CHAPTER 4

CR-SUBMANIFOLDS OF A SIX-DIMENSIONAL NEARLY KAEHLER MANIFOLD

It is well known that the six-dimensional sphere S^6 is a nearly Kaehler manifold which is not Kaehlerian (see [13], [15]). Gray [12] has shown that S^6 does not admit a 4-dimenisional holomorphic submanifold and Deshmukh-Ghazal [11] proved that there exists no 4-dimenisional CR-submanifold with integrable distribution $D^{\perp}$ of S^6. The natural question arises as to whether these results are valid for arbitrary non-Kaehlerian, 6-dimensional nearly Kaehler manifold.

The main purpose of this chapter is to show that certain classes of CR-submanifolds do not exist in a nearly Kaehler manifold of dimension six, which is not Kaehlerian.

Section 1 contains a review of some results on nearly Kaehler manifold. In Section 2 we first give an affirmative answer for the above question. We also proved that a 6-dimensional non-Kaehlerian, nearly Kaehler manifold has no 3-dimensional mixed geodesic proper CR-submanifold and real hypersurface with integrable holomorphic distribution D.

4.1 Some Results on Nearly Kaehler Manifolds

Let N be an almost Hermitian manifold. Then N is said to be of *constant type* at x ∈ N if it satisfies the following condition

$$\|(\tilde{\nabla}_X J)Y\| = \|(\tilde{\nabla}_X J)Z\|$$

for X, Y, Z ∈ $T_x N$ with ⟨X,Y⟩ = ⟨X,Z⟩ = ⟨X,JY⟩ = ⟨X,JZ⟩ = 0 and $\|Y\| = \|Z\|$.

We say that N is of *(pointwise) constant type* if it is of constant type at each point x ∈ N. Finally, if N is of pointwise constant type and $\|(\tilde{\nabla}_X J)Y\|$ is constant, for any unit vector fields X, Y ∈ Γ(TN) with ⟨X,Y⟩ = ⟨X,JY⟩ = 0, then we say that N is of *global constant type*.

We have the following simple characterization of a nearly Kaehler manifold of constant type.

<u>Proposition 4.1</u> (Gray [14])

Let N be a nearly Kaehler manifold. Then N is of pointwise constant type if and only if there exists $\alpha \in \mathcal{F}(N)$

such that

$$\|(\tilde{\nabla}_X J)Y\|^2 = \alpha\{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2 - \langle X,JY\rangle^2\} \qquad (4.1)$$

for all X, Y $\in \Gamma(TN)$. Furthermore, N is of global constant type if and only if (4.1) holds with a constant function α.

A nearly Kaehler manifold N is said to be *strict* if for all $x \in N$ and non-zero vector $X \in T_xN$ we have $(\tilde{\nabla}_X J) \neq 0$.

The following theorems are found in Gray [16].

<u>**Theoren 4.1**</u>

Let N be a nearly Kaehler manifold with dim N = 6. Assume that N is not Kaehlerian. Then it is of global constant type, that is

$$\|(\tilde{\nabla}_X J)Y\|^2 = \alpha\{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2 - \langle X,JY\rangle^2\}$$

where X, Y $\in \Gamma(TN)$ and α is a positive constant.

4.2 CR-submanifolds of a 6-dimensional Nearly Kaehler Manifold

We begin this section with the following lemmas.

Lemma 4.1

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then we have the following

(i) if $\dim \nu = 2$ then $(\tilde{\nabla}_{\xi} J)\eta = 0$

(ii) if $\dim D = 2$ then $(\tilde{\nabla}_{X} J)Y = 0$

for any ξ, $\eta \in \Gamma(\nu)$ and X, $Y \in \Gamma(D)$.

Proof:

Suppose that $\dim \nu = 2$. Then for any non-zero normal vector field $\xi \in \Gamma(\nu)$, $\{\xi, J\xi\}$ is a local frame on ν. Hence, a normal vector field $\eta \in \Gamma(\nu)$ can be written as $a\xi + bJ\xi$, where a, $b \in \mathcal{F}(M)$. Therefore

$$\begin{aligned}(\tilde{\nabla}_{\xi} J)\eta &= (\tilde{\nabla}_{\xi} J)(a\xi + bJ\xi) \\ &= a(\tilde{\nabla}_{\xi} J)\xi - bJ(\tilde{\nabla}_{\xi} J)\xi \\ &= 0.\end{aligned}$$

Similarly, if $\dim D = 2$, then for any X, $Y \in \Gamma(D)$ we have $(\tilde{\nabla}_{X} J)Y = 0$.■

Lemma 4.2

Let M be a CR-submanifold of a nearly Kaehler manifold N. Then we have the following

(i) if $\dim D^{\perp} = 1$ then $(\tilde{\nabla}_Z J)W = 0$

(ii) if $\dim D^{\perp} = 2$ and $D^{\perp}$ is integrable then $(\tilde{\nabla}_Z J)W \in \Gamma(\nu)$

for any $Z, W \in \Gamma(D^{\perp})$.

Proof:

Suppose that $\dim D^{\perp} = 1$. Then for any non-zero vector fields $Z, W \in \Gamma(D^{\perp})$, we have $W = \alpha Z$ where $\alpha \in \mathcal{F}(M)$. Therefore

$$(\tilde{\nabla}_Z J)W = \alpha(\tilde{\nabla}_Z J)Z = 0.$$

Now, suppose $D^{\perp}$ is integrable and it is of dimension 2. We observe that if $\{Z,W\}$ is a local frame on $D^{\perp}$, then $\{JZ,JW\}$ is a local frame on $JD^{\perp}$. Since N is nearly Kaehler, by using (1.2) and (1.3), we have

$$\langle(\tilde{\nabla}_Z J)W,Z\rangle = -\langle(\tilde{\nabla}_Z J)Z,W\rangle = 0$$

$$\langle(\tilde{\nabla}_Z J)W,W\rangle = 0$$

$$\langle(\tilde{\nabla}_Z J)W,JZ\rangle = -\langle(\tilde{\nabla}_W J)Z,JZ\rangle = 0$$

and $$\langle(\tilde{\nabla}_Z J)W,JW\rangle = 0.$$

Also, from Theorem 3.3 we have

$$\langle(\tilde{\nabla}_Z J)W,X\rangle = 0, \qquad \text{for any } X \in \Gamma(D).$$

Therefore, $$(\tilde{\nabla}_Z J)W \in \Gamma(\nu).\blacksquare$$

Lemma 4.3

Let M be a CR-submanifold of a nearly Kaehler manifold N. If dim D = 4 then

$$(\tilde{\nabla}_U J)V \perp D, \qquad \text{for any } U,\ V \in \Gamma(D).$$

Moreover, if D is integrable then $(\tilde{\nabla}_U J)V = 0$.

Proof:

Suppose dim D = 4 and let {X, JX, Y, JY} be any local frame on D. To prove the first part, since N is nearly Kaehler, it suffices to show that $(\tilde{\nabla}_X J)Y \perp D$. We can see that

$$\langle(\tilde{\nabla}_X J)Y, X\rangle = -\langle(\tilde{\nabla}_Y J)X, X\rangle = 0$$

$$\langle(\tilde{\nabla}_X J)Y, JX\rangle = -\langle(\tilde{\nabla}_Y J)X, JX\rangle = 0$$

and $$\langle(\tilde{\nabla}_X J)Y, Y\rangle = \langle(\tilde{\nabla}_X J)Y, JY\rangle = 0.$$

Hence, we obtain $(\tilde{\nabla}_X J)Y \perp D$.

Now, if D is integrable then from Theorem 3.1, we have

$$(\tilde{\nabla}_X J)Y \in \Gamma(D), \qquad \text{for any } X,\ Y \in \Gamma(D).$$

Since $(\tilde{\nabla}_X J)Y \perp D$, we have $(\tilde{\nabla}_X J)Y = 0$.■

We are now ready to prove the following proposition.

Proposition 4.2

Let N be a 6-dimensional non-Kaehlerian nearly Kaehler manifold. Then there exists no CR-submanifold M of N with integrable holomorphic distribution D of dimension 4.

Proof:

Suppose such a CR-submanifold M does exists. Let X, Y be two unit vector fields in D with $\langle X,Y\rangle = \langle X,JY\rangle = 0$. Then from Theorem 4.1 we obtain

$$\|(\tilde{\nabla}_X J)Y\|^2 = \alpha\{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2 - \langle X,JY\rangle^2\}$$
$$= \alpha$$

Lemma 4.3 tells us that $(\tilde{\nabla}_X J)Y = 0$. Thus, we obtain $\alpha = 0$, which is a contradiction because α is a positive constant. Thus the proposition is proved.■

We observe that if M is a real hypersurface of a 6-dimensional almost Hermitian manifold, the distribution D is necessary of dimension 4. Together with Proposition 4.2, this yields the following result.

Theorem 4.2

Let N be a 6-dimensional non-Kaehlerian nearly Kaehler manifold. Then

(i) it has no 4-dimensional holomorphic submanifold,

(ii) it has no real hypersurface with integrable holomorphic distribution D.

From Lemma 4.1, Lemma 4.2 and Theorem 4.1, we also obtain the following results.

Theorem 4.3

There does not exist a 4-dimensional CR-submanifold with integrable totally real distribution $D^{\perp}$ in a 6-dimensional non-Kaehlerian, nearly Kaehler manifold N.

Proof:

Let M be a 4-dimensional CR-submanifold of N. If D is of dimension 4, then M is a holomorphic submanifold. But this is impossible by Theorem 4.2. Therefore, both D and $D^{\perp}$ are of dimension 2. Now, suppose $D^{\perp}$ is integrable, then by Lemma 4.1 and Lemma 4.2, we get

$$(\tilde{\nabla}_X J)Y = 0 \tag{4.2}$$

and $$(\tilde{\nabla}_Z J)W \in \Gamma(\nu)$$

for any X, Y $\in \Gamma(D)$ and Z, W $\in \Gamma(D^{\perp})$. We note that M is an anti-holomorphic submanifold, that is $\nu = \{0\}$ and so

$$(\tilde{\nabla}_Z J)W = 0 \tag{4.3}$$

By using (4.2) and (4.3), we obtain

$$\langle(\tilde{\nabla}_X J)Z, Y\rangle = -\langle(\tilde{\nabla}_X J)Y, Z\rangle = 0$$

$$\begin{aligned}\langle(\tilde{\nabla}_X J)Z, W\rangle &= -\langle(\tilde{\nabla}_Z J)X, W\rangle \\ &= \langle(\tilde{\nabla}_Z J)W, X\rangle \\ &= 0\end{aligned}$$

and $$\begin{aligned}\langle(\tilde{\nabla}_X J)Z, JW\rangle &= -\langle(\tilde{\nabla}_Z J)X, JW\rangle \\ &= -\langle J(\tilde{\nabla}_Z J)W, X\rangle \\ &= 0\end{aligned}$$

for any X, Y $\in \Gamma(D)$ and Z, W $\in \Gamma(D^{\perp})$. Therefore

$$(\tilde{\nabla}_X J)Z = 0 \tag{4.4}$$

for any X $\in \Gamma(D)$ and Z $\in \Gamma(D^{\perp})$. By using Theorem 4.1 and (4.4), we obtain

$$\alpha\|X\|^2\|Z\|^2 = 0$$

In particular, for $\|X\| = \|Z\| = 1$ we have $\alpha = 0$. But this contradicts the fact that α is a positive constant and the theorem is proved.■

Theorem 4.4

There does not exist a 3-dimensional mixed geodesic proper CR-submanifold of a 6-dimensional non-Kaehlerian, nearly Kaehler manifold.

Proof:

Assume that M is a 3-dimensional mixed geodesic proper CR-submanifold of N. Then we have dim D = 2 and dim $D^{\perp}$ = 1. From Lemma 4.1 and Lemma 4.2 we have

$$(\tilde{\nabla}_X J)Y = 0$$

and

$$(\tilde{\nabla}_Z J)W = 0$$

for any X, Y $\in \Gamma(D)$ and Z, W $\in \Gamma(D^{\perp})$. It follows that

$$\begin{aligned} \langle(\tilde{\nabla}_Z J)X, Y\rangle &= -\langle(\tilde{\nabla}_X J)Z, Y\rangle \\ &= \langle(\tilde{\nabla}_X J)Y, Z\rangle \\ &= 0 \end{aligned}$$

$$\langle(\tilde{\nabla}_Z J)X, W\rangle = -\langle(\tilde{\nabla}_Z J)W, X\rangle = 0$$

$$\langle(\tilde{\nabla}_Z J)X, JW\rangle = \langle J(\tilde{\nabla}_Z J)W, X\rangle = 0.$$

Furthermore, for any $\xi \in \Gamma(\nu)$ we obtain

$$\begin{aligned} \langle(\tilde{\nabla}_Z J)X, \xi\rangle &= \langle f h(Z,\phi X) + f\nabla^{\perp}_Z \omega X - Ch(Z,X), \xi\rangle \quad \text{by (2.19)} \\ &= \langle h(Z,\phi X) - Ch(Z,X), \xi\rangle \\ &= 0, \qquad \text{since M is mixed geodesic.} \end{aligned}$$

Therefore, $$(\tilde{\nabla}_Z J)X = 0 \tag{4.5}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. By using Theorem 4.1 and (4.5) we obtain

$$\alpha\|X\|^2\|Z\|^2 = 0$$

Thus, we have $\alpha = 0$, which is a contradiction and this complete the proof.■

Remark: Theorem 4.4 implies that there exist no 3-dimensional mixed geodesic proper CR-submanifold of S^6. However, Sekigawa [25] has constructed an example of a 3-dimensional minimal proper CR-submanifold of S^6 with both distribution D and $D^{\perp}$ integrable.