CHAPTER 2

REGION ESTIMATES BASED ON LIKELIHOOD RATIO IN A NONLINEAR MODEL

2.1 Introduction

For a specific nonlinear model and a given value of $\boldsymbol{\theta}_f$, the coverage probability $I(\boldsymbol{\theta}_f, \sigma)$ of the nominally-100$(1-\alpha)$% confidence regions, based on likelihood ratio, for the parameter vector $\boldsymbol{\theta}$ (see (1.1.1)) may be treated as a function of σ. The quadratic approximation of the coverage probability, treated as a function of σ has been shown to be

$$(2.1.1) \qquad I(\boldsymbol{\theta}_f, \sigma) = 1 - \alpha + P_1\left[\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}(2a_{ijk}^2 - a_{ijj}a_{ikk})\right]\sigma^2 + o(\sigma^2)$$

where $P_1 = -\dfrac{n}{p(n-p)}\beta_{2,0}$,

$$\beta_{2,0} = \frac{\Gamma\left(\frac{n}{2}\right)(d^{*^2})^{p/2}}{\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{n-p}{2}\right)(1+d^{*2})^{n/2}}$$

and $d^{*^2} = \dfrac{pF_\alpha}{n-p}$.

The setback of (2.1.1) is that we do not know whether the quadratic approximation used is adequate or not.

In this chapter, the quartic approximation of the coverage probability shall be derived for a two-parameter nonlinear model. There are three major applications given by the quartic approximation.

Firstly, it gives a more accurate value of the coverage probability. Next, the closeness of the value based on quadratic approximation with that based on quartic approximation may be used to get an indication of the adequacy of the quadratic approximation. Finally, the quartic approximation can be used (see Chapter 3) to adjust the nominally-$100(1-\alpha)\%$ confidence regions such that the coverage probability of the adjusted confidence regions is given by

$$I(\boldsymbol{\theta}_f, \sigma) = 1 - \alpha + o(\sigma^4). \tag{2.1.2}$$

Equation (2.1.2) implies that to the extent that the quartic approximation of the coverage probability is adequate for all $\boldsymbol{\theta}_f \in \Omega_f$, the actual coverage probability of the adjusted confidence regions can be taken to be $1 - \alpha$.

2.2 Approximation of the Minimum Residual Sum of Squares

Let us confine our attention to models satisfying the following conditions:

(a) For each $\boldsymbol{\theta}_f \in \Omega_f$, there exists $\delta > 0$ such that $|\boldsymbol{\theta} - \boldsymbol{\theta}_f| < \delta$ implies that $\boldsymbol{\theta} \in \Omega_f$.

(b) If $[\eta(\boldsymbol{\xi}_1, \boldsymbol{\theta}_1), \eta(\boldsymbol{\xi}_2, \boldsymbol{\theta}_1), \ldots, \eta(\boldsymbol{\xi}_n, \boldsymbol{\theta}_1)] = [\eta(\boldsymbol{\xi}_1, \boldsymbol{\theta}_2), \eta(\boldsymbol{\xi}_2, \boldsymbol{\theta}_2), \ldots, \eta(\boldsymbol{\xi}_n, \boldsymbol{\theta}_2)]$, then $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$.

(c) $\boldsymbol{\eta}(\boldsymbol{\xi}_u, \boldsymbol{\theta})$ for $u = 1, 2, \ldots, n$, are functions of $\boldsymbol{\theta}$ with continuous derivatives up to the fifth order in Ω_f.

(d) The matrix $\{c_{uj}(\boldsymbol{\theta}_f)\}$ where $c_{uj}(\boldsymbol{\theta}_f) = \left[\dfrac{\partial \eta(\boldsymbol{\xi}_u, \boldsymbol{\theta})}{\partial \theta_j}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_f}\right]$ is of rank p for all $\boldsymbol{\theta}_f \in \Omega_f$.

Let $\{a_{ij}\}$ denote a matrix $\mathbf{A}$ of which the (i,j) entry is a_{ij} and γ denote the magnitude of the vector $\boldsymbol{\gamma}$.

With condition (c), $\boldsymbol{\eta}(\boldsymbol{\xi}_u, \boldsymbol{\theta})$ can be expressed as

(2.2.1)

$$\eta(\boldsymbol{\xi}_u, \boldsymbol{\theta}) = \eta(\boldsymbol{\xi}_u, \boldsymbol{\theta}_f) + \sum_{j=1}^{p} c_{uj} t_j + \mathbf{t}^T \mathbf{C}_u \mathbf{t} + \sum_{j=1}^{p} [\mathbf{t}^T \mathbf{C}_{uj} \mathbf{t}] t_j$$

$$+ \sum_{j=1}^{p} \sum_{k=1}^{p} [\mathbf{t}^T \mathbf{C}_{ujk} \mathbf{t}] t_j t_k + \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} [\mathbf{t}^T \mathbf{C}_{ujk\ell} \mathbf{t}] t_j t_k t_\ell + o(t^5)$$

$$u = 1, 2, \ldots, n$$

where

$$\mathbf{t} = \boldsymbol{\theta} - \boldsymbol{\theta}_f$$

and

$$c_{uj_1 j_2 \ldots j_r} = \frac{1}{r!} \left[\frac{\partial^r \eta(\boldsymbol{\xi}_u, \boldsymbol{\theta})}{\partial \theta_{j_1} \partial \theta_{j_2} \cdots \partial \theta_{j_r}} \bigg|_{\boldsymbol{\theta} = \boldsymbol{\theta}_f} \right], \qquad r = 1, 2, \ldots, 5.$$

Suppose $\mathbf{H}$ is an $(n \times n)$ orthogonal matrix such that $\mathbf{HC}$ is an upper triangular $(p \times p)$ matrix $\mathbf{D}$ with an $((n-p) \times p)$ zero matrix beneath it [cf. Businger and Golub (1965)], and $\boldsymbol{\eta}(\boldsymbol{\theta}) = [\eta(\boldsymbol{\xi}_1, \boldsymbol{\theta}), \eta(\boldsymbol{\xi}_2, \boldsymbol{\theta}), \ldots, \eta(\boldsymbol{\xi}_n, \boldsymbol{\theta})]^T$.

Let us apply an orthogonal transformation

(2.2.2)
$$\mathbf{H}(\mathbf{y} - \boldsymbol{\eta}(\boldsymbol{\theta}_f)) = \mathbf{z}$$

of coordinates in sample space so that the point $\boldsymbol{\eta}(\boldsymbol{\theta}_f)$ in the solution locus becomes the new origin $\mathbf{z} = \mathbf{0}$ and the tangent plane to the solution locus at $\boldsymbol{\eta}(\boldsymbol{\theta}_f)$ consists of points for which $z_i = 0$ for $i = p+1, p+2, \ldots, n$. The components of $\mathbf{z}$ are referred to as the rotated coordinates of the sample point $\mathbf{y}$.

The rotated coordinates z_i^* of a point $\boldsymbol{\eta}(\boldsymbol{\theta})$ in the solution locus with its uth coordinate given by (2.2.1) can be expressed in the following form:

$$
(2.2.3)\qquad z_i^* = \begin{cases} \displaystyle\sum_{j=1}^{p} d_{ij}t_j + \mathbf{t}^T\mathbf{D}_i\mathbf{t} + \sum_{j=1}^{p}[\mathbf{t}^T\mathbf{D}_{ij}\mathbf{t}]t_j \\ \displaystyle\qquad + \sum_{j=1}^{p}\sum_{k=1}^{p}[\mathbf{t}^T\mathbf{D}_{ijk}\mathbf{t}]t_jt_k \\ \displaystyle\qquad + \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p}[\mathbf{t}^T\mathbf{D}_{ijk\ell}\mathbf{t}]t_jt_kt_\ell + o(t^5), & i = 1, 2, \ldots, p \\ \displaystyle\mathbf{t}^T\mathbf{D}_i\mathbf{t} + \sum_{j=1}^{p}[\mathbf{t}^T\mathbf{D}_{ij}\mathbf{t}]t_j \\ \displaystyle\qquad + \sum_{j=1}^{p}\sum_{k=1}^{p}[\mathbf{t}^T\mathbf{D}_{ijk}\mathbf{t}]t_jt_k \\ \displaystyle\qquad + \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p}[\mathbf{t}^T\mathbf{D}_{ijk\ell}\mathbf{t}]t_jt_kt_\ell + o(t^5), & i = p+1, \ldots, n \end{cases}
$$

where $\mathbf{D}_i$, $\mathbf{D}_{ij}$, $\mathbf{D}_{ijk}$ and $\mathbf{D}_{ijk\ell}$ are $(p \times p)$ symmetric matrices [cf. Pooi (1992)]. (2.2.3) can next be transformed to

$$
(2.2.4)\qquad z_i^* = \begin{cases} \displaystyle\tau_i + \boldsymbol{\tau}^T\mathbf{F}_i\boldsymbol{\tau} + \sum_{j=1}^{p}[\boldsymbol{\tau}^T\mathbf{F}_{ij}\boldsymbol{\tau}]\tau_j + \sum_{j=1}^{p}\sum_{k=1}^{p}[\boldsymbol{\tau}^T\mathbf{F}_{ijk}\boldsymbol{\tau}]\tau_j\tau_k \\ \displaystyle\qquad + \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p}[\boldsymbol{\tau}^T\mathbf{F}_{ijk\ell}\boldsymbol{\tau}]\tau_j\tau_k\tau_\ell + o(\tau^5), & i = 1, 2, \ldots, p \\ \displaystyle\boldsymbol{\tau}^T\mathbf{F}_i\boldsymbol{\tau} + \sum_{j=1}^{p}[\boldsymbol{\tau}^T\mathbf{F}_{ij}\boldsymbol{\tau}]\tau_j + \sum_{j=1}^{p}\sum_{k=1}^{p}[\boldsymbol{\tau}^T\mathbf{F}_{ijk}\boldsymbol{\tau}]\tau_j\tau_k \\ \displaystyle\qquad + \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p}[\boldsymbol{\tau}^T\mathbf{F}_{ijk\ell}\boldsymbol{\tau}]\tau_j\tau_k\tau_\ell + o(\tau^5), & i = p+1, \ldots, n \end{cases}
$$

where

$$\boldsymbol{\tau} = \mathbf{D}\mathbf{t}$$

$$\mathbf{F}_i = (\mathbf{D}^{-1})^T \mathbf{D}_i \mathbf{D}^{-1} \qquad \text{with } \mathbf{D}^{-1} = \{d^{ij}\}$$

$$\mathbf{F}_{ij} = \sum_{k=1}^{j} d^{kj} \left[(\mathbf{D}^{-1})^T \mathbf{D}_{ik} \mathbf{D}^{-1}\right]$$

$$\mathbf{F}_{ijk} = \sum_{j_1=1}^{j} \sum_{k_1=1}^{k} d^{j_1 j} d^{k_1 k} \left[(\mathbf{D}^{-1})^T \mathbf{D}_{ij_1k_1} \mathbf{D}^{-1}\right]$$

and

$$\mathbf{F}_{ijk\ell} = \sum_{j_1=1}^{j} \sum_{k_1=1}^{k} \sum_{\ell_1=1}^{\ell} d^{j_1 j} d^{k_1 k} d^{\ell_1 \ell} \left[(\mathbf{D}^{-1})^T \mathbf{D}_{ij_1k_1\ell_1} \mathbf{D}^{-1}\right].$$

Finally, (2.2.4) can be expressed as

$$(2.2.5) \qquad z_i^* = \begin{cases} \phi_i, & i = 1, 2, \ldots, p \\ \boldsymbol{\phi}^T \mathbf{A}_i \boldsymbol{\phi} + \sum_{j=1}^{p} [\boldsymbol{\phi}^T \mathbf{A}_{ij} \boldsymbol{\phi}] \phi_j + \sum_{j=1}^{p} \sum_{k=1}^{p} [\boldsymbol{\phi}^T \mathbf{A}_{ijk} \boldsymbol{\phi}] \phi_j \phi_k & \\ \quad + \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} [\boldsymbol{\phi}^T \mathbf{A}_{ijk\ell} \boldsymbol{\phi}] \phi_j \phi_k \phi_\ell + o(\phi^5), & i = p+1, \ldots, n \end{cases}$$

where

$$\phi = \tau_i + \boldsymbol{\tau}^T \mathbf{F}_i \boldsymbol{\tau} + \sum_{j=1}^{p} [\boldsymbol{\tau}^T \mathbf{F}_{ij} \boldsymbol{\tau}] \tau_j + \sum_{j=1}^{p} \sum_{k=1}^{p} [\boldsymbol{\tau}^T \mathbf{F}_{ijk} \boldsymbol{\tau}] \tau_j \tau_k$$

$$+ \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} [\boldsymbol{\tau}^T \mathbf{F}_{ijk\ell} \boldsymbol{\tau}] \tau_j \tau_k \tau_\ell + o(\tau^5)$$

$$\mathbf{A}_i = \mathbf{F}_i$$

$$\mathbf{A}_{ij} = \mathbf{F}_{ij} - 2\sum_{k=1}^{p} f_{ijk}\mathbf{F}_k$$

$$\mathbf{A}_{ijk} = \mathbf{F}_{ijk} - \sum_{\ell=1}^{p}[2f_{ij\ell}\mathbf{F}_{\ell k} + f_{\ell jk}\mathbf{F}_{i\ell} + 2f_{ijk\ell}\mathbf{F}_{\ell}]$$

$$+\sum_{\ell=1}^{p}\sum_{m=1}^{p}[4f_{ij\ell}f_{\ell km}\mathbf{F}_m + f_{i\ell m}f_{mjk}\mathbf{F}_{\ell}]$$

and $\mathbf{A}_{ijk\ell}$ is a function of the $\mathbf{F}_{m_1}$, $\mathbf{F}_{m_1m_2}$, $\mathbf{F}_{m_1m_2m_3}$ and $\mathbf{F}_{m_1m_2m_3m_4}$ and it needs not be found out explicitly as it will not appear in the final expression for the quartic approximation of the coverage probability.

The residual sum of squares $S(\boldsymbol{\theta})$, can next be expressed in terms of $\boldsymbol{\phi}$ as follows

(2.2.6)

$$S(\boldsymbol{\theta}) = \sum_{i=1}^{p}\{z_i - \phi_i\}^2$$

$$+ \sum_{i=p+1}^{n}\left\{z_i - \boldsymbol{\phi}^T\mathbf{A}_j\boldsymbol{\phi} - \sum_{j=1}^{p}[\boldsymbol{\phi}^T\mathbf{A}_{ij}\boldsymbol{\phi}]\phi_j\right.$$

$$\left.-\sum_{j=1}^{p}\sum_{k=1}^{p}[\boldsymbol{\phi}^T\mathbf{A}_{ijk}\boldsymbol{\phi}]\phi_j\phi_k - \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p}[\boldsymbol{\phi}^T\mathbf{A}_{ijk\ell}\boldsymbol{\phi}]\phi_j\phi_k\phi_\ell - o(\phi^5)\right\}^2.$$

Minimizing $S(\boldsymbol{\theta})$ in (2.2.6) with respect to $\boldsymbol{\phi}$, we can derive the minimum residual sum of squares $S(\hat{\boldsymbol{\theta}})$. The result thus obtained has been shown in Pooi (1992) to be given by

$$S(\hat{\boldsymbol{\theta}}) = \sum_{j=p+1}^{n} z_j^2 + S^*(\hat{\boldsymbol{\theta}}) + o(z^6)$$

where $S^*(\hat{\boldsymbol{\theta}})$ is a sum of expressions each of which is of the following form

$$\sum_{i_1=p+1}^{n} \sum_{i_2=p+1}^{n} \cdots \sum_{i_{n_1}=p+1}^{n} \sum_{j_1=1}^{p} \sum_{j_2=1}^{p} \cdots \sum_{j_{n_2}=1}^{p} \text{constant} \times a'_{k_1k_2k_3}$$

$$\times a'_{k_4k_5k_6} \times a'_{k_7k_8k_9} \times a'_{k_{10}k_{11}k_{12}} \times a'_{\ell_1\ell_2\ell_3\ell_4} \times a'_{\ell_5\ell_6\ell_7\ell_8}$$

$$\times a'_{m_1m_2m_3m_4m_5} \times a'_{w_1w_2w_3w_4w_5w_6} \times z_{v_1}z_{v_2}\cdots z_{v_{n_4}}.$$

In the above expression,

$a'_{s_1s_2\ldots s_{n_5}}$ for $n_5 = 3,\ 4,\ 5$ or 6 is either 1 or $a_{s_1s_2\ldots s_{n_5}}$,

s_j for $j = 1, 2, \ldots$ or n_5 is a member of $\{i_1, i_2, \ldots, i_{n_1},\ j_1, j_2, \ldots, j_{n_2}\}$,

v_k for $k = 1, 2, \ldots$ or n_4 is also a member of $\{i_1, i_2, \ldots, i_{n_1},\ j_1, j_2, \ldots, j_{n_2}\}$.

The details of the ith expression of $S^*(\hat{\boldsymbol{\theta}})$ are given in a coded form in the $(2i-1)$th and $2i$th rows of Table 2-1.

In the $(2i-1)$th row the 1st entry is n_1, 2nd entry is n_2, 3rd entry is constant, 4th entry is the total number of subscripts in the product of the $a_{s_1s_2\ldots s_{n_5}}$ and the 5th entry is n_4.

The codes of the subscripts i_k and j_ℓ appearing in $z_{v_1}z_{v_2}\cdots z_{v_{n_4}}$ are $30k$ and 40ℓ respectively. The codes of the subscripts $v_1, v_2, \ldots, v_{n_4}$ are presented in the entries after the 5th entry in the $(2i-1)$th row.

The code of the subscript s_j in $a_{s_1s_2\ldots s_{n_5}}$ is $n_5 0$ immediately followed by the code of the subscript s_j in z_{s_j}. The codes of the subscripts in the product of the $a_{s_1s_2\ldots s_{n_5}}$ are presented in the $2i$th row.

Table 2-1 Expressions of Sstar(theta hat) in a coded form

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 1  2  -2.0          3 3 301 401 402
30301 30401 30402
 1  4   1.0          6 4 401 402 403 404
30301 30401 30402 30301 30403 30404
 2  3  -4.0          6 4 301 302 401 402
30301 30401 30403 30302 30402 30403
 1  3  -2.0          4 4 301 401 402 403
40301 40401 40402 40403
 3  4  16.0          9 5 301 402 302 303 404
30301 30402 30401 30302 30403 30401 30303 30404 30403
 2  4  12.0          7 5 301 402 302 403 404
30301 30402 30401 40302 40403 40404 40401
 2  5  -8.0          9 5 301 402 403 404 405
30301 30402 30401 30302 30403 30401 30302 30404 30405
 3  4 -16.0          9 5 301 401 302 303 404
30301 30401 30402 30302 30403 30402 30303 30404 30403
 2  4 -12.0          7 5 301 401 302 403 404
30301 30401 30402 40302 40403 40404 40402
 2  5   8.0          9 5 301 401 403 404 405
30301 30401 30402 30302 30403 30402 30302 30404 30405
 3  4  -8.0          9 5 301 302 403 303 404
30301 30401 30402 30302 30403 30401 30303 30404 30402
 2  4 -12.0          7 5 301 402 403 302 404
40301 40401 40402 40403 30302 30404 30401
 1  4  -2.0          5 5 301 403 404 401 402
50301 50401 50402 50403 50404
 2  5   8.0          9 5 401 402 302 403 405
30301 30401 30402 30301 30403 30404 30302 30405 30404
 1  5   2.0          7 5 401 402 404 405 403
30301 30401 30402 40301 40403 40404 40405
 4  5  32.0         12 6 301 402 302 303 304 405
30301 30402 30401 30302 30403 30401 30304 30404 30403 30303 30405 30404
 3  6 -16.0         12 6 301 402 302 404 405 406
30301 30402 30401 30302 30403 30401 30303 30404 30403 30303 30405 30406
 3  5  24.0         10 6 301 402 302 303 404 405
30301 30402 30401 30302 30403 30401 40303 40404 40405 40403
 3  5  24.0         10 6 301 402 302 303 403 405
30301 30402 30401 40302 40403 40404 40401 30303 30405 30404
 3  5  24.0         10 6 301 402 302 303 404 405
30301 30402 30401 40302 40403 40404 40401 30303 30405 30403
 3  6 -32.0         12 6 301 402 403 404 303 406
30301 30402 30401 30302 30403 30401 30302 30404 30405 30303 30406 30405
 3  6 -16.0         12 6 301 402 404 405 303 406
30301 30402 30401 30302 30403 30401 30302 30404 30405 30303 30406 30403
 2  6 -12.0         10 6 301 402 403 404 405 406
30301 30402 30401 40302 40403 40404 40401 30302 30405 30406
 2  6  -8.0         10 6 301 402 405 406 403 404
30301 30402 30401 40302 40403 40405 40406 30302 30404 30401
 2  5  16.0          8 6 301 402 302 403 404 405
30301 30402 30401 50302 50403 50404 50405 50401
 4  5  16.0         12 6 301 302 303 304 403 405
30301 30402 30401 30302 30403 30402 30303 30404 30401 30304 30405 30404
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  2  5   9.0           8 6 301 302 402 403 404 405
40301 40402 40403 40401 40302 40404 40405 40401
  2  7   4.0          12 6 402 403 404 405 406 407
30301 30402 30401 30301 30403 30404 30302 30406 30407 30302 30405 30401
  3  5  24.0          10 6 301 302 403 303 404 405
30301 30402 30401 30302 30403 30402 40303 40404 40405 40401
  3  6 -16.0          12 6 301 302 403 404 405 406
30301 30402 30401 30302 30403 30402 30303 30404 30401 30303 30405 30406
  2  6 -12.0          10 6 301 402 403 404 405 406
40301 40402 40403 40401 30302 30404 30401 30302 30405 30406
  4  5 -32.0          12 6 302 303 304 405 401 301
30302 30403 30402 30304 30404 30403 30303 30405 30404 30301 30401 30402
  3  6  16.0          12 6 302 404 405 406 401 301
30302 30403 30402 30303 30404 30403 30303 30405 30406 30301 30401 30402
  3  5 -24.0          10 6 302 303 404 405 401 301
30302 30403 30402 40303 40404 40405 40403 30301 30401 30402
  3  5 -24.0          10 6 302 303 403 405 401 301
40302 40403 40404 40402 30303 30405 30404 30301 30401 30402
  3  5 -24.0          10 6 302 303 404 405 401 301
40302 40403 40404 40402 30303 30405 30403 30301 30401 30402
  3  6  32.0          12 6 403 404 303 406 401 301
30302 30403 30402 30302 30404 30405 30303 30406 30405 30301 30401 30402
  3  6  16.0          12 6 404 405 303 406 401 301
30302 30403 30402 30302 30404 30405 30303 30406 30403 30301 30401 30402
  2  6  12.0          10 6 403 404 405 406 401 301
40302 40403 40404 40402 30302 30405 30406 30301 30401 30402
  2  6   8.0          10 6 405 406 403 404 401 301
40302 40403 40405 40406 30302 30404 30402 30301 30401 30402
  2  5 -16.0           8 6 302 403 404 405 401 301
50302 50403 50404 50405 50402 30301 30401 30402
  4  5 -32.0          12 6 302 303 404 304 405 301
30302 30403 30402 30303 30404 30403 30304 30405 30401 30301 30401 30402
  3  5 -24.0          10 6 302 403 404 303 405 301
40302 40403 40404 40402 30303 30405 30401 30301 30401 30402
  3  6  16.0          12 6 403 404 405 303 406 301
30302 30403 30402 30302 30404 30405 30303 30406 30401 30301 30401 30402
  3  5 -24.0          10 6 302 303 405 402 403 301
30302 30404 30401 30303 30405 30404 40301 40401 40402 40403
  2  5 -18.0           8 6 302 404 405 402 403 301
40302 40404 40405 40401 40301 40401 40402 40403
  2  6  12.0          10 6 404 405 406 402 403 301
30302 30404 30401 30302 30405 30406 40301 40401 40402 40403
  3  5 -24.0          10 6 401 301 302 404 303 405
40301 40401 40403 40402 30302 30404 30402 30303 30405 30403
  2  5 -16.0           8 6 403 301 401 402 302 405
50301 50401 50402 50403 50404 30302 30405 30404
  1  5  -2.0           6 6 301 401 402 403 404 405
60301 60401 60402 60403 60404 60405
  3  6  16.0          12 6 401 402 403 302 303 406
30301 30401 30402 30301 30403 30404 30302 30405 30404 30303 30406 30405
  2  6  12.0          10 6 401 402 403 302 405 406
30301 30401 30402 30301 30403 30404 40302 40405 40406 40404
  2  7  -8.0          12 6 401 402 403 405 406 407
30301 30401 30402 30301 30403 30404 30302 30405 30404 30302 30406 30407
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 3  6   8.0          12 6 401 402 302 405 303 406
30301 30401 30402 30301 30403 30404 30302 30405 30404 30303 30406 30403
 2  6  12.0          10 6 402 403 404 302 406 401
30301 30402 30403 40301 40401 40404 40405 30302 30406 30405
 1  6   2.0           8 6 403 404 405 406 401 402
30301 30403 30404 50301 50401 50402 50405 50406
 3  6  16.0          12 6 401 403 302 405 303 406
30301 30401 30402 30301 30403 30404 30302 30405 30404 30303 30406 30402
 2  6   8.0          10 6 402 302 406 404 405 401
30301 30402 30403 40301 40401 40404 40405 30302 30406 30403
 1  6   1.0           8 6 403 404 401 405 406 402
40301 40401 40403 40404 40301 40402 40405 40406
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2.3 Approximation of the Coverage Probability of the Region Estimates

Note that a confidence region given by (1.1.1) will cover $\boldsymbol{\theta}_f$ if and only if

(2.3.1)
$$S(\boldsymbol{\theta}_f) - S(\hat{\boldsymbol{\theta}}) \leq \frac{pF_\alpha}{n-p} S(\hat{\boldsymbol{\theta}}).$$

By substituting the expression of $S(\hat{\boldsymbol{\theta}})$ in terms of $\mathbf{z}$ and $S(\boldsymbol{\theta}_f) = \sum_{i=1}^{n} z_i^2$ into the inequality in (2.3.1), we can obtain an inequality in terms of $\mathbf{z}$. The resulting inequality can then be approximated by an inequality of the following form

(2.3.2)
$$\sum_{i=1}^{p} z_i'^2 \leq d^{*^2} \sum_{i=p+1}^{n} z_i'^2 + (1 + d^{*^2})\psi(z_1', z_2', \ldots, z_n', \mathbf{a}^+)$$

where

$$z_i' = \frac{z_i}{\sigma}, \qquad i = 1, 2, \ldots, n,$$

$\mathbf{a}^+$ is a vector whose components are the $a_{ijk}^+ = a_{ijk}\sigma$, $a_{ijk\ell}^+ = a_{ijk\ell}\sigma^2$, $a_{ijk\ell m}^+ = a_{ijk\ell m}\sigma^3$, and $a_{ijk\ell mv}^+ = a_{ijk\ell mv}\sigma^4$ where $i = 1, 2, \ldots, n$; $j, k, \ell, m, v = 1, 2, \ldots, p$; $j \leq k \leq \ell \leq m \leq v$,

ψ is a function which sums up a finite number of expressions, each of which is of the form

$$\text{constant} \times \pi_1 \times \pi_2$$

π_1 is a product of the z_i' and π_2 is some product of the nonlinear terms $a_{j_1 j_2 \ldots j_r}^+$, $r = 3, 4, 5, 6$.

Now let

(2.3.3)
$$z_i' = s_i \sqrt{z_i^{(s)}}$$

where

$$s_i = \begin{cases} -1 & \text{if } z_i' < 0 \\ 1 & \text{if } z_i' \geq 0. \end{cases} \qquad i = 1, 2, \ldots, p.$$

Then, the inequality in (2.3.2) becomes

$$\sum_{i=1}^{p} z_i^{(s)} \le d^{+^2} + (1 + d^{*^2})\psi\left(s_1\sqrt{z_1^{(s)}}, s_2\sqrt{z_2^{(s)}}, \ldots, s_p\sqrt{z_p^{(s)}}, z'_{p+1}, z'_{p+2}, \ldots, z'_n, \mathbf{a}^+\right) \tag{2.3.4}$$

where

$$d^{+^2} = d^{*^2} \sum_{i=p+1}^{n} z_i'^2.$$

Next, we apply the transformation

$$r^{(s)} = \sum_{i=1}^{p} z_i^{(s)}$$

and

$$\bar{z}_i^{(s)} = \frac{z_i^{(s)}}{r^{(s)}}, \qquad i = 1, 2, \ldots, p-1.$$

Then to the extent that the approximation given by (2.3.4) is adequate, we can express the coverage probability $I(\boldsymbol{\theta}_f, \sigma)$ as

(2.3.5)

$$I(\boldsymbol{\theta}_f, \sigma) = E_{z'_{p+1}, z'_{p+2}, \ldots, z'_n} \sum_{s_1=-1,+1} \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\cdot \int_{\bar{z}_1^{(s)}=0}^{1} \int_{\bar{z}_2^{(s)}=\bar{z}_1^{(s)}}^{1} \cdots \int_{\bar{z}_{p-1}^{(s)}=\bar{z}_{p-2}^{(s)}}^{1} \int_{r^{(s)}\in\kappa^*}$$

$$\frac{1}{2^p}\left[\prod_{i=1}^{p-1} \chi_1^2(\bar{z}_i^{(s)})\right] \chi_1^2\left(1 - \sum_{j=1}^{p-1} \bar{z}_j^{(s)}\right) \frac{\chi_p^2(r^{(s)})}{\chi_p^2(1)}\, dr^{(s)} d\bar{z}_{p-1}^{(s)} d\bar{z}_{p-2}^{(s)} \ldots d\bar{z}_1^{(s)}$$

where κ^* is the set of values of $r^{(s)}$ satisfying

(2.3.6)
$$r^{(s)} \leq d^{+^2} + (1 + d^{*^2})$$

$$\cdot \psi\left(s_1\sqrt{r^{(s)}\bar{z}_1^{(s)}}, s_2\sqrt{r^{(s)}\bar{z}_2^{(s)}}, \ldots, s_{p-1}\sqrt{r^{(s)}\bar{z}_{p-1}^{(s)}}, s_p\sqrt{r^{(s)}\left(1 - \sum_{i=1}^{p-1} \bar{z}_i^{(s)}\right)},\right.$$

$$\left. z'_{p+1}, z'_{p+2}, \ldots, z'_n, \mathbf{a}^+\right),$$

and $\chi_k(\cdot)$ is the probability density function of a chi-square distribution with k degrees of freedom.

Note that when the magnitude of $\mathbf{a}^+$ is sufficiently small, κ^* can be expressed as

(2.3.7)
$$\kappa^* = \left\{r^{(s)} : 0 \leq r^{(s)} \leq r^{(s)*}\right\}$$

where $r^{(s)*}$ is the value of $r^{(s)}$ which satisfies (2.3.6) with the inequality sign changed to an equal sign. The upper bound $r^{(s)*}$ of $r^{(s)}$ as given in (2.3.7) can be shown to be given by

$$r^{(s)*} = d^{+^2} + r^{(s)+}$$

where $r^{(s)+}$ is a sum of expressions each of which is of the following form

$$\sum_{i_1=p+1}^{n} \sum_{i_2=p+1}^{n} \cdots \sum_{i_{n_1}=p+1}^{n} \sum_{j_1=1}^{p} \sum_{j_2=1}^{p} \cdots \sum_{j_{n_2}=1}^{p} \text{constant} \times \left(1 + d^{*^2}\right)^{n_3} \left(d^{+^2}\right)^{n_4/2}$$

$$\times a'_{k_1k_2k_3} \times a'_{k_4k_5k_6} \times a'_{k_7k_8k_9} \times a'_{k_{10}k_{11}k_{12}}$$

$$\times a'_{\ell_1\ell_2\ell_3\ell_4} \times a'_{\ell_5\ell_6\ell_7\ell_8} \times a'_{m_1m_2m_3m_4m_5} \times a'_{w_1w_2w_3w_4w_5w_6}$$

$$\times \left(s_{v_1}\sqrt{\bar{z}_{v_1}^{(s)}}\right)\left(s_{v_2}\sqrt{\bar{z}_{v_2}^{(s)}}\right) \cdots \left(s_{v_{n_5}}\sqrt{\bar{z}_{v_{n_5}}^{(s)}}\right) z'_{u_1} z'_{u_2} \cdots z'_{u_{n_6}}$$

$a'_{t_1 t_2 \ldots t_{n_7}}$ for $n_7 = 3,\ 4,\ 5$ or 6 is either 1 or $a^+_{t_1 t_2 \ldots t_{n_7}}$,

t_j for $j = 1, 2, \ldots$ or n_7 is a member of $\{i_1, i_2, \ldots, i_{n_1},\ j_1, j_2, \ldots, j_{n_2}\}$

v_k for $k = 1, 2, \ldots$ or n_5 is a member of $\{j_1, j_2, \ldots, j_{n_2}\}$

and u_m for $m = 1, 2, \ldots$ or n_6 is a member of $\{i_1, i_2, \ldots, i_{n_1}\}$.

The details of the ith expression of $r^{(s)+}$ are given in a coded form in the $(2i-1)$th and $2i$th rows of Table 2-2.

Let m be the sum of the powers of σ when each of the terms of the form $a^+_{t_1 t_2 \ldots t_{n_7}}$ in the ith expression is expressed as $a_{t_1 t_2 \ldots t_{n_7}}(\sigma)^{n_7 - 2}$. In the $(2i-1)$th row the 1st entry is the value m, 2nd entry is n_1, 3rd entry is n_2, 4th entry is constant, 5th entry is n_3, 6th entry is the total number of subscripts in the product of the $a^+_{t_1 t_2 \ldots t_{n_7}}$, 7th entry is $(n_5 + n_6)$ and 8th entry is n_4.

The code of the subscript i_k appearing in $z'_{u_1} z'_{u_2} \cdots z'_{u_{n_6}}$ is $30k$ and that of the subscript j_ℓ appearing in $\left(s_{v_1}\sqrt{\bar{z}^{(s)}_{v_1}}\right)\left(s_{v_2}\sqrt{\bar{z}^{(s)}_{v_2}}\right)\cdots\left(s_{v_{n_5}}\sqrt{\bar{z}^{(s)}_{v_{n_5}}}\right)$ is 40ℓ. The codes of the subscripts $u_1, u_2, \ldots, u_{n_6},\ v_1, v_2, \ldots, v_{n_5}$ are presented in the entries after the 8th entry in the $(2i-1)$th row.

The code of the subscripts t_j in $a^+_{t_1 t_2 \ldots t_{n_7}}$ is $n_7 0$ immediately followed by the code of the subscript t_j in z'_{t_j} or $s_{t_j}\sqrt{\bar{z}^{(s)}_{t_j}}$. The codes of the subscripts in the product of the $a^+_{t_1 t_2 \ldots t_{n_7}}$ are presented in the $2i$th row.

Table 2-2 Expressions of r(s)+ in a coded form

```
1 1 2 -2.0000000000E+00 1 3 3 2 301 401 402
30301 30401 30402
2 2 4 4.0000000000E+00 2 6 6 2 301 401 402 302 403 404
30301 30401 30402 30302 30403 30404
2 1 4 1.0000000000E+00 1 6 4 4 401 402 403 404
30301 30401 30402 30301 30403 30404
2 2 3 -4.0000000000E+00 1 6 4 2 301 302 401 402
30301 30401 30403 30302 30402 30403
2 1 3 -2.0000000000E+00 1 4 4 3 301 401 402 403
40301 40401 40402 40403
3 3 6 -8.0000000000E+00 3 9 9 2 301 401 402 302 403 404 303 405 406
30301 30401 30402 30302 30403 30404 30303 30405 30406
3 2 6 -2.0000000000E+00 2 9 7 4 301 401 402 403 404 405 406
30301 30401 30402 30302 30403 30404 30302 30405 30406
3 3 5 8.0000000000E+00 2 9 7 2 301 401 402 302 303 403 404
30301 30401 30402 30302 30403 30405 30303 30404 30405
3 2 5 4.0000000000E+00 2 7 7 3 301 401 402 302 403 404 405
30301 30401 30402 40302 40403 40404 40405
3 2 6 -4.0000000000E+00 2 9 7 4 401 402 403 404 302 405 406
30301 30401 30402 30301 30403 30404 30302 30405 30406
3 3 5 8.0000000000E+00 2 9 7 2 301 302 401 402 303 404 405
30301 30401 30403 30302 30402 30403 30303 30404 30405
3 2 5 6.0000000000E+00 2 7 7 3 301 401 402 403 302 404 405
40301 40401 40402 40403 30302 30404 30405
3 3 4 1.6000000000E+01 1 9 5 2 301 402 302 303 404
30301 30402 30401 30302 30403 30401 30303 30404 30403
3 2 4 1.2000000000E+01 1 7 5 3 301 402 302 403 404
30301 30402 30401 40302 40403 40404 40401
3 2 5 -8.0000000000E+00 1 9 5 4 301 402 403 404 405
30301 30402 30401 30302 30403 30401 30302 30404 30405
3 3 4 -1.6000000000E+01 1 9 5 2 301 401 302 303 404
30301 30401 30402 30302 30403 30402 30303 30404 30403
3 2 4 -1.2000000000E+01 1 7 5 3 301 401 302 403 404
30301 30401 30402 40302 40403 40404 40402
3 2 5 8.0000000000E+00 1 9 5 4 301 401 403 404 405
30301 30401 30402 30302 30403 30402 30302 30404 30405
3 3 4 -8.0000000000E+00 1 9 5 2 301 302 403 303 404
30301 30401 30402 30302 30403 30401 30303 30404 30402
3 2 4 -1.2000000000E+01 1 7 5 3 301 402 403 302 404
40301 40401 40402 40403 30302 30404 30401
3 1 4 -2.0000000000E+00 1 5 5 4 301 403 404 401 402
50301 50401 50402 50403 50404
3 2 5 8.0000000000E+00 1 9 5 4 401 402 302 403 405
30301 30401 30402 30301 30403 30404 30302 30405 30404
3 1 5 2.0000000000E+00 1 7 5 5 401 402 404 405 403
30301 30401 30402 40301 40403 40404 40405
4 4 8 1.6000000000E+01 4 12 12 2 301 401 402 302 403 404 303 405 406 304 407 408
30301 30401 30402 30302 30403 30404 30303 30405 30406 30304 30407 30408
4 3 8 4.0000000000E+00 3 12 10 4 301 401 402 302 403 404 405 406 407 408
30301 30401 30402 30302 30403 30404 30303 30405 30406 30303 30407 30408
4 4 7 -1.6000000000E+01 3 12 10 2 301 401 402 302 403 404 303 304 405 406
30301 30401 30402 30302 30403 30404 30303 30405 30407 30304 30406 30407
```

```
4  3  7 -8.0000000000E+00  3 10 10  3  301  401  402  302  403  404  303  405  406  407
30301  30401  30402  30302  30403  30404  40303  40405  40406  40407
4  3  8  8.0000000000E+00  3 12 10  4  301  401  402  403  404  405  406  303  407  408
30301  30401  30402  30302  30403  30404  30302  30405  30406  30303  30407  30408
4  4  7 -1.6000000000E+01  3 12 10  2  301  401  402  302  303  403  404  304  406  407
30301  30401  30402  30302  30403  30405  30303  30404  30405  30304  30406  30407
4  3  7 -1.2000000000E+01  3 10 10  3  301  401  402  302  403  404  405  303  406  407
30301  30401  30402  40302  40403  40404  40405  30303  30406  30407
4  4  6 -3.2000000000E+01  2 12  8  2  301  401  402  302  404  303  304  406
30301  30401  30402  30302  30404  30403  30303  30405  30403  30304  30406  30405
4  3  6 -2.4000000000E+01  2 10  8  3  301  401  402  302  404  303  405  406
30301  30401  30402  30302  30404  30403  40303  40405  40406  40403
4  3  7  1.6000000000E+01  2 12  8  4  301  401  402  302  404  405  406  407
30301  30401  30402  30302  30404  30403  30303  30405  30403  30303  30406  30407
4  4  6  3.2000000000E+01  2 12  8  2  301  401  402  302  403  303  304  406
30301  30401  30402  30302  30403  30404  30303  30405  30404  30304  30406  30405
4  3  6  2.4000000000E+01  2 10  8  3  301  401  402  302  403  303  405  406
30301  30401  30402  30302  30403  30404  40303  40405  40406  40404
4  3  7 -1.6000000000E+01  2 12  8  4  301  401  402  302  403  405  406  407
30301  30401  30402  30302  30403  30404  30303  30405  30404  30303  30406  30407
4  4  6  1.6000000000E+01  2 12  8  2  301  401  402  302  303  405  304  406
30301  30401  30402  30302  30403  30404  30303  30405  30403  30304  30406  30404
4  3  6  2.4000000000E+01  2 10  8  3  301  401  402  302  404  405  303  406
30301  30401  30402  40302  40403  40404  40405  30303  30406  30403
4  2  6  4.0000000000E+00  2  8  8  4  301  401  402  302  405  406  403  404
30301  30401  30402  50303  50403  50404  50405  50406
4  3  7 -1.6000000000E+01  2 12  8  4  301  401  402  403  404  303  405  407
30301  30401  30402  30302  30403  30404  30302  30405  30406  30303  30407  30406
4  2  7 -4.0000000000E+00  2 10  8  5  301  401  402  403  404  406  407  405
30301  30401  30402  30302  30403  30404  40302  40405  40406  40407
4  3  8  8.0000000000E+00  3 12 10  4  401  402  403  404  302  405  406  303  407  408
30301  30401  30402  30301  30403  30404  30302  30405  30406  30303  30407  30408
4  2  8  2.0000000000E+00  2 12  8  6  401  402  403  404  405  406  407  408
30301  30401  30402  30301  30403  30404  30302  30405  30406  30302  30407  30408
4  3  7 -8.0000000000E+00  2 12  8  4  401  402  403  404  302  303  405  406
30301  30401  30402  30301  30403  30404  30302  30405  30407  30303  30406  30407
4  2  7 -4.0000000000E+00  2 10  8  5  401  402  403  404  302  405  406  407
30301  30401  30402  30301  30403  30404  40302  40405  40406  40407
4  3  8  4.0000000000E+00  3 12 10  4  401  402  403  404  302  405  406  303  407  408
30301  30401  30402  30301  30403  30404  30302  30405  30406  30303  30407  30408
4  4  7 -1.6000000000E+01  3 12 10  2  301  302  401  402  303  404  405  304  406  407
30301  30401  30403  30302  30402  30403  30303  30404  30405  30304  30406  30407
4  3  7 -4.0000000000E+00  2 12  8  4  301  302  401  402  404  405  406  407
30301  30401  30403  30302  30402  30403  30303  30404  30405  30303  30406  30407
4  4  6  1.6000000000E+01  2 12  8  2  301  302  401  402  303  304  404  405
30301  30401  30403  30302  30402  30403  30303  30404  30406  30304  30405  30406
4  3  6  8.0000000000E+00  2 10  8  3  301  302  401  402  303  404  405  406
30301  30401  30403  30302  30402  30403  40303  40404  40405  40406
4  3  7 -1.2000000000E+01  3 10 10  3  301  401  402  403  302  404  405  303  406  407
40301  40401  40402  40403  30302  30404  30405  30303  30406  30407
4  2  7 -3.0000000000E+00  2 10  8  5  301  401  402  403  404  405  406  407
40301  40401  40402  40403  30302  30404  30405  30302  30406  30407
4  3  6  1.2000000000E+01  2 10  8  3  301  401  402  403  302  303  404  405
40301  40401  40402  40403  30302  30404  30406  30303  30405  30406
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4  2  6  6.0000000000E+00  2  8  8  4  301  401  402  403  302  404  405  406
40301  40401  40402  40403  40302  40404  40405  40406
4  3  7 -3.0000000000E+00  3 10 10  3  301  401  402  403  302  404  405  303  406  407
40301  40401  40402  40403  30302  30404  30405  30303  30406  30407
4  4  6 -3.2000000000E+01  2 12  8  2  301  402  302  303  404  304  405  406
30301  30402  30401  30302  30403  30401  30303  30404  30403  30304  30405  30406
4  3  6 -3.6000000000E+01  2 10  8  3  301  402  302  403  404  303  405  406
30301  30402  30401  40302  40403  40404  40401  30303  30405  30406
4  3  7  3.2000000000E+01  2 12  8  4  301  402  403  404  405  303  406  407
30301  30402  30401  30302  30403  30401  30302  30404  30405  30303  30406  30407
4  4  6  3.2000000000E+01  2 12  8  2  301  401  302  303  404  304  405  406
30301  30401  30402  30302  30403  30402  30303  30404  30403  30304  30405  30406
4  3  6  3.6000000000E+01  2 10  8  3  301  401  302  403  404  303  405  406
30301  30401  30402  40302  40403  40404  40402  30303  30405  30406
4  3  7 -3.2000000000E+01  2 12  8  4  301  401  403  404  405  303  406  407
30301  30401  30402  30302  30403  30402  30302  30404  30405  30303  30406  30407
4  4  6  1.6000000000E+01  2 12  8  2  301  302  403  303  404  304  405  406
30301  30401  30402  30302  30403  30401  30303  30404  30402  30304  30405  30406
4  3  6  3.6000000000E+01  2 10  8  3  301  402  403  302  404  303  405  406
40301  40401  40402  40403  30302  30404  30401  30303  30405  30406
4  2  6  8.0000000000E+00  2  8  8  4  301  403  404  401  402  302  405  406
50301  50401  50402  50403  50404  30302  30405  30406
4  3  7 -3.2000000000E+01  2 12  8  4  401  402  302  403  405  303  406  407
30301  30401  30402  30301  30403  30404  30302  30405  30404  30303  30406  30407
4  2  7 -1.0000000000E+01  2 10  8  5  401  402  404  405  403  302  406  407
30301  30401  30402  40301  40403  40404  40405  30302  30406  30407
4  4  5  3.2000000000E+01  1 12  6  2  301  402  302  303  304  405
30301  30402  30401  30302  30403  30401  30304  30404  30403  30303  30405  30404
4  3  6 -1.6000000000E+01  1 12  6  4  301  402  302  404  405  406
30301  30402  30401  30302  30403  30401  30303  30404  30403  30303  30405  30406
4  3  5  2.4000000000E+01  1 10  6  3  301  402  302  303  404  405
30301  30402  30401  30302  30403  30401  40303  40404  40405  40403
4  3  5  2.4000000000E+01  1 10  6  3  301  402  302  303  403  405
30301  30402  30401  40302  40403  40404  40401  30303  30405  30404
4  3  5  2.4000000000E+01  1 10  6  3  301  402  302  303  404  405
30301  30402  30401  40302  40403  40404  40401  30303  30405  30403
4  3  6 -3.2000000000E+01  1 12  6  4  301  402  403  404  303  406
30301  30402  30401  30302  30403  30401  30302  30404  30405  30303  30406  30405
4  3  6 -1.6000000000E+01  1 12  6  4  301  402  404  405  303  406
30301  30402  30401  30302  30403  30401  30302  30404  30405  30303  30406  30403
4  2  6 -1.2000000000E+01  1 10  6  5  301  402  403  404  405  406
30301  30402  30401  40302  40403  40404  40401  30302  30405  30406
4  2  6 -8.0000000000E+00  1 10  6  5  301  402  405  406  403  404
30301  30402  30401  40302  40403  40405  40406  30302  30404  30401
4  2  5  1.6000000000E+01  1  8  6  4  301  402  302  403  404  405
30301  30402  30401  50302  50403  50404  50405  50401
4  4  5  1.6000000000E+01  1 12  6  2  301  302  303  304  403  405
30301  30402  30401  30302  30403  30402  30303  30404  30401  30304  30405  30404
4  2  5  9.0000000000E+00  1  8  6  4  301  302  402  403  404  405
40301  40402  40403  40401  40302  40404  40405  40401
4  2  7  4.0000000000E+00  1 12  6  6  402  403  404  405  406  407
30301  30402  30401  30301  30403  30404  30302  30406  30407  30302  30405  30401
4  3  5  2.4000000000E+01  1 10  6  3  301  302  403  303  404  405
30301  30402  30401  30302  30403  30402  40303  40404  40405  40401
```

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4 3 6 1.6000000000E+01 1 12 6 4 401 403 302 405 303 406
30301 30401 30402 30301 30403 30404 30302 30405 30404 30303 30406 30402
4 2 6 8.0000000000E+00 1 10 6 5 402 302 406 404 405 401
30301 30402 30403 40301 40401 40404 40405 30302 30406 30403
4 1 6 1.0000000000E+00 1 8 6 6 403 404 401 405 406 402
40301 40401 40403 40404 40301 40402 40405 40406
```

A truncated series expansion of $I(\boldsymbol{\theta}_f, \sigma)$ in (2.3.5) is given by

$$I(\boldsymbol{\theta}_f, \sigma) \simeq I - \alpha + I^{(1)} + I^{(2)} + I^{(3)} + I^{(4)} \tag{2.3.8}$$

where

$$I^{(1)} = \sum_{a_1 \in \mathcal{A}_1} \left[\frac{\partial I}{\partial a_1}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_1 \tag{2.3.9}$$

(2.3.10)

$$I^{(2)} = \frac{1}{2!} \sum_{a_1^{(1)}, a_1^{(2)} \in \mathcal{A}_1} \left[\frac{\partial^2 I}{\partial a_1^{(1)} \partial a_1^{(2)}}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_1^{(1)} a_1^{(2)} + \sum_{a_2 \in \mathcal{A}_2} \left[\frac{\partial I}{\partial a_2}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_2$$

(2.3.11)

$$I^{(3)} = \frac{1}{3!} \sum_{a_1^{(1)}, a_1^{(2)}, a_1^{(3)} \in \mathcal{A}_1} \left[\frac{\partial^3 I}{\partial a_1^{(1)} \partial a_1^{(2)} \partial a_1^{(3)}}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_1^{(1)} a_1^{(2)} a_1^{(3)}$$

$$+ \frac{1}{2!} \sum_{a_1 \in \mathcal{A}_1} \sum_{a_2 \in \mathcal{A}_2} \left[\frac{\partial^2 I}{\partial a_1 \partial a_2}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_1 a_2 + \sum_{a_3 \in \mathcal{A}_3} \left[\frac{\partial I}{\partial a_3}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_3$$

(2.3.12)

$$I^{(4)} = \frac{1}{4!} \sum_{a_1^{(1)}, a_1^{(2)}, a_1^{(3)} a_1^{(4)} \in \mathcal{A}_1} \left[\frac{\partial^4 I}{\partial a_1^{(1)} \partial a_1^{(2)} \partial a_1^{(3)} a_1^{(4)}}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_1^{(1)} a_1^{(2)} a_1^{(3)} a_1^{(4)}$$

$$+ \frac{1}{3!} \sum_{a_1^{(1)}, a_1^{(2)} \in \mathcal{A}_1} \sum_{a_2 \in \mathcal{A}_2} \left[\frac{\partial^3 I}{\partial a_1^{(1)} \partial a_1^{(2)} \partial a_2}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_1^{(1)} a_1^{(2)} a_2$$

$$+ \frac{1}{2!} \sum_{a_1 \in \mathcal{A}_1} \sum_{a_3 \in \mathcal{A}_3} \left[\frac{\partial^2 I}{\partial a_1 \partial a_3}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_1 a_3$$

$$+ \frac{1}{2!} \sum_{a_2^{(1)}, a_2^{(2)} \in \mathcal{A}_2} \left[\frac{\partial^2 I}{\partial a_2^{(1)} \partial a_2^{(2)}}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_2^{(1)} a_2^{(2)} + \sum_{a_4 \in \mathcal{A}_4} \left[\frac{\partial I}{\partial a_4}\bigg|_{\mathbf{a}=\mathbf{0}} \right] a_4$$

$\mathcal{A}_1$ is a set consisting of all the a^+_{ijk}, with $i = p+1, \ldots, n;\ j, k = 1, 2, \ldots, p,\ j \le k;$

$\mathcal{A}_2$ is a set consisting of all the $a^+_{ijk\ell}$, with $i = p+1, \ldots, n;\ j, k, \ell = 1, 2, \ldots, p,\ j \le k \le \ell;$

$\mathcal{A}_3$ is a set consisting of all the $a^+_{ijk\ell m}$, with $i = p+1, \ldots, n;\ j, k, \ell, m = 1, 2, \ldots, p,$ $j \le k \le \ell \le m;$

$\mathcal{A}_4$ is a set consisting of all the $a^+_{ijk\ell mv}$, with $i = p+1, \ldots, n;\ j, k, \ell, m, v = 1, 2, \ldots, p,$ $j \le k \le \ell \le m \le v.$

In (2.3.8), the series is truncated in a way such that for a specific model with a given value of $\boldsymbol{\theta}_f$, the coverage probability given by the right side of (2.3.8) is a quartic function of σ.

In a two-parameter nonlinear regression model, the result thus obtained is

(2.3.13)

$$
\begin{aligned}
I(\boldsymbol{\theta}_f, \sigma) \simeq 1 - \alpha \\
&- \sum_{i=3}^{n} \Big\{ \frac{2n}{(n-2)} \beta_{2,0} a_{i12}^2 + \frac{n}{2(n-2)} \beta_{2,0} (a_{i11} - a_{i22})^2 \Big\} \sigma^2 \\
&+ \Bigg\{ \sum_{i=3}^{n} \big[e_1 (a_{i11}^4 + a_{i22}^4) + e_2 a_{i12}^4 + e_3 a_{i11}^2 a_{i22}^2 + e_4 (a_{i11} a_{i22}^3 + a_{i11}^3 a_{i22}) \\
&\qquad + e_5 (a_{i11}^2 + a_{i22}^2) a_{i12}^2 + e_6 a_{i11} a_{i12}^2 a_{i22} \big] \\
&+ \sum_{\substack{i=3 \\ i<h}}^{n} \sum_{h=3}^{n} \big[e_7 (a_{i11}^2 a_{h11}^2 + a_{i22}^2 a_{h22}^2) + e_8 a_{i11} a_{i22} a_{h11} a_{h22} \\
&\qquad + e_9 a_{i12}^2 a_{h12}^2 + e_{10} a_{i11} a_{i12} a_{h11} a_{h12} \big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=3}^{n}\sum_{\substack{h=3 \\ i\neq h}}^{n} \Big[e_{11}(a_{i11}a_{i22}a_{h11}^2 + a_{i11}a_{i22}a_{h22}^2) \\
&\qquad\qquad + e_{12}(a_{i12}^2 a_{h11}^2 + a_{i12}^2 a_{h22}^2) + e_{13}a_{i12}^2 a_{h11}a_{h22} \\
&\qquad\qquad + e_{14}a_{i12}a_{i22}a_{h11}a_{h12} + e_{15}a_{i11}^2 a_{h22}^2 \Big] \\
&+ \sum_{i=3}^{n} \Big[e_{16}(a_{i111}^2 + a_{i222}^2) + e_{17}(a_{i122}^2 + a_{i112}^2) \\
&\qquad\qquad + e_{18}(a_{i111}a_{i122} + a_{i222}a_{i112}) + e_{19}(a_{i11}a_{i1111} + a_{i22}a_{i2222}) \\
&\qquad\qquad + e_{20}(a_{i11}a_{i1122} + a_{i22}a_{i1122}) + e_{21}(a_{i11}a_{i2222} + a_{i22}a_{i1111}) \\
&\qquad\qquad + e_{22}(a_{i12}a_{i1112} + a_{i12}a_{i1222}) \Big] \Big\}\sigma^4.
\end{aligned}
$$

The coefficient e_k in equation (2.3.13) is a sum of a number of terms each of which is of the following form

$$
\text{constant} \times \left[1 + \frac{pF_\alpha}{n-p}\right]^{v_1} \times \beta_{m_1,m_2,m_3}
$$

where

$$
\begin{aligned}
\beta_{m_1,m_2,m_3} &= E_{z'_{p+1},\ldots,z'_n}\left[[\chi_p^2(d^{+^2})]\,[d^{+^2}]^{m_1/2}[z'_j]^{m_2}[z'_k]^{m_3} \right] \\
&\qquad\qquad\qquad\qquad j \neq k,\ \ j,k \in \{p+1, p+2, \ldots, n\} \\
&= \frac{2^{(m_1+m_2+m_3)/2}\Gamma\left(\frac{m_2+1}{2}\right)\Gamma\left(\frac{m_3+1}{2}\right)\Gamma\left(\frac{m_1+m_2+m_3+n-2}{2}\right)\left(d^{*^2}\right)^{(m_1+p-2)/2}}{2\pi\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{m_2+m_3+n-p}{2}\right)\left(1+d^{*^2}\right)^{(m_1+m_2+m_3+n-2)/2}}
\end{aligned}
$$

and $p = 2$.

The details of the jth term of the coefficient e_k are given in a coded form in the jth row of Table 2-3-k. The symbol ** appearing in the headings of the tables means "to the power of".

In a typical row, the entries are row number, constant, v_1, m_1, m_2 and m_3 respectively.

From Tables 2-3-1 to 2-3-22, we can calculate the numerical values of the coefficient e_k for different values of n and α. The values thus computed are presented in Table 2-4.

Table 2-3-1 Coefficient of ai11**4 and ai22**4

no.,	constant,	v1,	m1,	m2,	m3
1	4.37500000000000E+0000	4	2	4	0
2	6.56250000000000E+0000	3	4	2	0
3	-1.50000000000000E+0001	3	2	4	0
4	1.80000000000000E+0001	2	2	4	0
5	-1.87500000000000E+0001	2	4	2	0
6	5.46875000000000E-0001	2	6	0	0
7	-8.00000000000000E+0000	1	2	4	0
8	1.50000000000000E+0001	1	4	2	0
9	-1.25000000000000E+0000	1	6	0	0
10	-2.27864583332575E-0002	4	8	4	0
11	-3.28125000000000E+0000	4	4	4	0
12	-2.18750000000000E+0000	3	6	2	0
13	7.50000000000000E+0000	3	4	4	0
14	-4.50000000000000E+0000	2	4	4	0
15	3.12500000000000E+0000	2	6	2	0
16	5.46875000000000E-0001	4	6	4	0
17	1.36718750000000E-0001	3	8	2	0
18	-6.25000000000000E-0001	3	6	4	0
19	-6.83593750000000E-0002	2	8	0	0

Table 2-3-2 Coefficient of ai12**4

no.,	constant,	v1,	m1,	m2,	m3
1	6.00000000000000E+0000	4	2	4	0
2	9.00000000000000E+0000	3	4	2	0
3	-2.40000000000000E+0001	3	2	4	0
4	3.20000000000000E+0001	2	2	4	0
5	-3.00000000000000E+0001	2	4	2	0
6	7.50000000000000E-0001	2	6	0	0
7	-1.60000000000000E+0001	1	2	4	0
8	2.80000000000000E+0001	1	4	2	0
9	-2.00000000000000E+0000	1	6	0	0
10	-3.12499999998863E-0002	4	8	4	0
11	-4.50000000000000E+0000	4	4	4	0
12	-3.00000000000000E+0000	3	6	2	0
13	1.20000000000000E+0001	3	4	4	0
14	-8.00000000000000E+0000	2	4	4	0
15	5.00000000000000E+0000	2	6	2	0
16	7.50000000000000E-0001	4	6	4	0
17	1.87500000000000E-0001	3	8	2	0
18	-1.00000000000000E+0000	3	6	4	0
19	-9.37500000000000E-0002	2	8	0	0

Table 2-3-3 Coefficient of (ai11**2)(ai22**2)

no.,	constant,	v1,	m1,	m2,	m3
1	2.25000000000000E+0000	4	2	4	0
2	3.37500000000000E+0000	3	4	2	0
3	-6.00000000000000E+0000	3	2	4	0

4	-7.50000000000000E+0000	2	4	2	0
5	2.81250000000000E-0001	2	6	0	0
6	4.00000000000000E+0000	2	2	4	0
7	4.00000000000000E+0000	1	4	2	0
8	-5.00000000000000E-0001	1	6	0	0
9	-1.17187499999574E-0002	4	8	4	0
10	-1.68750000000000E+0000	4	4	4	0
11	-1.12500000000000E+0000	3	6	2	0
12	3.00000000000000E+0000	3	4	4	0
13	1.25000000000000E+0000	2	6	2	0
14	2.81250000000000E-0001	4	6	4	0
15	7.03125000000000E-0002	3	8	2	0
16	-2.50000000000000E-0001	3	6	4	0
17	-3.51562500000000E-0002	2	8	0	0
18	-1.00000000000000E+0000	2	4	4	0

Table 2-3-4 Coefficient of (ai11)(ai22**3) and (ai11**3)(ai22)

no.,	constant,	v1,	m1,	m2,	m3
1	2.50000000000000E+0000	4	2	4	0
2	3.75000000000000E+0000	3	4	2	0
3	-6.00000000000000E+0000	3	2	4	0
4	4.00000000000000E+0000	2	2	4	0
5	-7.50000000000000E+0000	2	4	2	0
6	3.12500000000000E-0001	2	6	0	0
7	3.00000000000000E+0000	1	4	2	0
8	-5.00000000000000E-0001	1	6	0	0
9	-1.30208333332860E-0002	4	8	4	0
10	-1.87500000000000E+0000	4	4	4	0
11	-1.25000000000000E+0000	3	6	2	0
12	3.00000000000000E+0000	3	4	4	0
13	-1.00000000000000E+0000	2	4	4	0
14	1.25000000000000E+0000	2	6	2	0
15	3.12500000000000E-0001	4	6	4	0
16	7.81250000000000E-0002	3	8	2	0
17	-2.50000000000000E-0001	3	6	4	0
18	-3.90625000000000E-0002	2	8	0	0

Table 2-3-5 Coefficient of (ai11**2)(ai12**2) and (ai12**2)(ai22**2)

no.,	constant,	v1,	m1,	m2,	m3
1	1.50000000000000E+0001	4	2	4	0
2	2.25000000000000E+0001	3	4	2	0
3	-5.40000000000000E+0001	3	2	4	0
4	6.80000000000000E+0001	2	2	4	0
5	-6.75000000000000E+0001	2	4	2	0
6	1.87500000000000E+0000	2	6	0	0
7	-3.20000000000000E+0001	1	2	4	0
8	5.70000000000000E+0001	1	4	2	0
9	-4.50000000000000E+0000	1	6	0	0
10	-7.81249999997726E-0002	4	8	4	0
11	-1.12500000000000E+0001	4	4	4	0
12	-7.50000000000000E+0000	3	6	2	0

13	2.70000000000000E+0001	3	4	4	0
14	-1.70000000000000E+0001	2	4	4	0
15	1.12500000000000E+0001	2	6	2	0
16	1.87500000000000E+0000	4	6	4	0
17	4.68750000000000E-0001	3	8	2	0
18	-2.25000000000000E+0000	3	6	4	0
19	-2.34375000000000E-0001	2	8	0	0

Table 2-3-6 Coefficient of (ai11)(ai12**2)(ai22)

no.,	constant,	v1,	m1,	m2,	m3
1	1.80000000000000E+0001	4	2	4	0
2	2.70000000000000E+0001	3	4	2	0
3	-6.00000000000000E+0001	3	2	4	0
4	7.20000000000000E+0001	2	2	4	0
5	-7.30000000000000E+0001	2	4	2	0
6	2.25000000000000E+0000	2	6	0	0
7	-3.00000000000000E+0000	1	6	0	0
8	2.60000000000000E+0001	1	4	2	0
9	-9.37499999996589E-0002	4	8	4	0
10	-1.35000000000000E+0001	4	4	4	0
11	-9.00000000000000E+0000	3	6	2	0
12	3.10000000000000E+0001	3	4	4	0
13	-1.50000000000000E+0001	2	4	4	0
14	1.47500000000000E+0001	2	6	2	0
15	2.25000000000000E+0000	4	6	4	0
16	5.62500000000000E-0001	3	8	2	0
17	-3.00000000000000E+0000	3	6	4	0
18	-2.81250000000000E-0001	2	8	0	0
19	-1.60000000000000E+0001	1	2	4	0

Table 2-3-7 Coefficient of (ai11**2)(ah11**2) and (ai22**2)(ah22**2)

no.,	constant,	v1,	m1,	m2,	m3
1	2.62500000000000E+0001	4	2	2	2
2	6.56250000000000E+0000	3	4	2	0
3	-9.00000000000000E+0001	3	2	2	2
4	1.08000000000000E+0002	2	2	2	2
5	-1.87500000000000E+0001	2	4	2	0
6	6.56250000000000E+0000	3	4	0	2
7	1.09375000000000E+0000	2	6	0	0
8	-1.87500000000000E+0001	2	4	0	2
9	-4.80000000000000E+0001	1	2	2	2
10	1.50000000000000E+0001	1	4	2	0
11	-2.50000000000000E+0000	1	6	0	0
12	1.50000000000000E+0001	1	4	0	2
13	-1.36718749999545E-0001	4	8	2	2
14	-1.96875000000000E+0001	4	4	2	2
15	-2.18750000000000E+0000	3	6	2	0
16	4.50000000000000E+0001	3	4	2	2
17	-2.70000000000000E+0001	2	4	2	2
18	3.12500000000000E+0000	2	6	2	0
19	3.28125000000000E+0000	4	6	2	2

20	1.36718750000000E-0001	3	8	2	0
21	-3.75000000000000E+0000	3	6	2	2
22	-2.18750000000000E+0000	3	6	0	2
23	-1.36718750000000E-0001	2	8	0	0
24	3.12500000000000E+0000	2	6	0	2
25	1.36718750000000E-0001	3	8	0	2

Table 2-3-8 Coefficient of (ai11)(ai22)(ah11)(ah22)

no.,	constant,	v1,	m1,	m2,	m3
1	9.00000000000000E+0000	4	2	2	2
2	2.25000000000000E+0000	3	4	2	0
3	-2.40000000000000E+0001	3	2	2	2
4	-6.00000000000000E+0000	2	4	2	0
5	2.25000000000000E+0000	3	4	0	2
6	3.75000000000000E-0001	2	6	0	0
7	1.60000000000000E+0001	2	2	2	2
8	-6.00000000000000E+0000	2	4	0	2
9	4.00000000000000E+0000	1	4	2	0
10	-1.00000000000000E+0000	1	6	0	0
11	4.00000000000000E+0000	1	4	0	2
12	-4.68749999998295E-0002	4	8	2	2
13	-6.75000000000000E+0000	4	4	2	2
14	-7.50000000000000E-0001	3	6	2	0
15	1.20000000000000E+0001	3	4	2	2
16	1.00000000000000E+0000	2	6	2	0
17	1.12500000000000E+0000	4	6	2	2
18	4.68750000000000E-0002	3	8	2	0
19	-1.00000000000000E+0000	3	6	2	2
20	-7.50000000000000E-0001	3	6	0	2
21	-4.68750000000000E-0002	2	8	0	0
22	-4.00000000000000E+0000	2	4	2	2
23	1.00000000000000E+0000	2	6	0	2
24	4.68750000000000E-0002	3	8	0	2

Table 2-3-9 Coefficient of (ai12**2)(ah12**2)

no.,	constant,	v1,	m1,	m2,	m3
1	3.60000000000000E+0001	4	2	2	2
2	9.00000000000000E+0000	3	4	2	0
3	-1.44000000000000E+0002	3	2	2	2
4	1.92000000000000E+0002	2	2	2	2
5	-3.00000000000000E+0001	2	4	2	0
6	9.00000000000000E+0000	3	4	0	2
7	1.50000000000000E+0000	2	6	0	0
8	-3.00000000000000E+0001	2	4	0	2
9	-9.60000000000000E+0001	1	2	2	2
10	2.80000000000000E+0001	1	4	2	0
11	-4.00000000000000E+0000	1	6	0	0
12	2.80000000000000E+0001	1	4	0	2
13	-1.87499999999318E-0001	4	8	2	2
14	-2.70000000000000E+0001	4	4	2	2
15	-3.00000000000000E+0000	3	6	2	0

16	7.20000000000000E+0001	3	4	2	2
17	-4.80000000000000E+0001	2	4	2	2
18	5.00000000000000E+0000	2	6	2	0
19	4.50000000000000E+0000	4	6	2	2
20	1.87500000000000E-0001	3	8	2	0
21	-6.00000000000000E+0000	3	6	2	2
22	-3.00000000000000E+0000	3	6	0	2
23	-1.87500000000000E-0001	2	8	0	0
24	5.00000000000000E+0000	2	6	0	2
25	1.87500000000000E-0001	3	8	0	2

Table 2-3-10 Coefficient of (ail1)(ail2)(ahl1)(ahl2) and (ail2)(ai22)(ahl2)(ah22)

no.,	constant,	vl,	ml,	m2,	m3
1	6.00000000000000E+0001	4	2	2	2
2	1.50000000000000E+0001	3	4	2	0
3	-2.16000000000000E+0002	3	2	2	2
4	2.72000000000000E+0002	2	2	2	2
5	-4.80000000000000E+0001	2	4	2	0
6	1.50000000000000E+0001	3	4	0	2
7	2.50000000000000E+0000	2	6	0	0
8	-4.80000000000000E+0001	2	4	0	2
9	-1.28000000000000E+0002	1	2	2	2
10	-7.00000000000000E+0000	1	6	0	0
11	4.20000000000000E+0001	1	4	2	0
12	4.20000000000000E+0001	1	4	0	2
13	-3.12499999999091E-0001	4	8	2	2
14	-4.50000000000000E+0001	4	4	2	2
15	-5.00000000000000E+0000	3	6	2	0
16	1.08000000000000E+0002	3	4	2	2
17	-6.80000000000000E+0001	2	4	2	2
18	8.00000000000000E+0000	2	6	2	0
19	7.50000000000000E+0000	4	6	2	2
20	3.12500000000000E-0001	3	8	2	0
21	-9.00000000000000E+0000	3	6	2	2
22	-5.00000000000000E+0000	3	6	0	2
23	-3.12500000000000E-0001	2	8	0	0
24	8.00000000000000E+0000	2	6	0	2
25	3.12500000000000E-0001	3	8	0	2

Table 2-3-11 Coefficient of (ail1)(ai22)(ahl1**2) and (ail1)(ai22)(ah22**2)

no.,	constant,	vl,	ml,	m2,	m3
1	7.50000000000000E+0000	4	2	2	2
2	1.87500000000000E+0000	3	4	2	0
3	-1.80000000000000E+0001	3	2	2	2
4	1.87500000000000E+0000	3	4	0	2
5	3.12500000000000E-0001	2	6	0	0
6	-4.50000000000000E+0000	2	4	0	2
7	1.20000000000000E+0001	2	2	2	2
8	-3.00000000000000E+0000	2	4	2	0
9	-5.00000000000000E-0001	1	6	0	0
10	3.00000000000000E+0000	1	4	0	2

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11	-3.90624999998295E-0002	4	8	2	2
12	-5.62500000000000E+0000	4	4	2	2
13	-6.25000000000000E-0001	3	6	2	0
14	9.00000000000000E+0000	3	4	2	2
15	9.37500000000000E-0001	4	6	2	2
16	3.90625000000000E-0002	3	8	2	0
17	-7.50000000000000E-0001	3	6	2	2
18	-6.25000000000000E-0001	3	6	0	2
19	-3.90625000000000E-0002	2	8	0	0
20	7.50000000000000E-0001	2	6	0	2
21	3.90625000000000E-0002	3	8	0	2
22	-3.00000000000000E+0000	2	4	2	2
23	5.00000000000000E-0001	2	6	2	0

Table 2-3-12 Coefficient of (ail2**2)(ahl1**2) and (ail2**2)(ah22**2)

no.,	constant,	vl,	ml,	m2,	m3
1	1.50000000000000E+0001	4	2	2	2
2	3.75000000000000E+0000	3	4	2	0
3	-5.40000000000000E+0001	3	2	2	2
4	3.75000000000000E+0000	3	4	0	2
5	6.25000000000000E-0001	2	6	0	0
6	-9.00000000000000E+0000	2	4	0	2
7	-1.05000000000000E+0001	2	4	2	0
8	6.80000000000000E+0001	2	2	2	2
9	-3.20000000000000E+0001	1	2	2	2
10	9.00000000000000E+0000	1	4	2	0
11	-1.00000000000000E+0000	1	6	0	0
12	6.00000000000000E+0000	1	4	0	2
13	-7.81249999996589E-0002	4	8	2	2
14	-1.12500000000000E+0001	4	4	2	2
15	-1.25000000000000E+0000	3	6	2	0
16	2.70000000000000E+0001	3	4	2	2
17	1.87500000000000E+0000	4	6	2	2
18	7.81250000000000E-0002	3	8	2	0
19	-2.25000000000000E+0000	3	6	2	2
20	-1.25000000000000E+0000	3	6	0	2
21	-7.81250000000000E-0002	2	8	0	0
22	1.50000000000000E+0000	2	6	0	2
23	1.75000000000000E+0000	2	6	2	0
24	-1.70000000000000E+0001	2	4	2	2
25	7.81250000000000E-0002	3	8	0	2

Table 2-3-13 Coefficient of (ail2**2)(ahl1)(ah22)

no.,	constant,	vl,	ml,	m2,	m3
1	1.80000000000000E+0001	4	2	2	2
2	4.50000000000000E+0000	3	4	2	0
3	-6.00000000000000E+0001	3	2	2	2
4	7.20000000000000E+0001	2	2	2	2
5	-1.50000000000000E+0001	2	4	2	0
6	4.50000000000000E+0000	3	4	0	2
7	7.50000000000000E-0001	2	6	0	0

8	-1.20000000000000E+0001	2	4	0	2
9	-3.20000000000000E+0001	1	2	2	2
10	1.40000000000000E+0001	1	4	2	0
11	-2.00000000000000E+0000	1	6	0	0
12	8.00000000000000E+0000	1	4	0	2
13	-9.37499999996589E-0002	4	8	2	2
14	-1.35000000000000E+0001	4	4	2	2
15	-1.50000000000000E+0000	3	6	2	0
16	3.00000000000000E+0001	3	4	2	2
17	-1.80000000000000E+0001	2	4	2	2
18	2.50000000000000E+0000	2	6	2	0
19	2.25000000000000E+0000	4	6	2	2
20	9.37500000000000E-0002	3	8	2	0
21	-2.50000000000000E+0000	3	6	2	2
22	-1.50000000000000E+0000	3	6	0	2
23	-9.37500000000000E-0002	2	8	0	0
24	2.00000000000000E+0000	2	6	0	2
25	9.37500000000000E-0002	3	8	0	2

Table 2-3-14 Coefficient of (ai12)(ai22)(ah11)(ah12)

no.,	constant,	v1,	m1,	m2,	m3
1	3.60000000000000E+0001	4	2	2	2
2	9.00000000000000E+0000	3	4	2	0
3	-1.20000000000000E+0002	3	2	2	2
4	1.44000000000000E+0002	2	2	2	2
5	-2.40000000000000E+0001	2	4	2	0
6	9.00000000000000E+0000	3	4	0	2
7	1.50000000000000E+0000	2	6	0	0
8	-2.40000000000000E+0001	2	4	0	2
9	-6.40000000000000E+0001	1	2	2	2
10	1.80000000000000E+0001	1	4	2	0
11	-3.00000000000000E+0000	1	6	0	0
12	1.80000000000000E+0001	1	4	0	2
13	-1.87499999999318E-0001	4	8	2	2
14	-2.70000000000000E+0001	4	4	2	2
15	-3.00000000000000E+0000	3	6	2	0
16	6.00000000000000E+0001	3	4	2	2
17	-3.60000000000000E+0001	2	4	2	2
18	4.00000000000000E+0000	2	6	2	0
19	4.50000000000000E+0000	4	6	2	2
20	1.87500000000000E-0001	3	8	2	0
21	-5.00000000000000E+0000	3	6	2	2
22	-3.00000000000000E+0000	3	6	0	2
23	-1.87500000000000E-0001	2	8	0	0
24	4.00000000000000E+0000	2	6	0	2
25	1.87500000000000E-0001	3	8	0	2

Table 2-3-15 Coefficient of (ai11**2)(ah22**2)

no.,	constant,	v1,	m1,	m2,	m3
1	2.25000000000000E+0000	4	2	2	2
2	5.62500000000000E-0001	3	4	2	0

3	-6.00000000000000E+0000	3	2	2	2
4	5.62500000000000E-0001	3	4	0	2
5	9.37500000000000E-0002	2	6	0	0
6	-7.50000000000000E-0001	2	4	0	2
7	-7.50000000000000E-0001	2	4	2	0
8	4.00000000000000E+0000	2	2	2	2
9	-1.17187499999574E-0002	4	8	2	2
10	-1.68750000000000E+0000	4	4	2	2
11	-1.87500000000000E-0001	3	6	2	0
12	3.00000000000000E+0000	3	4	2	2
13	2.81250000000000E-0001	4	6	2	2
14	1.17187500000000E-0002	3	8	2	0
15	-2.50000000000000E-0001	3	6	2	2
16	-1.87500000000000E-0001	3	6	0	2
17	-1.17187500000000E-0002	2	8	0	0
18	1.25000000000000E-0001	2	6	0	2
19	1.25000000000000E-0001	2	6	2	0
20	-1.00000000000000E+0000	2	4	2	2
21	1.17187500000000E-0002	3	8	0	2

Table 2-3-16 Coefficient of (ai111**2) and (ai222**2)

no.,	constant,	v1,	m1,	m2,	m3
1	1.87500000000000E+0000	2	4	2	0
2	-3.37500000000000E+0000	1	4	2	0
3	3.12500000000000E-0001	1	6	0	0
4	-3.12500000000000E-0001	2	6	2	0

Table 2-3-17 Coefficient of (ai122**2) and (ai112**2)

no.,	constant,	v1,	m1,	m2,	m3
1	3.37500000000000E+0000	2	4	2	0
2	-7.87500000000000E+0000	1	4	2	0
3	5.62500000000000E-0001	1	6	0	0
4	-5.62500000000000E-0001	2	6	2	0

Table 2-3-18 Coefficient of (ai111)(ai122) and (ai112)(ai222)

no.,	constant,	v1,	m1,	m2,	m3
1	2.25000000000000E+0000	2	4	2	0
2	-2.25000000000000E+0000	1	4	2	0
3	3.75000000000000E-0001	1	6	0	0
4	-3.75000000000000E-0001	2	6	2	0

Table 2-3-19 Coefficient of (ai11)(ai1111) and (ai22)(ai2222)

no.,	constant,	v1,	m1,	m2,	m3
1	3.75000000000000E+0000	2	4	2	0
2	-6.00000000000000E+0000	1	4	2	0
3	6.25000000000000E-0001	1	6	0	0
4	-6.25000000000000E-0001	2	6	2	0

Table 2-3-20 Coefficient of (ai11)(ai1122) and (ai22)(ai1122)

no.,	constant,	v1,	m1,	m2,	m3
1	4.50000000000000E+0000	2	4	2	0
2	-6.00000000000000E+0000	1	4	2	0
3	7.50000000000000E-0001	1	6	0	0
4	-7.50000000000000E-0001	2	6	2	0

Table 2-3-21 Coefficient of (ai11)(ai2222) and (ai22)(ai1111)

no.,	constant,	v1,	m1,	m2,	m3
1	7.50000000000000E-0001	2	4	2	0
2	1.25000000000000E-0001	1	6	0	0
3	-1.25000000000000E-0001	2	6	2	0

Table 2-3-22 Coefficient of (ai12)(ai1112) and (ai12)(ai1222)

no.,	constant,	v1,	m1,	m2,	m3
1	6.00000000000000E+0000	2	4	2	0
2	-1.20000000000000E+0001	1	4	2	0
3	1.00000000000000E+0000	1	6	0	0
4	-1.00000000000000E+0000	2	6	2	0

Table 2-4 Coefficients e_k in the quartic approximation of the coverage probability of the unadjusted confidence regions

	$n=3$		$n=4$		$n=5$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
e_1	7.340E-01	1.480E-01	1.053E+00	2.310E-01	1.249E+00	3.106E-01
e_2	2.050E+00	4.130E-01	2.936E+00	6.451E-01	3.472E+00	8.665E-01
e_3	3.034E-01	6.115E-02	4.453E-01	9.578E-02	5.466E-01	1.306E-01
e_4	-1.395E-01	-2.787E-02	-1.924E-01	-4.366E-02	-2.144E-01	-5.739E-02
e_5	3.076E+00	6.198E-01	4.403E+00	9.676E-01	5.209E+00	1.300E+00
e_6	-7.059E+02	-3.561E+03	2.850E-02	-4.633E-02	1.541E+01	1.252E+01
e_7	—	—	2.105E+00	4.620E-01	2.497E+00	6.212E-01
e_8	—	—	9.548E-01	2.061E-01	1.165E+00	2.802E-01
e_9	—	—	5.871E+00	1.290E+00	6.945E+00	1.733E+00
e_{10}	—	—	6.954E+00	1.525E+00	8.252E+00	2.052E+00
e_{11}	—	—	-1.924E-01	-4.366E-02	-2.144E-01	-5.739E-02
e_{12}	—	—	9.263E-01	2.049E-01	1.082E+00	2.740E-01
e_{13}	—	—	1.340E+00	2.934E-01	1.593E+00	3.950E-01
e_{14}	—	—	1.596E+00	3.516E-01	1.879E+00	4.715E-01
e_{15}	—	—	-3.206E-02	-7.276E-03	-3.573E-02	-9.565E-03
e_{16}	-5.597E-01	-1.125E-01	-8.122E-01	-1.764E-01	-9.806E-01	-2.389E-01
e_{17}	-1.679E+00	-3.374E-01	-2.437E+00	-5.293E-01	-2.942E+00	-7.166E-01
e_{18}	-1.500E-13	1.246E-12	0.000E+00	-3.469E-18	1.822E-15	-1.507E-15
e_{19}	-8.395E-01	-1.687E-01	-1.218E+00	-2.646E-01	-1.471E+00	-3.583E-01
e_{20}	-5.597E-01	-1.125E-01	-8.123E-01	-1.764E-01	-9.806E-01	-2.389E-01
e_{21}	2.798E-01	5.624E-02	4.061E-01	8.821E-02	4.903E-01	1.194E-01
e_{22}	-2.239E+00	-4.499E-01	-3.249E+00	-7.057E-01	-3.923E+00	-9.555E-01

	$n=6$		$n=7$		$n=8$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
e_1	1.360E+00	3.760E-01	1.422E+00	4.281E-01	1.463E+00	4.693E-01
e_2	3.772E+00	1.048E+00	3.937E+00	1.191E+00	4.042E+00	1.305E+00
e_3	6.142E-01	1.607E-01	6.590E-01	1.858E-01	6.924E-01	2.066E-01
e_4	-2.193E-01	-6.750E-02	-2.168E-01	-7.468E-02	-2.120E-01	-7.971E-02
e_5	5.659E+00	1.571E+00	5.905E+00	1.787E+00	6.063E+00	1.957E+00
e_6	1.704E+01	1.240E+01	1.689E+01	1.181E+01	1.641E+01	1.131E+01
e_7	2.720E+00	7.520E-01	2.844E+00	8.563E-01	2.925E+00	9.385E-01
e_8	1.301E+00	3.438E-01	1.390E+00	3.966E-01	1.455E+00	4.397E-01
e_9	7.545E+00	2.095E+00	7.873E+00	2.383E+00	8.084E+00	2.609E+00
e_{10}	8.993E+00	2.484E+00	9.408E+00	2.829E+00	9.680E+00	3.102E+00
e_{11}	-2.193E-01	-6.750E-02	-2.168E-01	-7.468E-02	-2.120E-01	-7.971E-02
e_{12}	1.162E+00	3.294E-01	1.201E+00	3.725E-01	1.222E+00	4.058E-01
e_{13}	1.740E+00	4.788E-01	1.824E+00	5.459E-01	1.880E+00	5.991E-01
e_{14}	2.032E+00	5.688E-01	2.113E+00	6.455E-01	2.162E+00	7.054E-01
e_{15}	-3.655E-02	-1.125E-02	-3.613E-02	-1.245E-02	-3.534E-02	-1.329E-02
e_{16}	-1.086E+00	-2.916E-01	-1.151E+00	-3.348E-01	-1.198E+00	-3.696E-01
e_{17}	-3.257E+00	-8.748E-01	-3.453E+00	-1.400E+00	-3.593E+00	-1.109E+00
e_{18}	1.137E-14	4.337E-19	-1.025E-14	-2.429E-15	6.540E-15	-1.768E-15
e_{19}	-1.629E+00	-4.374E-01	-1.727E+00	-5.022E-01	-1.796E+00	-5.544E-01
e_{20}	-1.086E+00	-2.916E-01	-1.151E+00	-3.348E-01	-1.198E+00	-3.696E-01
e_{21}	5.429E-01	1.458E-01	5.756E-01	1.674E-01	5.988E-01	1.848E-01
e_{22}	-4.343E+00	-1.166E+00	-4.604E+00	-1.339E+00	-4.790E+00	-1.479E+00

	$n=10$		$n=12$		$n=14$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
e_1	1.501E+00	5.278E-01	1.518E+00	5.676E-01	1.522E+00	5.952E-01
e_2	4.135E+00	1.465E+00	4.176E+00	1.573E+00	4.180E+00	1.648E+00
e_3	7.332E-01	2.376E-01	7.589E-01	2.598E-01	7.733E-01	2.759E-01
e_4	-2.004E-01	-8.572E-02	-1.900E-01	-8.894E-02	-1.812E-01	-9.066E-02
e_5	6.203E+00	2.197E+00	6.264E+00	2.359E+00	6.270E+00	2.471E+00
e_6	1.540E+01	1.058E+01	1.459E+01	1.008E+01	1.396E+01	9.713E+00
e_7	3.001E+00	1.056E+00	3.037E+00	1.135E+00	3.045E+00	1.190E+00
e_8	1.533E+00	5.038E-01	1.581E+00	5.492E-01	1.607E+00	5.820E-01
e_9	8.270E+00	2.929E+00	8.352E+00	3.146E+00	8.361E+00	3.295E+00
e_{10}	9.937E+00	3.490E+00	1.006E+01	3.754E+00	1.009E+01	3.937E+00
e_{11}	-2.004E-01	-8.572E-02	-1.900E-01	-8.894E-02	-1.812E-01	-9.066E-02
e_{12}	1.234E+00	4.519E-01	1.234E+00	4.821E-01	1.226E+00	5.025E-01
e_{13}	1.934E+00	6.752E-01	1.961E+00	7.271E-01	1.969E+00	7.633E-01
e_{14}	2.201E+00	7.895E-01	2.215E+00	8.457E-01	2.211E+00	8.842E-01
e_{15}	-3.340E-02	-1.429E-02	-3.167E-02	-1.482E-02	-3.020E-02	-1.511E-02
e_{16}	-1.250E+00	-4.207E-01	-1.281E+00	-4.564E-01	-1.296E+00	-4.818E-01
e_{17}	-3.750E+00	-1.262E+00	-3.843E+00	-1.369E+00	-3.887E+00	-1.445E+00
e_{18}	-9.597E-16	6.525E-16	5.378E-15	-3.828E-15	1.737E-15	-5.843E-15
e_{19}	-1.875E+00	-6.310E-01	-1.921E+00	-6.846E-01	-1.944E+00	-7.227E-01
e_{20}	-1.250E+00	-4.207E-01	-1.281E+00	-4.564E-01	-1.296E+00	-4.818E-01
e_{21}	6.250E-01	2.103E-01	6.404E-01	2.282E-01	6.479E-01	2.409E-01
e_{22}	-5.000E+00	-1.683E+00	-5.123E+00	-1.826E+00	-5.183E+00	-1.927E+00

	$n=16$		$n=20$		$n=24$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
e_1	1.526E+00	6.177E-01	1.529E+00	6.460E-01	1.526E+00	6.636E-01
e_2	4.186E+00	1.708E+00	4.185E+00	1.785E+00	4.172E+00	1.832E+00
e_3	7.854E-01	2.893E-01	8.014E-01	3.069E-01	8.099E-01	3.184E-01
e_4	-1.741E-01	-9.191E-02	-1.632E-01	-9.281E-02	-1.555E-01	-9.298E-02
e_5	6.279E+00	2.563E+00	6.278E+00	2.677E+00	6.258E+00	2.747E+00
e_6	1.348E+01	9.452E+00	1.278E+01	9.056E+00	1.230E+01	8.782E+00
e_7	3.053E+00	1.235E+00	3.057E+00	1.292E+00	3.051E+00	1.327E+00
e_8	1.629E+00	6.091E-01	1.657E+00	6.448E-01	1.672E+00	6.678E-01
e_9	8.373E+00	3.417E+00	8.370E+00	3.569E+00	8.344E+00	3.663E+00
e_{10}	1.012E+01	4.087E+00	1.014E+01	4.276E+00	1.012E+01	4.393E+00
e_{11}	-1.741E-01	-9.191E-02	-1.632E-01	-9.281E-02	-1.555E-01	-9.298E-02
e_{12}	1.221E+00	5.190E-01	1.210E+00	5.389E-01	1.199E+00	5.509E-01
e_{13}	1.977E+00	7.930E-01	1.984E+00	8.304E-01	1.982E+00	8.538E-01
e_{14}	2.209E+00	9.155E-01	2.201E+00	9.541E-01	2.190E+00	9.778E-01
e_{15}	-2.901E-02	-1.532E-02	-2.721E-02	-1.547E-02	-2.591E-02	-1.550E-02
e_{16}	-1.309E+00	-5.028E-01	-1.325E+00	-5.300E-01	-1.331E+00	-5.474E-01
e_{17}	-3.926E+00	-1.508E+00	-3.974E+00	-1.590E+00	-3.994E+00	-1.642E+00
e_{18}	5.270E-15	-5.964E-15	-2.498E-15	4.155E-15	4.545E-16	1.323E-15
e_{19}	-1.963E+00	-7.542E-01	-1.987E+00	-7.950E-01	-1.997E+00	-8.211E-01
e_{20}	-1.309E+00	-5.028E-01	-1.325E+00	-5.300E-01	-1.331E+00	-5.474E-01
e_{21}	6.544E-01	2.514E-01	6.623E-01	2.650E-01	6.657E-01	2.737E-01
e_{22}	-5.235E+00	-2.011E+00	-5.298E+00	-2.120E+00	-5.326E+00	-2.190E+00

	$n=30$		$n=62$		$n=\infty$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
e_1	1.520E+00	6.829E-01	1.506E+00	7.168E-01	1.487E+00	7.503E-01
e_2	4.152E+00	1.883E+00	4.102E+00	1.973E+00	4.040E+00	2.061E+00
e_3	8.170E-01	3.311E-01	8.313E-01	3.552E-01	8.414E-01	3.798E-01
e_4	-1.473E-01	-9.312E-02	-1.296E-01	-9.199E-02	-1.124E-01	-9.030E-02
e_5	6.228E+00	2.825E+00	6.154E+00	2.959E+00	6.060E+00	3.092E+00
e_6	1.182E+01	8.518E+00	1.081E+01	7.930E+00	9.891E+00	7.408E+00
e_7	3.041E+00	1.366E+00	3.012E+00	1.434E+00	2.974E+00	1.501E+00
e_8	1.683E+00	6.933E-01	1.706E+00	7.412E-01	1.720E+00	7.897E-01
e_9	8.304E+00	3.766E+00	8.205E+00	3.946E+00	8.080E+00	4.122E+00
e_{10}	1.009E+01	4.522E+00	9.997E+00	4.748E+00	9.875E+00	4.972E+00
e_{11}	-1.473E-01	-9.312E-02	-1.296E-01	-9.199E-02	-1.124E-01	-9.030E-02
e_{12}	1.185E+00	5.639E-01	1.155E+00	5.852E-01	1.122E+00	6.055E-01
e_{13}	1.978E+00	8.795E-01	1.965E+00	9.251E-01	1.945E+00	9.703E-01
e_{14}	2.174E+00	1.004E+00	2.138E+00	1.048E+00	2.095E+00	1.091E+00
e_{15}	-2.456E-02	-1.552E-02	-2.159E-02	-1.533E-02	-1.874E-02	-1.505E-02
e_{16}	-1.336E+00	-5.665E-01	-1.344E+00	-6.019E-01	-1.346E+00	-6.374E-01
e_{17}	-4.008E+00	-1.700E+00	-4.032E+00	-1.806E+00	-4.039E+00	-1.912E+00
e_{18}	6.848E-16	3.524E-15	-2.490E-15	1.563E-15	7.156E-18	-2.128E-15
e_{19}	-2.004E+00	-8.498E-01	-2.016E+00	-9.028E-01	-2.020E+00	-9.562E-01
e_{20}	-1.336E+00	-5.665E-01	-1.344E+00	-6.019E-01	-1.346E+00	-6.374E-01
e_{21}	6.680E-01	2.833E-01	6.720E-01	3.009E-01	6.732E-01	3.187E-01
e_{22}	-5.344E+00	-2.266E+00	-5.376E+00	-2.407E+00	-5.385E+00	-2.550E+00