# CHAPTER 3

# CR-SUBMANIFOLDS OF A NEARLY KAEHLER MANIFOLD

The results of this chapter are generalized from the works of Bejancu and Chen. Bejancu [3] proved that under certain conditions, the normal subbundle $\nu$ of a CR-submanifold of a Kaehler manifold does not admit any D-parallel section. Chen [7] showed that if M is a mixed foliate CR-submanifold of a Kaehler manifold N then for any unit vector fields $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$,

$$\tilde{H}_B(X,Z) = -2\|A_{JZ}X\|^2.$$

Using this fact, some results on mixed foliate CR-submanifolds are obtained.

We will first discuss some integrability conditions of the distributions D and $D^{\perp}$, and some geometrical properties of their leaves in the first section. In Section 2 we extend the result of Bejancu to nearly Kaehler manifold. In the last section we will proved that one of Chen's result is also true when N is a nearly Kaehler manifold.

## 3.1 Integrability of Distributions on a CR-submanifold of a Nearly Kaehler Manifold

In this section we will discuss the integrability condition for the distributions D and $D^{\perp}$ as well as the geometry of their leaves. We first start with the following theorem.

Theorem 3.1 (Bejancu [4], p.27)

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distribution D is integrable if and only if

$$h(X,JY) = h(JX,Y)$$

and

$$(\tilde{\nabla}_X J)Y \in \Gamma(D)$$

for any X, Y $\in \Gamma(D)$.

Kon-Tan [19] proved a necessary and sufficient condition for D to be integrable and its leaves to be totally geodesic in M.

Theorem 3.2 (Kon-Tan [19])

Let M be a CR-submanifold of a nearly Kaehler manifold

N. Then the distribution D is integrable and its leaves are totally geodesic in M if and only if

$$h(X,JY) = Jh(X,Y) \tag{3.1}$$

and

$$(\tilde{\nabla}_X J)Y \in \Gamma(D)$$

for any X, Y $\in \Gamma(D)$.

We have the following simple result on CR-submanifolds of a nearly Kaehler manifold.

<u>Lemma 3.1</u>

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then we have

(i) $$\langle(\tilde{\nabla}_X J)Y,\xi\rangle = 0 \tag{3.2}$$

(ii) $$\langle h(X,JY) - Jh(X,Y),\xi\rangle = 0 \tag{3.3}$$

for any X, Y $\in \Gamma(D)$ and $\xi \in \Gamma(\nu)$.

Proof:

For X, Y $\in \Gamma(D)$, we have JX , JY $\in \Gamma(D)$ since D is holomorphic, and so all [X,Y], [JX,JY], [X,JY] and [JX,Y] are in $\Gamma(TM)$. Hence

$$\langle[J,J](X,Y),\xi\rangle = \langle[JX,JY] - [X,Y] - J[JX,Y] - J[X,JY],\xi\rangle$$

$$= \langle [JX,Y],J\xi \rangle + \langle [X,JY],J\xi \rangle$$

$$= 0, \qquad \text{for any } \xi \in \Gamma(\nu).$$

By using Proposition 1.3, we have

$$4\langle (\tilde{\nabla}_X J)Y,\xi \rangle = \langle 4J(\tilde{\nabla}_X J)Y, J\xi \rangle$$

$$= -\langle [J,J](X,Y), J\xi \rangle$$

$$= 0$$

for any X, Y $\in \Gamma(D)$ and $\xi \in \Gamma(\nu)$. Moreover, from (3.2) and taking account of (2.19) we obtain

$$\langle h(X,JY) - Jh(X,Y),\xi \rangle = \langle fh(X,\phi Y) - Ch(X,Y) + f\nabla^{\perp}_X \omega Y,\xi \rangle$$

$$= \langle (\tilde{\nabla}_X J)Y,\xi \rangle$$

$$= 0$$

for any X, Y $\in \Gamma(D)$ and $\xi \in \Gamma(\nu)$.■

The following corollary is derived from Theorem 3.2 and Lemma 3.1.

<u>Corollary 3.1</u>

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distribution D is integrable and its leaves are totally geodesic in M if and only if

$$(\tilde{\nabla}_X J)Y \in \Gamma(D)$$

and $$Bh(X,Y) = 0 \tag{3.4}$$

for any $X$, $Y \in \Gamma(D)$.

Proof:

It suffices to show that the equation (3.1) and (3.4) are equivalent. Suppose (3.1) is satisfied. Then by taking the tangential part of (3.1) we obtain (3.4).

Now suppose that $Bh(X,Y) = 0$ for all $X$, $Y \in \Gamma(D)$. After applying $\omega$ to the above equation, from Proposition 2.4 we obtain

$$-th(X,Y) = \omega Bh(X,Y) = 0$$

which means $h(X,Y) \in \Gamma(\nu)$ for any $X$, $Y \in \Gamma(D)$. It follows that both $h(X,JY)$ and $Jh(X,Y)$ are in $\Gamma(\nu)$ since $\nu$ is invariant by $J$. Hence, by Lemma 3.1 we obtain

$$h(X,JY) - Jh(X,Y) = 0 \ , \quad \text{for any } X,\ Y \in \Gamma(D).$$

Thus, the corollary is proved.■

The integrability of the distribution $D^{\perp}$ of a CR-submanifold is characterized in the following theorems.

Theorem 3.3 (Bejancu [4], p.28)

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distrubution $D^{\perp}$ is integrable if and only if

$$\langle(\tilde{\nabla}_Z J)W,X\rangle = 0$$

for $X \in \Gamma(D)$ and $W,Z \in \Gamma(D^{\perp})$.

Theoren 3.4 (Bejancu [4], p.28)

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distribution $D^{\perp}$ is integrable if and only if

$$\langle h(Z,X),JW\rangle = \langle h(W,X),JZ\rangle$$

for any $X \in \Gamma(D)$ and $Z$, $W \in \Gamma(D^{\perp})$ .

Theorem 3.5 (Bejancu [4], p.28)

Let M be a CR-submanifold of a nearly Kaehler manifold N. If $D^{\perp}$ is integrable then each leaf of $D^{\perp}$ is immersed in M as a totally geodesic submanifold if and only if

$$\langle h(Z,X),JW\rangle = 0 \qquad (3.5)$$

for any $X \in \Gamma(D)$ and $Z,W \in \Gamma(D^{\perp})$.

We note that if equation (3.5) is satisfied, then $D^{\perp}$ is integrable follows by Theorem 3.4. Together with Theorem 3.5 we have the following analog of a result of Chen ([7], Lemma 3.5).

Corollary 3.2

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distribution $D^{\perp}$ is integrable and its leaves are totally geodesic in M if and only if

$$\langle h(Z,X),JW\rangle = 0$$

for any $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$.

As an immediate consequence we have the following corollary.

Corollary 3.3

Let M be an anti-holomorphic submanifold of a nearly Kaehler manifold. Then M is mixed geodesic if and only if $D^{\perp}$ is integrable and its leaves are totally geodesic in M.

## 3.2 D-parallel Normal Sections on a CR-submanifold

The main purpose of this section is to obtain a generalization of a result of Bejancu [3] on CR-submanifold of a Kaehler manifold to the setting of nearly Keahler manifold. We begin with the following definition.

Definition

A normal section $\xi$ ($\neq 0$) is said to be *D-parallel* if $\nabla^{\perp}_X\xi = 0$, for each $X \in \Gamma(D)$.

We now give some simple results that will be needed later.

Lemma 3.2

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then

$$\phi A_\xi X = PA_{J\xi}X$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(\nu)$.

Proof:

For any $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(\nu)$,

$$\langle \phi A_\xi X - PA_{J\xi}X, Y\rangle = \langle JA_\xi X, Y\rangle - \langle A_{J\xi}X, Y\rangle$$

$$= -\langle A_\xi X, JY\rangle - \langle h(X,Y), J\xi\rangle$$

$$= -\langle h(X,JY),\xi\rangle + \langle Jh(X,Y),\xi\rangle$$

$$= -\langle h(X,JY) - Jh(X,Y),\xi\rangle$$

$$= 0 , \qquad \text{by Lemmn 3.1.}$$

Therefore, $\phi A_\xi X = PA_{J\xi} X.$■

<u>Lemma 3.3</u>

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then

$$\phi A_\xi X = -PA_\xi JX$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(\nu)$.

Proof:

For any X, $Y \in \Gamma(D)$, $\xi \in \Gamma(\nu)$ and by using Lemma 3.1

$$\langle \phi A_\xi X + PA_\xi JX, Y\rangle = \langle JA_\xi X, Y\rangle + \langle A_\xi JX, Y\rangle$$

$$= -\langle A_\xi X, JY\rangle + \langle h(Y,JX),\xi\rangle$$

$$= -\langle h(X,JY),\xi\rangle + \langle h(Y,JX),\xi\rangle$$

$$= -\langle Jh(Y,X),\xi\rangle + \langle Jh(Y,X),\xi\rangle$$

$$= 0.$$

Therefore, $\phi A_\xi X = -PA_\xi JX.$■

Lemma 3.4

Let M be a CR-submanifold of a nearly Kaehler manifold N. Suppose that $\xi \in \Gamma(\nu)$ is a D-parallel normal section then

$$QA_{\xi}JX = -2QA_{J\xi}X$$

for any $X \in \Gamma(D)$.

Proof:

For any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, since N is nearly Kaehler we have

$$\begin{aligned} \langle(\tilde{\nabla}_Z J)X,\xi\rangle &= -\langle(\tilde{\nabla}_X J)Z,\xi\rangle \\ &= \langle(\tilde{\nabla}_X J)\xi,Z\rangle. \end{aligned}$$

By using the fact that $\xi$ is D-parallel and equation (2.19) and (2.21), we obtain

$$\begin{aligned} \langle fh(Z,\phi X),\xi\rangle - \langle Ch(Z,X),\xi\rangle &= -\langle QA_{C\xi}X,Z\rangle \\ \langle h(Z,JX),\xi\rangle - \langle Jh(Z,X),\xi\rangle &= -\langle QA_{J\xi}X,Z\rangle \\ \langle A_{\xi}JX,Z\rangle + \langle h(Z,X),J\xi\rangle &= -\langle QA_{J\xi}X,Z\rangle \\ \langle A_{\xi}JX,Z\rangle + \langle QA_{J\xi}X,Z\rangle &= -\langle QA_{J\xi}X,Z\rangle \end{aligned}$$

or,

$$\langle QA_{\xi}JX,Z\rangle = -2\langle QA_{J\xi}X,Z\rangle.$$

Thus,

$$QA_{\xi}JX = -2QA_{J\xi}X. \blacksquare$$

From Lemma 3.2, Lemma 3.3 and Lemma 3.4, we are able to prove the following theorem.

<u>Theorem 3.6</u>

Let M be a CR-submanifold of a nearly Kaehler manifold N. Suppose the distribution D is integrable. If there exists a unit vector field $X \in \Gamma(D)$ such that for all normal sections $\xi \in \Gamma(\nu)$, the holomorphic bisectional curvatures $\tilde{H}_B(X,\xi)$ are positive, then the normal subbundle $\nu$ does not admit D-parallel normal section.

Proof:

Suppose $\xi$ is a D-parallel normal section in $\nu$. For any $X \in \Gamma(D)$ we have

$$R^{\perp}(X,JX)\xi = \nabla^{\perp}_X\nabla^{\perp}_{JX}\xi - \nabla^{\perp}_{JX}\nabla^{\perp}_X\xi - \nabla^{\perp}_{[X,JX]}\xi = 0.$$

From the Ricci equation,

$$\begin{aligned} \langle\tilde{R}(X,JX)\xi, J\xi\rangle &= \langle R^{\perp}(X,JX)\xi, J\xi\rangle + \langle[A_{J\xi}, A_{\xi}]X, JX\rangle \\ &= \langle[A_{J\xi}, A_{\xi}]X, JX\rangle \\ &= \langle A_{J\xi}A_{\xi}X, JX\rangle - \langle A_{\xi}A_{J\xi}X, JX\rangle \\ &= \langle A_{\xi}X, A_{J\xi}JX\rangle - \langle A_{J\xi}X, A_{\xi}JX\rangle \\ &= \langle PA_{\xi}X, PA_{J\xi}JX\rangle + \langle QA_{\xi}X, QA_{J\xi}JX\rangle \\ &\quad - \langle PA_{J\xi}X, PA_{\xi}JX\rangle - \langle QA_{J\xi}X, QA_{\xi}JX\rangle. \quad (3.6) \end{aligned}$$

By using Lemma 3.2 and Lemma 3.3, we have

$$
\begin{aligned}
\langle PA_\xi X, PA_{J\xi} JX\rangle &- \langle PA_{J\xi} X, PA_\xi JX\rangle \\
&= \langle PA_\xi X, \phi A_\xi JX\rangle - \langle PA_{J\xi} X, -\phi A_\xi X\rangle \\
&= \langle PA_\xi X, -PA_\xi J^2 X\rangle - \langle PA_{J\xi} X, -PA_{J\xi} X\rangle \\
&= \|PA_\xi X\|^2 + \|PA_{J\xi} X\|^2. \qquad (3.7)
\end{aligned}
$$

Also, by using Lemma 3.4 we obtain

$$
\begin{aligned}
\langle QA_\xi X, QA_{J\xi} JX\rangle &- \langle QA_{J\xi} X, QA_\xi JX\rangle \\
&= \langle QA_\xi X, -\frac{1}{2} QA_\xi J^2 X\rangle - \langle -\frac{1}{2} QA_\xi JX, QA_\xi JX\rangle \\
&= \frac{1}{2}\{\|QA_\xi X\|^2 + \|QA_\xi JX\|^2\}. \qquad (3.8)
\end{aligned}
$$

By substituting (3.7) and (3.8) into (3.6), we obtain

$$
\begin{aligned}
\langle \tilde{R}(X,JX)\xi, J\xi\rangle &= \|PA_\xi X\|^2 + \|PA_{J\xi} X\|^2 \\
&\quad + \frac{1}{2}\{\|QA_\xi X\|^2 + \|QA_\xi JX\|^2\} \\
&\geq 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{H}_B(X,\xi) &= \langle \tilde{R}(X,JX)J\xi, \xi\rangle \\
&= -\langle \tilde{R}(X,JX)\xi, J\xi\rangle \leq 0.
\end{aligned}
$$

But this contradicts the hypothesis. Thus, the theorem is proved.■

Remark: When M is mixed geodesic and N is a Kaehler manifold, then Theorem 3.6 becomes [3, Theorem 2.1].

## 3.3 Mixed Foliate CR-submanifolds of a Nearly Kaehler Manifold

The results of this section are straightfoward generalizations from the case of mixed foliate CR-submanifold of a Kaehler manifold to that of a nearly Kaehler manifold. The definition of a mixed foliate CR-submanifold is given by

Definition

A CR-submanifold of an almost Hermitian manifold is *mixed foliate* if it is mixed geodesic and D is integrable.

Firstly, we have the following simple result.

Lemma 3.5

Let M be a CR-submanifold of a nearly Keahler manifold N. If D is integrable then

$$P\nabla_X Z = \phi A_{JZ} X$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof:

For any $X$, $Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, since $D$ is integrable we have $(\tilde{\nabla}_X J)Y \in \Gamma(D)$ from Theorem 3.1. Thus we have

$$\langle(\tilde{\nabla}_X J)Z, Y\rangle = -\langle(\tilde{\nabla}_X J)Y, Z\rangle = 0$$

$$\langle P\nabla_X \phi Z - PA_{\omega Z}X - \phi\nabla_X Z, Y\rangle = 0 \qquad \text{by (2.16)}$$

$$-\langle A_{JZ}X, Y\rangle - \langle J\nabla_X Z, Y\rangle = 0$$

$$-\langle JA_{JZ}X, JY\rangle + \langle \nabla_X Z, JY\rangle = 0$$

$$-\langle \phi A_{JZ}X, JY\rangle + \langle P\nabla_X Z, JY\rangle = 0$$

$$\langle P\nabla_X Z - \phi A_{JZ}X, JY\rangle = 0.$$

Therefore, $$P\nabla_X Z = \phi A_{JZ} X. \blacksquare$$

Next, we will generalize a result on mixed foliate CR-submanifold (see [6] Lemma 3).

<u>Lemma 3.6</u>

Let M be a mixed foliate CR-submanifold of a nearly Kaehler manifold N. Then we have

$$A_\xi \phi U + \phi A_\xi U = 0$$

for any $U \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.

Proof:

From the hypothesis and by using Theorem 3.1, we have

$$h(X,\phi Y) = h(\phi X,Y)$$

and
$$h(X,Z) = 0$$

for any $X,\ Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$.

Moreover, we know that $PU \in \Gamma(D)$ and $QU \in \Gamma(D^\perp)$ for any vector field $U \in \Gamma(TM)$. Consequently, for any $U,\ V \in \Gamma(TM)$ we have

$$\begin{aligned} h(U,\phi V) &= h(PU,\phi V) + h(QU,\phi V) \\ &= h(PU,\phi PV) \\ &= h(\phi PU,PV) \\ &= h(\phi U,PV) \\ &= h(\phi U,V). \end{aligned}$$

Thus, for any $U,\ V \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$, we have

$$\begin{aligned} \langle A_\xi \phi U + \phi A_\xi U,V\rangle &= \langle A_\xi \phi U,V\rangle - \langle A_\xi U,\phi V\rangle \\ &= \langle h(\phi U,V),\xi\rangle - \langle h(U,\phi V),\xi\rangle \\ &= 0. \end{aligned}$$

Accordingly, $$A_\xi \phi U + \phi A_\xi U = 0.\blacksquare$$

The folowing lemma will also be needed for our investigation. This result generalizes [7, Lemma 9.1].

Lemma 3.7

Let M be a mixed foliate CR-submanifold of a nearly Kaehler manifold N. Then for any unit vector fields $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we have

$$\tilde{H}_B(X,Z) = -2\|A_{JZ}X\|^2.$$

Proof:

Consider unit vector fields $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. From the Codazzi equation

$$\begin{aligned}\tilde{R}(X,JX)Z^{\perp} &= (\nabla_X h)(JX,Z) - (\nabla_{JX} h)(X,Z)\\ &= \nabla^{\perp}_X h(JX,Z) - h(\nabla_X JX,Z) - h(JX,\nabla_X Z)\\ &\quad - \nabla^{\perp}_{JX} h(X,Z) + h(\nabla_{JX} X,Z) + h(X,\nabla_{JX} Z)\\ &= h(\nabla_{JX} X - \nabla_X JX,Z) - h(JX,\nabla_X Z) + h(X,\nabla_{JX} Z)\\ &= h([JX,X],Z) - h(JX,\nabla_X Z) + h(X,\nabla_{JX} Z). \quad (3.9)\end{aligned}$$

Since M is mixed geodesic and D is integrable, we have

$$h([JX,X],Z) = h(JX,Q\nabla_X Z) = h(X,Q\nabla_{JX} Z) = 0.$$

Thus, equation (3.9) becomes

$$\tilde{R}(X,JX)Z^{\perp} = -h(JX,P\nabla_X Z) + h(X,P\nabla_{JX}Z)$$

$$= -h(JX,\phi A_{JZ}X) + h(X,\phi A_{JZ}JX) \quad \text{by Lemma 3.5}$$

$$= -h(JX,\phi A_{JZ}X) + h(X,-A_{JZ}\phi JX) \quad \text{by Lemma 3.6}$$

$$= -h(JX,\phi A_{JZ}X) + h(X,A_{JZ}X). \qquad (3.10)$$

By using the fact that D is integrable and Theorem 3.1, we have

$$h(JX,\phi A_{JZ}X) = h(X,J\phi A_{JZ}X)$$

$$= -h(X,PA_{JZ}X)$$

$$= -h(X,A_{JZ}X).$$

Hence, equation (3.10) becomes

$$\tilde{R}(X,JX)Z^{\perp} = h(X,A_{JZ}X) + h(X,A_{JZ}X)$$

$$= 2h(X,A_{JZ}X).$$

Therefore,

$$\tilde{H}_B(X,Z) = \langle\tilde{R}(X,JX)JZ,Z\rangle$$

$$= -\langle\tilde{R}(X,JX)Z,JZ\rangle$$

$$= -2\langle h(X,A_{JZ}X),JZ\rangle$$

$$= -2\|A_{JZ}X\|^2. \blacksquare$$

The following theorem can now be obtained easily and generalizes [7, Theorem 9.2].

Theorem 3.7

Let N be a nearly Kaehler manifold with $\tilde{H}_B > 0$. Then N admits no mixed foliate proper CR-submanifold.

Proof:

Suppose that such a CR-submanifold M does exists. By Lemma 3.7 we have

$$\tilde{H}_B(X,Z) = -2\|A_{JZ}X\|^2 \leq 0$$

for any unit vector fields $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. But this is a contradiction and thus the theorem is proved.∎