CHAPTER 4

NEUTRINO OSCILLATIONS

4.1 Introduction

In principle, to show that a particle has mass, it is necessary to show that it travels at less than the speed of light. However, most neutrinos are produced travelling at almost the speed of light, and they interact extremely weakly with matter, so they are very hard to slow down. Instead, most searchers for neutrino mass rely on quantum-mechanical interference effect, the neutrino oscillation. If neutrinos have mass, and if the neutrinos produced by the weak interactions do not have definite mass, but are quantum-mechanical mixtures of states with different mass, then the composition of a beam of neutrinos will oscillate. In other words, they change flavour back and forth with distance from the source, i.e. they could oscillate from electron neutrino to other types of neutrinos. Then neutrino oscillations will occur if the weak neutrino eigenstates are not mass eigenstates, but superpositions of them. Flavour oscillations as well as matter-antimatter oscillations are also a possibility within the assumption that two neutrinos only are mixed. Theoretically, the formulation of the flavour mixings of Dirac neutrinos is identical to that of the quark sector. If the neutrinos are Majorana particles, then the number of parameters necessary to specify the mixing matrix differs from that required for Dirac neutrinos.

Some experiments like the Kamiokande and Super-Kamiokande in Japan [4,16] have tried to see the oscillations of the neutrinos inside the particle shower generated by the interaction of cosmic rays with atmospheric particles. The other main source of the neutrinos observed at Super-Kamiokande is the sun. Neutrinos pass through all forms of matter almost without interaction. This is why the detection of solar neutrinos was originally proposed as a way to 'see' directly into the core of the sun. Neutrinos are produced deep in its core, where the fusion reactions that power the sun take place and the investigation and understanding of the sun as a typical main sequence star is of outstanding importance for an understanding of stellar evolution. Neutrino mass has never been firmly found by evidence but however new evidence from the Super-Kamiokande [4] provides indications for the atmospheric $\nu_\mu \leftrightarrow \nu_\tau$ oscillations. The Super-Kamiokande result shows that muon neutrinos are disappearing into undetected tau neutrinos or perhaps some other type of neutrino, i.e. sterile neutrino. This can only occur if the neutrino possesses mass. The experiment, however, does not determine directly the masses of the neutrinos leading to this effect, but the rate of disappearance suggests that the difference in masses between the oscillating types is very small. Results from the Super-Kamiokande favour the mass range of $10^{-3}eV^2 \leq \Delta m^2 \leq 6\times10^{-3}eV^2$ [3].

Neutrino oscillations can occur when the mixed neutrino eigenstates propagates through vacuum or matter. In this chapter, we will discuss the physics of the oscillations in vacuum before moving on to the oscillations in matter. In the oscillations in matter, we will consider the adiabatic conversion and non-adiabatic conversion.

4.2 Neutrino Oscillations in Vacuum

If neutrinos have finite, non-degenerate masses then the neutrino flavour eigenstates v_f (f=e,μ,τ) are not necessarily coincide with the mass eigenstates v_i (i=1,2,3). Propagation of mass eigenstates are mixture of the flavour eigenstates or vice versa. Neutrino mass eigenstates are related to the flavour eigenstates by a unitary mixing U i.e.

$$v_f = \sum_i U_{li} v_i \qquad f = e, \mu, \tau \qquad i = 1,2,3 \ . \tag{4.1}$$

For two neutrino flavours v_e and v_l is the mixing characterised by a single mixing angle θ

$$\begin{pmatrix} v_e \\ v_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} . \tag{4.2}$$

The mass eigenstates are the eigenstates of the neutrino Hamiltonian in vacuum and hence they propagate independently with no $v_1 \leftrightarrow v_2$ process. They have different phases during propagation; $v_i e^{i\phi_i}$ due to different mass with the phase factor given as $\phi_i = E_i t - p_i x$ where E_i and p_i are the energy and momentum of the neutrino respectively. Then at any time t, we can describe the propagation of $|\nu_e\rangle$ after time t as

$$|\nu_e(t)\rangle = \left[e^{-iE_1 t}\cos^2\theta + e^{-iE_2 t}\cos^2\theta\right]|\nu_e(0)\rangle$$
$$+\left[e^{-iE_2 t}\cos^2\theta + e^{-iE_1 t}\right]\sin\theta\cos\theta|\nu_e(0)\rangle \quad . \tag{4.3}$$

The probability of the state $|\nu_e\rangle$ converting to the different flavour for instance the muon neutrino $|\nu_\mu\rangle$ is

$$P(\nu_e \to \nu_\mu) = |\langle\nu_\mu|\nu_e(t)\rangle|^2$$
$$= \frac{1}{2}\sin^2 2\theta[1 - \cos(E_2 - E_1)t] \; . \tag{4.4}$$

In normal astrophysical situations, temperatures are high and the neutrinos are usually relativistic and we can approximate the energy difference as

$$E_2 - E_1 \approx \frac{m_2 - m_1}{2E_\nu} = \frac{\Delta m^2}{2E_\nu} \; . \tag{4.5}$$

Inserting (4.5) into (4.4), we get

$$P(\nu_e \rightarrow \nu_\mu) = \frac{1}{2}\sin^2 2\theta[1 - \cos\Delta\phi] \tag{4.6}$$

where the phase difference is

$$\Delta\phi(t) = \frac{\Delta m^2}{2E_\nu}t \ . \tag{4.7}$$

According to (4.7), the phase difference increases monotonously with time which gives rise to the oscillations. The oscillation length in vacuum, L_ν can be found from (4.6) since the distance traveled by the neutrino $L \approx t$ and it is written as

$$L_\nu = \frac{4\pi E_\nu}{\Delta m^2} \ . \tag{4.8}$$

Eq (4.6) is usually written in a compact form by using (4.8) which is

$$P(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2}{4E_\nu}L\right)$$

$$= \sin^2 2\theta \sin^2\left(\pi\frac{L}{L_\nu}\right)$$

$$= \sin^2 2\theta \sin^2\left(1.27\Delta m^2 \frac{L}{E}\right) \tag{4.9}$$

where L, E and ΔM are in m MeV and eV^2 respectively. Survival of the electron neutrino is evidently given as

$$P(\nu_e \rightarrow \nu_e) = 1 - P(\nu_e \rightarrow \nu_\mu). \tag{4.10}$$

We conclude that for oscillations to occur $\Delta m^2 \neq 0$ i.e. at least one of the neutrino must be massive or their masses do not coincide and $\theta \neq 0$ which indicate that there must be a mixing between neutrinos. There are three conditions for neutrino oscillation observation:

1. If $L/L_\nu >> 1$, the oscillating term $\sin^2 (\pi \, L/L_\nu)$ would oscillate rapidly with L giving an average effect described by

$$\langle P(\nu_e \rightarrow \nu_e) \rangle = 1 - \frac{1}{2}\sin^2 2\theta \ . \tag{4.11}$$

For maximum mixing, only 50% of the original electron neutrinos will be observed.

2. If $L/L_\nu << 1$, hence $\sin^2 (\pi \, L/L_\nu) << 1$, then the oscillation pattern will vanish.

3. If $L = \frac{1}{2} L_\nu$ then $sin^2 (\pi L/L_\nu) = 1$ and hence

$$P(\nu_e \rightarrow \nu_e) = 1 - \sin^2 2\theta \ . \tag{4.12}$$

In the situation of maximum mixing, the full phenomenon of oscillations will be observed. Thus the oscillation pattern can be observed if the length of the experiment is of the order of magnitude of the oscillation length.

There are two general types of neutrino oscillation experiment namely the appearance experiments that look for $\nu_l \rightarrow \nu_{l'}$ $(l \neq l')$ and the disappearance experiment that look for any reduction in the flux of the original neutrino [3]. Up to 1998, experiments obtained negative results on the oscillation.

Recently, Super-Kamiokande experiment [4] reported possible positive results of atmospheric neutrino oscillation, the experiment would expect 29 ± 3 muon events but it was reported that only 17 ν_μ events were recorded [4]. The data is consistent with $\nu_\mu \rightarrow \nu_\tau$ oscillations while $\nu_\mu \rightarrow \nu_e$ oscillations do not fit the Super-Kamiokande data. Also oscillation to sterile neutrinos is disfavoured.

4.3 Neutrino Oscillations in Matter

In this section, we will consider the effect of matter on the propagation of neutrinos. Wolfenstein [17] pointed out that the coherent forward scattering of neutrinos in matter as discussed in Chapter 2 will modify the vacuum oscillations because the scattering of electron neutrinos with electrons has the charged current contribution apart from the neutral current which is the same for all neutrino flavors. Mikheyev and Smirnov [18] used the Wolfenstein matter oscillations to explain the solar neutrino problem. Oscillations can be greatly enhanced in a slowy changing electron density. Bethe [19] rederived the Mikheyev-Smirnov-Wolfenstein (MSW) enhancement mechanism using the level-crossing picture.

The main effect for a medium which is transparent to neutrinos and realised at low energies is refraction and the phase factor of the neutrino wavefunction changes from $ip_{\nu}x$ to $ip_{\nu}n_{\nu}x$ where n_{ν} is refractive index of matter of the neutrino-matter scattering. In this instance the wavefunction of the neutrino is described by

$$\nu_f(x,t) \sim e^{ip_{\nu}n_{\nu}x} e^{-iE_it} \tag{4.13}$$

where the proportional constants are the mixing matrix elements. The change in the neutrino wavefunction after propagating through a distance dx in matter is obtained by differentiating (4.13) i.e.

$$\frac{d\nu_f}{dx} = ip_\nu (n_\nu - 1)\nu_f . \qquad (4.14)$$

Index of refraction for the ν_e - e^- scattering contains the charged current f^{CC} and neutral current f^{NC} contribution. Thus for $\nu_e \rightarrow \nu_\mu$ oscillations, Eq (4.14) in terms of the scattering amplitude written in matrix form is,

$$i\frac{d}{dx}\begin{pmatrix}\nu_e \\ \nu_\mu\end{pmatrix} \sim \frac{1}{p_\nu}\begin{bmatrix} f^{CC} + f^{NC} & o \\ o & f^{NC}\end{bmatrix}\begin{pmatrix}\nu_e \\ \nu_\mu\end{pmatrix}. \qquad (4.15)$$

Thus we can see that the overall phase shift to the neutral current scattering is common to both neutrinos and such has no physical significant. For highly relativistic neutrinos, the time development of ν_e and ν_μ is described by a Schroedinger-like equation [19],

$$i\frac{d}{dt}\begin{pmatrix}\nu_e \\ \nu_\mu\end{pmatrix} = \frac{1}{2E_\nu}\begin{pmatrix} m_1{}^2\cos^2\theta + m_2{}^2\sin^2\theta + A & \Delta m^2 \sin\theta\cos\theta \\ \Delta m^2 \sin\theta\cos\theta & m_1{}^2\sin^2\theta + m_2\cos^2\theta \end{pmatrix}\begin{pmatrix}\nu_e \\ \nu_\mu\end{pmatrix} \qquad (4.16)$$

where we have ignored the neutral current scattering, $p_\nu \approx E_\nu$ and the mass squared arising from matter effect from the ν_e-e^- charged current scattering is

$$A = 2\sqrt{2}G_F N_e E_\nu \,. \tag{4.17}$$

Diagonalising the mass matrix in (4.16), the eigenstates are

$$M_\nu^{\,2} = \frac{1}{2}\left(m_1^{\,2} + m_2^{\,2} + A\right) \pm \frac{1}{2}\Delta M^2 \tag{4.18}$$

where

$$\Delta M^2 = \left[\left(\Delta m^2 \cos 2\theta - A\right)^2 + \left(\Delta m^2 \sin 2\theta\right)^2\right]^{1/2} . \tag{4.19}$$

The weak eigenstates for the propagation in matter are now described by a mixing angle in matter θ_M,

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_M = \begin{pmatrix} \cos\theta_M & -\sin\theta_M \\ \sin\theta_M & \cos\theta_M \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} . \tag{4.20}$$

As in the vacuum oscillation, the mass matrix in matter is given as

$$M_M = \frac{\Delta M^2}{4E_\nu} \begin{pmatrix} -\cos 2\theta_M & \sin 2\theta_M \\ \sin 2\theta_M & \cos 2\theta_M \end{pmatrix} . \tag{4.21}$$

and equating it with (4.16) gives the relationship between θ and θ_M,

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{1 - \dfrac{2A\cos 2\theta}{\Delta m^2} + \left(\dfrac{A}{\Delta m^2}\right)^2} . \tag{4.22}$$

The survival probability of the electron neutrino has the same form as in the vacuum oscillations i.e.

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta_M \sin^2\left(\frac{\pi x}{L_M}\right) \tag{4.23}$$

where the oscillation length in matter

$$L_M = \frac{4\pi E_\nu}{\Delta M^2} \tag{4.24}$$

$$= \frac{L_\nu}{\left(1 - 2\left(\dfrac{L_\nu}{L_O}\right)\cos 2\theta + \left(\dfrac{L_\nu}{L_O}\right)^2\right)^{1/2}} .$$

In (4.24), the Wolfenstein characteristic matter oscillation length is defined by equation (4.22)

$$L_O = \frac{2\pi}{\sqrt{2} G_F N_e} . \tag{4.25}$$

Thus (4.22) can be rewritten using L_M as

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{1-2\left(\frac{L_\nu}{L_O}\right)\cos 2\theta + \left(\frac{L_\nu}{L_O}\right)^2} \,. \tag{4.26}$$

We can describe the MSW effect using the plot of $\sin^2 2\theta_M$ as a function of $\frac{L_\nu}{L_e}$ and is shown in Fig 4.1 for matter with constant density,

1. When $\frac{L_\nu}{L_O}$ << 1, Eq (4.26) gives $\theta_M \approx \theta$ and from Eq (4.25), $L_m \approx L_M$. Hence the survival probability in matter (4.23) is the same as in vacuum. In this region the effect of matter is not important.

2. When $\frac{L_\nu}{L_O}$ >> 1, the oscillation amplitude is surpluses by $\frac{L_e}{L_\nu}$ since

$$P(\nu_e \to \nu_e) = \left(\frac{L_e}{L_\nu}\right)^2 \sin^2 2\theta_\nu \sin^2\left(\frac{\pi x}{L_M}\right).$$

3. When $\frac{L_\nu}{L_e}$ = cos $2\theta_m$, from (4.26) we have maximum mixing in matter since θ_M = $\pi/4$ i.e. the MSW resonance condition. In the case of matter with changing density as in the sun, the resonance takes place in a layer with density range where the adiabatic layer is larger that the oscillation length during resonance.

In the sun, since $\frac{L_\nu}{L_O} = \cos 2\theta$ at the MSW resonance the electron density at resonance is

$$N_e^{res} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2} G_F E_\nu} \quad . \tag{4.27}$$

This means that the electron neutrino produced in the sun will go through the resonance region if the density at its production site is greater than N_e^{res}. The neutrino has a minimum adiabatic energy E_ν^A for a adiabatic resonance which is

$$E_\nu^A \geq \frac{6.44 \times 10^6 \Delta m^2 \cos 2\theta}{\frac{N_e^c}{N_A}} \tag{4.28}$$

where Δm^2 is in eV and E_ν^A in MeV with N_e^c the electron density at the centre of the sun. We can obtain E_ν^A using the electron density as shown in Chapter 3.

4.3.1 Adiabatic Conversion

In the sun, the electron density is non-linear as shown in Fig (3.5). We can rewrite the propagation equations in matter in terms of mass eigenfunctions in matter ν_1^m and ν_2^m by diagonalising (4.16) to give

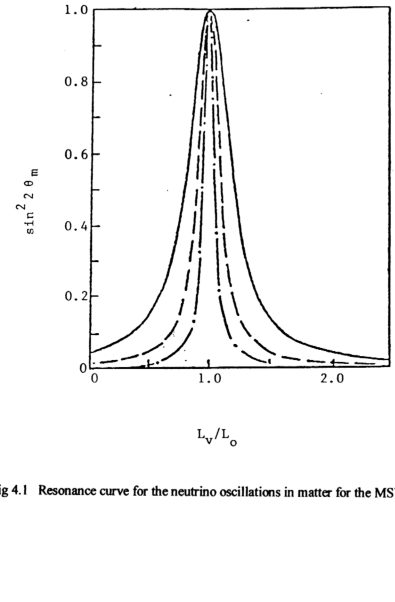

Fig 4.1 Resonance curve for the neutrino oscillations in matter for the MSW mechanism.

$$i\frac{d}{dx}\begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix} = \frac{1}{2E_\nu}\begin{pmatrix} M^2 & -i2E_\nu \dfrac{d\theta_M}{dx} \\ i2E_\nu \dfrac{d\theta_M}{dx} & M^2 \end{pmatrix}\begin{pmatrix} \nu_{1M} \\ \nu_{2M} \end{pmatrix} . \tag{4.29}$$

The change of θ_M can be obtained using Eqs (4.19) and (4.26) and setting the off-diagonal terms small compared to $\frac{1}{2}\Delta M^2$, the adiabatic condition requires that at resonance,

$$\frac{1}{N_e^{res}}\left|\frac{dN_e}{dx}\right|_{res} << \frac{\Delta m^2 \sin 2\theta}{2E_\nu \cos\theta} . \tag{4.30}$$

If we define the adiabatic parameter γ_A as

$$\gamma_A = \frac{\Delta m^2 N_e^{res} \sin^2 2\theta}{2E_\nu \cos 2\theta \left|\dfrac{dN_e}{dx}\right|_{res}} \tag{4.31}$$

then the adiabatic condition can be stated as

$$\gamma_A >> 1 . \tag{4.32}$$

During propagation of the neutrinos in the resonance region, the phase of the neutrino wavefunction will average to zero and this will reduce the probability to be the classical probability [20] which is

$$P(\nu_e \to \nu_e) = \frac{1}{2}\left(1 + \cos 2\theta \cos 2\dot{\theta}_M\right). \tag{4.33}$$

Bethe [19] described the MSW mechanism in the adiabatic situation using the level crossing concept. In matter, an electron neutrino gains an additional potential energy in the $\nu_e - e^-$ scattering which is given as

$$V = \left\langle e_\nu \middle| H_c^e \middle| e_\nu \right\rangle \tag{4.34}$$

where H_c^e is the effective charged current Hamiltonian

$$H_c^e = \frac{G_F}{\sqrt{2}} \overline{\nu_e} \gamma^\mu \left(1 - \gamma_5\right) \nu_e \bar{e} \gamma_\mu \left(1 - \gamma^5\right) e. \tag{4.35}$$

In the rest frame of the solar medium and the electrons are unpolarized, the additional potential energy is

$$V = \sqrt{2} G_F N_e \tag{4.36}$$

which is equivalent to an increase in the mass of the neutrino $A \approx 2E_\nu V$ as given by Eq (4.17). In the adiabatic limit, $(\Delta m^2 \cos 2\theta - A)^2$ in Eq (4.19) must be a minimum so that the eigenvalues are almost degenerate. This requires that $m_2^2 > m$, and in this case

$$A = \Delta m^2 \cos 2\theta . \qquad (4.37)$$

A plot of neutrino mass as a function of density is shown in Fig 4.2. At low density the mass of ν_e is smaller than ν_μ. But it increases when the electron neutrino interacts with the electrons while the ν_μ mass remains the same. When the two curves are at the minimum splitting given by (4.37), the MSW resonance is met. A complete crossing is prevented by $\Delta m^2 \cos 2\theta$. After going through the resonant region ν_e will follow the ν_μ curve and this gives the adiabatic conversion. From Eq (4.17), the relation of adiabatic density ρ^A and energy E_ν^A for the resonance to occur is

$$A = \Delta m^2 \cos 2\theta \approx 7.6 \times 10^{-8} \rho^A E_\nu^A \qquad (4.38)$$

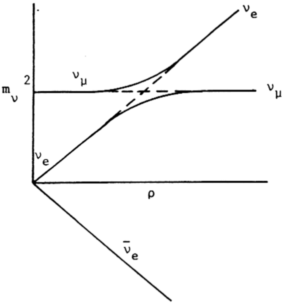

Fig 4.2 The masses of ν_e and ν_μ as a function of ρ [18]. The $\overline{\nu_e}$ does not have a resonance with $\overline{\nu_\mu}$ since it has an interaction of the opposite sign with matter.

where ρ^A is in g/cm^3. Using the standard solar model described in Chapter 3, the Bethe's solution can be calculated for the chlorine experiment. The chlorine detector detects 6.5% of the ^{8}B neutrinos in the solar model.

For Bethe's paper, the adiabatic conversion energy is $E_\nu^A \approx 5.1$ MeV and since $N_e^c / N_A \approx 93.04$ cm^{-3}, Eq (4.28) gives the adiabatic solution to the solar neutrino problem as

$$\Delta m^2 \approx 7.4 \times 10^{-5} \text{ eV}^2 . \tag{4.39}$$

Using this value, the seesaw model predicts the neutrino masses as

$$m_{\nu_e} : m_{\nu_\mu} : m_{\nu_\tau} = 2.0\times10^{-7} : 8.6\times10^{-3} : 2.4 \text{ eV} . \tag{4.40}$$

4.3.2 Non-Adiabatic Conversion

The adiabatic mechanism for the $\nu_e \rightarrow \nu_\mu$ conversion is no longer valid if the non-diagonal elements of the mass matrix in Eq (4.29) are comparable or larger than the neutrino masses. In this condition $\gamma_A \sim 1$. The diagonal eigenstates in matter will mix, preventing the $\nu_e \rightarrow \nu_\mu$ adiabatic conversion. To get the probability we have to solve the propagation Eqs (4.29) in order to obtain the conversion probability. Parke [21] derived an analytic electron neutrino survival probability that includes a correction term to the adiabatic conversion.

If the neutrinos are produced well above the resonance region, the probability of detecting an electron neutrino averaged over the production and detection positions is shown to be [21]

$$\langle P(\nu_e \rightarrow \nu_e) \rangle = \frac{1}{2} + \left(\frac{1}{2} - P_C \right) \cos 2\theta \cos 2\theta_m \qquad (4.41)$$

where $P_C = P(\nu_1 \Leftrightarrow \nu_2)$ is the non adiabatic correction probability for $|\nu_2\rangle \Leftrightarrow |\nu_1\rangle$. The form of P_C is analogous to the Landau-Zener level crossing [22] and if N_e^{res} is assumed to be constant then

$$P_C = e^{-\pi\gamma_A/2} \ . \qquad (4.42)$$

When $\gamma_A >> 1$, $P_C \rightarrow 0$ and Eq (4.41) is just the adiabatic conversion probability. When the neutrinos are produced at densities greater than the resonance density, $\cos 2\theta \approx -1$ because $\theta_M \approx \pi/2$. Then Eq (4.41) reduces to

$$\langle P(\nu_e \rightarrow \nu_e) \rangle \approx \sin^2\theta + P_C \cos 2\theta \ . \qquad (4.43)$$

Using the standard solar model in Chapter 3, the electron density in the exponential form can be derived numerically and has the form

$$\frac{N_R}{N_A} = 240\exp\left(-10.64\frac{R}{Ro}\right). \tag{4.44}$$

Differentiating and inserting into Eq (4.31) we have

$$\gamma_A \approx \frac{1.66\times10^8\Delta m^2\sin^2 2\theta}{\cos 2\theta E_\nu} \tag{4.45}$$

and thus we can calculate P_C. In the adiabatic situation, $P_C = 0$ and (4.43) reduces to

$$\langle P(\nu_e \rightarrow \nu_e)\rangle \approx \sin^2\theta \tag{4.46}$$

and this is the same given by Eq (4.33).