(x) = \begin{cases} \dfrac{x-2}{0.5}+1 & \text{for } 1.5 < x \le 2 \\ \dfrac{2-x}{0.5}+1 & \text{for } 2 < x \le 2.5 \\ 0 & \text{otherwise} \end{cases}$$

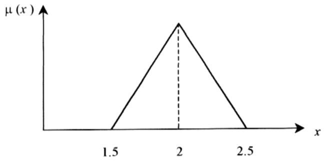

Figure 2.1 Membership function for fuzzy number 2

Definition 2-1

If X is a collection of object denoted by x, then a fuzzy set A in X is a set of ordered pairs:

$$A = \{ (x, \mu_A(x) \mid x \in X \}$$

$\mu_A(x)$ is called the membership function of x in A which maps X to the membership space M (when M contains only two points 0 and 1, A is non-fuzzy and $\mu_A(x)$ is same as characteristic function of a non-fuzzy set).

Definition 2-2

The support of a fuzzy set A, $S(A)$, is a crisp set of all $x \in X$ such that $\mu_A(x) > 0$.

Definition 2-3

The crisp set of elements which belong to the fuzzy set A at least to the degree α is called the α- level-set or α-cut:

$A_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha \}$

$A'_\alpha = \{ x \in X \mid \mu_A(x) > \alpha \}$ is called "strong α- level-set" or "strong α-cut."

Definition 2-4

The *height* of a fuzzy set is defined as the highest membership value of the set. If *height*(A) = 1, then set A is called a normalized fuzzy set.

Definition 2-5

A fuzzy set A is convex if:

$$\mu_A(\lambda x_1 + (1-\lambda))x_2 \geq \min(\mu_A(x_1), \mu_A(x_2)), \qquad x_1, x_2 \in X, \lambda \in [0, 1]$$

Alternatively, a fuzzy set is convex if all α- level-sets are convex

Definition 2-6

For a finite fuzzy set A, the cardinality $|A|$ is defined as:

$$|A| = \Sigma \mu_A(x), \qquad x \in X$$

$$\|A\| = \frac{|A|}{|X|}$$ is called the relative cardinality of A

Obviously, the relative cardinality of a fuzzy set depends on the cardinality of the universe. Therefore, one need to choose the same universe to compare fuzzy sets by their relative cardinality.

2.3 Fuzzy Set Operations

The membership function is the crucial component of a fuzzy set. Operations with fuzzy sets are defined via their membership functions. Assuming that A and B are two fuzzy sets with membership functions $\mu_A(x)$ and $\mu_B(x)$:

Definition 2-7

The *complement* of a fuzzy set A is a fuzzy set $\bar{A}$ with a membership function:

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \quad x \in X$$

Definition 2-8

The *Union* of A and B is a fuzzy set with the membership function:

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \quad x \in X$$

Definition 2-9

The *Intersection* of A and B is a fuzzy set with the following membership function:

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \quad x \in X$$

The use of these operations on fuzzy sets can be illustrated as follows:

The truth for a single statement has already been defined. Now, one should consider the truth of the combination of two statements, A AND B. A and B are both assertions; for example, John is a man, or Mary have to go to work this weekend. In conventional logic, of course both A and B must be either true or false. The statement (A AND B) is true only if both A and B is individually true; otherwise, the statement (A AND B) is false.

But, how to define the fuzzy truth value of the statement (A AND B) if the fuzzy truth values of A and B separately are known? Fuzzy logic gives a remarkably simple answer to this problem: the truth of (A AND B) together is the minimum of the truth value of A and the truth value of B.

Similarly, consider the statement (A OR B), which in conventional logic is true if statement A or statement B or both are true, and is only false if both A and B are false. In fuzzy logic, the truth of (A OR B) is simply the maximum of the truth value of A and the truth value of B.

Definition 2-10

Concentration is a unary operation which, when applied to a fuzzy set A results in a fuzzy subset of A in such a way that reduction in the higher grades of membership is much less than the reduction in lower grades of membership. By concentrating a fuzzy set, members with lower grades of membership will have even lower grades of membership. Thus, the fuzzy set becomes more concentrated. A common concentration operator is to square the membership function:

$$\mu_{CON\ (A)}(x) = \mu^2_A\ (x), \quad x \in X$$

For example, a typical concentration operator is the term *Very*, which when applied to a fuzzy label *Small* becomes *Very Small*.

Definition 2-11

The *Dilution* operator is the opposite of the concentration operator:

$$\mu_{DIL\ (A)}(x) = \sqrt{\mu_A\ (x)}, \quad x \in X$$

2.4 Fuzzy Logic Controller

Classical controllers have been used to control a variety of systems application and processes. These controllers are modeled after the systems and processes that they are designed for [4]. The performance of classical controllers depends a lot on the systems or processes being controlled. If the behavior of the system or process can be analytically modeled and known, the task of designing a controller becomes easier. Generally, traditional control theory is used and if applied correctly, the controller will perform as well as human operator. Unfortunately, not every system can be analyzed thoroughly and their behavior known. In many situations, sufficient knowledge about the workings of the systems or behavior of a process is not available. Despite this, control decisions have to be made based on incomplete information. A good example is the case in ATM congestion control whereby the traffic is unpredictable and yet decision has to be made. Of course the best strategy is to have a human operator but this would destroy the purpose of designing controllers.

Therefore, if one could convert and translate all the expertise of the human operator and incorporate into a controller, then the task will be easier. This is basically the idea of fuzzy control. Fuzzy control is based on the human operator experience or behavior. The main idea of fuzzy control is to build a model of a human control expert who is capable of effecting control without the need to think in complex mathematical model [4]. In recent years, fuzzy control has found a place in many industrial applications. It has been found that fuzzy controllers may perform better than conventional controllers especially for ill-defined or non-linear systems that is difficult to model, and where significant heuristic knowledge from human operators is available [4].

Different methods for developing fuzzy logic controllers have been suggested over the years. In the design of a fuzzy logic controller, one must identify the main control parameters and determine a term set which is at the right level of granularity for describing the values of each linguistics variables [15]. It should be mentioned here that a linguistic variable is a variable that can only take in linguistics values. Each system or domain will be different. For example, a term set including linguistics values such as {Slow, Medium, Fast} may not be sufficient in some domains and, might need extra term sets such as {Very Slow, Slow, Medium, Fast, Very Fast}.

Another issue is the type of fuzzy membership function chosen for the fuzzy controller. Different types of fuzzy membership function are available, but the four most common are [16, 17]:

(a) Monotonic
(b) Triangular
(c) Trapezoidal
(d) Bell-shaped

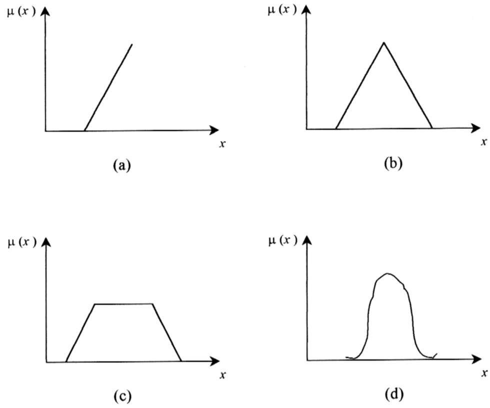

Figure 2.2 Different shaped of Membership functions

Selection of the types of fuzzy variables to be used in the fuzzy controller will affect the type of reasoning to be performed by the rules using these variables. The next step is to determine the values for the main control parameters. Subsequently, a knowledge base is built using the above control variables and the values that they may take. If the knowledge base is a rule base, there is a possibility of more than one rule firing. Therefore a conflict resolution method is required in order to make a wise decision.

Figure 2.3 below illustrates a simple architecture of a fuzzy logic controller:

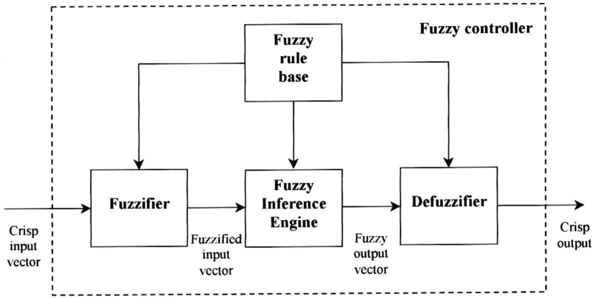

Figure 2.3 Architecture of a fuzzy logic controller

2.4.1 Fuzzification

This is the interface between the measurements or readings from the sensors and the fuzzy controller. The function of the fuzzification process is to encode the values from the sensors and transform the measurements in terms of linguistics variables used in the preconditions of the rules [16].

Usually, the sensor readings have a crisp value. Thus, the fuzzification process requires matching the sensor measurement against the membership function of the linguistic variable of fuzzy sets. As can be seen, the proper fuzzification of the input crisp values will greatly rely on the type of fuzzy membership function chosen.

2.4.2 Rule base (Knowledge base)

In developing the control rule base, there are mainly two tasks involved. Firstly, as mentioned earlier, a set of linguistics variables must be selected that describes the values of the main control parameters of the process. Each process or system to be controlled will be different. Therefore, one need to select proper term sets to linguistically represent the main input parameters and main output parameters. The selection of level of granularity of a term set for input or output variable plays an important part in providing smoothness of control.

Secondly, the linguistic description of the above main parameters is used to build a control rule base. Four methods have been suggested in developing the rule base [16]:

a) Expert's Experience and Knowledge
b) Modeling the Operator's Control Actions
c) Modeling a process
d) Self Organization

The most widely used method is to apply an expert's experience and knowledge in developing the rule base. The rule base holds a set of rules in the form of "if – then" statement [18]. This form of fuzzy control rules can be applied when expert human operators can express the heuristics or knowledge (experience) that they use in controlling the process in the same form.

An example of a fuzzy control rule with this form is:

IF Speed is slow and Error is small **THEN** Increase is high

Note that this form of fuzzy control rule has its conclusion expressed as another fuzzy variable. However, one can developed a rule whereby its conclusion is a function of the input parameters. For example, the following rule can be developed:

IF X is A_1 and Y is B_1 **THEN** $Z = f_1(X, Y)$

where the output action Z is expressed as a function of the X and Y values.

The second method above models the control actions of the operator. Instead of utilizing the experience or knowledge of expert, the types of control action taken by the operator are modeled.

Another alternative is to model the process being controlled, as stated in the third method. Through this method, one needs to build a fuzzy model of a process and a fuzzy controller is constructed to control the fuzzy model. An approximate model of the system is configured using implications that describe the possible states of the system. Since a fuzzy model of the process or system needs to be built, this approach is similar to the conventional control theory.

The main idea of the fourth method is to develop a set of rules that can be adjusted over time to improve performance. This method is similar to applying neural networks in the fuzzy controllers.

2.4.3 Conflict Resolution

Conflict resolution comes into place when there is a dispute in rule firing. Very often, more than one rule can be fired at one time. This is due to the partial matching attribute of fuzzy control rules and the preconditions of the rules overlapping. To resolve this kind of situation, one needs to decide what control action should be taken as result of the firing of several rules.

To illustrate this, assume that there are two rules of the form:

Rule 1 : **IF** X is A_1 and Y is B_1 **THEN** $Z = C_1$

Rule 2 : **IF** X is A_2 and Y is B_2 **THEN** $Z = C_2$

If x_0 and y_0 is the measurements for fuzzy variables X and Y, then their truth values are represented by $\mu_{A1}(x_0)$ and $\mu_{B1}(y_0)$ for Rule 1. μ_{A1} represents the membership function for A_1. For Rule 2, the truth values are $\mu_{A2}(x_0)$ and $\mu_{B2}(y_0)$. Therefore the strength of Rule 1 is calculated to be:

$$\alpha_1 = \mu_{A1}(x_0) \wedge \mu_{B1}(y_0)$$

And the strength of Rule 2 is:

$$\alpha_2 = \mu_{A2}(x_0) \wedge \mu_{B2}(y_0)$$

The control output of Rule 1 is calculated by matching strength of its preconditions on its conclusion:

$$\mu_{C'1}(w) = \alpha_1 \wedge \mu_{C1}(w),$$

For Rule 2:

$$\mu_{C'2}(w) = \alpha_2 \wedge \mu_{C2}(w)$$

where w range from the values that the rule conclusions can take.

Based on receiving inputs, x_0 and y_0 from the sensors, Rule 1 is suggesting a control action with $\mu_{C'1}(w)$ as its membership function while Rule 2 is recommending a control action with $\mu_{C'2}(w)$ as its membership function. To resolve this conflict, the combined conclusion of Rule 1 and Rule 2 is given by:

$$\begin{aligned}\mu_C(w) &= \mu_{C'1}(w) \vee \mu_{C'2}(w) \\ &= [\alpha_1 \wedge \mu_{C1}(w)] \vee [\alpha_2 \wedge \mu_{C2}(w)]\end{aligned}$$

$\mu_C(w)$ is a pointwise membership function for the combined conclusion of Rule 1 and Rule 2 The operator $\wedge$ and $\vee$ is called the min and max functions respectively. This method is also known as the max-min inference method [14]. Finally, $\mu_C(w)$ as the membership function of the combined conclusion has to be translated back to a crisp output value. This step is performed by the defuzzification process.

2.4.4 Defuzzification

The final process to be performed by the fuzzy controller is defuzzification of the inferred fuzzy control action. The defuzzification process will produce a non-fuzzy, crisp output value that best represents the membership function for the output control action.

Several defuzzification methods are available. Four methods most often used are described below [16]:

The Center Of Area (COA) Method

Assume that a control action with a pointwise membership function μ_C is produced. COA calculates the center of gravity of the distribution for the control action. A crisp control action can be calculated by:

$$Z^* = \frac{\sum_{j=1}^{q} z_j \mu_C(z_j)}{\sum_{j=1}^{q} \mu_C(z_j)}$$

where q is the number of quantization levels of the output, z_j is the amount of control output at the quantization level j and $\mu_C(z_j)$ represents its membership value in C

Tsukamoto's Defuzzification Method

Using this method, a crisp control action is calculated by:

$$Z^* = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

where n is the number of rules with firing strength (w_i) greater than 0 and x_i is the amount of control action recommended by rule i.

The Mean of Maximum (MOM) Method

For the MOM method, a crisp control action is generated by averaging the support values in which their membership values reach the maximum.

$$Z^* = \sum_{j=1}^{l} \frac{z_j}{l}$$

where l is the number of quantized z values that reach their maximum memberships.

Defuzzification when the output of rules are functions of their inputs

From the previous section, it can be seen that the fuzzy control rules may be written as a function of their inputs:

Rule i : **IF** X is A_i and Y is B_i **THEN** $Z = f_i\,(X, Y)$

Assume that α_i is the firing strength of rule i, the crisp output control action is:

$$Z^* = \frac{\sum_{i=1}^{n} \alpha_i f_i(x_i, y_i)}{\sum_{j=1}^{n} \alpha_i}$$

where n is the number of firing rules.

2.5 Advantages of Fuzzy Logic Controller

The widespread use of fuzzy logic controller in real world applications shows that it does have its advantages over the classical controllers. Among some of the benefits of fuzzy logic controller are [4]:

a) Fuzzy logic controller is successful in controlling ill-defined and non-linear system which is difficult to model.
b) Fuzzy logic controller incorporates expert's knowledge and experience into a set of rules, and these rules will formed a rule base to be used in controlling the system or process.
c) The use of linguistics variables replacing numerical values makes the controller easier to implement and modify. Simple and fewer rules can replaced complex instructions in the control operation thus resulting in faster and more efficient decision making.

2.6 Applications of Fuzzy Logic Controller

Recent years have seen significant increase in the number of applications for fuzzy logic control. Among some of the applications are in the field of [4]:

a) Telecommunications
b) Automotive Engineering
c) Medicine
d) Environmental Control
e) Chemical Process Control
f) Electrical home appliances
g) Automatic train control

The successful implementation in these areas has proven that fuzzy logic control is very flexible and adaptable. It is foreseen that in the near future, fuzzy control will be used in more and more real world applications.

The flexibility and adaptability of fuzzy logic controller has also attracted the interests of researchers in ATM congestion control. As mentioned earlier, one of the main challenges of ATM congestion control is the unpredictable input traffic. It is often difficult for a network to acquire complete statistics of the input traffic accurately. Despite this, control decisions have to be made by the traffic and congestion control functions to either pass or discard cells. Thus, there is a possibility of making less than satisfactory decisions due to incomplete information.

The success of fuzzy logic control in handling nonlinear and ill-defined system is seen desirable in developing a possible solution to effectively control congestion in ATM. Therefore, a lot of work is currently being done to utilize fuzzy logic controller to perform traffic and congestion control functions.