# INTRODUCTION

The main purpose of this thesis is to consider some aspects of the geometry of CR-submanifolds of nearly Kaehler and Kaehler manifolds.

Some basic concepts and well known results on Riemannian geometry are reviewed in the first chapter. For conveniece, we shall also fix notations and terminologies for later use.

In Chapter 2 we shall characterize a CR-submanifold of an almost Hermitian manifold. We also include some basic formulas in CR-submanifolds.

The next chapter is focused on CR-submanifolds of a nearly Kaehler manifold. A survey of some results on the integrability conditions for the distributions D and $D^{\perp}$ as well as the geometry of their leaves is given here. We also generalize some results on D-parallel normal sections and mixed foliate CR-submanifolds of a Kaehler manifold to the setting of nearly Kaehler manifold.

In Chapter 4 we shall be mainly concern with the geometry of CR-submanifolds of a 6-dimensional nearly

Kaehler manifold N. Some facts about nearly Kaehler manifolds of constant type are given in the first section. By using these results, we shall show that certain classes of CR-submanifolds do not exist in a 6-dimensional nearly Kaehler manifold.

In Chapter 5 we shall study some sufficient conditions for a totally umbilical CR-submanifold of a nearly Kaehler manifold to admit a nearly Sasakian structure. This leads us to classify all connected totally umbilical CR-submanifolds of a nearly Kaehler manifold.

In Chapter 6 we shall first characterize a CR-product of a Kaehler manifold. We also have a section on the geometry of normal CR-submanifolds of a Kaehler manifold. The last section of this chapter is a discussion on Sasakian CR-submanifolds of a Kaehler manifold. Here, we obtain some consequences of the results of Sun-Li on Sasakian anti-holomorphic submanifolds of a Kaehler manifold.

$$(\tilde{\nabla}_X K)(X_1 \ldots, X_k) = \tilde{\nabla}_X K(X_1, \ldots, X_k) - \sum_{i=1}^{k} K(X_1, \ldots, \tilde{\nabla}_X X_i, \ldots, X_k)$$

for any $X_i \in \Gamma(TN)$, $i = 1, \ldots, k$.

For $k \geq 0$, we define the *exterior differentiation* map $d : \bigwedge^k(N) \longrightarrow \bigwedge^{k+1}(N)$, where $\bigwedge^k(N)$ is the set of differentiable k-forms on N, in the following manner:

(a) If $f \in \bigwedge^0(N)$ and $X \in \Gamma(TN)$, then $df(X) = Xf$.

(b) For $k > 1$, letting $\omega$ be a $(k-1)$-form on N and $X_1, \ldots, X_k \in \Gamma(TN)$, then

$$d\omega(X_1, \ldots, X_k) = \frac{1}{k} \sum_{j=1}^{k} (-1)^{j+1} X_j \omega(X_1, \ldots, \hat{X}_j, \ldots, X_k)$$

$$+ \frac{1}{k} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, \ldots, X_k)$$

where $\hat{X}$ indicates that the field X is omitted as an argument and $[X,Y]$ is the Lie bracket of the vector fields X and Y.

Next, we shall define the *curvature tensor* $\tilde{R}$ of type (1,3) by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z$$

for any $X, Y, Z \in \Gamma(TN)$.

It can be easily verified that $\tilde{R}$ satisfies the following properties: (Hicks [17], p.72)

$$\langle\tilde{R}(X,Y)Z,W\rangle = -\langle\tilde{R}(Y,X)Z,W\rangle$$

$$\langle\tilde{R}(X,Y)Z,W\rangle = -\langle\tilde{R}(X,Y)W,Z\rangle$$

$$\langle\tilde{R}(X,Y)Z,W\rangle = \langle\tilde{R}(Z,W)X,Y\rangle$$

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$$

for any $X, Y, Z, W \in \Gamma(TN)$.

The *Ricci tensor field* is defined by

$$\widetilde{Ric}(X,Y) = \sum_{i=1}^{n} \langle\tilde{R}(E_i,X)Y,E_i\rangle$$

where $\{E_1,\dots,E_n\}$ is a local field of orthonormal frames on N and $X, Y \in \Gamma(TN)$.

A Riemannian manifold N is called an *Einstein space* if

$$\widetilde{Ric}(X,Y) = \lambda\langle X,Y\rangle, \qquad \text{for any } X,\ Y \in \Gamma(TN)$$

where $\lambda$ is a constant.

For each plane $\gamma$ spanned by orthogonal unit vectors X and Y in the tangent space $T_xN$, $x \in N$, the *sectional curvature* $\tilde{K}(\gamma)$ is defined by

$$\tilde{K}(\gamma) = \tilde{K}(X,Y) = \langle\tilde{R}(X,Y)Y,X\rangle.$$

We note that $\tilde{K}(\gamma)$ is independent of the choice of the orthonormal basis {X, Y} of $\gamma$. If $\tilde{K}(\gamma)$ is a constant for all planes $\gamma$ and for all points $x \in N$, then N is called a *space of constant curvature.*

The following proposition is known.

Proposition 1.1 (Kobayashi-Nomizu [18], Vol.I, p.293)

Every 3-dimensional Einstein space is a space of constant curvature.

## 1.2 Submanifolds of a Riemannian Manifold

Let N be an n-dimensional Riemannian manifold and let M be an m-dimensional manifold isometrically immersed in N. Denote by $\langle\ ,\ \rangle$ both the Riemannian metric of N and M. Let $\tilde{\nabla}$ and $\nabla$ be the Levi-Civita connections on N and M respectively. Then the *Gauss* and *Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$
$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$$

for any X, Y $\in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$, where $T^\perp M$ denotes the normal bundle of M in N, $\nabla^\perp$ the linear connection on the normal bundle $T^\perp M$, called the *normal connection* and h the *second fundamental form* of M. The linear operator $A_\xi$ is called the *fundamental tensor of Weingarten* with respect to the normal section $\xi$, and is related to h by

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle.$$

The covariant derivative of h is defined by

$$(\nabla_X h)(Y,Z) = \nabla^\perp_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for all X, Y, Z $\in \Gamma(TM)$.

The equations of *Gauss* and *Codazzi* are then given respectively by

$$\langle \tilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \langle h(X,Z), h(Y,W)\rangle - \langle h(Y,Z), h(X,W)\rangle$$

$$(\tilde{R}(X,Y)Z)^\perp = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)$$

for all X, Y, Z, W $\in \Gamma(TM)$, where $U^\perp$ denotes the normal part of the vector U in $T_x N$, $x \in M$ and R the curvature tensor of M.

We define the curvature tensor of the normal connection $\nabla^\perp$ by

$$R^\perp(X,Y)\xi = \nabla^\perp_X \nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_X \xi - \nabla^\perp_{[X,Y]}\xi$$

for all X, Y, $\in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$. For $\xi$, $\zeta \in \Gamma(T^\perp M)$ we define

$$[A_\xi, A_\zeta] = A_\xi \circ A_\zeta - A_\zeta \circ A_\xi$$

and by using the Gauss and Weingarten formulas, we have

$$\langle \tilde{R}(X,Y)\xi,\zeta\rangle = \langle R^{\perp}(X,Y)\xi,\zeta\rangle + \langle [A_{\zeta},A_{\xi}]X,Y\rangle$$

for any X, Y $\in \Gamma(TM)$ and $\xi$, $\zeta \in \Gamma(T^{\perp}M)$. The above equation is called the *Ricci equation*.

If $R^{\perp} = 0$, we say that the normal connection on M is *flat*. A normal vector field $\xi$ on M is said to be *parallel* if $\nabla^{\perp}_{X}\xi = 0$ for any X $\in \Gamma(TM)$. The following proposition gives a necessary and sufficient condition for a submanifold to have a flat normal connection (see Chen [8], p.99).

Proposition 1.2

Let M be an m-dimensional submanifold of an n-dimensional Riemannian manifold N. Then the normal connection $\nabla^{\perp}$ of M in N is flat if and only if there exist locally n − m mutually orthogonal unit normal vector fields $\xi_i$ such that each of the $\xi_i$ is parallel in the normal bundle.

A submanifold M is *totally geodesic* if its second fundamental form vanishes identically, that is $h = 0$ or for any $\xi \in \Gamma(T^{\perp}M)$ we have $A_{\xi} = 0$.

Let $\{E_1, \ldots, E_m\}$ be an orthonormal basis in $T_xM$. The *mean curvature vector* $H$ of $M$ is defined by

$$H = \frac{1}{m} \mathrm{Tr}(h)$$

where $\mathrm{Tr}(h) = \sum_{i=1}^{m} h(E_i, E_i)$, which is independent of the choice of basis. We say that $M$ is a *totally umbilical submanifold* if

$$h(X,Y) = \langle X,Y\rangle H$$

for any $X$, $Y \in \Gamma(TM)$. We note that $M$ is totally umbilical if and only if

$$A_{\xi}X = \langle H,\xi\rangle X$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$.

Finally, $M$ is called an *extrinsic sphere* if it is totally umbilical and has non-zero parallel mean curvature vector.

## 1.3 Distributions on a Manifold

A k-dimensional *distribution* on a manifold $N$ is a mapping D that assigns to each $x \in N$ a k-dimensional vector subspace $D_x$ of $T_xN$. We say that D is differentiable if for each $x \in N$ there are k independent differentiable vector fields $X_1,\dots,X_k$ which span $D_y$ for all y in some neighborhood of x. A vector field X is said to belong to D if $X_x \in D_x$ for all $x \in N$. We denote this by $X \in \Gamma(D)$. A distribution D is said to be *involutive* if for X, $Y \in \Gamma(D)$, we have $[X,Y] \in \Gamma(D)$.

A submanifold M of N is an *integral manifold* of D if $T_xM = D_x$ for all $x \in M$. If there exists no integral manifold of D which properly contains M, then M is called a *maximal integral manifold* or *leaf* of D. A distribution D is said to be *integrable* if for every $x \in N$, there exists an integral manifold of D containing x. It is well known that a distribution is integrable if and only if it is involutive. The following is the classical theorem of Frobenius (see [10], p.94).

Theorem 1.1

Let D be an involutive distribution on a manifold N.

Through each point $x \in N$, there passess a unique maximal integral manifold of D. Any integral manifold through x is an open submanifold of the maximal one.

## 1.4 Almost Hermitian Manifolds

Let N be a real differentiable manifold. An *almost complex structure* on N is a tensor field J of type (1,1) which is at every point $x \in N$, an endomorphism of $T_xN$ such that $J^2 = -I$. A manifold N with a fixed almost complex structure is called an *almost complex manifold.* Every almost complex manifold is of even dimension and is orientable (see Kobayashi-Nomizu [18], Vol.II, p.121).

Next we define the *torsion* (or *Nijenhuis*) tensor field of type (1,2) of an almost complex structure J by

$$[J,J](X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$$

for all $X, Y \in \Gamma(TN)$. If the torsion tensor of J vanishes identically on N, then J is called a *complex structure* and N is called a *complex manifold.*

A *Hermitian metric* on an almost complex manifold N is a Riemannian metric $\langle \ , \ \rangle$ such that

$$\langle JX, JY\rangle = \langle X, Y\rangle$$

for any X, $Y \in \Gamma(TN)$. An almost complex manifold with a Hermitian metric is called an *almost Hermitian manifold*. It is easy to show that every almost complex manifold with a Riemannian metric g admits a Hermitian metric $\langle\ ,\ \rangle$.

The *holomorphic bisectional curvature* of an almost Hermitian manifold N is defined for any pair of unit vectors X and Y on N by

$$\tilde{H}_B(X,Y) = \langle\tilde{R}(X,JX)JY,Y\rangle.$$

The fundamental 2-form $\Omega$ of an almost Hermitian manifold N is defined by

$$\Omega(X,Y) = \langle X,JY\rangle, \qquad \text{for any } X,\ Y \in \Gamma(TN).$$

Then we have

$$3d\Omega(X,Y,Z) = \langle(\tilde{\nabla}_X J)Z,Y\rangle + \langle(\tilde{\nabla}_Y J)X,Z\rangle + \langle(\tilde{\nabla}_Z J)Y,X\rangle$$

$$(\tilde{\nabla}_X J)JY + J(\tilde{\nabla}_X J)Y = 0 \tag{1.1}$$

$$\langle(\tilde{\nabla}_X J)Y,Z\rangle + \langle(\tilde{\nabla}_X J)Z,Y\rangle = 0 \tag{1.2}$$

and

$$\langle(\tilde{\nabla}_X J)Y,Y\rangle = \langle(\tilde{\nabla}_X J)JY,Y\rangle = 0 \tag{1.3}$$

for all X, Y, $Z \in \Gamma(TN)$.

An almost Hermitian manifold N is said to be a *Kaehler* *manifold* if $(\tilde{\nabla}_X J)Y = 0$ for any X, Y $\in \Gamma(TN)$. N is called a *nearly Kaehler manifold* if for any X, Y $\in \Gamma(TN)$ we have

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0$$

or equivalently, $$(\tilde{\nabla}_X J)X = 0.$$

Thus, we can see that every Kaehler manifold is nearly Kaehlerian but the converse is not true in general, for instance, the six-dimensional sphere $S^6$ is a non-Kaehlerian, nearly Kaehler manifold (see Gray [15]).

Before we end this section, we give some well known results on nearly Kaehler manifold (see Yano-Kon [31]).

Proposition 1.3

Let N be a nearly Kaehler manifold. Then we have the following

(i) $[J,J](X,Y) = -4J(\tilde{\nabla}_X J)Y$

(ii) $\langle\tilde{R}(X,Y)Z,W\rangle = \langle\tilde{R}(X,Y)JZ,JW\rangle - \langle(\tilde{\nabla}_X J)Y,(\tilde{\nabla}_Z J)W\rangle$

(iii) $\langle\tilde{R}(X,Y)Z,W\rangle = \langle\tilde{R}(JX,JY)JZ,JW\rangle$

for any X, Y, Z, W $\in \Gamma(TN)$.

## 1.5 Almost Contact Metric Manifolds

Let N be a real (2n+1)-dimensional manifold and $\phi$, $\xi$, $\eta$ be a tensor field of type (1,1), a vector field and a 1-form respectively on N satisfying

$$\eta(\xi) = 1; \qquad \phi^2 X = -X + \eta(X)\xi$$

for any $X \in \Gamma(TN)$. Then N is called an *almost contact manifold* and $(\phi, \xi, \eta)$ the *almost contact structure* on N.

Now, suppose there is given a Riemannian metric $\langle\ ,\ \rangle$ on N such that

$$\langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TN)$. Then N is said to have an *almost contact metric structure* $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ and is called an *almost contact metric manifold*. The Riemannian metric mentioned above is called an *associated metric* with respect to the almost contact structure $(\phi, \xi, \eta)$.

An almost contact metric manifold N is said to be a *nearly Sasakian manifold* if

$$(\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = \eta(Y)X + \eta(X)Y - 2\langle X, Y \rangle \xi$$

for any $X, Y \in \Gamma(TN)$. It is known that in a nearly Sasakian manifold the vector field $\xi$ is killing, that is

$$\langle \tilde{\nabla}_X \xi, Y\rangle + \langle X, \tilde{\nabla}_Y \xi\rangle = 0$$

for any X, $Y \in \Gamma(TN)$. Moreover, N is said to be a *Sasakian manifold* if

$$(\tilde{\nabla}_X \phi)Y = \eta(Y)X - \langle X,Y\rangle \xi$$

for any X, $Y \in \Gamma(TN)$.

We close this section with the following known result (see Okumura [23]; Yamaguchi, Nemoto and Kawabata [29]).

<u>Theorem 1.2</u>

Let N be a Riemannian manifold. If N admits a killing vector field $\xi$ of constant length satisfying

$$\lambda^2(\tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_{\tilde{\nabla}_X Y}\xi) = \langle Y,\xi\rangle X - \langle X,Y\rangle \xi$$

for a non-zero constant $\lambda$ and any X, $Y \in \Gamma(TN)$, then N is homothetic to a Sasakian manifold.

<u>Remark:</u> The definition of Sasakian manifold in some literatures differs from the definition of this thesis by a sign. However, it does not effect the result in the preceeding theorem.