CHAPTER 3

ADJUSTMENT OF REGION ESTIMATES BASED ON LIKELIHOOD RATIO IN A NONLINEAR MODEL

3.1 Introduction

We recall that the nominally-$100(1-\alpha)\%$ confidence regions, based on likelihood ratio, for the parameter vector $\boldsymbol{\theta}$ in a nonlinear regression model may be expressed in the following form

$$\mathcal{R}(\mathbf{y}) = \left\{\boldsymbol{\theta} : S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}) \leq \frac{pF_\alpha}{n-p} S(\hat{\boldsymbol{\theta}})\right\}$$

[see (1.1.1)].

In Pooi (1991b), the regions given in (1.1.1) were adjusted to the following form

$$\left\{\boldsymbol{\theta} : S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}) \leq \frac{pF_{\alpha^*}}{n-p} S(\hat{\boldsymbol{\theta}})\right\} \tag{3.1.1}$$

where α^* is a value such that the coverage probability based on quadratic approximation of the nominally-$100(1-\alpha^*)\%$ confidence regions evaluated at $\hat{\boldsymbol{\theta}}$ and a multiple

$$c = \left[\frac{n-p+pF_\alpha}{n}\right]^{1/2} \tag{3.1.2}$$

of $\left[\frac{S(\hat{\boldsymbol{\theta}})}{(n-p)}\right]^{1/2}$ is equal to $1-\alpha$. For a specific model and a given value of $\hat{\boldsymbol{\theta}}$, the value of F_{α^*} corresponding to the above α^* has been shown to be given by

$$F_{\alpha^*} = F_\alpha - \frac{\beta_{2.0}}{pF(F_\alpha)} \hat{\rho} c^2 \frac{S(\hat{\boldsymbol{\theta}})}{(n-p)} + o\big(S(\hat{\boldsymbol{\theta}})\big) \tag{3.1.3}$$

where $\hat{\rho} = -\frac{n}{n-p}\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}(2\hat{a}_{ijk}^2 - \hat{a}_{ijj}\hat{a}_{ikk})$, $\hat{a}_{ijk}$ is the value of a_{ijk} when $\boldsymbol{\theta}_f = \hat{\boldsymbol{\theta}}$ and $F(\cdot)$ is the probability density function of an F distribution with p and $n-p$ degrees of freedom.

It has also been shown that for a specific model and a given value of $\boldsymbol{\theta}_f$, the coverage probability of the adjusted regions evaluated at $\boldsymbol{\theta}_f$ is given by $1-\alpha+o(\sigma^2)$.

In this chapter, a similar adjustment of the nominally-$100(1-\alpha)\%$ confidence regions based on likelihood ratio for a two-parameter nonlinear model is proposed.

There are two stages involved in the present adjustment of the nominally-$100(1-\alpha)\%$ confidence regions. In the first stage, the original confidence regions are adjusted to regions with a coverage probability of $1-\alpha+o(\sigma^2)$. In the second stage, the adjusted regions in the first stage are adjusted further to regions with a coverage probability of $1-\alpha+o(\sigma^4)$.

This signifies that the coverage probability of the present adjusted regions is even closer to $1-\alpha$ than that of the previous adjusted regions proposed in Pooi (1991b).

3.2 Estimate of Error Variance and Nonlinearity

The quadratic approximation of the coverage probability of the nominally-$100(1-\alpha)\%$ confidence regions in a two-parameter nonlinear model is

$$I(\boldsymbol{\theta}_f, \sigma) = 1 - \alpha - \frac{n}{2(n-2)}\beta_{2,0}(\rho_1 + \rho_2 - 2\rho_3 + 4\rho_4)\sigma^2 + o(\sigma^2)$$

where

$$\rho_1 = \sum_{i=3}^{n} a_{i11}^2, \quad \rho_2 = \sum_{i=3}^{n} a_{i22}^2, \quad \rho_3 = \sum_{i=3}^{n} a_{i11}a_{i22} \text{ and } \rho_4 = \sum_{i=3}^{n} a_{i12}^2.$$

Let $\hat{\rho}_v$ be the value of ρ_v when the a_{ijk} are changed to the corresponding $\hat{a}_{ijk}$. Obviously $\hat{\rho}_v$ and $\frac{S(\hat{\boldsymbol{\theta}})}{n-p}$ are estimates of ρ_v and σ^2 respectively. These estimates have been used in Pooi (1991b) to obtain F_{α^*} for the adjusted confidence regions (see (3.1.3)). We notice that the estimate $\frac{S(\hat{\boldsymbol{\theta}})}{n-p}$ used is not unbiased for σ^2. In this section, we shall find an unbiased estimate $(\sigma^2)^{(e)}$ for σ^2. From the result

$$
\begin{aligned}
(3.2.1) \qquad S(\hat{\boldsymbol{\theta}}) = & \sum_{i=p+1}^{n} z_i^2 - 2 \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} a_{ijk} z_i z_j z_k \\
& + \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} \sum_{m=1}^{p} a_{ijk} a_{i\ell m} z_j z_k z_\ell z_m \\
& - 4 \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a_{hj\ell} a_{ik\ell} z_h z_i z_j z_k \\
& - 2 \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a_{ijk\ell} z_i z_j z_k z_\ell + o(z^4)
\end{aligned}
$$

[cf. Beale (1960)] we get

$$
\begin{aligned}
(3.2.2) \qquad \frac{1}{n-p} \sum_{i=p+1}^{n} z_i^2 \simeq & \frac{1}{n-p} S(\hat{\boldsymbol{\theta}}) + \frac{1}{n-p} \Bigg\{ 2 \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} a_{ijk} z_i z_j z_k \\
& - \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} \sum_{m=1}^{p} a_{ijk} a_{i\ell m} z_j z_k z_\ell z_m \\
& + 4 \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a_{hj\ell} a_{ik\ell} z_h z_i z_j z_k \\
& + 2 \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a_{ijk\ell} z_i z_j z_k z_\ell \Bigg\}.
\end{aligned}
$$

The left side of (3.2.2) has an expected value equal to σ^2. This means we may attempt to derive an unbiased estimate of σ^2 from the right side of (3.2.2) by setting the z_i to some suitable constants and the a_{ijk} as well as a_{ijkl} to their respective suitable estimates. We notice that $E(z_i) = 0$, $E(z_i^3) = 0$, $E(z_i^2 z_j^2) = \sigma^4$ for $i \neq j$, and $E(z_i^4) = 3\sigma^4$. Therefore, we may set z_i, z_i^3, $z_i^2 z_j^2$ and z_i^4 to be respectively 0, 0, $c_1^*\left(\frac{S(\hat{\boldsymbol{\theta}})}{n-p}\right)^2$ and $3c_1^*\left(\frac{S(\hat{\boldsymbol{\theta}})}{n-p}\right)^2$, where c_1^* is a constant. The right side of (3.2.2) is now equal to

$$\frac{S(\hat{\boldsymbol{\theta}})}{n-p} + \frac{1}{n-p}\{\rho_1 + \rho_2 - 2\rho_3 + 4\rho_4\}c_1^*\left(\frac{S(\hat{\boldsymbol{\theta}})}{n-p}\right)^2.$$

Let $\rho_v^{(e)}$ be an estimate of ρ_v such that

$$E(\rho_v^{(e)}) = \rho_v + \text{ terms of order higher than } \rho_v. \tag{3.2.3}$$

After replacing ρ_v in the above expression derived from the right side of (3.2.2) by $\rho_v^{(e)}$, we get the following estimate for σ^2

$$(\sigma^2)^{(e)} = \frac{S(\hat{\boldsymbol{\theta}})}{n-p} + \frac{1}{n-p}\{\rho_1^{(e)} + \rho_2^{(e)} - 2\rho_3^{(e)} + 4\rho_4^{(e)}\}c_1^*\left(\frac{S(\hat{\boldsymbol{\theta}})}{n-p}\right)^2. \tag{3.2.4}$$

On setting the expectation of the right side of (3.2.4) to σ^2 and neglecting terms of order higher than ρ_v, we get

$$c_1^* = \frac{n-p}{n-p+2}. \tag{3.2.5}$$

This means to the extent that terms of order higher than ρ_v are negligible, $(\sigma^2)^{(e)}$ is an unbiased estimate of σ^2.

We next find an estimate $\rho_v^{(e)}$ satisfying (3.2.3) for ρ_v such that in the first stage of adjustment of confidence regions in a two-parameter nonlinear model (i.e. one with $p = 2$), the quartic terms in the coverage probability of the adjusted confidence regions are simplified.

First, let $\hat{c}_{uj_1j_2\ldots j_r}$ $(r = 1, 2, \ldots, 5)$ be the value of $c_{uj_1j_2\ldots j_r}$ when $\boldsymbol{\theta}_f = \hat{\boldsymbol{\theta}}$

and

$$\hat{t} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_f.$$

Differentiating both sides of (2.2.1) with respect to θ_j and subsequently setting $\boldsymbol{\theta}$ to $\hat{\boldsymbol{\theta}}$, we get

$$\hat{c}_{uj} = c_{uj} + 2\sum_{k=1}^{p} c_{ujk}\hat{t}_k + 3\sum_{k=1}^{p}\sum_{\ell=1}^{p} c_{ujk\ell}\hat{t}_k\hat{t}_\ell + 4\sum_{k=1}^{p}\sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{ujk\ell m}\hat{t}_k\hat{t}_\ell\hat{t}_m + 5\sum_{k=1}^{p}\sum_{\ell=1}^{p}\sum_{m=1}^{p}\sum_{v=1}^{p} c_{ujk\ell mv}\hat{t}_k\hat{t}_\ell\hat{t}_m\hat{t}_v + o(\hat{t}^4). \tag{3.2.6}$$

Differentiating both sides of (2.2.1) with respect to θ_j and θ_k, setting $\boldsymbol{\theta}$ to $\hat{\boldsymbol{\theta}}$ and multiplying both sides of the resulting equation by half, we get

$$\hat{c}_{ujk} = c_{ujk} + 3\sum_{\ell=1}^{p} c_{ujk\ell}\hat{t}_\ell + 6\sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{ujk\ell m}\hat{t}_\ell\hat{t}_m + 10\sum_{\ell=1}^{p}\sum_{m=1}^{p}\sum_{v=1}^{p} c_{ujk\ell mv}\hat{t}_\ell\hat{t}_m\hat{t}_v + o(\hat{t}^3). \tag{3.2.7}$$

Let $\mathbf{H}$ be as defined in Chapter 2, $\mathbf{I}$ be an $(n \times n)$ identity matrix and $\mathbf{v}^{(j)}$ be an $(n \times 1)$ column vector, $j = 1, 2, \ldots, p$. Now we find $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ such that

$$[\mathbf{I} - \mathbf{v}^{(2)}\mathbf{v}^{(2)T}][\mathbf{I} - \mathbf{v}^{(1)}\mathbf{v}^{(1)T}]\mathbf{H}\hat{\mathbf{C}}$$

is an upper triangular $(p \times p)$ matrix $-\tilde{\mathbf{D}} = \{-\tilde{d}_{ij}\}$ with an $((n-p) \times p)$ zero matrix beneath it.

Next, we premultiple $-\tilde{\mathbf{D}}$ by

$$\left[\mathbf{I}-\begin{pmatrix}0\\ \sqrt{2}\\ 0\\ \vdots\\ 0\end{pmatrix}(0\ \sqrt{2}\ 0\ \dots\ 0)\right]\left[\mathbf{I}-\begin{pmatrix}\sqrt{2}\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}(\sqrt{2}\ 0\ 0\ \dots\ 0)\right]$$

to change the matrix $-\tilde{\mathbf{D}}$ to $\tilde{\mathbf{D}}$.

Therefore, $\tilde{\mathbf{H}}$ given by

$$\tilde{\mathbf{H}}=\left[\mathbf{I}-\begin{pmatrix}0\\ \sqrt{2}\\ 0\\ \vdots\\ 0\end{pmatrix}(0\ \sqrt{2}\ 0\ \dots\ 0)\right]\left[\mathbf{I}-\begin{pmatrix}\sqrt{2}\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}(\sqrt{2}\ 0\ 0\ \dots\ 0)\right] \tag{3.2.8}$$

$$\times\left[\mathbf{I}-\mathbf{v}^{(2)}\mathbf{v}^{(2)T}\right]\left[\mathbf{I}-\mathbf{v}^{(1)}\mathbf{v}^{(1)T}\right]\mathbf{H}$$

is a matrix such that $\tilde{\mathbf{H}}\hat{\mathbf{C}}$ is an upper triangular $(p\times p)$ matrix $\tilde{\mathbf{D}}=\{\tilde{d}_{ij}\}$ with an $((n-p)\times p)$ zero matrix beneath it.

Let $\hat{\mathbf{c}}_{jk}$ be a column vector of which the uth component is $\hat{c}_{ujk}$, and $\tilde{d}_{ijk}$ be the ith component of $\tilde{\mathbf{H}}\hat{\mathbf{c}}_{jk}$. Let us define

$$\tilde{a}_{ijk}=(j,k)\text{ entry of }(\tilde{\mathbf{D}}^{-1})^T\tilde{\mathbf{D}}_i\tilde{\mathbf{D}}^{-1}.$$

It has been shown (see Pooi (*)) that for $p=2$,

$$\tilde{a}_{ijk}=a_{ijk}+3f_{ijk\cdot}-2\sum_{m=1}^{p}a_{mjk}a_{im\cdot}-2\sum_{v=1}^{k}a_{ivj}a^{*}_{vk}-2\sum_{v=1}^{j}a_{ivk}a^{*}_{vj}+o(z) \tag{3.2.9}$$

where

$$i = p+1, p+2, \ldots, n; \quad j, k = 1, 2, \ldots, p$$

$$a_{ij\cdot} = \sum_{k=1}^{p} a_{ijk} z_k$$

$$f_{ijk\cdot} = \sum_{\ell=1}^{p} f_{ijk\ell} z_\ell$$

$$a_{11}^* = a_{11\cdot}, \quad a_{22}^* = a_{22\cdot}$$

and

$$a_{12}^* = a_{12\cdot} + a_{21\cdot}$$

From (3.2.9), we get

(3.2.10)
$$\sum_{i=3}^{n} \tilde{a}_{ijk}^2 = \sum_{i=3}^{n} a_{ijk}^2 + 6\sum_{i=3}^{n} a_{ijk} f_{ijk\cdot} - 4\sum_{i=3}^{n}\sum_{m=1}^{2} a_{ijk} a_{mjk} a_{im\cdot}$$
$$- 4\sum_{i=3}^{n}\sum_{v=1}^{k} a_{ijk} a_{ivj} a_{vk}^* - 4\sum_{i=3}^{n}\sum_{v=1}^{j} a_{ivk} a_{vj}^* + o(z)$$

$$j, k = 1, 2, \ldots, p$$

and

(3.2.11)
$$\sum_{i=3}^{n} \tilde{a}_{i11}\tilde{a}_{i22} = \sum_{i=3}^{n} a_{i11} a_{i22} + \sum_{i=3}^{n} a_{i11}\left[3f_{i22\cdot} - 2\sum_{m=1}^{2} a_{m22} a_{im\cdot} - 4\sum_{v=1}^{2} a_{iv2} a_{v2}^*\right]$$
$$+ \sum_{i=3}^{n} a_{i22}\left[3f_{i11\cdot} - 2\sum_{m=1}^{2} a_{m11} a_{im\cdot} - 4a_{i11} a_{11}^*\right] + o(z).$$

The results in (3.2.10) and (3.2.11) can be extended by using higher order expansion of $\hat{c}_{ujk}$ to

(3.2.12)

$$\sum_{i=3}^{n} \tilde{a}_{i11}^2 = \sum_{i=3}^{n} a_{i11}^2$$

$$+ \sum_{i=3}^{n} \Big\{ 6a_{i11} f_{i11\cdot} - 4a_{111} a_{i11} a_{i1\cdot} - 4a_{211} a_{i11} a_{i2\cdot} - 8a_{11\cdot} a_{i11}^2 + 9 f_{i11\cdot}^2$$

$$+ 4a_{111}^2 a_{i1\cdot}^2 + 4a_{211}^2 a_{i2\cdot}^2 + 16 a_{11\cdot}^2 a_{i11}^2 - 12 a_{111} a_{i1\cdot} f_{i11\cdot}$$

$$- 12 a_{211} a_{i2\cdot} f_{i11\cdot} - 24 a_{11\cdot} a_{i11} f_{i11\cdot} + 8 a_{111} a_{211} a_{i1\cdot} a_{i2\cdot}$$

$$+ 16 a_{111} a_{11\cdot} a_{i11} a_{i1\cdot} + 16 a_{211} a_{11\cdot} a_{i11} a_{i2\cdot}$$

$$+ \left(2 \sum_{h=3}^{n} a_{h1\cdot} z_h - \sum_{j=1}^{2} a_{1j\cdot} z_j \right)$$

$$\cdot (6 a_{i11} f_{i111} - 12 a_{111} a_{i11}^2 - 4 a_{211} a_{i11} a_{i12})$$

$$+ \left(2 \sum_{h=3}^{n} a_{h2\cdot} z_h - \sum_{j=1}^{2} a_{2j\cdot} z_j \right)$$

$$\cdot (6 a_{i11} f_{i112} - 4 a_{111} a_{i11} a_{i12} - 4 a_{211} a_{i11} a_{i22} - 8 a_{112} a_{i11} a_{i12})$$

$$- 12 f_{111\cdot} a_{i11} a_{i1\cdot} - 24 a_{11\cdot} a_{i11} f_{i11\cdot} - 12 a_{i11}^2 a_{i1\cdot}^2$$

$$- 6 a_{111} a_{i11} f_{i1\cdot\cdot} + 24 a_{111} a_{11\cdot} a_{i11} a_{i1\cdot} + 16 a_{211} a_{11\cdot} a_{i11} a_{i2\cdot}$$

$$+ 24 a_{11\cdot}^2 a_{i11}^2 - 8 a_{21\cdot}^2 a_{i11}^2 - 6 a_{211} a_{i11} f_{i2\cdot\cdot}$$

$$- 12 f_{211\cdot} a_{i11} a_{i2\cdot} + 8 a_{211} a_{22\cdot} a_{i11} a_{i2\cdot} - 4 a_{i11}^2 a_{i2\cdot}^2$$

$$+ 8 a_{211} a_{12\cdot} a_{i11} a_{i1\cdot} + 8 a_{111} a_{21\cdot} a_{i11} a_{i2\cdot} - 12 f_{11\cdot\cdot} a_{i11}^2 \Big\}$$

$$+\sum_{\substack{i=3\\ i\neq h}}^{n}\sum_{h=3}^{n}\left\{-4a_{i11}a_{i1\cdot}a_{h11}a_{h1\cdot}-8a_{i11}^{2}a_{h1\cdot}^{2}-4a_{i11}a_{i2\cdot}a_{h11}a_{h2\cdot}\right\}$$

$$+o(z^2)$$

where $f_{ij\cdot\cdot}=\sum_{k=1}^{p}\sum_{\ell=1}^{p}f_{ijk\ell}z_kz_\ell,\ j=1,2,\ldots,p.$

(3.2.13)

$$\sum_{i=3}^{n}\tilde{a}_{i22}^{2}=\sum_{i=3}^{n}a_{i22}^{2}$$

$$+\sum_{i=3}^{n}\Big\{6a_{i22}f_{i22\cdot}-8a_{i12}a_{i22}a_{12\cdot}-8a_{i12}a_{i22}a_{21\cdot}$$

$$-4a_{122}a_{i22}a_{i1\cdot}-4a_{222}a_{i22}a_{i2\cdot}-8a_{i22}^{2}a_{22\cdot}$$

$$+16a_{i12}^{2}a_{12\cdot}^{2}+16a_{i12}^{2}a_{21\cdot}^{2}+4a_{122}^{2}a_{i1\cdot}^{2}$$

$$+4a_{222}^{2}a_{i2\cdot}^{2}+40a_{i22}^{2}a_{22\cdot}^{2}+9f_{i22\cdot}^{2}$$

$$+\left(2\sum_{h=3}^{n}a_{h1\cdot}z_h-\sum_{j=1}^{2}a_{1j\cdot}z_j\right)$$

$$\cdot(6f_{i122}-8a_{112}a_{i12}-8a_{211}a_{i12}$$

$$-4a_{122}a_{i11}-4a_{222}a_{i12}-8a_{212}a_{i22})a_{i22}$$

$$+\left(2\sum_{h=3}^{n}a_{h2\cdot}z_h-\sum_{j=1}^{2}a_{2j\cdot}z_j\right)$$

$$\cdot(6f_{i222}-12a_{122}a_{i12}-8a_{212}a_{i12}-12a_{222}a_{i22})a_{i22}$$

$$- 24a_{i22}a_{21\cdot}f_{i12\cdot} - 12a_{i12}a_{i22}(f_{21\cdot\cdot} + f_{12\cdot\cdot})$$

$$- 24a_{i22}a_{12\cdot}f_{i12\cdot} - 12a_{i22}a_{i1\cdot}f_{122\cdot}$$

$$- 6a_{122}a_{i22}f_{i1\cdot\cdot} - 6a_{222}a_{i22}f_{i2\cdot\cdot} - 12a_{i22}a_{i2\cdot}f_{222\cdot}$$

$$- 48a_{i22}a_{22\cdot}f_{i22\cdot} - 12a_{i22}^2 f_{22\cdot\cdot} + 8a_{i11}a_{i22}(a_{21\cdot}^2 + a_{12\cdot}^2)$$

$$+ 16a_{i11}a_{i22}a_{12\cdot}a_{21\cdot} + 16a_{112}a_{i22}a_{21\cdot}a_{i1\cdot}$$

$$+ 32a_{i12}a_{i22}a_{11\cdot}a_{21\cdot} + 16a_{112}a_{i22}a_{12\cdot}a_{i1\cdot}$$

$$+ 16a_{212}a_{i22}a_{21\cdot}a_{i2\cdot} - 16a_{i12}a_{i22}a_{i1\cdot}a_{i2\cdot}$$

$$+ 16a_{i12}a_{i22}a_{11\cdot}a_{12\cdot} + 48a_{i12}a_{i22}a_{21\cdot}a_{22\cdot} - 4a_{i22}^2 a_{i1\cdot}^2$$

$$+ 8a_{122}a_{i22}a_{11\cdot}a_{i1\cdot} + 8a_{i22}^2 a_{21\cdot}^2 + 16a_{212}a_{i22}a_{12\cdot}a_{i2\cdot}$$

$$+ 64a_{i12}a_{i22}a_{12\cdot}a_{22\cdot} + 8a_{222}a_{i22}a_{12\cdot}a_{i1\cdot} + 8a_{122}a_{i22}a_{21\cdot}a_{i2\cdot}$$

$$+ 32a_{122}a_{i22}a_{22\cdot}a_{i1\cdot} + 16a_{i22}^2 a_{12\cdot}a_{21\cdot}$$

$$+ 40a_{222}a_{i22}a_{22\cdot}a_{i2\cdot} - 12a_{i22}^2 a_{i2\cdot}^2 + 32a_{i12}^2 a_{12\cdot}a_{21\cdot}$$

$$+ 16a_{122}a_{i12}a_{12\cdot}a_{i1\cdot} + 16a_{222}a_{i12}a_{12\cdot}a_{i2\cdot} - 24a_{i12}a_{12\cdot}f_{i22\cdot}$$

$$+ 16a_{122}a_{i12}a_{21\cdot}a_{i1\cdot} + 16a_{222}a_{i12}a_{21\cdot}a_{i2\cdot} - 24a_{i12}a_{21\cdot}f_{i22\cdot}$$

$$+ 8a_{122}a_{222}a_{i1\cdot}a_{i2\cdot} - 12a_{122}a_{i1\cdot}f_{i22\cdot} - 12a_{222}a_{i2\cdot}f_{i22\cdot}\Big\}$$

$$+ \sum_{i=3}^{n}\sum_{\substack{h=3 \\ i\neq h}}^{n}\Big\{-16a_{i12}a_{i22}a_{h1\cdot}a_{h2\cdot} - 4a_{i22}a_{i1\cdot}a_{h22}a_{h1\cdot}$$

$$- 4a_{i22}a_{i2\cdot}a_{h22}a_{h2\cdot} - 8a_{i22}^2 a_{h2\cdot}^2\Big\}$$

$$+ o(z^2)$$

(3.2.14)

$$\sum_{i=3}^{n} \bar{a}_{i12}^2 = \sum_{i=3}^{n} a_{i12}^2$$

$$+ \sum_{i=3}^{n} \Big\{ 6a_{i12}f_{i12\cdot} - 4a_{i11}a_{i12}(a_{12\cdot} + a_{21\cdot}) - 4a_{112}a_{i12}a_{i1\cdot}$$

$$- 4a_{212}a_{i12}a_{i2\cdot} - 4a_{i12}^2(a_{11\cdot} + a_{22\cdot}) + 4a_{i11}^2(a_{12\cdot}^2 + a_{21\cdot}^2)$$

$$+ 4a_{112}^2 a_{i1\cdot}^2 + 4a_{212}^2 a_{i2\cdot}^2 + 12a_{i12}^2(a_{11\cdot}^2 + a_{22\cdot}^2) + 9f_{i12\cdot}^2$$

$$+ 8a_{i11}^2 a_{12\cdot}a_{21\cdot} + 8a_{112}a_{i11}a_{12\cdot}a_{i1\cdot} + 8a_{212}a_{i11}a_{12\cdot}a_{i2\cdot}$$

$$+ 24a_{i11}a_{i12}a_{11\cdot}a_{12\cdot} + 16a_{i11}a_{i12}a_{12\cdot}a_{22\cdot} - 12a_{i11}a_{12\cdot}f_{i12\cdot}$$

$$+ 8a_{112}a_{i11}a_{21\cdot}a_{i1\cdot} + 8a_{212}a_{i11}a_{21\cdot}a_{i2\cdot} + 32a_{i11}a_{i12}a_{11\cdot}a_{21\cdot}$$

$$+ 8a_{i11}a_{i12}a_{21\cdot}a_{22\cdot} - 12a_{i11}a_{21\cdot}f_{i12\cdot} + 8a_{112}a_{212}a_{i1\cdot}a_{i2\cdot}$$

$$+ 24a_{112}a_{i12}a_{11\cdot}a_{i1\cdot} + 16a_{112}a_{i12}a_{22\cdot}a_{i1\cdot} - 12a_{112}a_{i1\cdot}f_{i12\cdot}$$

$$+ 16a_{212}a_{i12}a_{11\cdot}a_{i2\cdot} + 24a_{212}a_{i12}a_{22\cdot}a_{i2\cdot} - 12a_{212}a_{i2\cdot}f_{i12\cdot}$$

$$+ 16a_{i12}^2 a_{11\cdot}a_{22\cdot} - 24a_{i12}a_{11\cdot}f_{i12\cdot} - 24a_{i12}a_{22\cdot}f_{i12\cdot}$$

$$+ \left(2\sum_{h=3}^{n} a_{h1\cdot}z_h - \sum_{j=1}^{2} a_{1j\cdot}z_j \right)$$

$$\cdot (6f_{i112} - 8a_{112}a_{i11} - 4a_{211}a_{i11} - 8a_{212}a_{i12} - 4a_{111}a_{i12})a_{i12}$$

$$+\left(2\sum_{h=3}^{n} a_{h2\cdot} z_h - \sum_{j=1}^{2} a_{2j\cdot} z_j\right)$$

$$\cdot (6f_{i122} - 4a_{122}a_{i11} - 4a_{212}a_{i11}$$

$$- 8a_{112}a_{i12} - 4a_{212}a_{i22} - 4a_{222}a_{i12})a_{i12}$$

$$- 12a_{i12}a_{12\cdot}f_{i11\cdot} - 6a_{i11}a_{i12}f_{12\cdot\cdot} - 12a_{i12}a_{i1\cdot}f_{112\cdot}$$

$$- 6a_{112}a_{i12}f_{i1\cdot\cdot} - 6a_{i12}^2 f_{11\cdot\cdot} - 6a_{212}a_{i12}f_{i2\cdot\cdot}$$

$$- 12a_{i12}a_{i2\cdot}f_{212\cdot} - 6a_{i12}^2 f_{22\cdot\cdot} - 12a_{i12}a_{21\cdot}f_{i11\cdot}$$

$$- 6a_{i11}a_{i12}f_{21\cdot\cdot} + 8a_{111}a_{i12}a_{12\cdot}a_{i1\cdot} + 8a_{211}a_{i12}a_{21\cdot}a_{i2\cdot}$$

$$- 8a_{i11}a_{i12}a_{i1\cdot}a_{i2\cdot} - 8a_{i12}^2 a_{i1\cdot}^2 + 8a_{211}a_{i12}a_{12\cdot}a_{i2\cdot}$$

$$+ 8a_{212}a_{i12}a_{12\cdot}a_{i1\cdot} + 8a_{112}a_{i12}a_{21\cdot}a_{i2\cdot} + 8a_{i12}^2 a_{12\cdot}a_{21\cdot}$$

$$- 8a_{i12}^2 a_{i2\cdot}^2 + 8a_{111}a_{i12}a_{21\cdot}a_{i1\cdot}$$

$$+ \sum_{\substack{i=3 \\ i\neq h}}^{n} \sum_{h=3}^{n} \Big\{ -8a_{i11}a_{i12}a_{h1\cdot}a_{h2\cdot} - 4a_{i12}a_{i1\cdot}a_{h12}a_{h1\cdot} - 4a_{i12}^2 a_{h1\cdot}^2$$

$$- 4a_{i12}a_{i2\cdot}a_{h12}a_{h2\cdot} - 4a_{i12}^2 a_{h2\cdot}^2 \Big\}$$

$$+ o(z^2)$$

and

(3.2.15)

$$\sum_{i=3}^{n} \bar{a}_{i11}\bar{a}_{i22} = \sum_{i=3}^{n} a_{i11}a_{i22}$$

$$+ \sum_{i=3}^{n} \Big\{ 3a_{i11}f_{i22\cdot} + 3a_{i22}f_{i11\cdot} - 4a_{i11}a_{i12}(a_{12\cdot} + a_{21\cdot})$$

$$- 2a_{122}a_{i11}a_{i1\cdot} - 2a_{222}a_{i11}a_{i2\cdot} - 4a_{i11}a_{i22}(a_{11\cdot} + a_{22\cdot})$$

$$- 2a_{111}a_{i22}a_{i1\cdot} - 2a_{211}a_{i22}a_{i2\cdot}$$

$$+ \left(2\sum_{h=3}^{n} a_{h1\cdot}z_h - \sum_{j=1}^{2} a_{1j\cdot}z_j \right)$$

$$\cdot (3a_{i11}f_{i122} + 3a_{i22}f_{i111} - 4a_{112}a_{i11}a_{i12}$$

$$- 4a_{211}a_{i11}a_{i12} - 2a_{122}a_{i11}^2 - 2a_{222}a_{i11}a_{i12}$$

$$- 4a_{212}a_{i11}a_{i22} - 6a_{111}a_{i11}a_{i22} - 2a_{211}a_{i12}a_{i22})$$

$$+ \left(2\sum_{h=3}^{n} a_{h2\cdot}z_h - \sum_{j=1}^{2} a_{2j\cdot}z_j \right)$$

$$\cdot (3a_{i11}f_{i222} + 3a_{i22}f_{i112} - 6a_{122}a_{i11}a_{i12}$$

$$- 4a_{212}a_{i11}a_{i12} - 6a_{222}a_{i11}a_{i22} - 2a_{111}a_{i12}a_{i22}$$

$$- 2a_{211}a_{i22}^2 - 4a_{112}a_{i11}a_{i22})$$

$$- 12a_{i11}a_{21\cdot}f_{i12\cdot} - 6a_{i11}a_{i12}(f_{21\cdot\cdot} + f_{12\cdot\cdot})$$

$$- 12a_{i11}a_{12\cdot}f_{i12\cdot} - 6a_{i11}(a_{i1\cdot}f_{122\cdot} + a_{i2\cdot}f_{222\cdot})$$

$$- 3a_{122}a_{i11}f_{i1\cdot\cdot} - 3a_{222}a_{i11}f_{i2\cdot\cdot} - 12a_{i11}(a_{11\cdot} + a_{22\cdot})f_{i22\cdot}$$

$$- 6a_{i11}a_{i22}(f_{11\cdot\cdot} + f_{22\cdot\cdot}) + 9f_{i11\cdot}f_{i22\cdot}$$

$$- 12a_{i12}(a_{12\cdot} + a_{21\cdot})f_{i11\cdot} - 6a_{122}a_{i1\cdot}f_{i11\cdot} - 6a_{222}a_{i2\cdot}f_{i11\cdot}$$

$$- 12a_{i22}(a_{11\cdot} + a_{22\cdot})f_{i11\cdot} - 6a_{111}a_{i1\cdot}f_{i22\cdot} - 6a_{211}a_{i2\cdot}f_{i22\cdot}$$

$$- 6a_{i22}a_{i1\cdot}f_{111\cdot} - 3a_{111}a_{i22}f_{i1\cdot\cdot} - 3a_{211}a_{i22}f_{i2\cdot\cdot}$$

$$- 6a_{i22}a_{i2\cdot}f_{211\cdot} + 4a_{i11}^2(a_{21\cdot}^2 + a_{12\cdot}^2) + 8a_{i11}^2 a_{12\cdot}a_{21\cdot}$$

$$+ 8a_{112}a_{i11}a_{21\cdot}a_{i1\cdot} + 32a_{i11}a_{i12}a_{11\cdot}a_{21\cdot} + 8a_{112}a_{i11}a_{12\cdot}a_{i1\cdot}$$

$$+ 8a_{212}a_{i11}a_{21\cdot}a_{i2\cdot} - 8a_{i11}a_{i12}a_{i1\cdot}a_{i2\cdot} + 24a_{i11}a_{i12}a_{11\cdot}a_{12\cdot}$$

$$+ 8a_{i11}a_{i12}a_{21\cdot}a_{22\cdot} - 8a_{i11}a_{i22}a_{i1\cdot}^2 + 12a_{122}a_{i11}a_{11\cdot}a_{i1\cdot}$$

$$+ 4a_{122}a_{i11}a_{21\cdot}a_{i2\cdot} + 8a_{212}a_{i11}a_{12\cdot}a_{i2\cdot} + 16a_{i11}a_{i12}a_{12\cdot}a_{22\cdot}$$

$$+ 4a_{222}a_{i11}a_{12\cdot}a_{i1\cdot} + 8a_{122}a_{i11}a_{22\cdot}a_{i1\cdot} + 8a_{i11}a_{i22}a_{12\cdot}a_{21\cdot}$$

$$+ 12a_{222}a_{i11}a_{22\cdot}a_{i2\cdot} - 8a_{i11}a_{i22}a_{i2\cdot}^2 + 12a_{i11}a_{i22}a_{22\cdot}^2$$

$$+ 8a_{111}a_{i12}(a_{12\cdot} + a_{21\cdot})a_{i1\cdot} + 4a_{111}a_{122}a_{i1\cdot}^2 + 4a_{111}a_{222}a_{i1\cdot}a_{i2\cdot}$$

$$+ 8a_{111}a_{i22}a_{22\cdot}a_{i1\cdot} + 8a_{211}a_{i12}(a_{12\cdot} + a_{21\cdot})a_{i2\cdot} + 4a_{122}a_{211}a_{i1\cdot}a_{i2\cdot}$$

$$+ 4a_{211}a_{222}a_{i2\cdot}^2 + 12a_{211}a_{i22}a_{22\cdot}a_{i2\cdot} + 8a_{222}a_{i11}a_{11\cdot}a_{i2\cdot}$$

$$+ 16a_{i11}a_{i22}a_{11\cdot}a_{22\cdot} + 12a_{111}a_{i22}a_{11\cdot}a_{i1\cdot} + 8a_{211}a_{i22}a_{11\cdot}a_{i2\cdot}$$

$$+ 12a_{i11}a_{i22}a_{11\cdot}^2 + 4_{211}a_{i22}a_{12\cdot}a_{i1\cdot} + 4a_{111}a_{i22}a_{21\cdot}a_{i2\cdot}\Big\}$$

$$+ \sum_{i=3}^{n}\sum_{\substack{h=3 \\ i\neq h}}^{n}\Big\{-8a_{i11}a_{i12}a_{h1\cdot}a_{h2\cdot} - 2a_{i11}a_{h22}(a_{i1\cdot}a_{h1\cdot} + a_{i2\cdot}a_{h2\cdot})$$

$$- 4a_{i11}a_{i22}(a_{h1\cdot}^2 + a_{h2\cdot}^2) - 2a_{i22}a_{h11}(a_{i1\cdot}a_{h1\cdot} + a_{i2\cdot}a_{h2\cdot})\Big\}$$

$$+ o(z^2).$$

We note that $E(z_i) = 0$ and $E(z_i^2) = \sigma^2$. After replacing z_i and z_i^2 appearing in the right sides of (3.2.12), (3.2.13), (3.2.14) and (3.2.15) respectively by 0 and a multiple c_2^* of $(\sigma^2)^{(e)}$ and subsequently setting the a_{ijk} and $f_{ijk\ell}$ to $\bar{a}_{ijk}$ and $\bar{f}_{ijk\ell}$ respectively, we get

(3.2.16)

$$
\begin{aligned}
\rho_1^{(e)} = \sum_{i=3}^{n} \bar{a}_{i11}^2 \\
& - \sum_{i=3}^{n} \Big\{ 9(\bar{f}_{i111}^2 + \bar{f}_{i112}^2) + 96\bar{a}_{111}^2\bar{a}_{i11}^2 + 4\bar{a}_{111}^2\bar{a}_{i12}^2 + 4\bar{a}_{211}^2\bar{a}_{i12}^2 \\
& + 4\bar{a}_{211}^2\bar{a}_{i22}^2 + 40\bar{a}_{112}^2\bar{a}_{i11}^2 - 72\bar{a}_{111}\bar{a}_{i11}\bar{f}_{i111} \\
& - 12\bar{a}_{111}\bar{a}_{i12}\bar{f}_{i112} - 12\bar{a}_{211}\bar{a}_{i12}\bar{f}_{i111} - 12\bar{a}_{211}\bar{a}_{i22}\bar{f}_{i112} \\
& - 48\bar{a}_{112}\bar{a}_{i11}\bar{f}_{i112} + 56\bar{a}_{111}\bar{a}_{211}\bar{a}_{i11}\bar{a}_{i12} + 8\bar{a}_{111}\bar{a}_{211}\bar{a}_{i12}\bar{a}_{i22} \\
& + 40\bar{a}_{111}\bar{a}_{112}\bar{a}_{i11}\bar{a}_{i12} + 32\bar{a}_{112}\bar{a}_{211}\bar{a}_{i11}\bar{a}_{i22} - 6\bar{a}_{112}\bar{a}_{i11}\bar{f}_{i111} \\
& + 12\bar{a}_{111}\bar{a}_{112}\bar{a}_{i11}^2 + 12\bar{a}_{112}\bar{a}_{211}\bar{a}_{i11}\bar{a}_{i12} + 4\bar{a}_{211}^2\bar{a}_{i11}\bar{a}_{i22} \\
& - 6\bar{a}_{222}\bar{a}_{i11}\bar{f}_{i112} + 4\bar{a}_{111}\bar{a}_{222}\bar{a}_{i11}\bar{a}_{i12} + 12\bar{a}_{211}\bar{a}_{222}\bar{a}_{i11}\bar{a}_{i22} \\
& + 8\bar{a}_{112}\bar{a}_{222}\bar{a}_{i11}\bar{a}_{i12} - 12\bar{a}_{i11}^2\bar{f}_{1111} - 12\bar{a}_{i11}\bar{a}_{i12}\bar{f}_{1112} \\
& - 12\bar{a}_{i11}^4 - 16\bar{a}_{i11}^2\bar{a}_{i12}^2 - 6\bar{a}_{111}\bar{a}_{i11}\bar{f}_{i122} \\
& - 8\bar{a}_{211}^2\bar{a}_{i11}^2 - 8\bar{a}_{212}^2\bar{a}_{i11}^2 - 6\bar{a}_{211}^2\bar{a}_{i11}\bar{f}_{i112} - 6\bar{a}_{211}\bar{a}_{i11}\bar{f}_{i222} \\
& - 12\bar{a}_{i11}\bar{a}_{i12}\bar{f}_{2111} - 12\bar{a}_{i11}\bar{a}_{i22}\bar{f}_{2112} + 8\bar{a}_{211}\bar{a}_{212}\bar{a}_{i11}\bar{a}_{i12}
\end{aligned}
$$

$$- 4\tilde{a}_{i11}^2\tilde{a}_{i22}^2 + 8\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i11}^2 + 8\tilde{a}_{122}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i12}$$

$$+ 8\tilde{a}_{111}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i22} - 12\tilde{a}_{i11}^2(\tilde{f}_{1111} + \tilde{f}_{1122})\Big\} c_2^* \frac{S(\hat{\boldsymbol{\theta}})}{n-p}$$

$$+ \sum_{\substack{i=3\\ i\neq h}}^{n}\sum_{h=3}^{n}\Big\{12\tilde{a}_{i11}^2\tilde{a}_{h11}^2 + 8\tilde{a}_{i11}\tilde{a}_{i12}\tilde{a}_{h11}\tilde{a}_{h12} + 8\tilde{a}_{i11}^2\tilde{a}_{h12}^2$$

$$+ 4\tilde{a}_{i11}\tilde{a}_{i22}\tilde{a}_{h11}\tilde{a}_{h22}\Big\} c_2^* \frac{S(\hat{\boldsymbol{\theta}})}{n-p}$$

(3.2.17)

$$\rho_2^{(e)} = \sum_{i=3}^{n}\tilde{a}_{i22}^2$$

$$- \sum_{i=3}^{n}\Big\{16\tilde{a}_{112}^2\tilde{a}_{i12}^2 + 36\tilde{a}_{122}^2\tilde{a}_{i12}^2 + 16\tilde{a}_{211}^2\tilde{a}_{i12}^2 + 16\tilde{a}_{212}^2\tilde{a}_{i12}^2$$

$$+ 4\tilde{a}_{122}^2\tilde{a}_{i11}^2 + 4\tilde{a}_{222}^2\tilde{a}_{i12}^2 + 96\tilde{a}_{222}^2\tilde{a}_{i22}^2 + 40\tilde{a}_{212}^2\tilde{a}_{i22}^2$$

$$+ 24\tilde{a}_{111}\tilde{a}_{112}\tilde{a}_{i12}\tilde{a}_{i22} + 40\tilde{a}_{111}\tilde{a}_{211}\tilde{a}_{i12}\tilde{a}_{i22} + 12\tilde{a}_{111}\tilde{a}_{122}\tilde{a}_{i11}\tilde{a}_{i22}$$

$$+ 4\tilde{a}_{111}\tilde{a}_{222}\tilde{a}_{i12}\tilde{a}_{i22} + 8\tilde{a}_{111}\tilde{a}_{212}\tilde{a}_{i22}^2 + 48\tilde{a}_{112}\tilde{a}_{122}\tilde{a}_{i12}\tilde{a}_{i22}$$

$$+ 28\tilde{a}_{122}\tilde{a}_{211}\tilde{a}_{i12}\tilde{a}_{i22} + 12\tilde{a}_{122}^2\tilde{a}_{i11}\tilde{a}_{i22} + 144\tilde{a}_{122}\tilde{a}_{222}\tilde{a}_{i12}\tilde{a}_{i22}$$

$$+ 48\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i22}^2 + 72\tilde{a}_{211}\tilde{a}_{212}\tilde{a}_{i12}\tilde{a}_{i22} + 12\tilde{a}_{211}\tilde{a}_{222}\tilde{a}_{i22}^2$$

$$+ 112\tilde{a}_{212}\tilde{a}_{222}\tilde{a}_{i12}\tilde{a}_{i22} + 8\tilde{a}_{211}^2\tilde{a}_{i11}\tilde{a}_{i22} + 8\tilde{a}_{212}^2\tilde{a}_{i11}\tilde{a}_{i22}$$

$$+ 32\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i22} + 48\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i22} + 128\tilde{a}_{112}\tilde{a}_{212}\tilde{a}_{i12}\tilde{a}_{i22}$$

$$+ 24\tilde{a}_{112}^2\tilde{a}_{i11}\tilde{a}_{i22} + 24\tilde{a}_{212}^2\tilde{a}_{i22}^2 - 16\tilde{a}_{i11}\tilde{a}_{i12}^2\tilde{a}_{i22}$$

$$
\begin{aligned}
& - 32\tilde{a}_{i12}^2\tilde{a}_{i22}^2 - 4\tilde{a}_{i11}^2\tilde{a}_{i22}^2 + 8\tilde{a}_{211}^2\tilde{a}_{i22}^2 \\
& + 8\tilde{a}_{112}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i22} + 16\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i22}^2 - 12\tilde{a}_{i22}^4 \\
& + 32\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i12}^2 + 48\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i12}^2 + 16\tilde{a}_{112}\tilde{a}_{122}\tilde{a}_{i11}\tilde{a}_{i12} \\
& + 16\tilde{a}_{112}\tilde{a}_{222}\tilde{a}_{i12}^2 + 16\tilde{a}_{122}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i12} + 16\tilde{a}_{211}\tilde{a}_{222}\tilde{a}_{i12}^2 \\
& + 8\tilde{a}_{122}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i12} + 9(\tilde{f}_{i122}^2 + \tilde{f}_{i222}^2) - 6\tilde{a}_{111}\tilde{a}_{i22}\tilde{f}_{i122} \\
& - 36\tilde{a}_{122}\tilde{a}_{i22}\tilde{f}_{i122} - 6\tilde{a}_{211}\tilde{a}_{i22}\tilde{f}_{i222} - 72\tilde{a}_{222}\tilde{a}_{i22}\tilde{f}_{i222} \\
& - 24\tilde{a}_{211}\tilde{a}_{i22}\tilde{f}_{i112} - 72\tilde{a}_{212}\tilde{a}_{i22}\tilde{f}_{i122} - 12\tilde{a}_{i12}\tilde{a}_{i22}\tilde{f}_{2111} \\
& - 24\tilde{a}_{i12}\tilde{a}_{i22}\tilde{f}_{2122} - 24\tilde{a}_{112}\tilde{a}_{i22}\tilde{f}_{i112} - 12\tilde{a}_{i12}\tilde{a}_{i22}\tilde{f}_{1112} \\
& - 24\tilde{a}_{i12}\tilde{a}_{i22}\tilde{f}_{1222} - 12\tilde{a}_{i11}\tilde{a}_{i22}\tilde{f}_{1122} - 6\tilde{a}_{122}\tilde{a}_{i22}\tilde{f}_{i111} \\
& - 6\tilde{a}_{222}\tilde{a}_{i22}\tilde{f}_{i112} - 24\tilde{a}_{i22}^2\tilde{f}_{2222} - 12\tilde{a}_{i22}^2\tilde{f}_{2112} - 24\tilde{a}_{112}\tilde{a}_{i12}\tilde{f}_{i122} \\
& - 36\tilde{a}_{122}\tilde{a}_{i12}\tilde{f}_{i222} - 24\tilde{a}_{211}\tilde{a}_{i12}\tilde{f}_{i122} - 24\tilde{a}_{212}\tilde{a}_{i12}\tilde{f}_{i222} \\
& \left. - 12\tilde{a}_{122}\tilde{a}_{i11}\tilde{f}_{i122} - 12\tilde{a}_{222}\tilde{a}_{i12}\tilde{f}_{i122} \right\} c_2^* \frac{S(\hat{\boldsymbol{\theta}})}{n-p} \\
+ & \sum_{i=3}^{n} \sum_{\substack{h=3 \\ i \neq h}}^{n} \Big\{ 16\tilde{a}_{i12}\tilde{a}_{i22}\tilde{a}_{h11}\tilde{a}_{h12} + 24\tilde{a}_{i12}\tilde{a}_{i22}\tilde{a}_{h12}\tilde{a}_{h22} \\
& \qquad \left. + 4\tilde{a}_{i11}\tilde{a}_{i22}\tilde{a}_{h11}\tilde{a}_{h22} + 12\tilde{a}_{i22}^2\tilde{a}_{h22}^2 + 8\tilde{a}_{i22}^2\tilde{a}_{h12}^2 \right\} c_2^* \frac{S(\hat{\boldsymbol{\theta}})}{n-p}.
\end{aligned}
$$

(3.2.18)

$$
\begin{aligned}
\rho_3^{(e)} &= \sum_{i=3}^{n} \tilde{a}_{i12}^2 \\
&- \sum_{i=3}^{n} \Big\{ 16\tilde{a}_{112}^2\tilde{a}_{i11}^2 + 4\tilde{a}_{122}^2\tilde{a}_{i11}^2 + 4\tilde{a}_{211}^2\tilde{a}_{i11}^2 + 4\tilde{a}_{212}^2\tilde{a}_{i11}^2 \\
&\quad + 40\tilde{a}_{112}^2\tilde{a}_{i12}^2 + 40\tilde{a}_{212}^2\tilde{a}_{i12}^2 + 4\tilde{a}_{212}^2\tilde{a}_{i22}^2 + 16\tilde{a}_{111}^2\tilde{a}_{i12}^2 \\
&\quad + 16\tilde{a}_{222}^2\tilde{a}_{i12}^2 + 8\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i11}^2 + 8\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i11}^2 \\
&\quad + 40\tilde{a}_{112}\tilde{a}_{122}\tilde{a}_{i11}\tilde{a}_{i12} + 96\tilde{a}_{112}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i12} + 8\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i22} \\
&\quad + 64\tilde{a}_{111}\tilde{a}_{112}\tilde{a}_{i11}\tilde{a}_{i12} + 20\tilde{a}_{122}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i12} + 8\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i11}^2 \\
&\quad + 20\tilde{a}_{211}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i12} + 8\tilde{a}_{212}^2\tilde{a}_{i11}\tilde{a}_{i22} + 44\tilde{a}_{111}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i12} \\
&\quad + 12\tilde{a}_{212}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i12} + 32\tilde{a}_{112}\tilde{a}_{212}\tilde{a}_{i12}\tilde{a}_{i22} + 24\tilde{a}_{112}\tilde{a}_{222}\tilde{a}_{i12}^2 \\
&\quad + 48\tilde{a}_{111}\tilde{a}_{212}\tilde{a}_{i12}^2 + 28\tilde{a}_{212}\tilde{a}_{222}\tilde{a}_{i12}\tilde{a}_{i22} + 8\tilde{a}_{122}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i12} \\
&\quad + 24\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i12}^2 + 4\tilde{a}_{111}\tilde{a}_{122}\tilde{a}_{i12}^2 + 32\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i12}^2 \\
&\quad + 12\tilde{a}_{211}\tilde{a}_{212}\tilde{a}_{i12}\tilde{a}_{i22} + 4\tilde{a}_{211}\tilde{a}_{222}\tilde{a}_{i12}^2 + 8\tilde{a}_{111}\tilde{a}_{122}\tilde{a}_{i12}^2 \\
&\quad + 8\tilde{a}_{211}^2\tilde{a}_{i12}^2 - 16\tilde{a}_{i11}^2\tilde{a}_{i12}^2 - 16\tilde{a}_{i12}^4 \\
&\quad + 8\tilde{a}_{122}\tilde{a}_{211}\tilde{a}_{i12}\tilde{a}_{i22} + 16\tilde{a}_{112}\tilde{a}_{222}\tilde{a}_{i12}^2 - 8\tilde{a}_{i12}^2\tilde{a}_{i22}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 9(\tilde{f}_{i112}^2 + \tilde{f}_{i122}^2) - 24\tilde{a}_{112}\tilde{a}_{i11}\tilde{f}_{i112} - 12\tilde{a}_{122}\tilde{a}_{i11}\tilde{f}_{i122} \\
&- 12\tilde{a}_{211}\tilde{a}_{i11}\tilde{f}_{i112} - 12\tilde{a}_{212}\tilde{a}_{i11}\tilde{f}_{i122} - 30\tilde{a}_{112}\tilde{a}_{i12}\tilde{f}_{i122} \\
&- 54\tilde{a}_{212}\tilde{a}_{i12}\tilde{f}_{i112} - 12\tilde{a}_{212}\tilde{a}_{i22}\tilde{f}_{i122} - 30\tilde{a}_{111}\tilde{a}_{i12}\tilde{f}_{i112} \\
&- 30\tilde{a}_{222}\tilde{a}_{i12}\tilde{f}_{i122} - 18\tilde{a}_{122}\tilde{a}_{i12}\tilde{f}_{i112} - 6\tilde{a}_{211}\tilde{a}_{i12}\tilde{f}_{i122} \\
&- 18\tilde{a}_{112}\tilde{a}_{i12}\tilde{f}_{i111} - 18\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{1112} - 6\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{1222} \\
&- 18\tilde{a}_{i12}^2\tilde{f}_{1122} - 12\tilde{a}_{112}\tilde{a}_{i12}\tilde{f}_{i122} \\
&- 6\tilde{a}_{i12}^2\tilde{f}_{1111} - 6\tilde{a}_{212}\tilde{a}_{i12}\tilde{f}_{i222} - 18\tilde{a}_{i12}^2\tilde{f}_{2112} \\
&- 12\tilde{a}_{i12}\tilde{a}_{i22}\tilde{f}_{2122} - 6\tilde{a}_{i12}^2\tilde{f}_{2222} - 12\tilde{a}_{211}\tilde{a}_{i12}\tilde{f}_{i111} \\
&- 6\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{2111} - 6\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{2122}\Big\} c_2^* \frac{S(\hat{\boldsymbol{\theta}})}{n-p}
\end{aligned}
$$

$$
\begin{aligned}
+ \sum_{i=3}^{n}\sum_{\substack{h=3 \\ i\neq h}}^{n} \Big\{ & 12\tilde{a}_{i11}\tilde{a}_{i12}\tilde{a}_{h11}\tilde{a}_{h12} + 16\tilde{a}_{i12}^2\tilde{a}_{h12}^2 + 4\tilde{a}_{i12}^2\tilde{a}_{h11}^2 \\
& + 4\tilde{a}_{i12}\tilde{a}_{i22}\tilde{a}_{h12}\tilde{a}_{h22} + 4\tilde{a}_{i12}^2\tilde{a}_{h22}^2\Big\} c_2^* \frac{S(\hat{\boldsymbol{\theta}})}{n-p}
\end{aligned}
$$

(3.2.19)

$$
\begin{aligned}
\rho_4^{(e)} = \sum_{i=3}^{n} \tilde{a}_{i11}^2 \tilde{a}_{i22}
\\
- \sum_{i=3}^{n} \Big\{ & 36\tilde{a}_{111}\tilde{a}_{112}\tilde{a}_{i11}\tilde{a}_{i12} + 44\tilde{a}_{111}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i12} + 18\tilde{a}_{111}\tilde{a}_{122}\tilde{a}_{i11}^2 \\
& + 14\tilde{a}_{111}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i12} + 28\tilde{a}_{111}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i22} + 30\tilde{a}_{111}^2\tilde{a}_{i11}\tilde{a}_{i22} \\
& + 16\tilde{a}_{111}\tilde{a}_{211}\tilde{a}_{i12}\tilde{a}_{i22} + 48\tilde{a}_{112}\tilde{a}_{122}\tilde{a}_{i11}\tilde{a}_{i12} + 18\tilde{a}_{122}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i12} \\
& + 6\tilde{a}_{122}^2\tilde{a}_{i11}^2 + 36\tilde{a}_{122}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i12} + 24\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i22} \\
& + 6\tilde{a}_{111}\tilde{a}_{122}\tilde{a}_{i11}\tilde{a}_{i22} + 18\tilde{a}_{122}\tilde{a}_{211}\tilde{a}_{i12}\tilde{a}_{i22} + 20\tilde{a}_{211}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i12} \\
& + 6\tilde{a}_{211}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i22} + 2\tilde{a}_{211}^2\tilde{a}_{i22}^2 + 16\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i11}\tilde{a}_{i22} \\
& + 24\tilde{a}_{212}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i12} + 30\tilde{a}_{222}^2\tilde{a}_{i11}\tilde{a}_{i22} + 14\tilde{a}_{111}\tilde{a}_{222}\tilde{a}_{i12}\tilde{a}_{i22} \\
& + 18\tilde{a}_{211}\tilde{a}_{222}\tilde{a}_{i22}^2 + 28\tilde{a}_{112}\tilde{a}_{222}\tilde{a}_{i11}\tilde{a}_{i22} + 4\tilde{a}_{211}^2\tilde{a}_{i11}^2 \\
& + 4\tilde{a}_{212}^2\tilde{a}_{i11}^2 + 12\tilde{a}_{112}^2\tilde{a}_{i11}^2 + 16\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i11}^2 + 16\tilde{a}_{122}\tilde{a}_{212}\tilde{a}_{i11}^2 \\
& + 64\tilde{a}_{112}\tilde{a}_{212}\tilde{a}_{i11}\tilde{a}_{i12} + 20\tilde{a}_{212}^2\tilde{a}_{i11}\tilde{a}_{i22} - 8\tilde{a}_{i11}^2\tilde{a}_{i12}^2 \\
& - 24\tilde{a}_{i11}\tilde{a}_{i12}^2\tilde{a}_{i22} - 8\tilde{a}_{i11}^3\tilde{a}_{i22} \\
& + 4\tilde{a}_{112}\tilde{a}_{222}\tilde{a}_{i11}^2 - 8\tilde{a}_{i11}\tilde{a}_{i22}^3 + 12\tilde{a}_{111}\tilde{a}_{122}\tilde{a}_{i12}^2 \\
& + 8\tilde{a}_{111}\tilde{a}_{212}\tilde{a}_{i12}^2 + 8\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i12}^2 + 8\tilde{a}_{211}^2\tilde{a}_{i12}^2 \\
& + 20\tilde{a}_{211}\tilde{a}_{212}\tilde{a}_{i12}\tilde{a}_{i22} + 4\tilde{a}_{211}\tilde{a}_{222}\tilde{a}_{i12}^2 \\
& + 12\tilde{a}_{111}\tilde{a}_{112}\tilde{a}_{i12}\tilde{a}_{i22} + 8\tilde{a}_{112}\tilde{a}_{211}\tilde{a}_{i22}^2 \\
& + 12\tilde{a}_{112}^2\tilde{a}_{i11}\tilde{a}_{i22} + 4\tilde{a}_{111}\tilde{a}_{212}\tilde{a}_{i22}^2
\end{aligned}
$$

$$
\begin{aligned}
&- 31\tilde{a}_{111}\tilde{a}_{i11}\tilde{f}_{i122} - 18\tilde{a}_{122}\tilde{a}_{i11}\tilde{f}_{i122} - 18\tilde{a}_{111}\tilde{a}_{i22}\tilde{f}_{i111} \\
&- 3\tilde{a}_{122}\tilde{a}_{i22}\tilde{f}_{i111} - 3\tilde{a}_{211}\tilde{a}_{i11}\tilde{f}_{i222} - 18\tilde{a}_{222}\tilde{a}_{i11}\tilde{f}_{i222} \\
&- 6\tilde{a}_{211}\tilde{a}_{i22}\tilde{f}_{i112} - 21\tilde{a}_{222}\tilde{a}_{i22}\tilde{f}_{i112} - 12\tilde{a}_{211}\tilde{a}_{i11}\tilde{f}_{i112} \\
&- 24\tilde{a}_{212}\tilde{a}_{i11}\tilde{f}_{i122} - 6\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{2111} - 12\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{2122} \\
&- 6\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{1112} - 12\tilde{a}_{i11}\tilde{a}_{i12}\tilde{f}_{1222} - 12\tilde{a}_{112}\tilde{a}_{i11}\tilde{f}_{i112} \\
&- 6\tilde{a}^2_{i11}\tilde{f}_{1122} - 12\tilde{a}_{i11}\tilde{a}_{i22}\tilde{f}_{2222} - 9\tilde{a}_{122}\tilde{a}_{i11}\tilde{f}_{i111} \\
&- 3\tilde{a}_{222}\tilde{a}_{i11}\tilde{f}_{i112} - 6\tilde{a}_{i11}\tilde{a}_{i22}\tilde{f}_{2112} + 9\tilde{f}_{3111}\tilde{f}_{3122} + 9\tilde{f}_{3112}\tilde{f}_{3222} \\
&- 12\tilde{a}_{112}\tilde{a}_{i12}\tilde{f}_{i111} - 18\tilde{a}_{122}\tilde{a}_{i12}\tilde{f}_{i112} - 12\tilde{a}_{211}\tilde{a}_{i12}\tilde{f}_{i111} \\
&- 12\tilde{a}_{212}\tilde{a}_{i12}\tilde{f}_{i112} - 6\tilde{a}_{222}\tilde{a}_{i12}\tilde{f}_{i111} - 12\tilde{a}_{212}\tilde{a}_{i22}\tilde{f}_{i111} \\
&- 16\tilde{a}_{111}\tilde{a}_{i12}\tilde{f}_{i222} - 6\tilde{a}_{211}\tilde{a}_{i12}\tilde{f}_{i122} - 9\tilde{a}_{211}\tilde{a}_{i22}\tilde{f}_{i222} \\
&- 12\tilde{a}_{112}\tilde{a}_{i11}\tilde{f}_{i222} - 12\tilde{a}_{i11}\tilde{a}_{i22}\tilde{f}_{1111} - 6\tilde{a}_{i12}\tilde{a}_{i22}\tilde{f}_{1112} \\
&- 12\tilde{a}_{112}\tilde{a}_{i22}\tilde{f}_{i112} - 3\tilde{a}_{111}\tilde{a}_{i22}\tilde{f}_{i122} - 6\tilde{a}_{i12}\tilde{a}_{i22}\tilde{f}_{2111} \\
&- 6\tilde{a}^2_{i22}\tilde{f}_{2112} - 6\tilde{a}_{i11}\tilde{a}_{i22}\tilde{f}_{1122} \Big\} c_2^* \, \frac{S(\hat{\boldsymbol{\theta}})}{n-p}
\end{aligned}
$$

$$
\begin{aligned}
+ \sum_{i=3}^{n} \sum_{\substack{h=3 \\ i \neq h}}^{n} \Big\{ & 8\tilde{a}_{i11}\tilde{a}_{i12}\tilde{a}_{h11}\tilde{a}_{h12} + 12\tilde{a}_{i11}\tilde{a}_{i12}\tilde{a}_{h12}\tilde{a}_{h22} + 2\tilde{a}^2_{i11}\tilde{a}_{h11}\tilde{a}_{h22} \\
&+ 6\tilde{a}_{i11}\tilde{a}_{i22}\tilde{a}^2_{h22} + 8\tilde{a}_{i11}\tilde{a}_{i22}\tilde{a}^2_{h12} + 6\tilde{a}_{i11}\tilde{a}_{i22}\tilde{a}^2_{h11} \\
&+ 4\tilde{a}_{i12}\tilde{a}_{i22}\tilde{a}_{h11}\tilde{a}_{h12} + 2\tilde{a}^2_{i22}\tilde{a}_{h11}\tilde{a}_{h22} \Big\} c_2^* \, \frac{S(\hat{\boldsymbol{\theta}})}{n-p}
\end{aligned}
$$

The constant c_2^* is to be chosen such that the quartic terms in the coverage probability of the confidence regions after the first stage of adjustment are simplified. Now as

$$\tilde{a}_{ijk} = \hat{a}_{ijk}, \qquad i = 1, 2$$

$$\tilde{f}_{ijk\ell} = \hat{f}_{ijk\ell}, \quad i = 1, 2$$

$$\begin{bmatrix} \tilde{a}_{3jk} \\ \tilde{a}_{4jk} \\ \vdots \\ \tilde{a}_{njk} \end{bmatrix} = \mathbf{H}^* \begin{bmatrix} \hat{a}_{3jk} \\ \hat{a}_{4jk} \\ \vdots \\ \hat{a}_{njk} \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{f}_{3jk\ell} \\ \tilde{f}_{4jk\ell} \\ \vdots \\ \tilde{f}_{njk\ell} \end{bmatrix} = \mathbf{H}^* \begin{bmatrix} \hat{f}_{3jk\ell} \\ \hat{f}_{4jk\ell} \\ \vdots \\ \hat{f}_{njk\ell} \end{bmatrix}$$

for some orthogonal $(n-2) \times (n-2)$ matrix $\mathbf{H}^*$ (see Pooi (*)), we can replace all the $\tilde{a}_{ijk}$ and $\tilde{f}_{ijk\ell}$ appearing in the right sides of (3.2.16)–(3.2.19) by $\hat{a}_{ijk}$ and $\hat{f}_{ijk\ell}$ respectively so that $\rho_i^{(e)}$ $(i = 1, 2, 3, 4)$ can be computed in practice.

3.3 First Stage of Adjustment of Confidence Regions

In the first stage, we adjust the nominally-$100(1-\alpha)\%$ confidence regions to

$$\left\{\boldsymbol{\theta} : S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}) \leq \frac{pF_{\alpha+}}{n-p} S(\hat{\boldsymbol{\theta}})\right\} \tag{3.3.1}$$

where $F_{\alpha+}$ is given by

$$F_{\alpha+} = F_\alpha + \frac{n}{2(n-2)} \frac{\beta_{2.0}}{F(F_\alpha)} \left[\rho_1^{(e)} + \rho_2^{(e)} - 2\rho_3^{(e)} + 4\rho_4^{(e)}\right] c_1(\sigma^2)^{(e)}. \tag{3.3.2}$$

The coverage probability of the regions given by (3.3.1) can be shown to be given by

(3.3.3) $$I^+(\boldsymbol{0}_f, \sigma) = 1 - \alpha + \sigma^2\left\{1 - \frac{c_1\beta_{4.0}}{pF_\alpha^2 F(F_\alpha)}\right\}\delta_1 + \sigma^4\delta_2 + o(\sigma^4)$$

where

$$\delta_1 = -\sum_{i=3}^{n}\left\{\frac{2n}{(n-2)}\beta_{2.0}a_{i12}^2 + \frac{n}{2(n-2)}\beta_{2.0}(a_{i11} - a_{i22})^2\right\} \cdot$$

and δ_2 can be simplified by choosing $c_2^* = \frac{pF_\alpha}{2}$ to

(3.3.4)

$$\begin{aligned}
\delta_2 = \sum_{i=3}^{n} &[(e_1 + f_1)(a_{i11}^4 + a_{i22}^4) + (e_2 + f_2)a_{i12}^4 \\
&+ (e_3 + f_3)a_{i11}^2 a_{i22}^2 + (e_4 + f_4)(a_{i11}a_{i22}^3 + a_{i11}^3 a_{i22}) \\
&+ (e_5 + f_5)(a_{i11}^2 + a_{i22}^2)a_{i12}^2 + (e_6 + f_6)a_{i11}a_{i12}^2 a_{i22}] \\
+ \sum_{i=3}^{n}\sum_{\substack{h=3 \\ i<h}}^{n} &\Big[(e_7 + f_7)(a_{i11}^2 a_{h11}^2 + a_{i22}^2 a_{h22}^2) \\
&+ (e_8 + f_8)a_{i11}a_{i22}a_{h11}a_{h22} \\
&+ (e_9 + f_9)a_{i12}^2 a_{h12}^2 + e_{10}a_{i11}a_{i12}a_{h11}a_{h12}\Big] \\
+ \sum_{i=3}^{n}\sum_{\substack{h=3 \\ i\neq h}}^{n} &\Big[(e_{11} + f_{10})(a_{i11}a_{i22}a_{h11}^2 + a_{i11}a_{i22}a_{h22}^2) \\
&+ (e_{12} + f_{11})(a_{i12}^2 a_{h11}^2 + a_{i12}^2 a_{h22}^2) \\
&+ (e_{13} + f_{12})a_{i12}^2 a_{h11}a_{h22} + e_{14}a_{i12}a_{i22}a_{h11}a_{h12} \\
&+ (e_{15} + f_{13})a_{i11}^2 a_{h22}^2\Big]
\end{aligned}$$

$$+\sum_{i=3}^{n}[(e_{16}(a_{i111}^2+a_{i222}^2)+e_{17}(a_{i122}^2+a_{i112}^2)$$

$$+e_{18}(a_{i111}a_{i122}+a_{i222}a_{i112})+e_{19}(a_{i11}a_{i1111}+a_{122}a_{i2222})$$

$$+e_{20}(a_{i11}a_{i1122}+a_{i22}a_{i1122})+e_{21}(a_{i11}a_{i2222}+a_{i22}a_{i1111})$$

$$+e_{22}(a_{i12}a_{i1112}+a_{i12}a_{i1222})].$$

In (3.3.4), the coefficient e_k has been given in Chapter 2; whereas the coefficient f_k is a sum of a number of terms each of which is given by

$$\frac{\text{constant}\times\left[-\frac{c_1\beta_{2,0,0}}{F(F_\alpha)}\right]^{w_1}\times[c_1^*]^{w_2}\times\left[1+\frac{pF_\alpha}{n-p}\right]^{v_1}\times\beta_{m_1,m_2,m_3}}{[p]^{u_1}[F_\alpha]^{u_2}[n-p]^{u_3}}. \tag{3.3.5}$$

The details of the jth term of the coefficient f_k are given in a coded form in the jth row of Table 3-1-k. In a typical row, the entries are row number, constant, w_1, w_2, u_1, u_2, u_3, v_1, m_1, m_2 and m_3 respectively.

From (3.3.3), we see that if we choose c_1 to be

$$c_1=\frac{pF_\alpha^2F(F_\alpha)}{\beta_{4,0}}=\frac{n-p+pF_\alpha}{n} \tag{3.3.6}$$

then

$$I^+(\boldsymbol{\theta}_f,\sigma)=1-\alpha+\sigma^4\delta_2+o(\sigma^4). \tag{3.3.7}$$

From Tables 3-1-1 to 3-1-13, we can calculate the numerical values of the f_k in (3.3.4) for different values of n and α. The values thus computed are presented in Table 3-2.

Table 3-1-1 Coefficient of ai11**4 and ai22**4

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	5.00000000000000E-0001	1	0	1	2	0	1	4	2	0
2	-7.50000000000000E+0000	1	0	0	1	1	1	4	2	0
3	-7.50000000000000E-0001	1	0	1	2	0	1	6	0	0
4	-7.50000000000000E-0001	1	0	0	1	1	0	6	0	0
5	4.00000000000000E+0000	1	0	0	1	1	0	4	2	0
6	-1.00000000000000E+0000	1	1	2	3	1	0	6	0	0
7	-1.50000000000000E+0000	1	0	-1	0	2	0	4	2	0
8	1.50000000000000E+0000	1	0	1	2	0	2	6	2	0
9	1.50000000000000E+0000	1	0	0	1	1	1	6	2	0
10	-1.87500000000000E-0001	1	0	1	2	0	2	8	2	0
11	-2.50000000000000E-0001	2	0	2	4	0	0	8	0	0
12	1.87500000000000E-0001	1	0	1	2	0	1	8	0	0
13	-1.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-2 Coefficient of ai12**4

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	8.00000000000000E+0000	1	0	1	2	0	1	4	2	0
2	-4.00000000000000E+0001	1	0	0	1	1	1	4	2	0
3	-4.00000000000000E+0000	1	0	1	2	0	1	6	0	0
4	-4.00000000000000E+0000	1	0	0	1	1	0	6	0	0
5	3.20000000000000E+0001	1	0	0	1	1	0	4	2	0
6	-1.60000000000000E+0001	1	1	2	3	1	0	6	0	0
7	-8.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
8	8.00000000000000E+0000	1	0	1	2	0	2	6	2	0
9	8.00000000000000E+0000	1	0	0	1	1	1	6	2	0
10	-1.00000000000000E+0000	1	0	1	2	0	2	8	2	0
11	-4.00000000000000E+0000	2	0	2	4	0	0	8	0	0
12	1.00000000000000E+0000	1	0	1	2	0	1	8	0	0
13	-8.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-3 Coefficient of (ai11**2)(ai22**2)

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	3.00000000000000E+0000	1	0	1	2	0	1	4	2	0
2	-5.00000000000000E+0000	1	0	0	1	1	1	4	2	0
3	-5.00000000000000E-0001	1	0	1	2	0	1	6	0	0
4	-5.00000000000000E-0001	1	0	0	1	1	0	6	0	0
5	8.00000000000000E+0000	1	0	0	1	1	0	4	2	0
6	-6.00000000000000E+0000	1	1	2	3	1	0	6	0	0
7	-1.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
8	1.00000000000000E+0000	1	0	1	2	0	2	6	2	0
9	1.00000000000000E+0000	1	0	0	1	1	1	6	2	0
10	-1.25000000000000E-0001	1	0	1	2	0	2	8	2	0
11	-1.50000000000000E+0000	2	0	2	4	0	0	8	0	0
12	1.25000000000000E-0001	1	0	1	2	0	1	8	0	0
13	-2.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-4 Coefficient of (ai11)(ai22**3) and (ai11**3)(ai22)

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	-2.00000000000000E+0000	1	0	1	2	0	1	4	2	0
2	1.00000000000000E+0001	1	0	0	1	1	1	4	2	0
3	1.00000000000000E+0000	1	0	1	2	0	1	6	0	0
4	1.00000000000000E+0000	1	0	0	1	1	0	6	0	0
5	-8.00000000000000E+0000	1	0	0	1	1	0	4	2	0
6	4.00000000000000E+0000	1	1	2	3	1	0	6	0	0
7	2.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
8	-2.00000000000000E+0000	1	0	1	2	0	2	6	2	0
9	-2.00000000000000E+0000	1	0	0	1	1	1	6	2	0
10	2.50000000000000E-0001	1	0	1	2	0	2	8	2	0
11	1.00000000000000E+0000	2	0	2	4	0	0	8	0	0
12	-2.50000000000000E-0001	1	0	1	2	0	1	8	0	0
13	2.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-5 Coefficient of (ai11**2)(ai12**2) and (ai12**2)(ai22**2)

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	4.00000000000000E+0000	1	0	1	2	0	1	4	2	0
2	-4.00000000000000E+0001	1	0	0	1	1	1	4	2	0
3	-4.00000000000000E+0000	1	0	1	2	0	1	6	0	0
4	-4.00000000000000E+0000	1	0	0	1	1	0	6	0	0
5	2.40000000000000E+0001	1	0	0	1	1	0	4	2	0
6	-8.00000000000000E+0000	1	1	2	3	1	0	6	0	0
7	-8.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
8	8.00000000000000E+0000	1	0	1	2	0	2	6	2	0
9	8.00000000000000E+0000	1	0	0	1	1	1	6	2	0
10	-1.00000000000000E+0000	1	0	1	2	0	2	8	2	0
11	-2.00000000000000E+0000	2	0	2	4	0	0	8	0	0
12	1.00000000000000E+0000	1	0	1	2	0	1	8	0	0
13	-6.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-6 Coefficient of (ai11)(ai12**2)(ai22)

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	-8.00000000000000E+0000	1	0	1	2	0	1	4	2	0
2	-1.60000000000000E+0001	1	0	0	1	1	0	4	2	0
3	1.60000000000000E+0001	1	1	2	3	1	0	6	0	0
4	4.00000000000000E+0000	2	0	2	4	0	0	8	0	0
5	4.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-7 Coefficient of (ai11**2)(ah11**2) and (ai22**2)(ah22**2)

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	5.00000000000000E-0001	1	0	1	2	0	1	4	2	0
2	5.00000000000000E-0001	1	0	1	2	0	1	4	0	2
3	-7.50000000000000E+0000	1	0	0	1	1	1	4	2	0
4	-7.50000000000000E+0000	1	0	0	1	1	1	4	0	2
5	-1.50000000000000E+0000	1	0	1	2	0	1	6	0	0

6	-1.50000000000000E+0000	1	0	0	1	1	0	6	0	0
7	4.00000000000000E+0000	1	0	0	1	1	0	4	2	0
8	4.00000000000000E+0000	1	0	0	1	1	0	4	0	2
9	-2.00000000000000E+0000	1	1	2	3	1	0	6	0	0
10	-1.50000000000000E+0000	1	0	-1	0	2	0	4	0	2
11	-1.50000000000000E+0000	1	0	-1	0	2	0	4	2	0
12	1.50000000000000E+0000	1	0	1	2	0	2	6	0	2
13	1.50000000000000E+0000	1	0	1	2	0	2	6	2	0
14	1.50000000000000E+0000	1	0	0	1	1	1	6	0	2
15	1.50000000000000E+0000	1	0	0	1	1	1	6	2	0
16	-1.87500000000000E-0001	1	0	1	2	0	2	8	0	2
17	-1.87500000000000E-0001	1	0	1	2	0	2	8	2	0
18	-5.00000000000000E-0001	2	0	2	4	0	0	8	0	0
19	3.75000000000000E-0001	1	0	1	2	0	1	8	0	0
20	-1.00000000000000E+0000	1	0	1	2	0	1	6	0	2
21	-1.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-8 Coefficient of (ai11*ai22)(ah11*ah22)

no.,	constant,	w1,	w2,	u1,	u2,	u3,	v1,	m1,	m2,	m3
1	2.00000000000000E+0000	1	0	1	2	0	1	4	0	2
2	2.00000000000000E+0000	1	0	1	2	0	1	4	2	0
3	1.00000000000000E+0001	1	0	0	1	1	1	4	0	2
4	1.00000000000000E+0001	1	0	0	1	1	1	4	2	0
5	2.00000000000000E+0000	1	0	1	2	0	1	6	0	0
6	2.00000000000000E+0000	1	0	0	1	1	0	6	0	0
7	-8.00000000000000E+0000	1	1	2	3	1	0	6	0	0
8	2.00000000000000E+0000	1	0	-1	0	2	0	4	0	2
9	2.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
10	-2.00000000000000E+0000	1	0	1	2	0	2	6	0	2
11	-2.00000000000000E+0000	1	0	1	2	0	2	6	2	0
12	-2.00000000000000E+0000	1	0	0	1	1	1	6	0	2
13	-2.00000000000000E+0000	1	0	0	1	1	1	6	2	0
14	2.50000000000000E-0001	1	0	1	2	0	2	8	0	2
15	2.50000000000000E-0001	1	0	1	2	0	2	8	2	0
16	-2.00000000000000E+0000	2	0	2	4	0	0	8	0	0
17	-5.00000000000000E-0001	1	0	1	2	0	1	8	0	0

Table 3-1-9 Coefficient of (ai12**2)(ah12**2)

no.,	constant,	w1,	w2,	u1,	u2,	u3,	v1,	m1,	m2,	m3
1	8.00000000000000E+0000	1	0	1	2	0	1	4	0	2
2	8.00000000000000E+0000	1	0	1	2	0	1	4	2	0
3	-4.00000000000000E+0001	1	0	0	1	1	1	4	0	2
4	-4.00000000000000E+0001	1	0	0	1	1	1	4	2	0
5	-8.00000000000000E+0000	1	0	1	2	0	1	6	0	0
6	-8.00000000000000E+0000	1	0	0	1	1	0	6	0	0
7	3.20000000000000E+0001	1	0	0	1	1	0	4	0	2
8	3.20000000000000E+0001	1	0	0	1	1	0	4	2	0
9	-3.20000000000000E+0001	1	1	2	3	1	0	6	0	0
10	-8.00000000000000E+0000	1	0	-1	0	2	0	4	0	2
11	-8.00000000000000E+0000	1	0	-1	0	2	0	4	2	0

12	8.00000000000000E+0000	1	0	1	2	0	2	6	0	2
13	8.00000000000000E+0000	1	0	1	2	0	2	6	2	0
14	8.00000000000000E+0000	1	0	0	1	1	1	6	0	2
15	8.00000000000000E+0000	1	0	0	1	1	1	6	2	0
16	-1.00000000000000E+0000	1	0	1	2	0	2	8	0	2
17	-1.00000000000000E+0000	1	0	1	2	0	2	8	2	0
18	2.00000000000000E+0000	1	0	1	2	0	1	8	0	0
19	-8.00000000000000E+0000	1	0	1	2	0	1	6	0	2
20	-8.00000000000000E+0000	1	0	1	2	0	1	6	2	0
21	-8.00000000000000E+0000	2	0	2	4	0	0	8	0	0

Table 3-1-10 Coefficient of (ai11)(ai22)(ah11**2) and (ai11)(ai22)(ah22**2)

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	-1.00000000000000E+0000	1	0	1	2	0	1	4	0	2
2	1.50000000000000E+0001	1	0	0	1	1	1	4	0	2
3	-1.00000000000000E+0000	1	0	1	2	0	1	4	2	0
4	-5.00000000000000E+0000	1	0	0	1	1	1	4	2	0
5	1.00000000000000E+0000	1	0	1	2	0	1	6	0	0
6	1.00000000000000E+0000	1	0	0	1	1	0	6	0	0
7	-8.00000000000000E+0000	1	0	0	1	1	0	4	0	2
8	4.00000000000000E+0000	1	1	2	3	1	0	6	0	0
9	3.00000000000000E+0000	1	0	-1	0	2	0	4	0	2
10	-1.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
11	-3.00000000000000E+0000	1	0	1	2	0	2	6	0	2
12	1.00000000000000E+0000	1	0	1	2	0	2	6	2	0
13	-3.00000000000000E+0000	1	0	0	1	1	1	6	0	2
14	1.00000000000000E+0000	1	0	0	1	1	1	6	2	0
15	3.75000000000000E-0001	1	0	1	2	0	2	8	0	2
16	-1.25000000000000E-0001	1	0	1	2	0	2	8	2	0
17	1.00000000000000E+0000	2	0	2	4	0	0	8	0	0
18	-2.50000000000000E-0001	1	0	1	2	0	1	8	0	0
19	2.00000000000000E+0000	1	0	1	2	0	1	6	0	2

Table 3-1-11 Coefficient of (ai12**2)(ah11**2) and (ai12**2)(ah22**2)

no.	constant	w1	w2	u1	u2	u3	v1	m1	m2	m3
1	2.00000000000000E+0000	1	0	1	2	0	1	4	0	2
2	-3.00000000000000E+0001	1	0	0	1	1	1	4	0	2
3	2.00000000000000E+0000	1	0	1	2	0	1	4	2	0
4	-1.00000000000000E+0001	1	0	0	1	1	1	4	2	0
5	-4.00000000000000E+0000	1	0	1	2	0	1	6	0	0
6	-4.00000000000000E+0000	1	0	0	1	1	0	6	0	0
7	1.60000000000000E+0001	1	0	0	1	1	0	4	0	2
8	8.00000000000000E+0000	1	0	0	1	1	0	4	2	0
9	-8.00000000000000E+0000	1	1	2	3	1	0	6	0	0
10	-2.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
11	-6.00000000000000E+0000	1	0	-1	0	2	0	4	0	2
12	6.00000000000000E+0000	1	0	1	2	0	2	6	0	2
13	2.00000000000000E+0000	1	0	1	2	0	2	6	2	0
14	6.00000000000000E+0000	1	0	0	1	1	1	6	0	2
15	2.00000000000000E+0000	1	0	0	1	1	1	6	2	0

16	-7.50000000000000E-0001	1	0	1	2	0	2	8	0	2
17	-2.50000000000000E-0001	1	0	1	2	0	2	8	2	0
18	-2.00000000000000E+0000	2	0	2	4	0	0	8	0	0
19	1.00000000000000E+0000	1	0	1	2	0	1	8	0	0
20	-2.00000000000000E+0000	1	0	1	2	0	1	6	2	0
21	-4.00000000000000E+0000	1	0	1	2	0	1	6	0	2

Table 3-1-12 Coefficient of (ai12**2)(ah11)(ah22)

no.,	constant,	w1,	w2,	u1,	u2,	u3,	v1,	m1,	m2,	m3
1	-4.00000000000000E+0000	1	0	1	2	0	1	4	0	2
2	-4.00000000000000E+0000	1	0	1	2	0	1	4	2	0
3	-2.00000000000000E+0001	1	0	0	1	1	1	4	0	2
4	2.00000000000000E+0001	1	0	0	1	1	1	4	2	0
5	-1.60000000000000E+0001	1	0	0	1	1	0	4	2	0
6	1.60000000000000E+0001	1	1	2	3	1	0	6	0	0
7	-4.00000000000000E+0000	1	0	-1	0	2	0	4	0	2
8	4.00000000000000E+0000	1	0	-1	0	2	0	4	2	0
9	4.00000000000000E+0000	1	0	1	2	0	2	6	0	2
10	-4.00000000000000E+0000	1	0	1	2	0	2	6	2	0
11	4.00000000000000E+0000	1	0	0	1	1	1	6	0	2
12	-4.00000000000000E+0000	1	0	0	1	1	1	6	2	0
13	-5.00000000000000E-0001	1	0	1	2	0	2	8	0	2
14	5.00000000000000E-0001	1	0	1	2	0	2	8	2	0
15	4.00000000000000E+0000	2	0	2	4	0	0	8	0	0
16	0.00000000000000E+0000	1	0	1	2	0	1	8	0	0
17	4.00000000000000E+0000	1	0	1	2	0	1	6	2	0

Table 3-1-13 Coefficient of (ai11**2)(ah22**2)

no.,	constant,	w1,	w2,	u1,	u2,	u3,	v1,	m1,	m2,	m3
1	5.00000000000000E-0001	1	0	1	2	0	1	4	0	2
2	5.00000000000000E-0001	1	0	1	2	0	1	4	2	0
3	-7.50000000000000E+0000	1	0	0	1	1	1	4	0	2
4	-7.50000000000000E+0000	1	0	0	1	1	1	4	2	0
5	-1.50000000000000E+0000	1	0	1	2	0	1	6	0	0
6	-1.50000000000000E+0000	1	0	0	1	1	0	6	0	0
7	4.00000000000000E+0000	1	0	0	1	1	0	4	0	2
8	4.00000000000000E+0000	1	0	0	1	1	0	4	2	0
9	-2.00000000000000E+0000	1	1	2	3	1	0	6	0	0
10	-1.50000000000000E+0000	1	0	-1	0	2	0	4	0	2
11	-1.50000000000000E+0000	1	0	-1	0	2	0	4	2	0
12	1.50000000000000E+0000	1	0	1	2	0	2	6	0	2
13	1.50000000000000E+0000	1	0	1	2	0	2	6	2	0
14	1.50000000000000E+0000	1	0	0	1	1	1	6	0	2
15	1.50000000000000E+0000	1	0	0	1	1	1	6	2	0
16	-1.87500000000000E-0001	1	0	1	2	0	2	8	0	2
17	-1.87500000000000E-0001	1	0	1	2	0	2	8	2	0
18	-5.00000000000000E-0001	2	0	2	4	0	0	8	0	0
19	3.75000000000000E-0001	1	0	1	2	0	1	8	0	0
20	-1.00000000000000E+0000	1	0	1	2	0	1	6	2	0
21	-1.00000000000000E+0000	1	0	1	2	0	1	6	0	2

Table 3-2 The values of f_k in the quartic approximation of the coverage probability of the adjusted confidence regions

	$n=3$		$n=4$		$n=5$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
f_1	-4.151E-02	-8.332E-03	-6.947E-02	-1.478E-02	-8.461E-02	-1.957E-02
f_2	-6.642E-01	-1.333E-01	-1.112E+00	-2.364E-01	-1.354E+00	-3.132E-01
f_3	-2.491E-01	-4.999E-02	-4.168E-01	-8.865E-02	-5.077E-01	-1.174E-01
f_4	1.660E-01	3.333E-02	2.779E-01	5.910E-02	3.384E-01	7.830E-02
f_5	-3.321E-01	-6.665E-02	-5.557E-01	-1.182E-01	-6.769E-01	-1.566E-01
f_6	6.642E-01	1.333E-01	1.112E+00	2.364E-01	1.354E+00	3.132E-01
f_7	—	—	-1.389E-01	-2.955E-02	-1.692E-01	-3.915E-02
f_8	—	—	-5.558E-01	-1.182E-01	-6.769E-01	-1.566E-01
f_9	—	—	-2.223E+00	-4.728E-01	-2.708E+00	-6.264E-01
f_{10}	—	—	2.779E-01	5.910E-02	3.384E-01	7.830E-02
f_{11}	—	—	-5.558E-01	-1.182E-01	-6.769E-01	-1.566E-01
f_{12}	—	—	1.111E+00	2.364E-01	1.354E+00	3.132E-01
f_{13}	—	—	-1.389E-01	-2.955E-02	-1.692E-01	-3.915E-02

	$n=6$		$n=7$		$n=8$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
f_1	-9.201E-02	-2.280E-02	-9.521E-02	-2.492E-02	-9.670E-02	-2.628E-02
f_2	-1.472E+00	-3.648E-01	-1.523E+00	-3.987E-01	-1.547E+00	-4.204E-01
f_3	-5.521E-01	-1.368E-01	-5.713E-01	-1.495E-01	-5.802E-01	-1.577E-01
f_4	3.681E-01	9.120E-02	3.808E-01	9.968E-02	3.868E-01	1.051E-01
f_5	-7.361E-01	-1.824E-01	-7.617E-01	-1.994E-01	-7.736E-01	-2.102E-01
f_6	1.472E+00	3.648E-01	1.523E+00	3.987E-01	1.547E+00	4.204E-01
f_7	-1.840E-01	-4.560E-02	-1.904E-01	-4.984E-02	-1.934E-01	-5.255E-02
f_8	-7.361E-01	-1.824E-01	-7.617E-01	-1.994E-01	-7.736E-01	-2.102E-01
f_9	-2.944E+00	-7.296E-01	-3.047E+00	-7.974E-01	-3.094E+00	-8.408E-01
f_{10}	3.681E-01	9.120E-02	3.808E-01	9.968E-02	3.868E-01	1.051E-01
f_{11}	-7.361E-01	-1.824E-01	-7.617E-01	-1.994E-01	-7.736E-01	-2.102E-01
f_{12}	1.472E+00	3.648E-01	1.523E+00	3.987E-01	1.547E+00	4.204E-01
f_{13}	-1.840E-01	-4.560E-02	-1.904E-01	-4.984E-02	-1.934E-01	-5.255E-02

	$n=10$		$n=12$		$n=14$	
	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.01$
f_1	-9.657E-02	-2.763E-02	-9.546E-02	-2.812E-02	-9.368E-02	-2.818E-02
f_2	-1.545E+00	-4.420E-01	-1.527E+00	-4.499E-01	-1.499E+00	-4.509E-01
f_3	-5.794E-01	-1.658E-01	-5.727E-01	-1.687E-01	-5.621E-01	-1.691E-01
f_4	3.863E-01	1.105E-01	3.818E-01	1.125E-01	3.747E-01	1.127E-01
f_5	-7.725E-01	-2.210E-01	-7.637E-01	-2.249E-01	-7.494E-01	-2.254E-01
f_6	1.545E+00	4.420E-01	1.527E+00	4.499E-01	1.499E+00	4.509E-01
f_7	-1.931E-01	-5.525E-02	-1.909E-01	-5.623E-02	-1.874E-01	-5.636E-02
f_8	-7.725E-01	-2.210E-01	-7.637E-01	-2.249E-01	-7.494E-01	-2.254E-01
f_9	-3.090E+00	-8.840E-01	-3.055E+00	-8.997E-01	-2.998E+00	-9.018E-01
f_{10}	3.863E-01	1.105E-01	3.818E-01	1.125E-01	3.747E-01	1.127E-01
f_{11}	-7.725E-01	-2.210E-01	-7.637E-01	-2.249E-01	-7.494E-01	-2.254E-01
f_{12}	1.545E+00	4.420E-01	1.527E+00	4.499E-01	1.499E+00	4.509E-01
f_{13}	-1.931E-01	-5.525E-02	-1.909E-01	-5.623E-02	-1.874E-01	-5.636E-02

	$n = 16$		$n = 20$		$n = 24$	
	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$
f_1	-9.237E-02	-2.819E-02	-9.011E-02	-2.780E-02	-8.810E-02	-2.734E-02
f_2	-1.478E+00	-4.510E-01	-1.442E+00	-4.449E-01	-1.410E+00	-4.374E-01
f_3	-5.542E-01	-1.691E-01	-5.406E-01	-1.668E-01	-5.286E-01	-1.640E-01
f_4	3.695E-01	1.127E-01	3.604E-01	1.112E-01	3.524E-01	1.094E-01
f_5	-7.389E-01	-2.255E-01	-7.209E-01	-2.224E-01	-7.048E-01	-2.187E-01
f_6	1.478E+00	4.510E-01	1.442E+00	4.449E-01	1.410E+00	4.374E-01
f_7	-1.847E-01	-5.637E-02	-1.802E-01	-5.561E-02	-1.762E-01	-5.468E-02
f_8	-7.389E-01	-2.255E-01	-7.209E-01	-2.224E-01	-7.048E-01	-2.187E-01
f_9	-2.956E+00	-9.020E-01	-2.883E+00	-8.897E-01	-2.819E+00	-8.749E-01
f_{10}	3.695E-01	1.127E-01	3.604E-01	1.112E-01	3.524E-01	1.094E-01
f_{11}	-7.389E-01	-2.255E-01	-7.209E-01	-2.224E-01	-7.048E-01	-2.187E-01
f_{12}	1.478E+00	4.510E-01	1.442E+00	4.449E-01	1.410E+00	4.374E-01
f_{13}	-1.847E-01	-5.637E-02	-1.802E-01	-5.561E-02	-1.762E-01	-5.468E-02

	$n = 30$		$n = 62$		$n = \infty$	
	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$
f_1	-8.579E-02	-2.684E-02	-8.059E-02	-2.513E-02	-7.509E-02	-2.321E-02
f_2	-1.373E+00	-4.295E-01	-1.289E+00	-4.020E-01	-1.201E+00	-3.713E-01
f_3	-5.147E-01	-1.610E-01	-4.835E-01	-1.508E-01	-4.505E-01	-1.392E-01
f_4	3.432E-01	1.074E-01	3.223E-01	1.005E-01	3.004E-01	9.282E-02
f_5	-6.863E-01	-2.147E-01	-6.447E-01	-2.010E-01	-6.007E-01	-1.856E-01
f_6	1.373E+00	4.295E-01	1.289E+00	4.020E-01	1.201E+00	3.713E-01
f_7	-1.716E-01	-5.368E-02	-1.612E-01	-5.025E-02	-1.502E-01	-4.641E-02
f_8	-6.863E-01	-2.147E-01	-6.447E-01	-2.010E-01	-6.007E-01	-1.856E-01
f_9	-2.745E+00	-8.589E-01	-2.579E+00	-8.041E-01	-2.403E+00	-7.426E-01
f_{10}	3.432E-01	1.074E-01	3.223E-01	1.005E-01	3.004E-01	9.282E-02
f_{11}	-6.863E-01	-2.147E-01	-6.447E-01	-2.010E-01	-6.007E-01	-1.856E-01
f_{12}	1.373E+00	4.295E-01	1.289E+00	4.020E-01	1.201E+00	3.713E-01
f_{13}	-1.716E-01	-5.368E-02	-1.612E-01	-5.025E-02	-1.502E-01	-4.641E-02

3.4 Second Stage of Adjustment of Confidence Regions

In the second stage, we adjust the regions in the first stage (see (3.3.1)) to

$$\left\{\boldsymbol{\theta} : S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}) \leq \frac{pF_{\alpha++}}{n-p} S(\hat{\boldsymbol{\theta}})\right\} \tag{3.4.1}$$

where $F_{\alpha++}$ is given by

$$F_{\alpha++} = F_{\alpha+} - (\delta_2)^{(e)} c_2 \left(\frac{S(\hat{\boldsymbol{\theta}})}{n-p}\right)^2, \tag{3.4.2}$$

c_2 is a constant and $(\delta_2)^{(e)}$ is an estimate of δ_2 obtained by setting the a_{ijk}, $a_{ijk\ell}$ and $a_{ijk\ell m}$ in the right side of (3.3.4) to $\hat{a}_{ijk}$, $\hat{a}_{ijk\ell}$ and $\hat{a}_{ijk\ell m}$ respectively.

The coverage probability of the regions given by (3.4.1) can be shown to be given by

$$I^{++}(\boldsymbol{\theta}_f, \sigma) = 1 - \alpha + \sigma^4\left\{1 - \frac{c_2}{p^2 F_\alpha^3}\beta_{6,0}\right\}\delta_2 + o(\sigma^4). \tag{3.4.3}$$

Therefore, if c_2 is chosen to be

$$c_2 = \frac{p^2 F_\alpha^3}{\beta_{6,0}}$$

then

$$I^{++}(\boldsymbol{\theta}_f, \sigma) = 1 - \alpha + o(\sigma^4).$$