CHAPTER 2

CR-SUBMANIFOLDS

The concept of CR-submanifolds of an almost Hermitian manifold is introduced in Section 1. In the next section we are able to improve on Bejancu's characterization theorem for a CR-submanifold in an almost Hermitian manifold by dropping one of his conditions. Some basic formulas in CR-submanifolds are given in the last section.

2.1 Basic Definitions

Let N be an n-dimensional almost Hermitian manifold and let M be an m-dimensional manifold isometrically immersed in N. M is called a *complex (holomorphic) submanifold* of N if T_xM is invariant by J, that is

$$J(T_xM) = T_xM, \qquad \text{for each } x \in M.$$

Also, M is said to be a *totally real (anti-invariant) submanifold* of N if

$$J(T_xM) \subseteq T_x^{\perp}M, \qquad \text{for each } x \in M.$$

There are many results in the theory of holomorphic and totally real submanifolds. For instance, results on the geometry of totally real submanifolds can be found in Yano-Kon [30] while a survey on the geometry of holomorphic submanifolds can be found in Ogiue [22]. In [2], Bejancu generalized the above two classes of submanifold to a new class of submanifolds, which is situated between the above two classes, called the CR-submanifolds.

<u>Definition</u>

M is said to be a *CR-submanifold* of N if there exists a differentiable distribution

$$D : x \longrightarrow D_x \subseteq T_x M$$

on M that satisfies the following conditions:

(i) D is holomorphic, that is,

$$J(D_x) = D_x , \qquad \text{for each } x \in M .$$

(ii) the complementary orthogonal distribution

$$D^{\perp} : x \longrightarrow D^{\perp}_x \subseteq T^{\perp}_x M$$

is anti-invariant, that is,

$$J(D^{\perp}) \subseteq T^{\perp}_x M , \qquad \text{for each } x \in M.$$

In the sequel, we put dim D = 2p and dim $D^{\perp}$ = q. If p =

0 then M becomes a totally real submanifold and when $q = 0$, M becomes a holomorphic submanifold. If $\dim T_x^{\perp}M = q$, the CR-submanifold M is called an *anti-holomorphic submanifold*. A *proper CR-submanifold* is a CR-submanifold which is neither a holomorphic submanifold nor a totally real submanifold.

Finally, a CR-submanifold is said to be *mixed geodesic* if

$$h(X,Z) = 0, \quad \text{for any } X \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}).$$

The following lemma characterizes a mixed geodesic CR-submanifold of an almost Hermitian manifold.

<u>Lemma 2.1</u>

Let M be a CR-submanifold of an almost Hermitian manifold N. Then the following statements are equivalent:

(i) M is mixed geodesic

(ii) $A_{\xi}X \in \Gamma(D)$

(iii) $A_{\xi}Z \in \Gamma(D^{\perp})$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$ and $\xi \in \Gamma(T^{\perp}M)$.

Proof :

First, we shall suppose that (i) holds. Then

$$\langle A_\xi X, Z\rangle = \langle h(X,Z),\xi\rangle = 0$$

for $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$ and $\xi \in \Gamma(T^\perp M)$. Hence, $A_\xi X \in \Gamma(D)$ and so (i) implies (ii).

We shall now show that (ii) implies (iii). For any $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$ and $\xi \in \Gamma(T^\perp M)$,

$$\langle A_\xi Z, X\rangle = \langle A_\xi X, Z\rangle = 0.$$

Hence, $A_\xi Z \in \Gamma(D^\perp)$.

Suppose that (iii) holds. For any $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$ and $\xi \in \Gamma(T^\perp M)$, we have

$$\langle h(X,Z),\xi\rangle = \langle A_\xi Z, X\rangle = 0.$$

This shows that M is mixed geodesic and the proof is completed.■

<u>Remark</u>: A similar characterization for mixed geodesic CR-submanifold of a Kaehler manifold can be found in Bejancu, Kon and Yano [6].

2.2 Characterization of a CR-submanifold

Let M be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold N.

For each $X \in \Gamma(TM)$, we put

$$JX = \phi X + \omega X \qquad (2.1)$$

where ϕX is the tangential part and ωX is the normal part of JX. Similarly, for each $\eta \in \Gamma(T^{\perp}M)$, we put

$$J\eta = B\eta + C\eta \qquad (2.2)$$

where $B\eta$ is the tangential part and $C\eta$ is the normal part of $J\eta$.

By applying J to (2.1) and using (2.1), also (2.2), we have

$$-X = \phi^2 X + \omega\phi X + B\omega X + C\omega X.$$

By comparing the tangential and normal parts of the above equation, we have

$$-I - B\circ\omega = \phi^2 \qquad (2.3)$$

$$\omega\circ\phi + C\circ\omega = 0. \qquad (2.4)$$

Similarly, by applying J to (2.2) and using (2.1), also (2.2) again we have

$$-\eta = \phi B\eta + \omega B\eta + BC\eta + C^2\eta.$$

By comparing the tangential and normal parts of the above equation, we have

$$-I - \omega \circ B = C^2 \qquad (2.5)$$

$$\phi \circ B + B \circ C = 0. \qquad (2.6)$$

We shall first recall two characterization theorems of a CR-submanifold, which can be found in Bejancu [4].

Theorem 2.1 (Bejancu [4], p.21)

The submanifold M of N is a CR-submanifold if and only if

$$\text{rank}(\phi) = \text{constant}$$

and

$$\omega \circ \phi = 0.$$

Theorem 2.2 (Bejancu [4], p.22)

The submanifold M of N is a CR-submanifold if and only if

$$\text{rank}(B) = \text{constant}$$

and

$$\phi \circ B = 0.$$

The following proposition is found in Ng [21], it is an improvement of Theorem 2.1.

<u>Proposition 2.1</u> (Ng [21], p.27)

The submanifold M of N is a CR-submanifold if and only if

$$\omega \circ \phi = 0. \tag{2.7}$$

Proof:

If M is a CR-submanifold, then (2.7) is clearly true by Theorem 2.1. Conversely, suppose $\omega \circ \phi = 0$. We now prove that the rank of ϕ is a constant and therefore, by Theorem 2.1, M is a CR-submanifold.

Since $\omega \circ \phi = 0$, (2.4) becomes $C \circ \omega = 0$. For any $X \in \Gamma(TM)$ and $\eta \in \Gamma(T^{\perp}M)$, we have

$$\begin{aligned} \langle \omega X, \eta \rangle + \langle X, B\eta \rangle &= \langle JX - \phi X, \eta \rangle + \langle X, J\eta - C\eta \rangle \\ &= \langle JX, \eta \rangle + \langle X, J\eta \rangle = 0 \end{aligned}$$

Thus, we have

$$\langle X, B\eta \rangle = -\langle \omega X, \eta \rangle$$

By replacing η by $C\eta$ in the above equation, it becomes

$$\begin{aligned} \langle X, BC\eta \rangle &= -\langle \omega X, C\eta \rangle \\ &= -\langle \omega X, J\eta - B\eta \rangle \\ &= -\langle \omega X, J\eta \rangle + \langle \omega X, B\eta \rangle \\ &= \langle J\omega X, \eta \rangle \end{aligned}$$

$$= \langle B\omega X + C\omega X, \eta \rangle$$

$$= 0, \qquad \text{since } C \circ \omega = 0.$$

Hence, $B \circ C = 0$. Together with (2.6) we have

$$\phi \circ B = 0.$$

From (2.3), $$\phi^2 = -I - B \circ \omega$$

$$\phi^3 = -\phi - \phi \circ (B \circ \omega)$$

$$= -\phi, \qquad \text{since } \phi \circ B = 0.$$

Therefore, $$\phi^3 + \phi = 0.$$

By a result of Stong [26], the rank of ϕ is a constant and by Theorem 2.1, M is a CR-submanifold.■

As an easy consequence of Proposition 2.1 and (2.4) we have the following proposition.

Proposition 2.2 (Ng [21], p.30)

The submanifold M of N is a CR-submanifold if and only if $$C \circ \omega = 0.$$

Now, by using Proposition 2.1, we can improve on Theorem 2.2.

Proposition 2.3

The submanifold M of N is a CR-submanifold if and only if

$$\phi \circ B = 0. \qquad (2.8)$$

Proof:

Suppose M is a CR-submanifold. Then by using Theorem 2.2 we obtain that $\phi \circ B = 0$.

For any $X \in \Gamma(TM)$ and $\eta \in \Gamma(T^{\perp}M)$, we have

$$\begin{aligned} \langle \eta, \omega\phi X\rangle &= \langle \eta, J\phi X\rangle - \langle \eta, \phi^2 X\rangle \\ &= -\langle J\eta, \phi X\rangle \\ &= -\langle B\eta, \phi X\rangle - \langle C\eta, \phi X\rangle \\ &= -\langle B\eta, JX - \omega X\rangle \\ &= -\langle B\eta, JX\rangle \\ &= \langle JB\eta, X\rangle \\ &= \langle \phi B\eta, X\rangle + \langle \omega B\eta, X\rangle \\ &= \langle \phi B\eta, X\rangle = 0, \quad \text{by (2.8)}. \end{aligned}$$

Thus, $\omega \circ \phi = 0$. According to Proposition 2.1, M is a CR-submanifold.■

From Proposition 2.3 and (2.6), we obtain the following corollary.

Corollary 2.1

The submanifold M of N is a CR-submanifold if and only if

$$B \circ C = 0.$$

2.3 Some Basic Formulas in CR-submanifolds

Throughout this section, let N be an almost Hermitian manifold and let M be a CR-submanifold of N. Denote by P and Q the projections of TM to D and $D^{\perp}$ respectively. Thus, from (2.1) we have

$$J(PX + QX) = \phi X + \omega X$$

$$JPX + JQX = \phi X + \omega X.$$

Since $PX \in \Gamma(D)$ and $QX \in \Gamma(D^{\perp})$, $JPX \in \Gamma(D)$ and $JQX \in \Gamma(T^{\perp}M)$. Thus by comparing the tangential and normal parts of the above equation, we have

$$JPX = \phi X \qquad (2.9)$$

$$JQX = \omega X. \qquad (2.10)$$

By applying ϕ on (2.9), we obtain

$$\phi(\phi X) = \phi(JPX) = JP(JPX) = J(JPX) = -PX.$$

Hence, we have

$$\phi^2 = -P.$$

Also, by applying J on (2.10), we obtain

$$J(JQX) = J\omega X$$

$$-QX = B\omega X + C\omega X.$$

From Proposition 2.2, we have $C\circ\omega = 0$. Thus, we obtain

$$B\circ\omega = -Q. \qquad (2.11)$$

Now, from (2.9) we get that

$$\mathrm{Im}\ \phi_x \subseteq D_x, \qquad \text{for each } x \in M.$$

On the other hand, since $P = -\phi^2$ we have $D_x \subseteq \mathrm{Im}\ \phi_x$. Accordingly, we conclude that

$$D_x = \mathrm{Im}\ \phi_x, \qquad \text{for each } x \in M.$$

Moreover, from (2.11) we obtain that $D_x^{\perp} \subseteq \mathrm{Im}\ B_x$. Conversely, by using (2.1) and taking into account (2.8), we obtain that $\mathrm{Im}\ B_x$ is orthogonal to D_x. In other words, $\mathrm{Im}\ B_x \subseteq D_x^{\perp}$. Hence, we have

$$\mathrm{Im}\ B_x = D_x^{\perp}, \qquad \text{for each } x \in M.$$

Next, we let ν be the complementary orthogonal subbundle of $JD^{\perp}$ in $T^{\perp}M$ and denoted by t and f the projections of $T^{\perp}M$ on $JD^{\perp}$ and ν respectively.

Now we shall show that ν is invariant by J, that is $J(\nu_x) = \nu_x$, for $x \in M$.

For any $\xi \in \Gamma(\nu)$ and $Z \in \Gamma(D^{\perp})$,

$$0 = \langle J\xi, Z\rangle = \langle B\xi + C\xi, Z\rangle = \langle B\xi, Z\rangle.$$

Thus, $B\xi = 0$ and so $J\xi = C\xi \in \Gamma(T^{\perp}M)$ for any $\xi \in \Gamma(\nu)$. We note that, for any $\xi \in \Gamma(\nu)$ and $Z \in \Gamma(D^{\perp})$

$$\langle C\xi, JZ\rangle = \langle J\xi, JZ\rangle = \langle \xi, Z\rangle = 0.$$

That is $C\xi \in \Gamma(\nu)$. Therefore we have

$$J(\nu_x) \subseteq \nu_x, \qquad \text{for each } x \in M.$$

Since $J\xi = C\xi \in \Gamma(\nu)$, we have $\xi = -JC\xi \in \Gamma(J\nu)$. It follows that $\nu_x \subseteq J(\nu_x)$ for each $x \in M$. Hence $J(\nu_x) = \nu_x$.

From (2.2), we have

$$J(t\eta + f\eta) = B\eta + C\eta$$

$$Jt\eta + Jf\eta = B\eta + C\eta.$$

By comparing the tangential and normal parts, we have

$$Jt\eta = B\eta \tag{2.12}$$

$$Jf\eta = C\eta. \tag{2.13}$$

By applying J to (2.12), we obtain

$$-t\eta = \phi B\eta + \omega B\eta.$$

By using Proposition 2.3, we obtain that

$$\omega B = -t. \tag{2.14}$$

Similarly, after applying J on (2.13), we have

$$-f\eta = BC\eta + C^2\eta.$$

By using Corollary 2.1, we obtain that

$$C^2 = -f. \tag{2.15}$$

We summarize our observations in the following proposition.

<u>Proposition 2.4</u>

(i) $D_x = \mathrm{Im}\,\phi_x$

(ii) $D_x^{\perp} = \mathrm{Im}\,B_x$

(iii) $J(\nu_x) = \nu_x$

(iv) $\phi^2 = -P$

(v) $B \circ \omega = -Q$

(vi) $\omega \circ B = -t$

(vii) $C^2 = -f$.

Let Z be any vector field tangent to N, we denote by $Z^{\top}$ and $Z^{\perp}$ its tangential part and normal part to M respectively.

For any U, V $\in \Gamma(TM)$ and taking account of the Gauss and Weingarten equations, we have

$$\begin{aligned}(\tilde{\nabla}_U J)V &= \tilde{\nabla}_U JV - J\tilde{\nabla}_U V \\ &= \tilde{\nabla}_U \phi V + \tilde{\nabla}_U \omega V - J(\nabla_U V + h(U,V))\end{aligned}$$

$$= \nabla_U \phi V + h(U,\phi V) - A_{\omega V} U + \nabla^{\perp}_U \omega V - J\nabla_U V - Jh(U,V)$$

$$= P\nabla_U \phi V + Q\nabla_U \phi V + t h(U,\phi V) + f h(U,\phi V) - PA_{\omega V} U$$

$$- QA_{\omega V} U + t\nabla^{\perp}_U \omega V + f\nabla^{\perp}_U \omega V - \phi\nabla_U V - \omega\nabla_U V - Bh(U,V)$$

$$- Ch(U,V).$$

By comparing the tangential and normal parts, we obtain the following equations:-

$$P(\tilde{\nabla}_U J)V^{\top} = P\nabla_U \phi V - \phi\nabla_U V - PA_{\omega V} U \qquad (2.16)$$

$$Q(\tilde{\nabla}_U J)V^{\top} = Q\nabla_U \phi V - QA_{\omega V} U - Bh(U,V) \qquad (2.17)$$

$$t(\tilde{\nabla}_U J)V^{\perp} = t h(U,\phi V) + t\nabla^{\perp}_U \omega V - \omega\nabla_U V \qquad (2.18)$$

$$f(\tilde{\nabla}_U J)V^{\perp} = f h(U,\phi V) + f\nabla^{\perp}_U \omega V - Ch(U,V). \qquad (2.19)$$

Similarly, for any $U \in \Gamma(TM)$, $\eta \in \Gamma(T^{\perp}M)$ and taking account of the Gauss and Weingarten equations, we have

$$(\tilde{\nabla}_U J)\eta = \tilde{\nabla}_U J\eta - J\tilde{\nabla}_U \eta$$

$$= \tilde{\nabla}_U B\eta + \tilde{\nabla}_U C\eta - J(\nabla^{\perp}_U \eta - A_{\eta} U)$$

$$= \nabla_U B\eta + h(U,B\eta) - A_{C\eta} U + \nabla^{\perp}_U C\eta - J\nabla^{\perp}_U \eta + JA_{\eta} U$$

$$= P\nabla_U B\eta + Q\nabla_U B\eta + t h(U,B\eta) + f h(U,B\eta) - PA_{C\eta} U$$

$$- QA_{C\eta} U + t\nabla^{\perp}_U C\eta + f\nabla^{\perp}_U C\eta - B\nabla^{\perp}_U \eta - C\nabla^{\perp}_U \eta + \phi A_{\eta} U$$

$$+ \omega A_{\eta} U .$$

By comparing the tangential and normal parts, we obtain the following:-

$$P(\tilde{\nabla}_U J)\eta^{\top} = P\nabla_U B\eta - PA_{C\eta}U + \phi A_{\eta}U \tag{2.20}$$

$$Q(\tilde{\nabla}_U J)\eta^{\top} = Q\nabla_U B\eta - QA_{C\eta}U - B\nabla^{\perp}_U\eta \tag{2.21}$$

$$t(\tilde{\nabla}_U J)\eta^{\perp} = th(U,B\eta) + t\nabla^{\perp}_U C\eta + \omega A_{\eta}U \tag{2.22}$$

$$f(\tilde{\nabla}_U J)\eta^{\perp} = fh(U,B\eta) + f\nabla^{\perp}_U C\eta - C\nabla^{\perp}_U\eta . \tag{2.23}$$