CHAPTER 4
CR-SUBMANIFOLDS OF A SIX-DIMENSIONAL NEARLY KAehler MANIFOLD

It is well known that the six-dimensional sphere $S^6$ is a nearly Kaehler manifold which is not Kaehlerian (see [13], [15]). Gray [12] has shown that $S^6$ does not admit a 4-dimensional holomorphic submanifold and Deshmukh-Ghazal [11] proved that there exists no 4-dimensional CR-submanifold with integrable distribution $\mathcal{D}$ of $S^6$. The natural question arises as to whether these results are valid for arbitrary non-Kaehlerian, 6-dimensional nearly Kaehler manifold.

The main purpose of this chapter is to show that certain classes of CR-submanifolds do not exist in a nearly Kaehler manifold of dimension six, which is not Kaehlerian.

Section 1 contains a review of some results on nearly Kaehler manifold. In Section 2 we first give an affirmative answer for the above question. We also proved that a 6-dimensional non-Kaehlerian, nearly Kaehler manifold has no 3-dimensional mixed geodesic proper CR-submanifold and real
hypersurface with integrable holomorphic distribution $D$.

4.1 Some Results on Nearly Kaehler Manifolds

Let $N$ be an almost Hermitian manifold. Then $N$ is said to be of constant type at $x \in N$ if it satisfies the following condition

$$\| (\tilde{\nabla}_x J) Y \| = \| (\tilde{\nabla}_x J) Z \|$$

for $X, Y, Z \in T_x N$ with $\langle X, Y \rangle = \langle X, Z \rangle = \langle X, J Y \rangle = \langle X, J Z \rangle = 0$ and $\| Y \| = \| Z \|$.

We say that $N$ is of (pointwise) constant type if it is of constant type at each point $x \in N$. Finally, if $N$ is of pointwise constant type and $\| (\tilde{\nabla}_x J) Y \|$ is constant, for any unit vector fields $X, Y \in \Gamma(TN)$ with $\langle X, Y \rangle = \langle X, J Y \rangle = 0$, then we say that $N$ is of global constant type.

We have the following simple characterization of a nearly Kaehler manifold of constant type.

**Proposition 4.1** (Gray [14])

Let $N$ be a nearly Kaehler manifold. Then $N$ is of pointwise constant type if and only if there exists $\alpha \in \mathcal{F}(N)$

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such that
\[
\| (\tilde{\nabla}_X J) Y \| ^2 = \alpha ( \| X \| ^2 \| Y \| ^2 - \langle X, Y \rangle ^2 - \langle X, JY \rangle ^2 )
\] (4.1)
for all \( X, Y \in \Gamma (TN) \). Furthermore, \( N \) is of global constant type if and only if (4.1) holds with a constant function \( \alpha \).

A nearly Kaehler manifold \( N \) is said to be strict if for all \( x \in N \) and non-zero vector \( X \in T_x N \) we have \( (\tilde{\nabla}_X J) \neq 0 \).

The following theorems are found in Gray [16].

**Theorem 4.1**

Let \( N \) be a nearly Kaehler manifold with \( \text{dim } N = 6 \). Assume that \( N \) is not Kaehlerian. Then it is of global constant type, that is
\[
\| (\tilde{\nabla}_X J) Y \| ^2 = \alpha ( \| X \| ^2 \| Y \| ^2 - \langle X, Y \rangle ^2 - \langle X, JY \rangle ^2 )
\]
where \( X, Y \in \Gamma (TN) \) and \( \alpha \) is a positive constant.

**4.2 CR-submanifolds of a 6-dimensional Nearly Kaehler Manifold**

We begin this section with the following lemmas.
Lemma 4.1

Let $M$ be a CR-submanifold of a nearly Kaehler manifold $N$. Then we have the following

(i) if $\dim \nu = 2$ then $(\tilde{\nabla}_\xi J)\eta = 0$

(ii) if $\dim D = 2$ then $(\tilde{\nabla}_X J)Y = 0$

for any $\xi$, $\eta \in \Gamma(\nu)$ and $X$, $Y \in \Gamma(D)$.

Proof:

Suppose that $\dim \nu = 2$. Then for any non-zero normal vector field $\xi \in \Gamma(\nu)$, $\{\xi, J\xi\}$ is a local frame on $\nu$. Hence, a normal vector field $\eta \in \Gamma(\nu)$ can be written as $a\xi + bJ\xi$, where $a$, $b \in \mathcal{F}(M)$. Therefore

$$(\tilde{\nabla}_\xi J)\eta = (\tilde{\nabla}_\xi J)(a\xi + bJ\xi)$$

$$= a(\tilde{\nabla}_\xi J)\xi - bJ(\tilde{\nabla}_\xi J)\xi$$

$$= 0.$$

Similarly, if $\dim D = 2$, then for any $X$, $Y \in \Gamma(D)$ we have

$$(\tilde{\nabla}_X J)Y = 0.$$  

Lemma 4.2

Let $M$ be a CR-submanifold of a nearly Kaehler manifold $N$. Then we have the following

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(i) if $\dim D = 1$ then $(\nabla_Z J)W = 0$
(ii) if $\dim D = 2$ and $D$ is integrable then
$$ (\nabla_Z J)W \in \Gamma(D) $$
for any $Z, W \in \Gamma(D)$.

Proof:
Suppose that $\dim D = 1$. Then for any non-zero vector fields $Z, W \in \Gamma(D)$, we have $W = \alpha Z$ where $\alpha \in \mathcal{F}(M)$. Therefore
$$ (\nabla_Z J)W = \alpha (\nabla_Z J)Z = 0. $$

Now, suppose $D$ is integrable and it is of dimension 2. We observe that if $\{Z, W\}$ is a local frame on $D$, then $\{JZ, JW\}$ is a local frame on $JD$. Since $N$ is nearly Kaehler, by using (1.2) and (1.3), we have
$$ \langle (\nabla_Z J)W, Z \rangle = -\langle (\nabla_Z J)Z, W \rangle = 0 $$
$$ \langle (\nabla_Z J)W, W \rangle = 0 $$
$$ \langle (\nabla_Z J)W, JZ \rangle = -\langle (\nabla_Z J)Z, JZ \rangle = 0 $$
and
$$ \langle (\nabla_Z J)W, JW \rangle = 0. $$

Also, from Theorem 3.3 we have
$$ \langle (\nabla_Z J)W, X \rangle = 0, \quad \text{for any } X \in \Gamma(D). $$

Therefore,
$$ (\nabla_Z J)W \in \Gamma(D). $$
Lemma 4.3

Let $M$ be a CR-submanifold of a nearly Kaehler manifold $N$. If $\dim D = 4$ then

$$(\nabla_{U}J)V \perp D, \quad \text{for any } U, V \in \Gamma(D).$$

Moreover, if $D$ is integrable then $(\nabla_{U}J)V = 0$.

Proof:

Suppose $\dim D = 4$ and let $\{X, JX, Y, JY\}$ be any local frame on $D$. To prove the first part, since $N$ is nearly Kaehler, it suffices to show that $(\tilde{\nabla}_{X}J)Y \perp D$. We can see that

$$
\langle (\tilde{\nabla}_{X}J)Y, X \rangle = -\langle (\tilde{\nabla}_{Y}J)X, X \rangle = 0
$$

$$
\langle (\tilde{\nabla}_{X}J)Y, JX \rangle = -\langle (\tilde{\nabla}_{Y}J)X, JX \rangle = 0
$$

and

$$
\langle (\tilde{\nabla}_{X}J)Y, Y \rangle = \langle (\tilde{\nabla}_{X}J)Y, JY \rangle = 0.
$$

Hence, we obtain $(\tilde{\nabla}_{X}J)Y \perp D$.

Now, if $D$ is integrable then from Theorem 3.1, we have

$$(\tilde{\nabla}_{X}J)Y \in \Gamma(D), \quad \text{for any } X, Y \in \Gamma(D).$$

Since $(\tilde{\nabla}_{X}J)Y \perp D$, we have $(\tilde{\nabla}_{X}J)Y = 0$. 

We are now ready to prove the following proposition.
Proposition 4.2

Let $N$ be a 6-dimensional non-Kaehlerian nearly Kaehler manifold. Then there exists no CR-submanifold $M$ of $N$ with integrable holomorphic distribution $D$ of dimension 4.

Proof:

Suppose such a CR-submanifold $M$ does exists. Let $X$, $Y$ be two unit vector fields in $D$ with $\langle X, Y \rangle = \langle X, JY \rangle = 0$. Then from Theorem 4.1 we obtain

$$\| (\tilde{\nabla}_X J) Y \|^2 = \alpha \left( \| X \|^2 \| Y \|^2 - \langle X, Y \rangle^2 - \langle X, JY \rangle^2 \right)$$

$$= \alpha$$

Lemma 4.3 tells us that $(\tilde{\nabla}_X J) Y = 0$. Thus, we obtain $\alpha = 0$, which is a contradiction because $\alpha$ is a positive constant. Thus the proposition is proved.

We observe that if $M$ is a real hypersurface of a 6-dimensional almost Hermitian manifold, the distribution $D$ is necessary of dimension 4. Together with Proposition 4.2, this yields the following result.
Theorem 4.2

Let $N$ be a 6-dimensional non-Kaehlerian nearly Kaehler manifold. Then

(i) it has no 4-dimensional holomorphic submanifold,

(ii) it has no real hypersurface with integrable holomorphic distribution $D$.

From Lemma 4.1, Lemma 4.2 and Theorem 4.1, we also obtain the following results.

Theorem 4.3

There does not exist a 4-dimensional CR-submanifold with integrable totally real distribution $\perp$ in a 6-dimensional non-Kaehlerian, nearly Kaehler manifold $N$.

Proof:

Let $M$ be a 4-dimensional CR-submanifold of $N$. If $D$ is of dimension 4, then $M$ is a holomorphic submanifold. But this is impossible by Theorem 4.2. Therefore, both $D$ and $\perp$ are of dimension 2. Now, suppose $D$ is integrable, then by Lemma 4.1 and Lemma 4.2, we get

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\[ (\tilde{\nabla}_X J)Y = 0 \quad (4.2) \]
and
\[ (\tilde{\nabla}_Z J)W \in \Gamma(\nu) \]
for any \( X, Y \in \Gamma(D) \) and \( Z, W \in \Gamma(D^\perp) \). We note that \( M \) is an anti-holomorphic submanifold, that is \( \nu = \{0\} \) and so
\[ (\tilde{\nabla}_Z J)W = 0 \quad (4.3) \]
By using (4.2) and (4.3), we obtain
\[
\langle (\tilde{\nabla}_X J)Z, Y \rangle = -\langle (\tilde{\nabla}_X J)Y, Z \rangle = 0
\]
\[
\langle (\tilde{\nabla}_X J)Z, W \rangle = -\langle (\tilde{\nabla}_X J)X, W \rangle
\]
\[
= \langle (\tilde{\nabla}_Z J)W, X \rangle
\]
\[
= 0
\]
and
\[
\langle (\tilde{\nabla}_X J)Z, JW \rangle = -\langle (\tilde{\nabla}_Z J)X, JW \rangle
\]
\[
= -\langle J(\tilde{\nabla}_Z J)W, X \rangle
\]
\[
= 0
\]
for any \( X, Y \in \Gamma(D) \) and \( Z, W \in \Gamma(D^\perp) \). Therefore
\[ (\tilde{\nabla}_X J)Z = 0 \quad (4.4) \]
for any \( X \in \Gamma(D) \) and \( Z \in \Gamma(D^\perp) \). By using Theorem 4.1 and (4.4), we obtain
\[
\alpha \|X\|^2 \|Z\|^2 = 0
\]
In particular, for \( \|X\| = \|Z\| = 1 \) we have \( \alpha = 0 \). But this contradicts the fact that \( \alpha \) is a positive constant and the theorem is proved. \( \square \)
Theorem 4.4

There does not exist a 3-dimensional mixed geodesic proper CR-submanifold of a 6-dimensional non-Kaehlerian, nearly Kaehler manifold.

Proof:

Assume that M is a 3-dimensional mixed geodesic proper CR-submanifold of N. Then we have \( \dim D = 2 \) and \( \dim D^\perp = 1 \).

From Lemma 4.1 and Lemma 4.2 we have

\[
(\tilde{\nabla}_X J)Y = 0
\]

and

\[
(\tilde{\nabla}_Z J)W = 0
\]

for any \( X, Y \in \Gamma(D) \) and \( Z, W \in \Gamma(D^\perp) \). It follows that

\[
<(\tilde{\nabla}_Z J)X, Y> = -<(\tilde{\nabla}_X J)Z, Y>
\]

\[
= <(\tilde{\nabla}_X J)Y, Z>
\]

\[
= 0
\]

\[
<(\tilde{\nabla}_Z J)X, W> = -<(\tilde{\nabla}_Z J)W, X> = 0
\]

\[
<(\tilde{\nabla}_Z J)X, JW> = <J(\tilde{\nabla}_Z J)W, X> = 0.
\]

Furthermore, for any \( \xi \in \Gamma(\nu) \) we obtain

\[
<(\tilde{\nabla}_Z J)X, \xi> = <fh(Z, \phi X) + f\tilde{\nabla}_Z \omega X - Ch(Z, X), \xi> \quad \text{by (2.19)}
\]

\[
= <h(Z, \phi X) - Ch(Z, X), \xi>
\]

\[
= 0, \quad \text{since } M \text{ is mixed geodesic.}
\]
Therefore, \( (\nabla_Z J)X = 0 \) \( (4.5) \)
for any \( X \in \Gamma(D) \) and \( Z \in \Gamma(D^\perp) \). By using Theorem 4.1 and (4.5) we obtain
\[
\alpha \|X\|^2 \|Z\|^2 = 0
\]
Thus, we have \( \alpha = 0 \), which is a contradiction and this complete the proof. \( \Box \)

Remark: Theorem 4.4 implies that there exist no 3-dimensional mixed geodesic proper CR-submanifold of \( S^6 \).
However, Sekigawa [25] has constructed an example of a 3-dimensional minimal proper CR-submanifold of \( S^6 \) with both \( D \) and \( D^\perp \) integrable.