CHAPTER 5

TOTALLY UMBILICAL CR-SUBMANIFOLDS OF A NEARLY KAEHLER MANIFOLD

Totally umbilical CR-submanifolds of a Kaehler manifold have been studied by Chen [9], Toyonari-Nemoto [28] and Bashir [1]. In [20], Kon-Tan initiated the study of totally umbilical CR-submanifolds of a nearly Kaehler manifold. In this chapter we continue their work and give a classification of all connected totally umbilical CR-submanifolds of a nearly Kaehler manifold. We also show that a 3-dimensional Einstein non-totally geodesic proper CR-submanifold of a nearly Kaehler manifold is an extrinsic sphere.

5.1 On the Classification of a Totally Umbilical CR-submanifold of a Nearly Kaehler Manifold

In their paper [20], Kon-Tan have proved a classification theorem, that is, a totally umbilical CR-submanifold M of a nearly Kaehler manifold is either totally geodesic, or totally real, or dim $D^{\perp} = 1$. In this section we consider the case when dim $D^{\perp} = 1$ and then give a complete classification of a connected totally umbilical CR-submanifold of a nearly Kaehler manifold.

Let M be a totally umbilical CR-submanifold of a nearly Kaehler manifold N. Then the equation of Gauss and Codazzi take the forms

$$\langle \tilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle$$
$$+ \{\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle,\langle Y,Z\rangle\}\|H\|^2 \qquad (5.1)$$

$$\tilde{R}(X,Y)Z^{\perp} = \langle Y,Z\rangle\nabla^{\perp}_X H - \langle X,Z\rangle\nabla^{\perp}_Y H \qquad (5.2)$$

for any X, Y, Z and W $\in \Gamma(TM)$.

The following theorem classifies a totally umbilical CR-submanifold of a nearly Kaehler manifold.

<u>Theorem 5.1</u> (Kon-Tan [20])

Let M be a totally umbilical CR-submanifold of a nearly Kaehler manifold. Then either

(i) M is totally geodesic; or

(ii) M is totally real; or

(iii) dim $D^{\perp} = 1$.

We note that Theorem 5.1 has been proved by Chen [9] when N is a Kaehler manifold.

Suppose M is a non-totally geodesic, totally umbilical proper CR-submanifold of a nearly Kaehler manifold N. Then we have dim $D^{\perp} = 1$. We first start with the following simple results.

Lemma 5.1

Let M be a totally umbilical proper CR-submanifold of a nearly Kaehler manifold N. Then the mean curvature vector $H \in \Gamma(JD^{\perp})$.

Proof:

For any unit vector fields $X \in \Gamma(D)$, $\xi \in \Gamma(\nu)$, by taking Y = X in (3.3) we obtain

$$\langle h(X,JX),\xi\rangle - \langle Jh(X,X),\xi\rangle = 0$$

$$\langle X,JX\rangle\langle H,\xi\rangle - \langle X,X\rangle\langle JH,\xi\rangle = 0$$

$$-\langle CH,\xi\rangle = 0$$

which means CH = 0 and hence $H \in \Gamma(JD^{\perp})$.■

Lemma 5.2

Let M be a non-totally geodesic totally umbilical proper CR-submanifold of a nearly Kaehler manifold N. Then we have

$$\nabla^{\perp}_{U} JZ \in \Gamma(JD^{\perp})$$

for any $U \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$.

Proof:

For any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$, by using (2.19) we have

$$f(\tilde{\nabla}_X J)Z^{\perp} = -f(\tilde{\nabla}_Z J)X^{\perp}$$

$$fh(X,\phi Z) + f\nabla^{\perp}_X \omega Z - Ch(X,Z) = -fh(Z,\phi X) - f\nabla^{\perp}_Z \omega X + Ch(Z,X)$$

$$f\nabla^{\perp}_X JZ - \langle X,Z\rangle CH = -\langle Z,\phi X\rangle fH + \langle Z,X\rangle CH$$

$$f\nabla^{\perp}_X JZ = 0. \qquad (5.3)$$

Next, for any Z, $W \in \Gamma(D^{\perp})$, since $\dim D^{\perp} = 1$ we have $(\tilde{\nabla}_Z J)W = 0$ by Lemma 4.2. Then by using (2.19) again we obtain

$$fh(W,\phi Z) + f\nabla^{\perp}_W \omega Z - Ch(W,Z) = f(\tilde{\nabla}_W J)Z^{\perp} = 0$$

$$f\nabla^{\perp}_W JZ - \langle W.Z\rangle CH = 0.$$

From Lemma 5.1 we know that CH = 0. Therefore we have

$$f\nabla^{\perp}_{W}JZ = 0. \tag{5.4}$$

By combining (5.3) and (5.4) we obtain

$$f\nabla^{\perp}_{U}JZ = 0, \qquad \text{for any } U \in \Gamma(TM) \text{ and } Z \in \Gamma(D^{\perp}).$$

That is, $\nabla^{\perp}_{U}JZ = t\nabla^{\perp}_{U}JZ \in \Gamma(JD^{\perp})$.■

Next we have the following result which gives a sufficient condition for a totally umbilical CR-submanifold of a nearly Kaehler manifold to admit a nearly Sasakian structure.

Proposition 5.1

Let M be a proper CR-submanifold of a nearly Kaehler manifold N. If M is an extrinsic sphere then it admits a nearly Sasakian structure.

Proof:

Suppose M is an extrinsic sphere. Then the mean curvature $\mu = \|H\|$ is a non-zero constant. Since $H \in \Gamma(JD^{\perp})$, $\xi = -\frac{1}{\mu}JH$ is a unit vector field in $D^{\perp}$ globally defined in M. Moreover we have $\dim D^{\perp} = 1$.

Now, we define a new Riemannian metric g on M by

$$g(X,Y) = \mu^2\langle X,Y\rangle \qquad (5.5)$$

Moreover, we put $\zeta = \frac{1}{\mu}\xi$ and $\eta(X) = \mu\langle X,\xi\rangle$. Then for any $X, Y \in \Gamma(TM)$ we have

$$\eta(\zeta) = \mu\langle\zeta,\xi\rangle = 1$$

$$\begin{aligned} X &= PX + QX \\ &= -\phi^2X + \langle X,\xi\rangle\xi \\ &= -\phi^2X + \mu\langle X,\xi\rangle\frac{1}{\mu}\xi \\ &= -\phi^2X + \eta(X)\zeta \end{aligned}$$

and

$$\begin{aligned} g(X,Y) &= \mu^2\langle X,Y\rangle \\ &= \mu^2\langle PX,PY\rangle + \mu^2\langle QX,QY\rangle \\ &= -\mu^2\langle\phi^2X,PY\rangle + \mu^2\langle X,\xi\rangle\langle Y,\xi\rangle \\ &= \mu^2\langle\phi X,\phi PY\rangle + \mu\langle X,\xi\rangle\mu\langle Y,\xi\rangle \\ &= \mu^2\langle\phi X,\phi Y\rangle + \eta(X)\eta(Y) \\ &= g(\phi X,\phi Y) + \eta(X)\eta(Y). \end{aligned}$$

Therefore, (ϕ, ζ, η, g) is an almost contact metric structure on M.

Next, by (2.16) and (2.17) we obtain

$$\begin{aligned} (\tilde{\nabla}_XJ)Y^T &= P(\tilde{\nabla}_XJ)Y^T + Q(\tilde{\nabla}_XJ)Y^T \\ &= \nabla_X\phi Y - \phi\nabla_XY - A_{\omega Y}X - Bh(X,Y). \end{aligned}$$

Therefore, by using the fact that N is nearly Kaehler we obtain

$$(\tilde{\nabla}_X J)Y^T + (\tilde{\nabla}_Y J)X^T = 0$$

$$(\nabla_X \phi)Y - A_{\omega Y}X - Bh(X,Y) + (\nabla_Y \phi)X - A_{\omega X}Y - Bh(Y,X) = 0$$

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X - A_{\omega Y}X - A_{\omega X}Y - 2Bh(X,Y) = 0$$

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X - \langle H,\omega Y\rangle X - \langle H,\omega X\rangle Y - 2\langle X,Y\rangle BH = 0$$

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X + \langle JH,Y\rangle X + \langle JH,X\rangle Y - 2\langle X,Y\rangle JH = 0$$

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X - \mu\langle \xi,Y\rangle X - \mu\langle \xi,X\rangle Y + 2\mu\langle X,Y\rangle \xi = 0$$

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X - \eta(Y)X - \eta(X)Y + 2\mu^2\langle X,Y\rangle \frac{1}{\mu}\xi = 0$$

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X - \eta(Y)X - \eta(X)Y + 2g(X,Y)\zeta = 0.$$

That is,

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = \eta(X)Y + \eta(Y)X - 2g(X,Y)\zeta$$

for any $X, Y \in \Gamma(TM)$.

We observe that since g is homothetic to the induced metric $\langle\ ,\ \rangle$, then ∇ the Levi-Civita connection on M is determined by g, hence (ϕ, ζ, η, g) defined a nearly Sasakian structure on M.■

In particular, when the mean curvature $\mu = 1$ we have the following corollary.

Corollary 5.1

Let M be a totally umbilical proper CR-submanifold of a nearly Kaehler manifold N. If M is an extrinsic sphere with mean curvature $\mu = 1$, then M is immersed in N as a nearly Sasakian submanifold.

Proof:

If $\mu = 1$ then (5.5) becomes

$$g(X,Y) = \langle X,Y\rangle, \qquad \text{for any } X,\ Y \in \Gamma(TM).$$

Thus, M admits a nearly Sasakian structure with the induced metric $\langle\ ,\ \rangle$ as the associated Riemannian metric and so it is immersed in N as a nearly Sasakian submanifold.■

We are now able to classify a connected totally umbilical CR-submanifold of a nearly Kaehler manifold in detail.

Theorem 5.2

Let M be a connected totally umbilical CR-submanifold of a nearly Kaehler manifold N. Then either

(i) M is totally geodesic; or

(ii) M is totally real; or

(iii) H is not D-parallel; or

(iv) M is an extrinsic sphere and so admits a nearly Sasakian structure.

Proof:

Let M be a connected non-totally geodesic and non-totally real, totally umbilical CR-submanifold of N. Then by Theorem 5.1, dim $D^{\perp} = 1$ and so it is proper. Now suppose that the mean curvature vector H is D-parallel. By using the Codazzi equation (5.2), for any unit vector fields $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ we obtain

$$\begin{aligned}\langle\tilde{R}(Z,X)X,JZ\rangle &= \langle X,X\rangle\langle\nabla^{\perp}_{Z}H,JZ\rangle - \langle Z,X\rangle\langle\nabla^{\perp}_{X}H,JZ\rangle \\ &= \langle\nabla^{\perp}_{Z}H,JZ\rangle. \qquad (5.6)\end{aligned}$$

From Proposition 1.3, we get

$$\begin{aligned}\langle\tilde{R}(Z,X)X,JZ\rangle &= \langle\tilde{R}(JZ,JX)JX,J^{2}Z\rangle \\ &= -\langle\tilde{R}(JZ,JX)JX,Z\rangle \\ &= -\langle\tilde{R}(Z,JX)JX,JZ\rangle.\end{aligned}$$

By using the Codazzi equation (5.2) again we obtain

$$\begin{aligned}\langle\tilde{R}(Z,X)X,JZ\rangle &= -\langle JX,JX\rangle\langle\nabla^{\perp}_{Z}H,JZ\rangle + \langle Z,JX\rangle\langle\nabla^{\perp}_{JX}H,JZ\rangle \\ &= -\langle\nabla^{\perp}_{Z}H,JZ\rangle. \qquad (5.7)\end{aligned}$$

From (5.6) and (5.7) we obtain that $\langle\nabla^{\perp}_{Z}H, JZ\rangle = 0$.

Since $\dim D^{\perp} = 1$, we obtain

$$\nabla^{\perp}_{Z}H \in \Gamma(\nu), \qquad \text{for any } Z \in \Gamma(D^{\perp}).$$

Lemma 5.2 tells us that $\nabla^{\perp}_{Z}H \in \Gamma(JD^{\perp})$. Thus, we have proved

$$\nabla^{\perp}_{Z}H = 0, \qquad \text{for any } Z \in \Gamma(D^{\perp}).$$

This shows that H is parallel since it is D-parallel. As M is connected, it follows that $\mu = \|H\|$ is a constant. Since M is non-totally geodesic, μ is a non-zero constant, that is M is an extrinsic sphere and hence it admits a nearly Sasakian structure by Proposition 5.1.■

<u>Remark</u>: When N is a Kaehler manifold and M is of dimension greater than 4, then part (iii) of Theorem 5.2 will never happens and part (iv) of Theorem 5.2 is replaced by M admits a Sasakian structure (see Toyonari-Nemoto [28]).

Before we close this section, we consider a particular case of Theorem 5.2.

<u>Theorem 5.3</u>

Let M be a 5-dimensional connected totally umbilical

CR-submanifold of a nearly Kaehler manifold N. Then either

(i) M is totally geodesic; or

(ii) M is totally real; or

(iii) M admits a nearly Sasakian structure.

Proof:

Suppose M is a 5-dimensional connected non-totally geodesic and non-totally real, totally umbilical CR-submanifold of N. Then we have dim D = 4 and dim $D^{\perp}$ = 1. From Lemma 4.2 amd Lemma 4.3, we have

$$(\tilde{\nabla}_X J)Y \perp D \tag{5.8}$$

and

$$(\tilde{\nabla}_Z J)W = 0 \tag{5.9}$$

for any X, Y $\in \Gamma(D)$ and Z, W $\Gamma(D^{\perp})$. By using (5.9) we obtain

$$\langle(\tilde{\nabla}_Z J)X,W\rangle = -\langle(\tilde{\nabla}_Z J)W,X\rangle = 0$$

$$\langle(\tilde{\nabla}_Z J)X,JW\rangle = \langle J(\tilde{\nabla}_Z J)W,X\rangle = 0.$$

Furthermore, from (2.19) we have

$$\begin{aligned} f(\tilde{\nabla}_Z J)X^{\perp} &= fh(Z,\phi X) + f\nabla^{\perp}_Z \omega X - Ch(Z,X) \\ &= \langle Z,\phi X\rangle fH - \langle Z,X\rangle CH \\ &= 0. \end{aligned}$$

Therefore, we have

$$(\tilde{\nabla}_Z J)X \in \Gamma(D), \qquad \text{for any } Z \in \Gamma(D^{\perp}) \text{ and } X \in \Gamma(D).$$

By using the above equation and (5.8) we have

$$\langle (\tilde{\nabla}_X J)Y, J(\tilde{\nabla}_Z J)X\rangle = 0 \tag{5.10}$$

for any $X,\ Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Consider unit vector fields $X,\ Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, since $\dim D = 4$, we may assume that $\langle X,Y\rangle = \langle X,JY\rangle = 0$. By using the Codazzi equation (5.2) we have

$$\begin{aligned}\langle \tilde{R}(Y,X)X,JZ\rangle &= \langle X,X\rangle\langle \nabla^{\perp}_Y H,JZ\rangle - \langle Y,X\rangle\langle \nabla^{\perp}_X H,JZ\rangle \\ &= \langle \nabla^{\perp}_Y H,JZ\rangle.\end{aligned}$$

We observe that

$$\begin{aligned}\langle \tilde{R}(Y,X)X,JZ\rangle &= \langle \tilde{R}(JY,JX)X,JZ\rangle - \langle (\tilde{\nabla}_Y J)X,(\tilde{\nabla}_X J)JZ\rangle \\ &= \langle \tilde{R}(JY,JX)X,JZ\rangle + \langle (\tilde{\nabla}_X J)Y,J(\tilde{\nabla}_Z J)X\rangle \\ &= \langle \tilde{R}(JY,JX)X,JZ\rangle \qquad \text{by (5.10)}.\end{aligned}$$

By using the Codazzi equation (5.2) again we have

$$\begin{aligned}\langle \tilde{R}(Y,X)X,JZ\rangle &= \langle JX,X\rangle\langle \nabla^{\perp}_{JY} H,JZ\rangle - \langle JY,X\rangle\langle \nabla^{\perp}_{JX} H,JZ\rangle \\ &= 0.\end{aligned}$$

Therefore, we have

$$\langle \nabla^{\perp}_Y H,JZ\rangle = 0, \qquad \text{for any } Y \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}).$$

Since $\dim D^{\perp} = 1$, by Lemma 5.1 and Lemma 5.2 we have

$$\nabla^{\perp}_Y H = 0, \qquad \text{for any } Y\ \Gamma(D).$$

In other words, the mean curvature vector H is D-parallel. Consequently, M admits a nearly Sasakian structure.■

5.2 3-dimensional Totally Umbilical CR-submanifolds of a Nearly Kaehler manifold

In this section we obtain a classification theorem for 3-dimensional connected totally umbilical submanifolds of a nearly Kaehler manifold, which is almost similar to Theorem 5.2 and also generalize a result of Bashir [1] to nearly Kaehler manifold. We first consider a particular case for Proposition 5.1.

<u>Proposition 5.2</u>

Let M be a 3-dimensional totally umbilical proper CR-submanifold of a nearly Kaehler manifold N. If M is an extrinsic sphere then it admits a Sasakian structure.

Proof:

We already know that M admits a nearly Sasakian strcture from Proposition 5.1. Since M is of dimension 3, by a result of Olszak ([24], Theorem 5.1), the nearly Sasakian structure is Sasakian.■

From Theorem 5.2 and Proposition 5.2. The following result can be proven.

Proposition 5.3

Let M be a 3-dimensional connected totally umbilical CR-submanifold of a nearly Kaehler manifold N. Then either

(i) M is totally geodesic; or

(ii) M is totally real; or

(iii) H is not D-parallel; or

(iv) M admits a Sasakian structure.

The following theorem generalizes [1, Theorem 2] to nearly Kaehler manifold.

Theorem 5.4

Let M be an 3-dimensional Einstein non-totally geodesic totally umbilical proper CR-submanifold of a nearly Kaehler manifold N. Then M is an extrinsic sphere.

Proof:

Let M be a 3-dimensional non-totally geodesic, totally umbilical proper CR-submanifold of N. Then $\dim D^{\perp} = 1$ and $\dim D = 2$. From Lemma 4.1 and Lemma 4.2, for any vector fields $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$ we have

$$(\tilde{\nabla}_X J)Y = 0 \qquad (5.11)$$

$$(\tilde{\nabla}_Z J)W = 0. \qquad (5.12)$$

By using (5.11) and (5.12) we obtain

$$\langle(\tilde{\nabla}_Z J)X, Y\rangle = \langle(\tilde{\nabla}_X J)Y, Z\rangle = 0$$

$$\langle(\tilde{\nabla}_Z J)X, W\rangle = -\langle(\tilde{\nabla}_Z J)W, X\rangle = 0$$

$$\langle(\tilde{\nabla}_Z J)X, JW\rangle = \langle J(\tilde{\nabla}_Z J)W, X\rangle = 0.$$

Moreover, by taking $Z = U$ and $X = V$ in (2.19) we obtain

$$\begin{aligned} f(\tilde{\nabla}_Z J)X^{\perp} &= fh(Z,\phi X) + f\nabla^{\perp}_Z \omega X - Ch(Z,X) \\ &= \langle Z,\phi X\rangle fH - \langle Z,X\rangle CH \\ &= 0. \end{aligned}$$

It follows that

$$(\tilde{\nabla}_Z J)X = 0, \qquad \text{for any } X \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}).$$

For any unit vector fields $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$, by using Proposition 1.3 and the Codazzi equation (5.2), we obtain

$$\begin{aligned} \langle\tilde{R}(Z,X)X,Z\rangle &= \langle\tilde{R}(Z,X)JX,JZ\rangle - \langle(\tilde{\nabla}_Z J)X,(\tilde{\nabla}_X J)Z\rangle \\ &= \langle\tilde{R}(Z,X)JX,JZ\rangle \\ &= \langle X,JX\rangle\langle\nabla^{\perp}_Z H,JZ\rangle - \langle Z,JX\rangle\langle\nabla^{\perp}_X H,JZ\rangle \\ &= 0. \end{aligned}$$

Thus, by using the above equation and taking into account the Gauss equation (5.1), we obtain

$$\langle R(Z,X)X,Z\rangle + \langle Z,X\rangle\langle X,Z\rangle\|H\|^2 - \langle Z,Z\rangle\langle X,X\rangle\|H\|^2 = 0$$

$$\langle R(Z,X)X,Z\rangle = \|H\|^2. \qquad (5.13)$$

If M is an Einstein space, then it is a space of constant curvature c by Proposition 1.1. Thus, from (5.13) we have $\|H\|^2 = c$, is a constant on M. Since M is non-totally geodesic, $\|H\|^2$ is a non-zero constant. Therefore, we have

$$\langle \nabla^{\perp}_U H, H\rangle = \frac{1}{2} U\langle H,H\rangle = 0$$

for any $U \in \Gamma(TM)$. That is, $\nabla^{\perp}_U H \in \Gamma(\nu)$. Lemma 5.2 tells us that $\nabla^{\perp}_U H \in \Gamma(JD^{\perp})$ and hence $\nabla^{\perp}_U H = 0$. Thus, we have proved that M is an extrinsic sphere.■

The following corollary is an easy consequence of the preceeding theorem and Propostion 5.2.

Corollary 5.2

Under the hypothesis of Theorem 5.4, M admits a Sasakian structure.