CHAPTER 6

CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

The main purpose of this chapter is to consider some aspects of the geometry of the CR-submanifolds of a Kaehler manifold.

In Section 1 we shall study CR-products. We are particularly interested in necessary and sufficient conditions for a CR-submanifold of a Kaehler manifold to be a CR-product.

Next in Section 2 we shall discuss some characteristics of normal CR-submanifold of a Kaehler manifold. Most of the results obtained here is needed in the last section.

The geometry of Sasakian CR-submanifold of a Kaehler manifold will be discussed in the last section. We shall see that under certain contidions, a Sasakian anti-holomorphic submanifold of a Kaehler manifold N contains a submanifold which is homothetic to a Sasakian manifold and immersed in N as a Sasakian CR-submanifold.

6.1 CR-products of a Kaehler Manifold

In this section, we will give two propositions which are a slight modification of Chen's characterization theorem for a CR-product of a Kaehler manifold.

Let N be a Kaehler manifold. Then for any U, V $\in$ $\Gamma(TN)$ we have $(\tilde{\nabla}_U J)V = 0$. If M is a CR-submanifold of N then equations (2.16)-(2.23) becomes

$$P\nabla_U \phi V - PA_{\omega V}U - \phi\nabla_U V = 0 \qquad (6.1)$$

$$Q\nabla_U \phi V - QA_{\omega V}U - Bh(U,V) = 0 \qquad (6.2)$$

$$th(U,\phi V) + t\nabla^{\perp}_U \omega V - \omega\nabla_U V = 0 \qquad (6.3)$$

$$fh(U,\phi V) + f\nabla^{\perp}_U \omega V - Ch(U,V) = 0 \qquad (6.4)$$

$$P\nabla_U B\eta - PA_{C\eta}U + \phi A_\eta U = 0 \qquad (6.5)$$

$$Q\nabla_U B\eta - QA_{C\eta}U - B\nabla^{\perp}_U \eta = 0 \qquad (6.6)$$

$$th(U,B\eta) + t\nabla^{\perp}_U C\eta + \omega A_\eta U = 0 \qquad (6.7)$$

$$fh(U,B\eta) + f\nabla^{\perp}_U C\eta - C\nabla^{\perp}_U \eta = 0. \qquad (6.8)$$

The following well known results can be found in Bejancu [4].

Theorem 6.1 (Bejancu [4], p.39)

Let M be a CR-submanifold of a Kaehler manifold N. Then we have the following:

(i) The distribution $D^{\perp}$ is integrable.

(ii) The distribution D is integrable if and only if

$$h(X,JY) = h(JX,Y) \tag{6.9}$$

for any X, $Y \in \Gamma(D)$.

Let M be a CR-submanifold of an almost Hermitian manifold N. M is a *CR-product* if both the distribution D and $D^{\perp}$ are integrable, and M is locally a Riemannian product of a holomorphic submanifold M_1 and a totally real submanifold M_2 of N. In other words, M is a CR-product if and only if the distribution D and $D^{\perp}$ are integrable and their leaves are totally geodesic in M (see Bejancu [4], p.10).

Definition

A distribution $\mathcal{D}$ on M is said to be auto-parallel if

$$\nabla_X Y \in \Gamma(\mathcal{D})$$

for any X, $Y \in \Gamma(\mathcal{D})$.

It is not hard to see that a distribution on M is auto-parallel if and only if it is integrable and each of its leaf is totally geodesic in M. Hence, we have the following proposition.

<u>Proposition 6.1</u>

Let M be a CR-submanifold of a Kaehler manifold N. Then M is a CR-product if and only if both the the distribution D and $D^{\perp}$ are auto-parallel.

The next result which gives a necessary and sufficient condition for a CR-submanifold of a Kaehler manifold to be a CR-product, is due to Chen [7].

<u>Theorem 6.2</u>

Let M be a CR-submanifold of a Kaehler manifold N. Then M is a CR-product if and only if

$$(\nabla_U \phi)V = 0$$

for any U, V $\in \Gamma(TM)$.

We modify this result to the following form.

Proposition 6.2

Let M be a CR-submanifold of a Kaehler manifold N. Then M is a CR-product if and only if

$$(\nabla_U \phi)U = 0 \tag{6.10}$$

for any $U \in \Gamma(TM)$.

Proof:

If M is a CR-product then (6.10) is clearly true by Theorem 6.2. Now suppose (6.10) is satisfied. Then for any $X \in \Gamma(D)$ we have

$$\nabla_X \phi X - \phi \nabla_X X = (\nabla_X \phi)X$$

$$Q\nabla_X \phi X + (P\nabla_X \phi X - \phi \nabla_X X) = 0.$$

Since the first term is in $D^{\perp}$ and the last two terms are in D, we have

$$Q\nabla_X \phi X = 0.$$

By using the above equation and by taking X = U = V in (6.2) we obtain

$$-QA_{\omega X}X - Bh(X,X) = 0$$

$$-Bh(X,X) = 0.$$

That is $Bh(X,X) = 0$, for any $X \in \Gamma(D)$. Since h is symmetric we obtain

$$Bh(X,Y) = 0, \qquad \text{for any } X,\ Y \in \Gamma(D).$$

By using (6.2) again, we obtain

$$Q\nabla_X \phi Y - QA_{\omega Y}X - Bh(X,Y) = 0$$

$$Q\nabla_X \phi Y = 0$$

for any X, Y $\in \Gamma(D)$. This tells us that the distribution D is auto-parallel.

Next, for any $Z \in \Gamma(D^{\perp})$ we have

$$\nabla_Z \phi Z - \phi \nabla_Z Z = (\nabla_Z \phi) Z$$

$$-\phi \nabla_Z Z = 0.$$

Therefore, for any Z, $W \in \Gamma(D^{\perp})$ we have

$$\phi \nabla_{Z+W} Z+W = 0$$

$$\phi \nabla_Z Z + \phi \nabla_Z W + \phi \nabla_W Z + \phi \nabla_W W = 0$$

$$\phi \nabla_Z W + \phi \nabla_W Z = 0. \qquad (6.11)$$

Since $D^{\perp}$ is integrable, we have

$$\phi \nabla_Z W - \phi \nabla_W Z = \phi[Z,W] = 0, \qquad \text{for any } Z,\ W \in \Gamma(D^{\perp}).$$

That is, $\phi \nabla_Z W = \phi \nabla_W Z$. Hence (6.11) becomes

$$2\phi \nabla_Z W = 0.$$

Consequently, the distribution $D^{\perp}$ is auto-parallel and hence M is a CR-product by Proposition 6.1.■

The following proposition is also a modification of Theorem 6.2.

Proposition 6.3

Let M be a CR-submanifold of a Kaehler manifold N. Then M is a CR-product if and only if

$$(\nabla_U\phi)V = (\nabla_V\phi)U \qquad (6.12)$$

for any U, $V \in \Gamma(TM)$.

Proof:

The necessary part is clearly true by Theorem 6.2. Conversely, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ we have

$$\nabla_Z\phi X - \phi\nabla_Z X - \nabla_X\phi Z + \phi\nabla_X Z = (\nabla_Z\phi)X - (\nabla_X\phi)Z$$

$$\nabla_Z\phi X - \phi\nabla_Z X + \phi\nabla_X Z = 0$$

$$Q\nabla_Z\phi X + P\nabla_Z\phi X - \phi\nabla_Z X + \phi\nabla_X Z = 0. \qquad (6.13)$$

By taking Z = U and X = V in (6.1), we obtain

$$P\nabla_Z\phi X - PA_{\omega X}Z - \phi\nabla_Z X = 0$$

$$P\nabla_Z\phi X - \phi\nabla_Z X = 0.$$

By substituting the above equation into (6.13), we obtain

$$Q\nabla_Z\phi X + \phi\nabla_X Z = 0.$$

We observe that $Q\nabla_Z\phi X \in \Gamma(D^{\perp})$ and $\phi\nabla_X Z \in \Gamma(D)$. Thus, we have

$$Q\nabla_Z\phi X = \phi\nabla_X Z = 0.$$

Therefore, for any X, $Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we have

$$\langle\nabla_X Y, Z\rangle = -\langle Y, \nabla_X Z\rangle$$

$$= -\langle \phi Y, \phi \nabla_X Z \rangle$$
$$= 0.$$

Similarly, for any $X \in \Gamma(D)$ and Z, $W \in \Gamma(D^\perp)$, we have

$$\langle \nabla_Z W, \phi X \rangle = -\langle W, \nabla_Z \phi X \rangle$$
$$= -\langle W, Q\nabla_Z \phi X \rangle$$
$$= 0.$$

Therefore, we obtain that both D and $D^\perp$ are auto-parallel. Accordingly, M is a CR-product.∎

6.2 Normal CR-submanifolds of a Kaehler Manifold

Let M be a CR-submanifold of a Kaehler manifold N. We define the *Nijenhuis tensor field* of ϕ by

$$[\phi,\phi](U,V) = [\phi U,\phi V] + \phi^2[U,V] - \phi[U,\phi V] - \phi[\phi U,V]$$

for any U, $V \in \Gamma(TM)$.

The exterior derivative of ω is given by

$$d\omega(U,V) = \frac{1}{2}\{\nabla^\perp_U \omega V - \nabla^\perp_V \omega U - \omega[U,V]\}$$

for any U, $V \in \Gamma(TM)$.

We define the tensor field S by

$$S(U,V) = [\phi,\phi](U,V) - 2Bd\omega(U,V)$$

for any U, $V \in \Gamma(TM)$.

The CR-submanifold M is said to be *normal* if the tensor field S vanishes identically on M.

Bejancu [5] proved a necessary and sufficient condition for M to be normal.

Theorem 6.3

The CR-submanifold M of a Kaehler manifold N is normal if and only if

$$A_{\omega Z}\phi X = \phi A_{\omega Z}X$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Observe that for any Z, $W \in \Gamma(D^{\perp})$ we have

$$A_{\omega Z}\phi W - \phi A_{\omega Z}W = -\phi A_{\omega Z}W \in \Gamma(D).$$

Moreover, for any $X \in \Gamma(D)$ we have

$$\begin{aligned}\langle A_{\omega Z}\phi W - \phi A_{\omega Z}W, X\rangle &= \langle A_{\omega Z}\phi W, X\rangle - \langle \phi A_{\omega Z}W, X\rangle \\ &= \langle \phi W, A_{\omega Z}X\rangle + \langle A_{\omega Z}W, \phi X\rangle \\ &= -\langle W, \phi A_{\omega Z}X\rangle + \langle W, A_{\omega Z}\phi X\rangle \\ &= \langle W, A_{\omega Z}\phi X - \phi A_{\omega Z}X\rangle.\end{aligned}$$

Hence, if M is normal then

$$A_{\omega Z}\phi W - \phi A_{\omega Z}W = 0.$$

Combining our observation and Theorem 6.3, we have the following proposition.

Proposition 6.4

The CR-submanifold M of a Kaehler manifold N is normal if and only if

$$A_{\omega Z}\phi U = \phi A_{\omega Z}U$$

for any $U \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$.

By using Proposition 6.4 we have the following corollary.

Corollary 6.1

The CR-submanifold M of a Kaehler manifold N is normal if and only if

$$Bh(U,\phi V) + Bh(\phi U,V) = 0$$

for any U, $V \in \Gamma(TM)$.

Proof:

For any U, $V \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$

$$\langle Bh(U,\phi V) + Bh(\phi U,V),Z\rangle = \langle Jh(U,\phi V),Z\rangle + \langle Jh(\phi U,V),Z\rangle$$

$$= -\langle h(U,\phi V),JZ\rangle - \langle h(\phi U,V),JZ\rangle$$

$$= -\langle A_{JZ}U,\phi V\rangle - \langle A_{JZ}\phi U,V\rangle$$

$$= \langle \phi A_{JZ}U,V\rangle - \langle A_{\omega Z}\phi U,V\rangle$$

$$= \langle \phi A_{\omega Z}U - A_{\omega Z}\phi U,V\rangle.$$

From Proposition 6.4, it is clear that M is normal if and only if $Bh(U,\phi V) + Bh(\phi U,V) = 0$.■

If M is an anti-holomorphic submanifold then $J\eta = B\eta$ for any $\eta \in \Gamma(T^{\perp}M)$. As J is an isomorphism on $\Gamma(TN)$, by using Corollary 6.1 we have the following corollary.

Corollary 6.2

The anti-holomorphic submanifold M of a Kaehler manifold N is normal if and only if

$$h(U,\phi V) + h(\phi U,V) = 0$$

for any $U, V \in \Gamma(TM)$.

6.3 Sasakian CR-submanifolds of a Kaehler Manifold

The concept of Sasakian anti-holomorphic submanifolds of a Kaehler manifold was introduced by Bejancu [4] in analogy with the theory of Sasakian structure. Recently, Sun-Li [27] adapted the notion of anti-holomorphic submanifolds to CR-submanifolds and extended this study to Sasakian CR-submanfolds. In [27], Sun-Li proved that if M is a Sasakian anti-holomorphic submanifold of a Kaehler manifold N with flat normal connection and if dim D > 2 then the D-mean curvature tensor H_D is parallel. Using this fact, we shall show that under the some hypothesis and if M is connected then it admits a submanifold which is homothetic to a Sasakian manifold. Moreover, the submanifold is immersed in N as a Sasakian CR-submanifold. Thoughout this section, we let N be a Kaehler manifold and let M be a CR-submanifold of N.

Let $\{F_1,\ldots,F_p,JF_1,\ldots,JF_p\}$ be an arbitrary local field of orthonormal frames on D. We define the *D-mean curvature tensor* H_D of M by

$$H_D = \frac{1}{2p}\sum_{k=1}^{p}\{h(F_k,F_k) + h(JF_k,JF_k)\}.$$

We say that M is a *contact CR-submanifold* if $H_D \neq 0$ and we have (see Sun-Li [27])

$$d\omega(U,V) = \Omega(U,V)JBH_D$$

for any $U, V \in \Gamma(TM)$ where Ω is the fundamental 2-form of N.

Now, we would like to show that if M is a contact CR-submanifold of N then $H_D \in \Gamma(JD^\perp)$ or $CH_D = 0$.

For any $X \in \Gamma(D)$, by using (6.4) we have

$$fh(X,\phi X) - Ch(X,X) = 0. \qquad (6.14)$$

By replacing X by ϕX in (6.14) we obtain

$$-fh(\phi X,X) - Ch(\phi X,\phi X) = 0. \qquad (6.15)$$

By (6.14) and (6.15) we obtain

$$-Ch(X,X) - Ch(\phi X,\phi X) = 0$$

or

$$Ch(X,X) + Ch(\phi X,\phi X) = 0.$$

It follows that

$$CH_D = \frac{1}{2p}\sum_{k=1}^{p}\{Ch(F_k,F_k) + Ch(JF_k,JF_k)\} = 0.$$

Hence, $JBH_D = J^2H_D = -H_D$. Thus, we have

<u>Proposition 6.5</u>

The CR-submanifold M of a Kaehler manifold N is contact if and only if $H_D \neq 0$ and we have

$$d\omega(U,V) = -\Omega(U,V)H_D \qquad (6.16)$$

for any $U, V \in \Gamma(TM)$.

Observe that for any U, V $\in \Gamma(TM)$ we have

$$\begin{aligned}2d\omega(U,V) &= \overset{\perp}{\nabla}_U\omega V - \overset{\perp}{\nabla}_V\omega U - \omega[U,V]\\ &= \overset{\perp}{\nabla}_U\omega V - \overset{\perp}{\nabla}_V\omega U - \omega\nabla_U V + \omega\nabla_V U\\ &= \overset{\perp}{\nabla}_U\omega V - \omega\nabla_U V - \{\overset{\perp}{\nabla}_V\omega U - \omega\nabla_V U\}. \qquad (6.17)\end{aligned}$$

By (6.3) and (6.4) we obtain

$$h(U,\phi V) + \overset{\perp}{\nabla}_U\omega V - \omega\nabla_U V - Ch(U,V) = 0$$

or

$$\overset{\perp}{\nabla}_U\omega V - \omega\nabla_U V = -h(U,\phi V) + Ch(U,V)$$

for any U, V $\in \Gamma(TM)$. Hence, (6.17) becomes

$$2d\omega(U,V) = -h(U,\phi V) + Ch(U,V) - \{-h(V,\phi U) + Ch(U,V)\}$$

$$2d\omega(U,V) = h(\phi U,V) - h(U,\phi V).$$

Hence, we can see that (6.16) is equivalent to

$$h(\phi U,V) - h(U,\phi V) = -2\Omega(U,V)H_D. \qquad (6.18)$$

We can now extend a result of Bejancu on contact anti-holomorphic submanifold (see [4], p.68) to the setting of contact CR-submanifold.

<u>Proposition 6.6</u>

Let M be a contact CR-submanifold of N. Then M is mixed geodesic and D is not integrable.

Proof:

For any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, by taking $X = U$ and $Z = V$ in (6.18) we obtain

$$h(\phi X,Z) - h(X,\phi Z) = -2\Omega(X,Z)H_D$$
$$h(\phi X,Z) = -2\langle X,JZ\rangle H_D$$
$$= 0.$$

Thus, M is mixed geodesic.

Next, consider a unit vector field $X \in \Gamma(D)$. By taking $X = U$ and $V = \phi X$ in (6.18) we obtain

$$h(\phi X,\phi X) - h(X,\phi^2 X) = -2\Omega(X,\phi X)H_D$$
$$h(JX,\phi X) - h(X,J\phi X) = -2\langle X,J\phi X\rangle H_D$$
$$= 2\langle X,X\rangle H_D \neq 0.$$

That is, (6.9) is not satisfied. Thus D is not integrable.■

Definition

A *Sasakian CR-submanifold* is a normal contact CR-submanifold of N.

In the rest of this section, we will be mainly concerned with Sasakian anti-holomorphic submanifold. We first give a characterization for a Sasakian anti-holomorphic submanifold of a Kaehler manifold.

Theorem 6.4

Let M be an anti-holomorphic submanifold of N. If $H_D \neq 0$. Then the following statements are equivalent:

(i) M is a Sasakian anti-holomorphic submanifold

(ii) $h(\phi U,V) = \langle \phi U,V\rangle H_D$ (6.19)

(iii) $h(X,V) = \langle X,V\rangle H_D$

(iv) $A_{JZ}X = \langle H_D,JZ\rangle X$ (6.20)

(v) $A_{JZ}\phi U = \langle H_D,JZ\rangle \phi U$

for any U, V $\in \Gamma(TM)$, X $\in \Gamma(D)$ and Z $\in \Gamma(D^{\perp})$.

Proof:

First, we shall prove the equivalence of (i) and (ii). Suppose M is a Sasakian anti-holomorphic submanifold. Then by using Corollary 6.2 and (6.18) we have

$$h(\phi U,V) + h(U,\phi V) = 0$$

$$h(\phi U,V) - h(U,\phi V) = -2\Omega(U,V)H_D$$

for any U, V $\in \Gamma(TM)$.

By summing the above equations, we obtain

$$2h(\phi U,V) = -2\Omega(U,V)H_D$$

$$= -2\langle U,\phi V\rangle H_D$$

$$= 2\langle \phi U,V\rangle H_D.$$

That is, $$h(\phi U,V) = \langle \phi U,V\rangle H_D.$$

Conversely, for any $U,\ V \in \Gamma(TM)$ we have

$$\begin{aligned} h(\phi U,V) + h(U,\phi V) &= \langle \phi U,V\rangle H_D + \langle U,\phi V\rangle H_D \\ &= \langle \phi U,V\rangle H_D - \langle \phi U,V\rangle H_D \\ &= 0 \end{aligned}$$

and $$\begin{aligned} h(\phi U,V) - h(U,\phi V) &= \langle \phi U,V\rangle H_D - \langle U,\phi V\rangle H_D \\ &= -\langle U,\phi V\rangle H_D - \langle U,\phi V\rangle H_D \\ &= -2\Omega(U,V)H_D. \end{aligned}$$

According to Corollary 6.2, Proposition 6.5 and (6.18), M is a Sasakian anti-holomorphic submanifold.

If we put $U = \phi X$ in (6.19) where $X \in \Gamma(D)$, then

$$h(\phi^2 X,V) = \langle \phi^2 X,V\rangle H_D$$

$$-h(X,V) = -\langle X,V\rangle H_D.$$

Hence, (ii) implies (iii).

We shall now show that (iii) implies (iv). For any vector fields $X \in \Gamma(D)$, $U \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$,

$$\begin{aligned} \langle A_{JZ}X,U\rangle &= \langle h(X,U),JZ\rangle \\ &= \langle X,U\rangle\langle H_D,JZ\rangle \\ &= \langle\langle H_D,JZ\rangle X,U\rangle. \end{aligned}$$

This proves that (iii) implies (iv).

By taking $X = \phi U$ in (6.20), we obtain that (iv) implies (v). Finally, suppose (v) holds, then for any $U, V \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$ we have

$$\begin{aligned} \langle h(\phi U,V),JZ\rangle &= \langle A_{JZ}\phi U,V\rangle \\ &= \langle H_D,JZ\rangle\langle\phi U,V\rangle \\ &= \langle\langle\phi U,V\rangle H_D,JZ\rangle. \end{aligned}$$

Thus, we obtain

$$h(\phi U,V) = \langle\phi U,V\rangle H_D.$$

This shows that (v) implies that (ii) and hence the result is proved.■

<u>Remark</u>: Theorem 6.4 showed that the converse of Sun-Li [27, Corollary 2.1] is also true.

The following result on a Sasakian anti-holomorphic submanifold with flat normal connection is due to Sun-Li [27].

<u>Theorem 6.5</u>

Let M be a Sasakian anti-holomorphic submanifold of N with flat normal connection. If dim D > 2, then the D-mean curvature tensor H_D is parallel.

From now on, we assume that M is a connected Sasakian anti-holomorphic submanifold of N with flat normal connection and with dim D > 2. Then H_D is parallel by Theorem 6.5. If we put $\mu = \|H_D\|$, then

$$\begin{aligned} X\mu^2 &= X\langle H_D, H_D\rangle \\ &= 2\langle \nabla^{\perp}_X H_D, H_D\rangle \\ &= 0 \end{aligned}$$

for any $X \in \Gamma(TM)$. Therefore, μ^2 is a constant on some open subset of M and so is μ. As M is connected and $H_D \neq 0$, μ is a non-zero constant defined on M and hence $\xi = \frac{1}{\mu} JH_D$ is a unit vector field in $D^{\perp}$ defined on the whole of M. Furthermore, for any $U \in \Gamma(TM)$ we have

$$\begin{aligned} \nabla^{\perp}_U J\xi &= \nabla^{\perp}_U J\left(\frac{1}{\mu} JH_D\right) \\ &= -\frac{1}{\mu}\nabla^{\perp}_U H_D \\ &= 0, \qquad \text{since } H_D \text{ is parallel.} \end{aligned}$$

That is, $J\xi$ is also a parallel normal section.

Next, we define a distribution F on M by

$$F : x \longrightarrow D_x \oplus \langle \xi_x \rangle, \qquad \text{for } x \in M,$$

where $\langle \xi_x \rangle$ is the vector subspace of T_xM spanned by ξ_x.

For each $Z \in \Gamma(F)$ we put

$$\eta(Z) = \langle Z, \xi \rangle.$$

Then we have

$$Z = PZ + \eta(Z)\xi.$$

We now prove a useful Lemma.

Lemma 6.1

$$\nabla_U \xi = -\mu\phi U, \qquad \text{for all } U \in \Gamma(F).$$

Proof:

By taking a vector field $U \in \Gamma(TM)$ and $\xi = V$ in (6.3) we obtain

$$t h(U, \phi\xi) + t\nabla^{\perp}_U J\xi - \omega\nabla_U \xi = 0.$$

Since $J\xi$ is parallel and $\xi \in \Gamma(D^{\perp})$, we have

$$t h(U, \phi\xi) = t\nabla^{\perp}_U J\xi = 0.$$

It follows that $\omega\nabla_U \xi = 0$ or $\nabla_U \xi \in \Gamma(D)$. Next, for $X \in \Gamma(D)$ we have

$$\langle P\nabla_U \phi\xi - PA_{J\xi}U - \phi\nabla_U \xi, X\rangle = 0, \qquad \text{by (6.1)}$$

$$-\langle A_{J\xi}U, X\rangle - \langle \phi\nabla_U \xi, X\rangle = 0$$

$$-\langle A_{J\xi}X, U\rangle - \langle \phi\nabla_U \xi, X\rangle = 0$$

$$-\langle H_D, J\xi\rangle\langle X, U\rangle + \langle \nabla_U \xi, \phi X\rangle = 0, \qquad \text{by Theorem 6.4}$$

$$-\langle -\mu J\xi, J\xi\rangle\langle X, U\rangle + \langle \nabla_U \xi, \phi X\rangle = 0$$

$$\mu\langle \phi U, \phi X\rangle + \langle \nabla_U \xi, \phi X\rangle = 0.$$

Therefore, $$\nabla_U \xi = -\mu\phi U. \blacksquare$$

We now consider the following local field of orthonormal frames on $D^{\perp}$

$$\{\xi = E_1, E_2, \ldots, E_q\}.$$

Then we have

Theorem 6.6

The distribution F is auto-parallel and consequently is integrable.

Proof:

For any Z, W $\in \Gamma(F)$ **and j** $(2 \le j \le q)$**, we have**

$$\langle \nabla_Z W, E_j\rangle = \langle \nabla_Z PW, E_j\rangle + \langle \nabla_Z \{\eta(W)\xi\}, E_j\rangle$$

$$= \langle \nabla_Z PW, E_j\rangle + Z\eta(W)\langle \xi, E_j\rangle + \eta(W)\langle \nabla_Z \xi, E_j\rangle$$

$$= \langle \nabla_Z PW, E_j\rangle + \eta(W)\langle -\mu\phi Z, E_j\rangle, \qquad \text{by Lemma 6.1}$$

$$= \langle \nabla_Z PW, E_j \rangle.$$

From (6.3), we have

$$\langle t h(Z,\phi PW) + t\nabla_Z^{\perp}\omega PW - \omega\nabla_Z PW, JE_j \rangle = 0$$

$$\langle h(Z,\phi W), JE_j \rangle - \langle \omega\nabla_Z PW, JE_j \rangle = 0$$

$$\langle Z,\phi W\rangle\langle H_D, JE_j \rangle - \langle \nabla_Z PW, E_j \rangle = 0, \quad \text{by Theorem 6.4}$$

$$-\langle \nabla_Z PW, E_j \rangle = 0.$$

Hence, $$\langle \nabla_Z W, E_j \rangle = 0$$

That is, $\nabla_Z W \in \Gamma(F)$ and so F is auto-parallel.■

Before continuing, we shall introduce some notations. Let $\mathring{M}$ be a leaf of F. Denote by $\mathring{\nabla}$ the Levi-Civita connection induced by $\tilde{\nabla}$ on $\mathring{M}$, $\mathring{h}$ and $\mathring{A}$ respectively the second fundamental form and the fundamental tensor of Weingarten of $\mathring{M}$ in N. From the Gauss equation, for any Z, W $\in \Gamma(T\mathring{M})$ we have

$$\begin{aligned}
\mathring{\nabla}_Z W + \mathring{h}(Z,W) &= \tilde{\nabla}_Z W \\
&= \nabla_Z W + h(Z,W) \\
&= P\nabla_Z W + Q\nabla_Z W + h(Z,W) \\
&= P\nabla_Z W + \langle \nabla_Z W,\xi\rangle\xi + \sum_{j=2}^{q}\langle \nabla_Z W, E_j\rangle E_j + h(Z,W). \\
&= P\nabla_Z W + \langle \nabla_Z W,\xi\rangle\xi + h(Z,W).
\end{aligned}$$

By comparing the tangential and normal parts to $\mathring{M}$, we obtain

$$\mathring{\nabla}_Z W = P\nabla_Z W + \langle\nabla_Z W,\xi\rangle\xi \qquad (6.21)$$

$$\mathring{h}(Z,W) = h(Z,W).$$

By using Lemma 6.1 and (6.21) we obtain

$$\begin{aligned}\mathring{\nabla}_Z\xi &= P\nabla_Z\xi + \langle\nabla_Z\xi,\xi\rangle\xi \\ &= -\mu P\phi Z \\ &= -\mu\phi Z \qquad (6.23)\end{aligned}$$

for any $Z \in \Gamma(T\mathring{M})$.

The following result is a consequence of (6.23).

Proposition 6.7

ξ is a killing vector field on $\mathring{M}$.

Proof:

Let Z, W be any vector fields in $\Gamma(T\mathring{M})$. Then by using (6.23), we have

$$\begin{aligned}\langle\mathring{\nabla}_Z\xi,W\rangle + \langle Z,\mathring{\nabla}_W\xi\rangle &= -\mu\langle\phi Z,W\rangle - \mu\langle Z,\phi W\rangle \\ &= \mu\langle Z,\phi W\rangle - \mu\langle Z,\phi W\rangle \\ &= 0.\end{aligned}$$

Consequently, ξ is killing.■

We are now ready to prove the main result of this section. In fact, it divides into two parts.

<u>Theorem 6.7</u>

$\mathring{M}$ is homothetic to a Sasakian manifold.

Proof:

For any Z, W $\in \Gamma(T\mathring{M})$, by using (6.21), (6.23) and Lemma 6.1 we have

$$\begin{aligned} \mathring{\nabla}_Z\mathring{\nabla}_W\xi - \mathring{\nabla}_{\mathring{\nabla}_Z W}\xi &= -\mu\mathring{\nabla}_Z\phi W + \mu\phi\mathring{\nabla}_Z W \\ &= -\mu P\nabla_Z\phi W - \mu\langle\nabla_Z\phi W,\xi\rangle\xi + \mu\phi\nabla_Z W + \mu\langle\nabla_Z W,\xi\rangle\phi\xi \\ &= -\mu(P\nabla_Z\phi W - \phi\nabla_Z W) + \mu\langle\phi W,\nabla_Z\xi\rangle\xi \\ &= -\mu(P\nabla_Z\phi W - \phi\nabla_Z W) - \mu^2\langle\phi W,\phi Z\rangle\xi. \qquad (6.24) \end{aligned}$$

By using (6.1) we have

$$P\nabla_Z\phi W - \phi\nabla_Z W = PA_{\omega W}Z. \qquad (6.25)$$

Since M is a Sasakian anti-holomorphic submanifold, it is normal. From Proposition 6.4 we have

$$\phi A_{\omega W}Z = A_{\omega W}\phi Z.$$

By applying ϕ to the above equation, we get

$$-PA_{\omega W}Z = \phi^2 A_{\omega W}Z = \phi A_{\omega W}\phi Z.$$

By taking into account Theorem 6.4, we get

$$-PA_{\omega W}Z = \langle H_{D},\omega W\rangle\phi^2 Z$$
$$= -\langle H_D, JW\rangle PZ$$
$$= \langle JH_D, W\rangle PZ$$
$$= \mu\langle \xi, W\rangle PZ.$$

By substituing the above equation into (6.25), we obtain

$$P\nabla_Z\phi W - \phi\nabla_Z W = -\mu\langle\xi, W\rangle PZ.$$

Therefore, (6.24) becomes

$$\mathring{\nabla}_Z\mathring{\nabla}_W\xi - \mathring{\nabla}_{\mathring{\nabla}_Z W}\xi = \mu^2\langle\xi, W\rangle PZ - \mu^2\langle\phi W,\phi Z\rangle\xi$$
$$= \mu^2\eta(W)PZ - \mu^2\langle PW, PZ\rangle\xi$$
$$= \mu^2\eta(W)PZ - \mu^2\langle W, Z\rangle\xi + \mu^2\eta(W)\eta(Z)\xi$$
$$= \mu^2\eta(W)(PZ + \eta(Z)\xi) - \mu^2\langle W, Z\rangle\xi$$
$$= \mu^2\eta(W)Z - \mu^2\langle W, Z\rangle\xi.$$

From Proposition 6.7, we know that ξ is a killing vector field. Therefore, $\mathring{M}$ is homothetic to a Sasakian manifold by Theorem 1.2.■

Theorem 6.8

$\mathring{M}$ immersed in N as a Sasakian CR-submanifold.

Proof:

We define the disributions $\mathring{D}$ and $\mathring{D}^{\perp}$ on $\mathring{M}$ by $\mathring{D}_x = D_x$ and $\mathring{D}^{\perp}_x = \langle \xi_x \rangle$, for $x \in \mathring{M}$. We can see that $\mathring{D}$ is holomorphic and $\mathring{D}^{\perp}$ is anti-invariant, and hence $\mathring{M}$ is a CR-submanifold of N.

For each $X \in \Gamma(T\mathring{M})$, we put

$$JX = \mathring{\phi}X + \mathring{\omega}X$$

where $\mathring{\phi}X$ and $\mathring{\omega}X$ are the tangential and normal parts of JX respectively. Since $\mathring{D} = D$ on $\mathring{M}$, we have

$$\mathring{\phi}X = \phi X$$

and the $\mathring{D}$-mean curvature tensor of $\mathring{M}$,

$$\mathring{H}_D = H_D.$$

Since M is a Sasakian anti-holomorphic submanifold, by Theorem 6.4 and (6.22), for any Z, $W \in \Gamma(T\mathring{M})$ we have

$$\begin{aligned} \mathring{h}(\phi Z,W) &= h(\phi Z,W) \\ &= \langle \phi Z,W\rangle H_D. \end{aligned}$$

Therefore,

$$\begin{aligned} h(\phi Z,W) - h(Z,\phi W) &= \langle \phi Z,W\rangle H_D - \langle Z,\phi W\rangle H_D \\ &= -2\langle Z,\phi W\rangle H_D = -2\Omega(Z,W)H_D \end{aligned}$$

and

$$\begin{aligned} h(\phi Z,W) + h(Z,\phi W) &= \langle \phi Z,W\rangle H_D + \langle Z,\phi W\rangle H_D \\ &= \langle \phi Z,W\rangle H_D - \langle \phi Z,W\rangle H_D = 0. \end{aligned}$$

Therefore, $\mathring{M}$ is a Sasakian CR-submanifold of N by means of Corollary 6.1 and (6.18).■