1 Introduction

1.1 General Introduction

Combinatorial group theory studies groups from the perspective of their presentations, that is, their generators and relations. It is useful where finiteness assumptions are satisfied, for example, finitely generated groups. There are several natural questions arising about a group given by its presentation. The *word problem* asks whether two words are effectively the same group element. An equally important problem is the isomorphism problem which asks whether two groups given by different presentation are actually isomorphic.

In order to study the word problem, we study the property of residual finiteness in groups. Mostowski [13] showed that a finitely presented residually finite group has solvable word problem. The term residually finite was first introduced by Philip Hall in 1955 and the first systematic study on residually finite groups was done by Gruenberg. G. Baumslag [4] first studied the residual finiteness of generalised free products in detail. He proved that free products of residually finite groups amalgamating a finite subgroup or, under certain conditions, amalgamating a cyclic subgroup is again residually finite. Since then many mathematicians have done research on residual finiteness and its various extensions.

We will study a stronger residually finite property called weak potency. A group G is said to be weakly potent if for every element x of infinite order in G we can find a positive integer r with the property that for every positive integer n, there exists a normal subgroup N of finite index in G such that xN has order exactly rn. Clearly weak potency is a stronger form of residual finiteness.

In 1974, Evans[5] introduced the concept of weak potency with the name regular quotients. He showed that generalised free products of finite groups and generalised free products of weakly potent groups amalgamating finite subgroups are weakly potent. He then used it to show the cyclic subgroup separability of certain generalised free products.

In 1995, C.Y. Tang [18] defined weak potency independently and he proved that finite extensions of free groups and finitely generated torsion-free nilpotent groups are weakly potent. He then used it to determine the conjugacy separability of certain generalised free products of conjugacy separable groups. Tang's definition is derived from the stronger property of potency which was introduced by Allenby and Tang [1] in 1981 to prove the residual finiteness of certain one-relator groups with torsion.

In the first part of this thesis, we shall study the HNN extensions of weakly potent groups. In general, the weak potency of HNN extensions are not much known. Indeed one of the simplest type of HNN extensions, $\langle a, b | a^{-1}b^2a = b^3 \rangle$ is not even residually finite. However, B. Baumslag and Tretkoff [3] showed that HNN extensions of residually finite groups with certain conditions are residually finite.

Then we shall study the weak potency of generalised free products of weakly potent groups. A generalised free product of weakly potent groups need not be weakly potent or even residually finite (see [4], [8]). However, Evans [5] showed that if the amalgamated subgroup is cyclic or finite, then the generalised free product of two weakly potent groups is again weakly potent.

In our proofs, we shall use filters. The concept of filters was first introduced by G. Baumslag [4], called filtration, to prove the residual finiteness of generalised free products of nilpotent groups. More recently, Kim, Lee and McCarron [11] also used filters to show the residual p-finiteness of generalised free products of groups. In these papers, the filters are sufficient to show the residual finiteness (or p-finiteness). However, Shirvani in [17] questioned the necessity of the filters. He then showed that for residually finite groups satisfying a non-trivial relation, the filters are also necessary.

We now give a brief description of the chapters in this thesis.

In this chapter (Chapter 1), we give the general literature review and some basic facts about generalised free products, tree products and HNN extensions. We also introduce the notations used in this thesis.

In Chapter 2, we give the definitions for residual finiteness, subgroup separability, weak potency and characteristically weak potency. We also prove that finitely generated nilpotent groups are characteristically weakly potent. Since the main tools used in this thesis to prove the weak potency of certain generalised free products and HNN extensions are filters, we then give a brief discussion on the history and use of filters.

We study the weak potency of HNN extensions in Chapter 3. We first introduce the concept of h-filters and then we use it to prove the main criterion for weak potency of HNN extensions. Then we prove several characterisations for the weak potency of certain HNN extensions with cyclic associated subgroups. By applying the previous results, we can easily obtain a characterisation for the weak potency of the Baumslag-Solitar groups.

In Chapter 4, we extend the main criterion obtained in Chapter 3. We then apply this new criterion to the study of HNN extensions of finitely generated nilpotent groups. Next we shall give characterisations for certain HNN extensions of characteristically weakly potent groups with finitely generated central associated subgroups to be weakly potent. Finally, we prove some equivalent conditions for HNN extensions of free abelian group of finite rank to be weakly potent.

Finally in Chapter 5, we study the weak potency of generalised free products. We first give the definition of w-filter and prove a criterion for generalised free products to be weakly potent. Later, we apply this criterion to give characterisations for the weak potency of generalised free products with cyclic amalgamated subgroups and with central amalgamated subgroups. Then we extend the results to tree products of finitely many groups. Before we end this chapter, we show that certain one-relator groups are weakly potent.

1.2 Generalised Free Products

The concept of generalised free product was introduced by O. Schreier in 1927. Now we give a description of the generalised free product of two groups. Let A and B be groups and H and K be subgroups of A and B respectively with $\phi : H \to K$ an isomorphism. Then the generalised free product of A and B amalgamating the subgroups H of A and K of B via the isomorphism ϕ, is defined to be the group generated by the generators and relations of the groups A and B with the extra relations $\phi(h) = k$ where $h \in H, k \in K$. We let this generalised free product be the group G. Then we can write G as follows.

$$G = \langle A, B | \phi(h) = k \rangle, h \in H, k \in K.$$

Furthermore, G can also be written in another form which is widely used in this dissertation.

$$G = A \underset{H=K}{*} B$$

A and B are called the factors of the group G and H and K are the amalgamated subgroups.

We let g be an element in G. We say that g is in reduced form if $g = g_1 g_2 \dots g_n$ and no consecutive terms are from the same factor. The length of the reduced element $g = g_1 g_2 \dots g_n$ is denoted by $||g||$ and is defined as follows:

$$||g|| = \begin{cases} 0, & \text{if } n = 1 \text{ and } g_1 \in H \\ 1, & \text{if } n = 1 \text{ and } g_1 \in (A \cup B) - H \\ n, & \text{otherwise} \end{cases}$$

The reduced element $g = g_1 g_2 \dots g_n$ is called cyclically reduced if each of its cyclic permutations $g_i g_{i+1} \dots g_n g_1 g_2 \dots g_{i-1}$ is reduced.

1.3 Tree Products

Let us give some facts about tree products. Tree products were first introduced by Karrass and Solitar in 1970. Briefly tree products can be described as follows: Let T be a graph which is a tree. Assign to each vertex v of T a vertex group G_v and to each edge e, an edge group G_e together with monomorphisms α_e and β_e embedding G_e to the vertex groups at the end of e. The tree product G of T amalgamating the edge groups is defined to be the group generated by generators and relations of the vertex groups, together with the additional relations obtained by identifying $\alpha_e(g_e)$

and $\beta_e(g_e)$ for each $g_e \in G_e$.

By abused of notation, let G be a tree product of the vertex groups $A_1, A_2, \ldots, A_n$, $n \geq 2$, amalgamating the edge subgroups H_{ij} of A_i and H_{ji} of A_j. We shall denote $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$.

1.4 HNN Extensions

Let A be a group and let H and K be subgroups of A such that $\phi : H \to K$ is an isomorphism. The HNN extension G of A relative to the subgroups H and K with the isomorphism ϕ is defined to be the group generated by the generators and relations of the group A with an extra generator t and extra relations $t^{-1}ht = \phi(h)$ for each $h \in H$. We write

$$G = \langle t, A | t^{-1}ht = \phi(h), \forall h \in H \rangle$$

The group A is called the base group and t is called the stable letter. H and K are the associated subgroups and ϕ is the associated isomorphism of G.

Let $g \in G$. Then we have $g = g_0 t^{e_1} g_1 \ldots t^{e_n} g_n$ with $e_i = \pm 1$. The element g is said to be in reduced form if there are no consecutive terms $t^{-1} g_i t$ with $g_i \in H$ or $t g_n t^{-1}$ with $g_n \in K$. Note that each element of G can be written in reduced form. The length of a reduced element $g = g_0 t^{e_1} g_1 \ldots t^{e_n} g_n$ is denoted by $||g||$ and is defined as follows:

$$||g|| = \begin{cases} 0, & \text{if } g = g_0 \\ n, & \text{otherwise} \end{cases}$$

1.5 Notations and Definitions

Before we proceed to the next chapter, let us give some notations. The notations used in this dissertation are standard. In addition, the following will be used. Let G be a group.

$N \lhd_f G$ means N is a normal subgroup of finite index in the group G.

$N \ char_f \ A$ means N is a characteristic subgroup of finite index in A.

$Z(G)$ denotes the center of G.

$h \sim_G k$ means h is conjugate to k in G for $h, k \in G$.

2 Weak Potency and Filters

In this thesis, we shall study the weak potency of HNN extensions and generalised free products of weakly potent groups. We begin with the various definitions in the first part of this chapter. Since we shall use filters as the main method in our proofs, we shall offer a brief discussion on the history and use of filters in the second part.

2.1 Definitions

In this section we state the definitions of residual finiteness, subgroup separability, weak potency and characteristically weak potency. Then we shall prove that finitely generated nilpotent groups are characteristically weakly potent.

Definition 2.1. *A group G is said to be residually finite if, for each nontrivial $x \in G$, there exists $N \lhd_f G$ such that $x \notin N$.*

Definition 2.2. *A group G is called H-separable for the subgroup H if for each $x \in G \backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$. G is termed subgroup separable if G is H-separable for every finitely generated subgroup H.*

The well known subgroup separable groups are free groups and polycyclic groups ([7], [12]). Free-by-finite groups and polycyclic-by-finite groups are subgroup separable since a finite extension of a subgroup separable group is again subgroup separable.

Next we give the definitions of weak potency and characteristically weak potency. We also proved that finitely generated nilpotent groups are characteristically weakly potent.

Definition 2.3. *[18] A group G is said to be weakly potent if for every element x of infinite order in G we can find a positive integer r with the property that for every positive integer n, there exists a normal subgroup N of finite index in G such that xN has order exactly rn.*

Examples of weakly potent groups are free groups, polycyclic groups, free-by-finite groups, and polycyclic-by-finite groups. (see Evans [5], Tang [18])

Definition 2.4. *[5] A group G is said to be characteristically weakly potent if, for every element x of infinite order in G, we can find a positive integer r such that for every positive integer n, there exists a characteristic subgroup N of finite index such that xN has order exactly rn in G/N.*

Lemma 2.5. *([6]) Every finitely generated nilpotent group, with elements of finite order, is residually 'of order a power of one of $p_1, \dots, p_r$', where $p_1, \dots, p_r$ are the distinct primes genuinely occurring in the orders of the elements.*

Lemma 2.6. *Let A be a nilpotent group with $x \in A$ and let n be a positive integer which divides the order of x. Then we can find a positive integer r such that there exists a characteristic subgroup N of A such that xN has order rn.*

Proof. By [14], the lemma is true for $r = 1$. Now, suppose $r \neq 1$. Let's say the order of x is m. Then n divides m implies $m = kn$ for some positive integer k. We can find a prime number p that divides k. Then pn also divides m. Let $p = r$ and we are done. □

Theorem 2.7. *Let G be a finitely generated nilpotent group. Then G is characteristically weakly potent.*

Proof. Let $g \in G$ be an element of infinite order. By Lemma 2.5, there exists $N \triangleleft_f G$ such that xN has order p^n in G/N and $|G/N| = p^s$ for some integer $s \geq n$. The group G^{p^s} is a characteristic subgroup of finite index in G contained in N. The quotient G/G^{p^s} is a finite p-group and $|xG^{p^s}| \geq p^n$. Then the result follows from Lemma 2.6. □

2.2 Filters in generalised free products

The main tools used in this thesis to prove the weak potency of generalised free products and HNN extensions are filters, called h-filters and w-filters. They are defined and used to prove the main criterions in Chapters 3 and 5 respectively. We now give a brief discussion on the history and use of filters in the next three sections.

In the seminal paper [4], G. Baumslag introduced the concept of filters, called filtration, in order to prove a criterion for the residual finiteness of generalised free products of finitely generated nilpotent groups. This criterion has become a standard result in the study of residual finiteness of generalised free products.

To state the criterion of G. Baumslag, we let $G = A *_H B$ be the generalised free product amalgamating subgroup H and $\Delta = \{(M,N) | M \lhd_f A, N \lhd_f B \text{ and } M \cap H = N \cap H\}$.

The criterion is as follows:

Theorem 2.8. *[4] Let $G = A *_H B$. Suppose that*

(i) $\bigcap_{(M,N)\in\Delta} MH = H$ *and* $\bigcap_{(M,N)\in\Delta} NH = H$;

(ii) $\bigcap_{(M,N)\in\Delta} M = 1 = \bigcap_{(M,N)\in\Delta} N$.

Then G is residually finite.

For each $P = (M,N) \in \Delta$, let $\pi_M : A \to A/M$ and $\pi_N : B \to B/N$ be the natural maps. Let $G_P = A/M *_{\bar{H}} B/N$ be the generalised free product where $\bar{H} = HM/M \cong HN/N$. Since A/M and B/N are finite, G_P is residually finite by [5]. The maps π_M and π_N can be extended to a map π_P from G onto G_P. Then G is residually finite if and only if $\bigcap_{(M,N)\in\Delta} Ker\ \pi_P = 1$. Now $\bigcap_{(M,N)\in\Delta} Ker\ \pi_P = 1$ implies that $\bigcap_{(M,N)\in\Delta} M = A \cap \bigcap_{(M,N)\in\Delta} Ker\ \pi_P = 1$ and $\bigcap_{(M,N)\in\Delta} N = B \cap \bigcap_{(M,N)\in\Delta} Ker\ \pi_P = 1$. This shows that condition (ii) of the criterion is necessary for the residual finiteness of G. However, Shirvani [17] was able to construct a residually finite generalised free product where condition (i) is not necessary. This is given in the following example.

Example 2.9. *[17] There exists a residually finite generalised free product $G = A *_H B$ where A is torsion-free abelian and $H \subsetneq A, H \subsetneq B$ but A is not H-separable. Therefore condition* (i) *is not necessary.*

In the same paper [17], Shirvani also proved a theorem where the conditions (i) and (ii) are necessary and sufficient.

2.3 Filters in HNN extensions

In [3], B. Baumslag and Tretkoff in their study of HNN extensions proved a criterion for residual finiteness. That criterion has become a standard result in the study of residual finiteness of one-relator groups. We restate the criterion of B. Baumslag and Tretkoff using filters (see [16]). Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension and $\Delta = \{N \lhd_f A | \phi(N \cap H) = N \cap K\}$.

Then the criterion can be stated as follows:

Theorem 2.10. *[16] Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension. Suppose that*

(i) $\bigcap_{N \in \Delta} HN = H$ and $\bigcap_{N \in \Delta} KN = K$;

(ii) $\bigcap_{N \in \Delta} N = 1$.

Then G is residually finite.

For each $N \in \Delta$, let $\pi_N : A \to A/N$ be the natural map and $\phi_N : HN/N \to KN/N$ be the isomorphism induced by ϕ. Now we let $G_N = \langle t, A/N | t^{-1}(HN/N)t = KN/N, \phi_N \rangle$ be the HNN extension with base group A/N. Since A/N is finite, G_N is residually finite by [3]. The map π_N can be extended to a map, again denoted by π_N, from G onto G_N.

Then G is residually finite if and only if $\bigcap_{N \in \Delta} Ker\ \pi_N = 1$. Now $\bigcap_{N \in \Delta} Ker\ \pi_N = 1$ implies that $\bigcap_{N \in \Delta} N = \bigcap_{N \in \Delta} Ker\ \pi_N \cap A = 1$. This shows that condition (ii) of the criterion is necessary for the residual finiteness of G. However, Shirvani [16] proved a result which was used in constructing the following example showing that condition (i) is not necessary .

Example 2.11. *The group $\langle t, a | t^{-1}at = a^2 \rangle$ is a residually finite HNN extension. Then $\bigcap_{N \in \Delta} \langle a^2 \rangle N = \bigcap_{N \in \Delta} \langle a \rangle N \supseteq \langle a \rangle \supsetneq \langle a^2 \rangle$. Therefore, condition* (i) *is not necessary.*

In the same paper [16], Shirvani also proved three theorems where conditions (i) and (ii) are necessary and sufficient.

2.4 The basis of our method

Recently Kim, Lee and McCarron [11] introduced the concept of p-filters to study the residual p-finiteness of generalised free products of nilpotent groups. These filters are defined as follows.

Definition 2.12. *[11] Let $G = A \ast_H B$. Let $\Delta = \{(M_i, N_i) | i \in I\}$ be a non-empty family of pairs (M_i, N_i), where $M_i \lhd A$ and $N_i \lhd B$, satisfying the following:*

(F1) $M_i \cap H = N_i \cap H$ *for each* $i \in I$;

(F2) For each $i \in I$, $A/M_i \underset{\bar{H}}{\ast} B/N_i$ is residually p-finite, where $\bar{H} = HM_i/M_i \simeq HN_i/N_i$;

(F3) For each $i_1, i_2, \ldots, i_n \in I$ and $n \in \mathbb{Z}^+$, $(\bigcap_{k=1}^{n} M_{i_k}, \bigcap_{k=1}^{n} N_{i_k}) \in \Delta$;

(F4) $\bigcap_{i \in I} HM_i = H = \bigcap_{i \in I} HN_i$.

Such Δ is called a p-filter of the generalised free product $G = A \ast_H B$.

The filters used in this thesis are based on these p-filters.

3 HNN Extensions of Weakly Potent Groups

3.1 Introduction

In this chapter, we begin a systematic study on weak potency of HNN extensions by using filters. This chapter will be divided into two main parts. In the first part, we will introduce h-filters which will be the basis for the proof of the main criterion. Next we apply the main criterion to prove the weak potency of various HNN extensions with cyclic associated subgroups in the second part. Finally we prove a characterization for the Baumslag-Solitar groups to be weakly potent.

Notation:

For the rest of this chapter, we let A be a group with subgroups H and K and let $\phi : H \to K$ be an isomorphism.

The following lemma will be used to prove most of the results in this chapter.

Lemma 3.1. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where A is finite. Then G is free-by-finite (see [9]) and hence weakly potent (see [5]).*

3.2 Filters and weak potency

In this section we introduce h-filters (Definition 3.2) and then prove the main criterion (Theorem 3.3) for HNN extensions to be weakly potent. We then apply our criterion to HNN extensions with finite associated subgroups.

Definition 3.2. *Let A be a group with subgroups H and K and let $\phi : H \to K$ be an isomorphism. Let $\Delta = \{N_i \mid i \in I\}$ be a non-empty family of normal subgroups N_i of A. Then Δ is called an h-filter of A if the following hold:*

$(H1)$ *$\phi(N_i \cap H) = N_i \cap K$, for all $i \in I$;*

$(H2)$ *For each $i \in I$, the HNN extension with base group A/N_i and associated subgroups HN_i/N_i, KN_i/N_i is weakly potent;*

(*H*3) $\bigcap_{i\in I} N_i = 1$;

(*H*4) *For each* $i_1, i_2, \ldots, i_n \in I$ *and* $n \in \mathbb{Z}^+$, $\bigcap_{k=1}^{n} N_{i_k} \in \Delta$;

(*H*5) $\bigcap_{i\in I} HN_i = H$ *and* $\bigcap_{i\in I} KN_i = K$.

We now prove our main criterion.

Theorem 3.3. *Let* $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$. *Suppose*

(1) Δ *is an h-filter of A; and*

(2) *For each* $a \in A$ *of infinite order, we can find a positive integer* r *such that for each positive integer* n*, there exists* $N \in \Delta$ *such that* $N \cap \langle a\rangle = \langle a^{rn}\rangle$.

Then G is weakly potent.

Proof. Let $g \in G$ be a reduced element of infinite order.

Suppose $||g|| = 0$. Then $g \in A$. By assumption, we can find a positive integer r such that for each positive integer n, there exists $N \in \Delta$ such that $N \cap \langle g\rangle = \langle g^{rn}\rangle$. Let $\bar{G} = \langle t, A/N | t^{-1}(HN/N)t = KN/N, \bar{\phi}\rangle$ where $\bar{\phi} : HN/N \to KN/N$ is induced from ϕ. Then $\bar{g}, \bar{g}^2, \ldots, \bar{g}^{rn-1}$ are nontrivial in $\bar{G}$. Since $\bar{G}$ is residually finite by $(H2)$, there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{g}, \bar{g}^2, \ldots, \bar{g}^{rn-1} \notin \bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and gL has order rn.

Suppose $||g|| \geq 1$. We can assume $g = a_0 t^{e_1} a_1 t^{e_2} \ldots t^{e_n} a_n$ where $a_i \in A$ and $n \geq 1$. Let x_i be those $a_i \in A\backslash H$, y_i be those $a_i \in A\backslash K$ and $z_i \in H \cap K\backslash\{1\}$. By $(H5)$, there exist $N_x, N_y \in \Delta$ such that $x_i \notin HN_x$ and $y_i \notin KN_y$. By $(H3)$, there exists $N_z \in \Delta$ such that $z_i \notin N_z$. Let $N = \bigcap(N_x \cap N_y \cap N_z)$. Then $N \in \Delta$ and $\phi(N \cap H) = N \cap K$ by $(H1)$ and $(H4)$. Let $\bar{G} = \langle t, A/N | t^{-1}(HN/N)t = KN/N, \bar{\phi}\rangle$ as above. Then $||\bar{g}|| = ||g||$. Since $\bar{G}$ is weakly potent by $(H2)$, we can find a positive integer r such that for each positive integer n, there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{g}\bar{L}$ has order rn. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and gL has order rn. □

Theorem 3.4. *Let* $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$*, where A is weakly potent and H and K are finite subgroups of A. Then G is weakly potent.*

Proof. We use Theorem 3.3 here. First, we show that $\Delta = \{N \lhd_f A | N \cap H = 1 = N \cap K\}$ is an h-filter of A.

Since A is residually finite and H, K are finite, there exists $N \lhd_f A$ such that $N \cap H = 1 = N \cap K$. This implies $N \in \Delta$ and $\Delta \neq \emptyset$. By Lemma 3.1, $\langle t, A/N | t^{-1}(HN/N)t = KN/N \rangle$ is weakly potent. Thus $(H2)$ holds.

By the residual finiteness of A, for each nontrivial $a \in A$, there exists $M_a \lhd_f A$ such that $a \notin M_a$ and $M_a \cap H = 1 = M_a \cap K$. Thus, $M_a \in \Delta$. Therefore, $\bigcap_{N \in \Delta} N = 1$ and $(H3)$ holds.

Let $H = \{h_0 = 1, h_1, h_2, \ldots, h_n\}$ and $a \in A \backslash H$. Then $ah_i^{-1} \neq 1$ for each $h_i \in H, i = 0, 1, \ldots, n$. By the residual finiteness of A, there exists $N \lhd_f A$ such that $ah_i^{-1} \notin N$ for all i and $N \cap H = 1 = N \cap K$. Therefore $a \notin HN$ and $N \in \Delta$. This implies $a \notin \bigcap_{N \in \Delta} HN$ and hence $\bigcap_{N \in \Delta} HN \subseteq H$. Thus $\bigcap_{N \in \Delta} HN = H$. Similarly, $\bigcap_{N \in \Delta} KN = K$. Thus $(H5)$ holds.

Since $(H1)$ and $(H4)$ hold trivially, Δ is an h-filter of A.

Let $a \in A$ has infinite order. Since H, K are finite, there exists $N_1 \lhd_f A$ such that $N_1 \cap H = 1 = N_1 \cap K$. Suppose $N_1 \cap \langle a \rangle = \langle a^s \rangle$ for some positive integer s. By weak potency of A, we can find a positive integer r such that for each positive integer n, there exists $N_2 \lhd_f A$ such that $N_2 \cap \langle a \rangle = \langle a^{rsn} \rangle$. Let $N = N_1 \cap N_2$. Then $N \lhd_f A$ such that $N \cap \langle a \rangle = \langle a^{rsn} \rangle$ and $N \cap H = 1 = N \cap K$. Hence $N \in \Delta$ and we are done. $\square$

The two conditions in Theorem 3.3 are sufficient for an HNN extension to be weakly potent. We will also examine the converse, that is, if an HNN extension is weakly potent then these two conditions are necessary. For ease of exposition, we state the two conditions given in Theorem 3.3 as Condition 3.3 below.

Condition 3.3:

Let $\Delta = \{N \lhd_f A | \phi(N \cap H) = N \cap K\}$ such that

(1) Δ is an h-filter of A; and

(2) For each $a \in A$ of infinite order, we can find a positive integer r such that for each positive integer n, there exists $N \in \Delta$ such that $N \cap \langle a \rangle = \langle a^{rn} \rangle$.

3.3 HNN extensions with cyclic associated subgroups

In this section we study HNN extensions of weakly potent groups with cyclic associated subgroups.

Lemma 3.5. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ where $h, k \in A$ have infinite order and $h \sim_A k$. If G is weakly potent, then Condition 3.3 is satisfied.*

Proof. Since $A \in \Delta$, $\Delta \neq \emptyset$. For each $N \in \Delta$, $\langle t, A/N | t^{-1}(hN)t = kN \rangle$ is weakly potent by Lemma 3.1. Thus $(H2)$ holds.

Let $a \in A$ be a nontrivial element. Since G is residually finite, there exists $M \lhd_f G$ such that $a \notin M$. Let $N_a = M \cap A$. Then $N_a \lhd_f A$ and $a \notin N_a$. Furthermore $\phi(N_a \cap \langle h \rangle) = \phi(M \cap A \cap \langle h \rangle) = \phi(M \cap \langle h \rangle) = t^{-1}(M \cap \langle h \rangle)t = t^{-1}Mt \cap t^{-1}\langle h \rangle t = M \cap \langle k \rangle = M \cap A \cap \langle k \rangle = N_a \cap \langle k \rangle$. Therefore $N_a \in \Delta$. Hence $\bigcap_{N \in \Delta} N = 1$ and $(H3)$ holds.

Let $x \in A \backslash \langle h \rangle$. Suppose $x \in \langle h \rangle N$, whenever $N \in \Delta$. Since $h \sim_A k$, $h = a^{-1}ka$ for some $a \in A$. Then we let $g = [t^{-1}xt, axa^{-1}]$. Since $x \notin \langle h \rangle$, $g = (t^{-1}x^{-1}t)(ax^{-1}a^{-1})(t^{-1}xt)(axa^{-1}) \neq 1$. By the residual finiteness of G, there exists $M \lhd_f G$ such that $g \notin M$. Let $N = M \cap A$. Then $N \lhd_f A$ and $N \in \Delta$ as in above. We can form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{h}t = \bar{k} \rangle$ where $\bar{A} = A/N$, $\bar{h} = hN$ and $\bar{k} = kN$. Since $x \in \langle h \rangle N$, there exists some integer s such that $\bar{x} = \bar{h}^s$. Then $t^{-1}\bar{x}t = t^{-1}\bar{h}^s t = \bar{k}^s$ and $\bar{a}\bar{x}\bar{a}^{-1} = \bar{a}\bar{h}^s\bar{a} = \bar{k}^s$. This implies $\bar{g} = \bar{1}$, a contradiction.

Therefore $x \notin \langle h \rangle N$, for some $N \in \Delta$. This implies $x \notin \bigcap_{N \in \Delta} \langle h \rangle N$ and hence $\bigcap_{N \in \Delta} \langle h \rangle N \subseteq \langle h \rangle$. Thus $\bigcap_{N \in \Delta} \langle h \rangle N = \langle h \rangle$. Similarly, $\bigcap_{N \in \Delta} \langle k \rangle N = \langle k \rangle$. Then we have $(H5)$.

Since $(H1)$ and $(H4)$ hold trivially, Δ is an h-filter of A.

Let $a \in A$ be of infinite order. Since G is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $M \lhd_f G$ such that $M \cap \langle a \rangle = \langle a^{rn} \rangle$. Let $N = M \cap A$. Then $N \lhd_f A$ and $N \in \Delta$ as above. Furthermore $N \cap \langle a \rangle = M \cap A \cap \langle a \rangle = M \cap \langle a \rangle = \langle a^{rn} \rangle$. Hence, we are done. $\square$

Theorem 3.6. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ where A is weakly potent and $h, k \in A$ have infinite order such that $h \sim_A k$. Then G is weakly potent if and only if A is $\langle h \rangle$-separable and $\langle k \rangle$-separable.*

Proof. Suppose A is $\langle h \rangle$-separable and $\langle k \rangle$-separable. Since $A \in \Delta$, $\Delta \neq \emptyset$. For each $N \in \Delta$, $\langle t, A/N | t^{-1}(hN)t = kN \rangle$ is weakly potent by Lemma 3.1. Thus $(H2)$ holds.

Before we proceed with the proof, we show that for every $N \lhd_f A$, $N \in \Delta$. Let $N \cap \langle h \rangle = \langle h^s \rangle$ for some positive integer s. Then $N \cap \langle k \rangle = \langle k^s \rangle$ since $h = a^{-1}ka$ for some $a \in A$. Therefore $\phi(N \cap \langle h \rangle) = N \cap \langle k \rangle$ and hence $N \in \Delta$.

By the residual finiteness of A, for each nontrivial element $a \in A$, there exists $N_a \lhd_f A$ such that $a \notin N_a$. As shown above, $N_a \in \Delta$. Thus $\bigcap_{N \in \Delta} N = 1$ and $(H3)$ holds.

Since A is $\langle h \rangle$-separable, for any $a \in A \backslash \langle h \rangle$, there exists $N \lhd_f A$ such that $a \notin \langle h \rangle N$. As shown above, $N \in \Delta$. Therefore $a \notin \bigcap_{N \in \Delta} \langle h \rangle N$ and hence $\bigcap_{N \in \Delta} \langle h \rangle N \subseteq \langle h \rangle$. Thus $\bigcap_{N \in \Delta} \langle h \rangle N = \langle h \rangle$. Similarly, $\bigcap_{N \in \Delta} \langle k \rangle N = \langle k \rangle$. Hence, we have $(H5)$.

Since $(H1)$ and $(H4)$ hold trivially, Δ is an h-filter of A.

Let $a \in A$ be of infinite order. Since A is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $N \lhd_f A$ such that $N \cap \langle a \rangle = \langle a^{rn} \rangle$. As shown above, $N \in \Delta$. Therefore G is weakly potent by Theorem 3.3.

Conversely if G is weakly potent, then by Lemma 3.5, G satisfies Condition 3.3 and hence A is $\langle h \rangle$-separable and $\langle k \rangle$-separable. $\square$

Lemma 3.7. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ where $h, k \in A$ have infinite order and $h^m = k^{\pm m}$ for some $m > 0$. If G is weakly potent, then Condition 3.3 is satisfied .*

Proof. The proof is similar to Lemma 3.5 except for the following changes.

Let $x \in A\backslash H$. Suppose $x \in \langle h \rangle N$, whenever $N \in \Delta$. Now, let $g = [t^{-1}xt, h^m]$. Then $g = (t^{-1}x^{-1}t)(h^{-m})(t^{-1}xt)(h^m) = t^{-1}[x, h^{\pm m}]t \neq 1$ since $x \notin \langle h \rangle$.

By the residual finiteness of G, there exists $M \lhd_f G$ such that $g \notin M$. Let $N = M \cap A$. Then $N \lhd_f A$ and $\phi(N \cap \langle h \rangle) = \phi(M \cap A \cap \langle h \rangle) = \phi(M \cap \langle h \rangle) = t^{-1}(M \cap \langle h \rangle)t = t^{-1}Mt \cap t^{-1}\langle h \rangle t = M \cap \langle k \rangle = M \cap A \cap \langle k \rangle = N \cap \langle k \rangle$. Thus, $N \in \Delta$. We can form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{h}t = \bar{k} \rangle$ where $\bar{A} = A/N$, $\bar{h} = hN$ and $\bar{k} = kN$. Since $x \in \langle h \rangle N$, there exists some integer s such that $\bar{x} = \bar{h}^s$. Then $t^{-1}\bar{x}t = t^{-1}\bar{h}^s t = \bar{k}^s$ and $\bar{h}^m = \bar{k}^{\pm m}$. This implies that $\bar{g} = [\bar{k}^s, \bar{k}^{\pm m}] = \bar{1}$, a contradiction.

Therefore $x \notin \langle h \rangle N$, for some $N \in \Delta$. This implies $x \notin \bigcap_{N \in \Delta} \langle h \rangle N$ and hence $\bigcap_{N \in \Delta} \langle h \rangle N \subseteq \langle h \rangle$. Thus $\bigcap_{N \in \Delta} \langle h \rangle N = \langle h \rangle$. Similarly, $\bigcap_{N \in \Delta} \langle k \rangle N = \langle k \rangle$. Thus, we have $(H5)$. □

Theorem 3.8. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ where A is weakly potent and $h, k \in A$ have infinite order such that $h^m = k^{\pm m}$, for some $m > 0$. Then G is weakly potent if and only if A is $\langle h \rangle$-separable and $\langle k \rangle$-separable.*

Proof. The proof is similar to Theorem 3.6 except for the following changes.

Before we proceed with the proof, we construct $N^* \lhd_f A$ such that $N^* \in \Delta$. By weak potency of A, we find positive integers r_1, r_2 such that for each positive integer n, there exist $P_1 \lhd_f A, P_2 \lhd_f A$ such that $P_1 \cap \langle h \rangle = \langle h^{r_1 n} \rangle, P_2 \cap \langle k \rangle = \langle k^{r_2 n} \rangle$. Let $N_1 \lhd_f A, N_2 \lhd_f A$ be such that $N_1 \cap \langle h \rangle = \langle h^{r_1 r_2 m} \rangle$ and $N_2 \cap \langle k \rangle = \langle k^{r_1 r_2 m} \rangle$. Let $N^* = N_1 \cap N_2$. Then $N^* \lhd_f A$ and $N^* \cap \langle h \rangle = \langle h^{r_1 r_2 m} \rangle = \langle k^{r_1 r_2 m} \rangle = N^* \cap \langle k \rangle$. Hence, $N^* \in \Delta$.

Since A is residually finite, for each nontrivial element $a \in A$, there exists $N_a \lhd_f A$ such that $a \notin N_a$. Let $N_a^* = N_a \cap N^*$ where N^* is as constructed above. Then $N_a^* \lhd_f A$ and $a \notin N_a^*$. Furthermore $N_a^* \cap \langle h \rangle = N_a \cap N^* \cap \langle h \rangle = N_a \cap N^* \cap \langle k \rangle = N_a^* \cap \langle k \rangle$. Hence, $N_a^* \in \Delta$ and therefore $\bigcap_{N \in \Delta} N = 1$. Thus $(H3)$ holds.

Since A is $\langle h \rangle$-separable, for any $a \in A \backslash \langle h \rangle$, there exists $N_a \lhd_f A$ such that $a \notin \langle h \rangle N_a$. As shown in the previous paragraph, there exists $N_a^* \in \Delta$ such that $N_a^* \subseteq N_a$. Hence, $a \notin \langle h \rangle N_a^*$. It follows that $a \notin \bigcap_{N \in \Delta} \langle h \rangle N$ and hence $\bigcap_{N \in \Delta} \langle h \rangle N \subseteq \langle h \rangle$. Thus $\bigcap_{N \in \Delta} \langle h \rangle N = \langle h \rangle$. Similarly, $\bigcap_{N \in \Delta} \langle k \rangle N = \langle k \rangle$. We have $(H5)$.

Let $a \in A$ be of infinite order. Suppose $N^* \cap \langle a \rangle = \langle a^s \rangle$ where N^* is as constructed above and s is a positive integer. Since A is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $N_a \lhd_f A$ such that $N_a \cap \langle a \rangle = \langle a^{rsn} \rangle$. Let $N_a^* = N_a \cap N^*$. Then $N_a^* \in \Delta$ as above. Furthermore, $N_a^* \cap \langle a \rangle = N_a \cap N^* \cap \langle a \rangle = \langle a^{rsn} \rangle$. Hence we are done. $\square$

Corollary 3.9. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ where A is weakly potent and $h, k \in A$ have infinite order. Suppose $h \sim_A k$ or $h^m = k^{\pm m}$ for some positive integer m. Then G is weakly potent if and only if A is $\langle h \rangle$-separable and $\langle k \rangle$-separable.*

Corollary 3.10. *Let $G = \langle t, A | t^{-1}ht = k \rangle$ where A is polycyclic-by-finite or free-by-finite and $h, k \in A$ have infinite order. Suppose $h \sim_A k$ or $h^m = k^{\pm m}$ for some positive integer m. Then G is weakly potent.*

By adapting the proofs of the above theorems, it is not difficult to obtain the following results.

Lemma 3.11. *Let $G = \langle t, A | t^{-1}Ht = H, \phi \rangle$ where $\phi(h) = h$ or $\phi(h) = h^{-1}$, for all $h \in H$ (where H is abelian in the latter case). If G is weakly potent, then Condition 3.3 is satisfied.*

Proof. The proof is similar to Lemma 3.5 except for the following changes.

Let $x \in A \backslash H$. Suppose $x \in HN$, whenever $N \in \Delta$. Then we let $g = t^{-1}xtx^{\pm 1}$ where $g = t^{-1}xtx^{-1}$ if $\phi(h) = h$ and $g = t^{-1}xtx$ if $\phi(h) = h^{-1}$. Since $x \notin H$, $g \neq 1$. By the residual finiteness of G, there exists $M \lhd_f G$ such that $g \notin M$. Let $N = M \cap A$. Then $N \lhd_f A$ and $\phi(N \cap H) = \phi(M \cap A \cap H) = \phi(M \cap H) = t^{-1}(M \cap H)t = t^{-1}Mt \cap t^{-1}Ht = M \cap H = M \cap A \cap H = N \cap H$. Thus, $N \in \Delta$. We can form $\bar{G} = \langle t, \bar{A} | t^{-1}\bar{H}t = \bar{H}, \bar{\phi} \rangle$ where $\bar{A} = A/N$, $\bar{H} = HN/N$ and $\bar{\phi}(\bar{h}) = hN$ or $\bar{\phi}(\bar{h}) = h^{-1}N$. Since $x \in HN$, there exists some $h \in H$ such that $\bar{x} = \bar{h}$. Then $\bar{g} = t^{-1}\bar{h}t\bar{h}^{\pm 1} = \bar{1}$, a contradiction.

Therefore $x \notin HN$ for some $N \in \Delta$. This implies $x \notin \bigcap_{N \in \Delta} HN$ and hence $\bigcap_{N \in \Delta} HN \subseteq H$. Thus $\bigcap_{N \in \Delta} HN = H$. Then we have $(H5)$. $\square$

Theorem 3.12. *Let $G = \langle t, A | t^{-1}Ht = H, \phi \rangle$ as in Lemma 3.11. Then G is weakly potent if and only if A is H-separable.*

Proof. Suppose A is H-separable. The proof is similar to Theorem 3.6 except for the following changes.

If $\phi(h) = h$, then it is clear that $\phi(N \cap H) = N \cap H$. Now suppose $\phi(h) = h^{-1}$. Let $x \in N \cap H$. Then $t^{-1}xt = x^{-1} \in N \cap H$ since $x \in N \cap H$. Thus $t^{-1}(N \cap H)t \subseteq N \cap H$. On the other hand, $x = (x^{-1})^{-1} = t^{-1}x^{-1}t \in t^{-1}(N \cap H)t$. Therefore, $t^{-1}(N \cap H)t = N \cap H$ and this implies $\phi(N \cap H) = t^{-1}(N \cap H)t = N \cap H$. Thus, for any $N \lhd_f A$, $N \in \Delta$ in both cases.

Conversely if G is weakly potent, then by Lemma 3.11, G satisfies Condition 3.3 and hence A is H-separable. □

Corollary 3.13. *Let $G = \langle t, A | t^{-1}Ht = H, \phi \rangle$ be as in Lemma 3.11 with the additional condition that H is finitely generated. Suppose A is polycyclic-by-finite or free-by-finite. Then G is weakly potent.*

Next by extending the results above we will give a characterization for the Baumslag-Solitar groups to be weakly potent in Corollary 3.16.

Lemma 3.14. *Let A be a weakly potent group and $h \in A$ has infinite order. Suppose A is $\langle h \rangle$-separable. Then A is $\langle h^s \rangle$-separable for any positive integer s.*

Proof. Let $x \in A \backslash \langle h^s \rangle$.

Case 1. $x \notin \langle h \rangle$. Since A is $\langle h \rangle$-separable, there exists $N \lhd_f A$ such that $x \notin \langle h \rangle N$. Thus, $x \notin \langle h^s \rangle N$ for any positive integer s and the result follows.

Case 2. $x \in \langle h \rangle$.

Since A is weakly potent, we can find a positive integer r with the property that for each positive integer s, there exists $N \lhd_f A$ such that $N \cap \langle h \rangle = \langle h^{rs} \rangle$. Suppose $x \in \langle h^s \rangle N$. Then $x = h^{st}n$ for some positive integer t and $n \in N$. Since $x \in \langle h \rangle$, we have $n \in N \cap \langle h \rangle = \langle h^{rs} \rangle$. Now, we let $n = h^{\alpha rs}$ for some positive integer α. So, $x = h^{st}h^{\alpha rs} = h^{s(t+\alpha r)} \in \langle h^s \rangle$, a contradiction. Therefore $x \notin \langle h^s \rangle N$ and we are done. □

Corollary 3.15. *Let $G = \langle t, A | t^{-1}h^s t = h^s \rangle$ or $G = \langle t, A | t^{-1}h^s t = h^{-s} \rangle$ where A is weakly potent, $h \in A$ has infinite order and s is a positive integer. Then G is weakly potent if and only if A is $\langle h \rangle$-separable.*

Proof. We will only show the proof for $G = \langle t, A | t^{-1}h^s t = h^s \rangle$. Suppose A is $\langle h \rangle$-separable. By Lemma 3.14, A is $\langle h^s \rangle$-separable for any positive integer s. The result now follows from Theorem 3.12.

Suppose G is weakly potent but A is not $\langle h \rangle$-separable. We let $x \in A \backslash \langle h \rangle$ such that $x \in \langle h \rangle N$, whenever $N \lhd_f A$. Let $g = t^{-1}x^s t x^s$. Then $g \neq 1$ since $x \notin \langle h \rangle$. By the residual finiteness of G, there exists $M \lhd_f G$ such that $g \notin M$. Let $N = M \cap A$. Then $N \lhd_f A$ and $\phi(N \cap H) = t^{-1}(N \cap \langle h \rangle)t = N \cap \langle h \rangle$. We can form $\bar{G} = \langle t, A/N | t^{-1}(h^s N)t = h^{\pm s}N \rangle$. Since $x \in \langle h \rangle N$, there exists some r such that $\bar{x} = \bar{h}^r$. Then $\bar{g} = t^{-1}\bar{x}^s t \bar{x}^{\pm s} = t^{-1}\bar{h}^{rs} t \bar{h}^{\pm rs} = \bar{1}$, a contradiction. Thus A is $\langle h \rangle$-separable. □

Corollary 3.16. *The group* $G = \langle t, a | t^{-1}a^\alpha t a^\beta \rangle$ *is weakly potent if and only if* $|\alpha| = |\beta|$.

Proof. Suppose $|\alpha| = |\beta|$. Then the result follows from Corollary 3.15.

Now, suppose G is weakly potent. Then we can find a positive integer r such that for each positive integer n, there exists $M \lhd_f G$ such that aM has order rn in G/M. We choose $n = |\alpha||\beta|$ and the image of a in G/M denote by $\bar{a}$. Then $|\bar{a}^\alpha| = r|\beta|$ and $|\bar{a}^\beta| = r|\alpha|$. Since $\bar{a}^\alpha$ is conjugate to $\bar{a}^\beta$ in $\bar{G}$, $r|\alpha| = r|\beta|$. This implies that $|\alpha| = |\beta|$. □

4 HNN Extensions of Abelian Groups

4.1 Introduction

In this chapter, we continue the study on weak potency of HNN extensions in Chapter 3. We will prove various criterions for HNN extensions of finitely generated nilpotent groups and free abelian groups of finite rank to be weakly potent. This chapter will be divided into three parts. In the first part, we will show that under certain conditions the HNN extensions of finitely generated nilpotent groups are weakly potent. Then we concentrate on HNN extensions of characteristically weakly potent groups in the second part. Finally in the last part, we will give a characterization for the HNN extensions of free abelian groups of finite rank to be weakly potent.

Notation:

Throughout this chapter, we let A be a group with subgroups H, K and $\phi : H \to K$ be an isomorphism. We denote $\Delta = \{N \lhd_f A | \phi(N \cap H) = N \cap K\}$.

4.2 HNN extensions of finitely generated nilpotent groups

In this section we study the weak potency of HNN extensions of finitely generated nilpotent groups. We begin with the following result.

Theorem 4.1. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$. Suppose $\bigcap_{N \in \Delta} HN = H$ and $\bigcap_{N \in \Delta} KN = K$. If G is weakly potent, then Condition 3.3 is satisfied.*

Proof. Note that $\Delta \neq \emptyset$ since $A \in \Delta$. For each $N \in \Delta$, $\langle t, A/N | t^{-1}(HN/N)t = KN/N \rangle$ is weakly potent by Lemma 3.1. Thus $(H2)$ holds.

Let $a \in A$ be a nontrivial element. Since G is residually finite, there exists $M \lhd_f G$ such that $a \notin M$. Let $N_a = M \cap A$. Then $N_a \lhd_f A$ and $a \notin N_a$. Furthermore $\phi(N_a \cap H) = t^{-1}(N_a \cap H)t = t^{-1}(M \cap A \cap H)t = t^{-1}(M \cap H)t = t^{-1}Mt \cap t^{-1}Ht =$

$M \cap K = M \cap A \cap K = N_a \cap K$. Hence $N_a \in \Delta$ and therefore $\bigcap_{N \in \Delta} N = 1$. Thus $(H3)$ holds.

Since $\bigcap_{N\in\Delta} HN = H$ and $\bigcap_{N\in\Delta} KN = K$, we have $(H5)$. Thus Δ is an h-filter of A since $(H1)$ and $(H4)$ hold trivially.

Let $a \in A$ be of infinite order. Since G is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $M \lhd_f G$ such that $M \cap \langle a \rangle = \langle a^{rn} \rangle$. Let $N = M \cap A$. Then $N \lhd_f A$ and $N \in \Delta$ as above. Furthermore $N \cap \langle a \rangle = M \cap A \cap \langle a \rangle = M \cap \langle a \rangle = \langle a^{rn} \rangle$. Hence, we are done. $\square$

Lemma 4.2. *[21] Let A be a subgroup separable group and H, K be finitely generated normal subgroups of A. Suppose there exist subgroups $R \lhd_f H$ and $S \lhd_f K$ such that R and S are normal in A and $R \cap K = 1 = S \cap H$. Then there exists $N \lhd_f A$ such that $N \cap H = R$ and $N \cap K = S$.*

Lemma 4.3. *[21] Let A be a subgroup separable group. Let H and K be finitely generated normal subgroups in A and $\phi : H \to K$ be an isomorphism of H onto K. Suppose that*

(a) $H \cap K$ is finite; or

(b) there exists $S \lhd_f H$ such that $S \lhd A$ and $\phi(S) = S$.

Then $\bigcap_{N\in\Delta} HN = H$ and $\bigcap_{N\in\Delta} KN = K$.

Proof. (a) Suppose $H \cap K$ is finite. Since A is residually finite, there exists $M \lhd_f A$ such that $M \cap (H \cap K) = 1$. Let $x \in A \backslash H$. By the subgroup separability of A, there exists $P \lhd_f A$ such that $x \notin HP$. Let $N_H = P \cap M \cap H \cap \phi^{-1}(P \cap M \cap K)$. Then $N_H \cap K = 1$ and $\phi(N_H) \cap H = 1$ since $N_H \cap K$ and $\phi(N_H) \cap H$ are subgroups of $M \cap H \cap K = 1$. By Lemma 4.2, there exists $Q \lhd_f A$ such that $Q \cap H = N_H$ and $Q \cap K = \phi(N_H)$. Now let $N = Q \cap P \cap M$. Then $x \notin HN$. Furthermore $N \cap H = Q \cap P \cap M \cap H = N_H \cap P \cap M = N_H$. Similarly, $N \cap K = Q \cap P \cap M \cap K = \phi(N_H) \cap P \cap M = \phi(N_H)$. Therefore $\phi(N \cap H) = N \cap K$ and hence $N \in \Delta$. Thus $\bigcap_{N\in\Delta} HN = H$. Similarly, $\bigcap_{N\in\Delta} KN = K$.

(b) Suppose there exists $S \lhd_f H$ such that $S \lhd A$ and $\phi(S) = S$. Let $x \in A \backslash H$. By the subgroup separability of A, there exists $P \lhd_f A$ such that $x \notin HP$. Since

$P \cap S$ has finite index in S and S is finitely generated, there exists a subgroup $Q \subseteq P \cap S$ such that Q is a characteristic subgroup of finite index in S. Since ϕ is an automorphism of S, we have $\phi(Q) = Q$.

Now let $\bar{A} = A/Q$. Since $\bar{A}$ is residually finite and $\overline{HK}$ is finite, there exists $\bar{M} \lhd_f \bar{A}$ such that $\bar{M} \cap \overline{HK} = \bar{1}$. Let M be the preimage of $\bar{M}$. Then $M \cap HK = Q$. Now let $N = M \cap P$. Then $x \notin HN$. Next we show that $N \cap H = Q$. Let $\alpha \in N \cap H$. Then $\alpha \in M \cap H \subseteq M \cap HK = Q$ and hence $\alpha \in Q$. Now let $\beta \in Q$. Then $\beta \in M$ since $Q \subseteq M$. Furthermore $\beta \in P \cap H$ since $Q \subseteq P \cap S \subseteq P \cap H$. Therefore $\beta \in M \cap P \cap H = N \cap H$. Thus $N \cap H = Q$. In a similar way we can show that $N \cap K = \phi(Q)$. Therefore $\phi(N \cap H) = N \cap K$ and hence $N \in \Delta$. Thus $\bigcap_{N \in \Delta} HN = H$. Similarly we have $\bigcap_{N \in \Delta} KN = K$. □

Corollary 4.4. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where A is subgroup separable and H and K are finitely generated normal subgroups of A. Suppose*

(a) *$H \cap K$ is finite; or*

(b) *there exists $S \lhd_f H$ such that $S \lhd A$ and $\phi(S) = S$.*

If G is weakly potent, then Condition 3.3 is satisfied.

Proof. By Lemma 4.3, $\bigcap_{N \in \Delta} HN = H$ and $\bigcap_{N \in \Delta} KN = K$. The result now follows from Theorem 4.1. □

Theorem 4.5. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$, where A is a finitely generated nilpotent group and H and K are proper subgroups in the center of A. Suppose there exists $S \lhd_f H$ such that $\phi(S) = S$. Then G is weakly potent.*

Proof. We use Theorem 3.3 to show that G is weakly potent. We first show that Δ is an h-filter of A.

Since $A \in \Delta$, $\Delta \neq \emptyset$. For each $N \in \Delta$, $\langle t, A/N | t^{-1}(HN/N)t = KN/N \rangle$ is weakly potent by Lemma 3.1. Thus $(H2)$ holds.

By the residual finiteness of A, for each nontrivial element $a \in A$, there exists $P \lhd_f A$ such that $a \notin P$. Since $P \cap S$ has finite index in S and S is finitely generated, there exists $Q \subseteq P \cap S$ such that Q is a characteristic subgroup of finite index in S.

Since ϕ is an automorphism of S, $\phi(Q) = Q$. Let $\bar{A} = A/Q$. Since $\bar{A}$ is residually finite and $\overline{HK}$ is finite, there exists $\bar{M} \lhd_f \bar{A}$ such that $\bar{M} \cap \overline{HK} = \bar{1}$. Let M be the preimage of $\bar{M}$. Then $M \cap HK = Q$. Now let $N = M \cap P$. Then $N \lhd_f A$ and $a \notin N$. Next we show that $N \cap H = Q$. Let $\alpha \in N \cap H$. Then $\alpha \in M \cap H \subseteq M \cap HK = Q$ and hence $\alpha \in Q$. Now let $\beta \in Q$. Then $\beta \in M$ since $Q \subseteq M$. Furthermore $\beta \in P \cap H$ since $Q \subseteq P \cap S \subseteq P \cap H$. Therefore $\beta \in M \cap P \cap H = N \cap H$. Thus $N \cap H = Q$. In a similar way we can show that $N \cap K = \phi(Q)$. Thus $N \in \Delta$ for each nontrivial element $a \in A$. Hence $\bigcap_{N\in\Delta} N = 1$ and we have $(H3)$.

By Lemma 4.2, $\bigcap_{N\in\Delta} HN = H$ and $\bigcap_{N\in\Delta} KN = K$. Thus, $(H5)$ holds. Since $(H1)$ and $(H4)$ hold trivially, Δ is an h-filter of A.

Let $a \in A$ has infinite order.

Case 1. $S \cap \langle a \rangle = 1$. Then aS is of infinite order. Now, we form $\bar{G} = \langle t, A/S | t^{-1}(H/S)t = K/S\rangle$. Since A is finitely generated nilpotent and $S \lhd A$, A/S is finitely generated nilpotent and hence weakly potent. Therefore, $\bar{G}$ is weakly potent by Theorem 3.4. We denote $\bar{a} = aS$. Then we can find a positive integer r such that for each positive integer n, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{P} \cap \langle \bar{a} \rangle = \langle \bar{a}^{rn} \rangle$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ and $P \cap \langle a \rangle = \langle a^{rn} \rangle$. Let $M = P \cap A$. Then $M \lhd_f A$. Furthermore $\phi(M \cap H) = \phi(P \cap A \cap H) = \phi(P \cap H) = t^{-1}(P \cap H)t = t^{-1}Pt \cap t^{-1}Ht = P \cap K = P \cap A \cap K = M \cap K$. Hence $M \in \Delta$. Now $M \cap \langle a \rangle = P \cap A \cap \langle a \rangle = P \cap \langle a \rangle = \langle a^{rn} \rangle$. Hence, we are done.

Case 2. $S \cap \langle a \rangle = \langle a^{\alpha} \rangle$ for some integer α. Since S is characteristically weakly potent by [5], there exists a positive integer r such that for each positive integer n, there exists $R\ char_f\ S$ such that $R \cap \langle a \rangle = \langle a^{r\alpha n} \rangle$. Since R is characteristic in S, $\phi(R) = R$. We form $\bar{G} = \langle t, A/R | t^{-1}(H/R)t = K/R\rangle$. Then $\bar{G}$ is residually finite by Theorem 3.4. Hence there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{a}, \bar{a}^2, \ldots, \bar{a}^{r\alpha n-1} \notin \bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ and $P \cap \langle a \rangle = \langle a^{r\alpha n} \rangle$. Let $M = P \cap A$ as in Case 1 and the result follows.

Therefore by Theorem 3.3, G is weakly potent. □

Corollary 4.6. *Let $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ where A is finitely generated nilpotent and H and K are proper subgroups in the center of A. Suppose*

(a) $H = K$; *or*

(b) $H \cap K \lhd_f H$ *such that* $\phi(H \cap K) = H \cap K$.

Then G is weakly potent.

Proof. Let $S = H = K$ and $S = H \cap K$ in Theorem 4.5. □

Next we consider HNN extensions where the associated subgroups are retracts.

Lemma 4.7. *[10] Let A be a residually finite group with retract H. Then A is H-separable.*

Lemma 4.8. *[10] Let H be a retract of a group A. Then for each $R \lhd_f H$, there exists $N \lhd_f A$ such that $N \cap H = R$.*

Corollary 4.9. *Let $G = \langle t, A | t^{-1}Ht = H, \phi \rangle$ where A is residually finite and H is a finitely generated retract in A. If G is weakly potent, then Condition 3.3 is satisfied.*

Proof. We need to show that $\bigcap_{N \in \Delta} HN = H$, then the result follows from Theorem 4.1. Let $x \in A \backslash H$. Since A is H-separable by Lemma 4.7, there exists $N \lhd_f A$ such that $x \notin HN$. Let $R = H \cap N$. Then R is of finite index r in H. Since H is finitely generated, there exists a finite number of subgroups of index r in H. Let R^* be the intersection of all these subgroups of index r in H. Then $R^* \subset R$ and R^* is characteristic and of finite index in H. Since $\phi(H) = H$, $\phi(R^*) = R^*$. Then by Lemma 4.8, there exists $N^* \lhd_f A$ such that $N^* \cap H = R^*$. Hence $\phi(N^* \cap H) = \phi(R^*) = R^* = N^* \cap H$. Thus $N^* \in \Delta$ and $x \notin HN^*$. So $x \notin \bigcap_{N \in \Delta} HN$. Hence $\bigcap_{N \in \Delta} HN \subseteq H$. Thus $\bigcap_{N \in \Delta} HN = H$. The result now follows from Theorem 4.1. □

4.3 HNN extensions of characteristically weakly potent groups

In this section we study HNN extensions where the base group A is characteristically weakly potent and the associated isomorphism $\phi \in Aut(A)$. For these HNN extensions we show that the condition Δ is an h-filter of A is sufficient for

$G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ to be weakly potent (Theorem 4.11). We begin with the following.

Theorem 4.10. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$. Suppose H and K are proper subgroups in the center of A. If G is weakly potent, then Condition 3.3 is satisfied.*

Proof. We just need to show that $\bigcap_{N \in \Delta} HN = H$ and $\bigcap_{N \in \Delta} KN = K$ and apply Theorem 4.1.

Let $x \in \bigcap_{N \in \Delta} HN \backslash H$ and $y \in A \backslash K$. Then $z = [t^{-1}xt, y] \neq 1$. Since G is residually finite, there exists $L \lhd_f G$ such that $z \notin L$. Let $N = L \cap A$. Then $N \lhd_f A$ and $\phi(N \cap H) = t^{-1}(N \cap H)t = t^{-1}(L \cap A \cap H)t = t^{-1}(L \cap H)t = t^{-1}Lt \cap t^{-1}Ht = L \cap K = L \cap A \cap K = N \cap K$. Therefore, $N \in \Delta$. We can form $\bar{G} = \langle t, A/N | t^{-1}(HN/N)t = KN/N \rangle$. Since $x \in HN$, there exists some $h \in H$ such that $\bar{x} = \bar{h}$. It follows that $\bar{z} = [t^{-1}\bar{x}t, \bar{y}] = [t^{-1}\bar{h}t, \bar{y}] = [\bar{k}, \bar{y}] = \bar{1}$ since $\bar{K} \subset Z(\bar{A})$. This implies $z \in L$, a contradiction. Therefore $x \notin HN$, for some $N \in \Delta$. Thus $\bigcap_{N \in \Delta} HN = H$ and in a similar way we prove $\bigcap_{N \in \Delta} KN = K$. The result follows from Theorem 4.1. □

Theorem 4.11. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where A is characteristically weakly potent and $\phi \in Aut(A)$. Suppose H and K are subgroups in the center of A. If Δ is an h-filter of A, then G is weakly potent.*

Proof. Let $a \in A$ has infinite order. We now show that we can find a positive integer r such that for each positive integer n, there exists $N \in \Delta$ such that $N \cap \langle a \rangle = \langle a^{rn} \rangle$.

Since A is characteristically weakly potent, there exists a positive integer r such that for each positive integer n, there exists $N \; char_f \; A$ such that $N \cap \langle a \rangle = \langle a^{rn} \rangle$. Note that $N \lhd_f A$ and $\phi(N \cap H) = N \cap K$. Therefore $N \in \Delta$ and G is weakly potent by Theorem 3.3. □

Corollary 4.12. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where A is finitely generated characteristically weakly potent and $\phi \in Aut(A)$. Suppose H and K are subgroups in the center of A. Then G is weakly potent if and only if A is H-separable and K-separable.*

Proof. Suppose A is H-separable and K-separable. Since $A \in \Delta$, $\Delta \neq \emptyset$. For each $N \in \Delta$, $\langle t, A/N | t^{-1}(HN/N)t = KN/N \rangle$ is weakly potent by Lemma 3.1. Thus $(H2)$ holds.

Let $a \in A$ be a nontrivial element. Since A is residually finite, there exists $N_a \lhd_f A$ such that $a \notin N_a$. Suppose $[A : N_a] = r < \infty$. Since A is finitely generated, let N_a^* be the intersection of all normal subgroups of index r in A. Then N_a^* is a characteristic subgroup of finite index in A such that $N_a^* \subseteq N_a$. Hence $a \notin N_a^*$. Furthermore $\phi(N_a^* \cap H) = N_a^* \cap K$ and therefore, $N_a^* \in \Delta$. Thus, $\bigcap_{N \in \Delta} N = 1$ and we have $(H3)$.

Let $a \in A \backslash H$. Since A is H-separable, there exists $P \lhd_f A$ such that $a \notin HP$. As above, there exists a characteristic subgroup of finite index N in A such that $N \subseteq P$ and $N \in \Delta$. Hence $a \notin HN$ and so $a \notin \bigcap_{N \in \Delta} HN$. Therefore $\bigcap_{N \in \Delta} HN \subseteq H$ and thus $\bigcap_{N \in \Delta} HN = H$. Similarly, $\bigcap_{N \in \Delta} KN = K$. Thus $(H5)$ holds.

Since $(H1)$ and $(H4)$ hold trivially, Δ is an h-filter of A. Then by Theorem 4.11, G is weakly potent.

Suppose G is weakly potent. Then Δ is an h-filter of A by Theorem 4.10. This implies that $\bigcap_{N \in \Delta} HN = H$ and $\bigcap_{N \in \Delta} KN = K$. Thus, A is H-separable and K-separable. □

Theorem 4.13. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$, where A is characteristically weakly potent and $\phi \in Aut(A)$. Suppose H and K are subgroups in the center of A such that $H \nsubseteq K, K \nsubseteq H$. Further, suppose A is M-separable for every subgroup $M \lhd_f HK$. Then G is weakly potent if and only if $G_1 = \langle t, HK | t^{-1}Ht = K, \phi \rangle$ is weakly potent.*

Proof. Suppose G is weakly potent. Then G_1 is weakly potent since G_1 is a subgroup of G.

Suppose G_1 is weakly potent. Since $H \neq HK \neq K$, then by Theorem 4.10, Condition 3.3 is satisfied. Hence $\bigcap_{M \in \Delta_1} M = 1, \bigcap_{M \in \Delta_1} HM = H$ and $\bigcap_{M \in \Delta_1} KM = K$ where $\Delta_1 = \{M \lhd_f HK | \phi(M \cap H) = M \cap K\}$. In order to show that G is weakly potent, we show that Δ is an h-filter of A and the result follows from Theorem 4.11.

First, we show that we can construct a subgroup $N_M \in \Delta$ for each $M \in \Delta_1$. Let $M \in \Delta_1$ and $h_0 = 1, h_1, \ldots, h_m$ be coset representatives of M in HK. Since A is

M-separable, there exists $P_M \lhd_f A$ such that $h_i \notin P_M M$ for all $h_i, 1 \leq i \leq m$. Let $N_M = P_M M$. Then $N_M \lhd_f A$. Now we show that $N_M \cap HK = M$. Clearly we only need to show that $N_M \cap HK \subseteq M$. Suppose $a \in (N_M \cap HK)\backslash M$. Since $a \notin M$, then $a = h_i m_1$ where $h_i \neq 1$ is a coset representative of M in HK and $m_1 \in M$. Since $a \in N_M = P_M M$, we have $a = pm_2$ where $p \in P_M$ and $m_2 \in M$. But this implies that $h_i \in P_M M$, a contradiction. Therefore $N_M \cap HK = M$. Similarly we can show that $N_M \cap H = M \cap H$ and $N_M \cap K = M \cap K$. Hence $N_M \in \Delta$.

Note that $\Delta \neq \emptyset$ since $A \in \Delta$. For each $N \in \Delta$, $\langle t, A/N | t^{-1}(HN/N)t = KN/N\rangle$ is weakly potent by Lemma 3.1. Thus $(H2)$ holds.

Let $a \in A$ be nontrivial. Suppose $a \notin HK$. Since A is HK-separable, there exists $M \lhd_f A$ such that $a \notin HKM$. Then $HKM \lhd_f A$ and $HKM \in \Delta$. Suppose $a \in HK$. Since $\bigcap_{M \in \Delta_1} M = 1$, there exists $M \in \Delta_1$ such that $a \notin M$. Let $N_M = P_M M$ as constructed above. Then $N_M \in \Delta$ and $N_M \cap HK = M$. If $a \in N_M$, then $a \in N_M \cap HK = M$, a contradiction. So, $a \notin N_M$. Therefore $\bigcap_{N \in \Delta} N = 1$ and we have $(H3)$

Now we show $(H5)$. Let $x \in A\backslash H$. Suppose $x \notin HK$. Since A is HK-separable, there exists $M \lhd_f A$ such that $x \notin HKM$. Then $HKM \lhd_f A$ and $HKM \in \Delta$. Now, suppose $x \in HK$. Since $x \notin H$ and $\bigcap_{M \in \Delta_1} HM = H$, we can find $M \in \Delta_1$ such that $x \notin HM$. Let $N_M = P_M M$ as constructed above. Suppose $x \in HN_M$. Then $x = hn$ for some $h \in H$ and $n \in N_M$. This implies that $n \in HK \cap N_M = M$ and thus $x \in HM$, a contradiction. So, $x \notin HN_M$. Therefore $\bigcap_{N \in \Delta} HN = H$. Similarly, we can show $\bigcap_{N \in \Delta} KN = K$. Thus $(H5)$ holds.

Since $(H1)$, $(H4)$ hold trivially, Δ is an h-filter of A. By Theorem 4.11, G is weakly potent. □

Corollary 4.14. *Let $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ be as in Theorem 4.13. Let $\Delta_1 = \{M \lhd_f HK | \phi(M \cap H) = M \cap K\}$. Then Δ is an h-filter of A if and only if Δ_1 is an h-filter of HK.*

Proof. Suppose Δ is an h-filter of A. Then by Theorem 4.11, G is weakly potent. By Theorem 4.13, G_1 is weakly potent . Therefore Δ_1 is an h-filter of HK by Theorem 4.10.

Now, suppose Δ_1 is an h-filter of HK. This implies that $\bigcap_{M\in\Delta_1} M = 1, \bigcap_{M\in\Delta_1} HM = H$ and $\bigcap_{M\in\Delta_1} KM = K$. We can now follow the proof in Theorem 4.13, to show that Δ is an h-filter of A. □

Lemma 4.15. *[15] Let A be a finitely generated nilpotent group, H, K be proper subgroups of A and $\phi : H \to K$ an isomorphism. The HNN extension $G = \langle t, A | t^{-1}Ht = K, \phi\rangle$ of A with associated subgroups H and K is residually finite if and only if there exists a torsion-free normal subgroup $N \lhd_f A$ such that $\phi(H\cap N) = K\cap N$ and $H\cap N, K\cap N$ are isolated in N.*

Theorem 4.16. *Let $G = \langle t, A|t^{-1}Ht = K, \phi\rangle$ be as in Theorem 4.13 with the additional condition that H and K are finitely generated. Then G is weakly potent if and only if there exists a torsion-free $M \in \Delta_1 = \{M \lhd_f HK | \phi(M\cap H) = M\cap K\}$ such that $M\cap H, M\cap K$ are isolated in M.*

Proof. Suppose G is weakly potent. Then $G_1 = \langle t, HK|\, t^{-1}Ht = K, \phi\rangle$ is a subgroup of G and hence G_1 is weakly potent. Since $H \nsubseteq K, K \nsubseteq H$ and HK is finitely generated abelian, by Lemma 4.15, there exists a torsion-free $M \in \Delta_1 = \{M \lhd_f HK | \phi(M\cap H) = M\cap K\}$ such that $M\cap H, M\cap K$ are isolated in M.

Conversely, suppose that there exists a torsion-free $M \in \Delta_1 = \{M\lhd_f HK|\phi(M\cap H) = M\cap K\}$ such that $M\cap H, M\cap K$ are isolated in M. By Lemma 4.15, G_1 is residually finite. We first show that Δ_1 is an h-filter of HK.

Since $HK \in \Delta_1$, $\Delta_1 \neq \emptyset$. For each $M \in \Delta_1$, $\langle t, HK/M|t^{-1}(HM/M)t = KM/M\rangle$ is weakly potent by Lemma 3.1. Thus $(H2)$ holds.

Let $a \in HK$ be a nontrivial element. Since G_1 is residually finite, there exists $P \lhd_f G_1$ such that $a \notin P$. Let $M = P\cap HK$. Then $M \lhd_f HK$ and $\phi(M\cap H) = \phi(P\cap HK\cap H) = \phi(P\cap H) = t^{-1}(P\cap H)t = t^{-1}Pt\cap t^{-1}Ht = P\cap K = P\cap HK\cap K = M\cap K$. Thus, $M \in \Delta_1$ and $a \notin M$. Hence, $\bigcap_{M\in\Delta_1} M = 1$ and we have $(H3)$.

Let $x \in HK\backslash H$. Suppose $x \in HM$, whenever $M \in \Delta_1$. Let $z = [t^{-1}xt, k]$ where $k \in K$. Then $z \neq 1$. Since G_1 is residually finite, there exists $P \lhd_f G_1$ such that $g \notin P$. Let $M = P\cap HK$. Then $M \lhd_f HK$ and $M \in \Delta_1$ as above. We can form $\bar{G_1} = \langle t, HK/M|t^{-1}(HM/M)t = HM/M\rangle$. Since $x \in HM$, then there exists $\bar{h} \in HM/M$ such that $\bar{x} = \bar{h}$. Thus $[t^{-1}\bar{x}t, \bar{k}] = [t^{-1}\bar{h}t, \bar{k}] = \bar{1}$, a contradiction.

Therefore, $x \notin HM$, for some $M \in \Delta_1$. This implies $x \notin \bigcap_{M \in \Delta_1} HM$ and hence $\bigcap_{M \in \Delta_1} HM \subseteq H$. Thus $\bigcap_{M \in \Delta_1} HM = H$. Similarly, $\bigcap_{M \in \Delta_1} KM = K$. Thus $(H5)$ holds.

Since $(H1)$ and $(H4)$ hold trivially, Δ_1 is an h-filter of HK. Then by Corollary 4.14, Δ is an h-filter of A. Thus, G is weakly potent by Theorem 4.11. □

4.4 HNN extensions of free abelian groups of finite rank

In this section we give a characterisation for HNN extensions of a free abelian group of finite rank to be weakly potent.

Definition 4.17. *Let G be a group and $H \leq G$. Then the subgroup H is isolated in G if whenever $g^n \in H$ for $g \in G$ and $n \in \mathbb{N}$, we have $g \in H$.*

Theorem 4.18. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where A is a free abelian group with finite rank. Then the following are equivalent:*

(i) G is weakly potent;

(ii) Δ is an h-filter of A;

(iii) there exists $N \in \Delta$ such that $H \cap N, K \cap N$ are isolated in N;

(iv) there exists a free abelian group X of finite rank and an automorphism $\bar{\phi} \in Aut\ X$ such that $A \subseteq X$ and $\bar{\phi}\mid_H = \phi$.

First we prove (iv) → (i) in Theorem 4.18.

Lemma 4.19. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where A is an abelian group with finite rank. If there exists an abelian group with finite rank X and an automorphism $\bar{\phi} \in Aut\ X$ such that $A \subseteq X$ and $\bar{\phi}\mid_H = \phi$, then G is weakly potent.*

Proof. Let $G^* = \langle t, X | t^{-1}Ht = K, \phi \rangle$. By Corollary 4.12, G^* is weakly potent. Since $G \leq G^*$, G is weakly potent. □

We now show (i) $\to$ (ii).

Lemma 4.20. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where A is an abelian group with finite rank. If G is weakly potent, then Δ is an h-filter of A.*

Proof. Since A is characteristically weakly potent and abelian, the result holds by Theorem 4.10. $\square$

We now prove (ii) $\to$ (iii) in the following lemma.

Lemma 4.21. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ be an HNN extension where A is an abelian group with finite rank. If Δ is an h-filter of A, then there exists $N \in \Delta$ such that $H \cap N, K \cap N$ are isolated in N.*

Proof. By the definition of h-filter, $\bigcap_{N \in \Delta} HN = H$ and $\bigcap_{N \in \Delta} KN = K$. Let S/H and T/K be the torsion parts of A/H and A/K respectively. Since A is finitely generated, S/H and T/K are finite. For each non-trivial element $xH \in S/H$, there exists $N_x \in \Delta$ such that $N_x \cap xH = \emptyset$. Similarly, for each non-trivial element $yK \in T/K$, there exists $N_y \in \Delta$ such that $N_y \cap yK = \emptyset$. Let $N = (\cap N_x) \cap (\cap N_y)$ where the intersection extends over all the finitely many elements of S/H and T/K. Clearly $N \lhd_f A$ and $\phi(H \cap N) = K \cap N$. By the construction of $N, N/H \cap N$ and $N/K \cap N$ are torsion free. Thus $H \cap N$ and $K \cap N$ are isolated in N. Hence N is the required normal subgroup. $\square$

To prove (iii) $\to$ (iv), we shall need the following lemmas in Andreadakis, Raptis and Varsos [2].

Lemma 4.22. *[2] Let A be a free abelian group of finite rank, H, K subgroups of A which are direct factors of A and $\phi : H \longrightarrow K$ an isomorphism. Then there exists an automorphism $\theta \in Aut\ A$ such that $\theta \mid_H = \phi$.*

Lemma 4.23. *[2] Let A, K be free abelian groups and $\theta_1, \theta_2 : A \longrightarrow K$ monomorphisms such that $\theta_1 \mid_H = \theta_2 \mid_H$ for some subgroup H of finite index in A. Then $\theta_1 = \theta_2$.*

Lemma 4.24. *[2] Let A be a free abelian group of finite rank. Let H, K be subgroups of A of finite index in A and $\phi : H \longrightarrow K$ an isomorphism. Suppose that there exists a subgroup $P \leq_f A$ with $P < H \cap K$ and $\phi(P) = P$. Then there exists a free abelian group X with finite rank and an automorphism $\bar{\phi} \in Aut\ X$ such that $\bar{\phi} \mid_H = \phi$.*

Lemma 4.25. *Let A be a free abelian group of finite rank. Let H, K be subgroups of A and $\phi : H \longrightarrow K$ an isomorphism. Suppose that there exists $N \lhd_f A$ such that $\phi(H \cap N) = K \cap N$ and $H \cap N, K \cap N$ are isolated in N. Then there exists a free abelian group X with finite rank and an automorphism $\bar{\phi} \in Aut\ X$ such that $A \subseteq X$ and $\bar{\phi} \mid_H = \phi$.*

Proof. Since $H \cap N, K \cap N$ are isolated in $N, N/H \cap N$ and $N/K \cap N$ are torsion free. Thus $N/H \cap N$ is free abelian and there exists $L < N$ such that $N = (H \cap N) \times L$. Similarly, $N = (K \cap N) \times M$ for some $M < N$. Since $\phi(H \cap N) = K \cap N$, by Lemma 4.22, there exists an automorphism τ of N with $\tau \mid_{H \cap N} = \phi \mid_{H \cap N}$. Furthermore, by Lemma 4.24, there exists a free abelian group X with finite rank and an automorphism $\bar{\phi} \in$ Aut X such that $A \subseteq X$ and $\bar{\phi} \mid_N = \tau$. So there are two monomorphism $\bar{\phi} \mid_H, \phi : H \longrightarrow X$ and $H \cap N \lhd_f H$. Also, $\bar{\phi} \mid_{H \cap N} = \tau \mid_{H \cap N} = \phi \mid_{H \cap N}$. Therefore by Lemma 4.23, $\bar{\phi} \mid_H = \phi$ and our result follows. □

Theorem 4.18 now follows from Lemma 4.19, 4.20, 4.21, and 4.25.

Corollary 4.26. *Let $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$ where A is free abelian of finite rank. If H, K are isolated in A, then G is weakly potent.*

Proof. Let $N = A^n$ where $n \in \mathbb{Z}^+$ and $n > 1$. Then $N \lhd_f A$ and $\phi(N \cap H) = \phi(A^n \cap H) = \phi(H^n) = (\phi(H))^n = K^n = A^n \cap K = N \cap K$. Thus, $N \in \Delta$. Since H, K are isolated in A, then $N \cap H, N \cap K$ are isolated in N. Therefore, G is weakly potent by Theorem 4.18. □

Corollary 4.27. *Let $G = \langle t, a_1, a_2, \ldots, a_n | t^{-1} a_i^{p_i} t = a_i^{q_i}, i = 1, 2, \ldots, n, [a_i, a_j] = 1 \rangle$ and $A = \langle a_1, a_2, \ldots, a_n | [a_i, a_j] = 1 \rangle$. Then the following are equivalent:*

(i) G is weakly potent;

(ii) $|p_i| = |q_i|$, $i = 1, 2, \ldots n$;

(iii) the map ϕ which sends $a_i^{p_i}$ to $a_i^{q_i}, i = 1, 2, \ldots n$ comes from an automorphism of A.

Proof. Let $A = \langle a_1, a_2, \ldots, a_n | [a_i, a_j] = 1 \rangle$ be a free abelian group of rank n. Let $H = \langle a_1^{p_1}, a_2^{p_2}, \ldots, a_n^{p_n} \rangle$, $K = \langle a_1^{q_1}, a_2^{q_2}, \ldots, a_n^{q_n} \rangle$ be subgroups of A and $\phi : H \to K$ be the isomorphism defined by $\phi(a_i^{p_i}) = a_i^{q_i}$, $i = 1, 2, \ldots, n$. Then $G = \langle t, A | t^{-1}Ht = K, \phi \rangle$.

We begin by showing (i) $\Longrightarrow$ (ii). By Theorem 4.18, Δ is an h-filter of A. For each $x \in A \backslash H$, there exists $N_x \in \Delta$ such that $xH \cap N_x = \emptyset$ by $(H5)$. Similarly, for each $y \in A \backslash K$, there exists $N_y \in \Delta$ such that $yK \cap N_y = \emptyset$. Let $S = (\bigcap N_x) \cap (\bigcap N_y)$. Then $S \in \Delta$ by $(H4)$. Note that $S \subseteq H, S \subseteq K$. Therefore $S \cap H = S = S \cap K$ and $S \lhd_f H, S \lhd_f K$. This implies that $S \leq_f A$ and $\phi(S) = S$.

Now, we can let $S = \langle a_1^{s_1}, a_2^{s_2}, \ldots, a_n^{s_n} \rangle$ with $p_i | s_i$ and $q_i | s_i$. Note that $\phi(a_i^{s_i}) = \phi(a^{s_i p_i / p_i}) = (\phi(a^{p_i}))^{s_i/p_i} = a^{q_i s_i / p_i} = (a^{s_i})^{q_i/p_i}$. Since $\phi(S) = S$, we obtained $|p_i| = |q_i|$, $i = 1, 2, \ldots, n$.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i) follows from Corollary 4.12 since A is characteristically weakly potent and subgroup separable. $\square$

5 Weak Potency of Generalised Free Products

5.1 Introduction

In this chapter, we shall study the weak potency of generalised free products of weakly potent groups by using filters. This chapter will be divided into five parts. In the first part, we will introduce w-filters which will be the basis for the main criterion. Then we apply the main criterion to prove various criterions for the weak potency of generalised free products with cyclic amalgamated subgroups in the second part. Next in the third part, we prove characterizations for the generalised free products with central amalgamated subgroups to be weakly potent. In the fourth part, we extend our previous results to tree products. Finally in the last part we prove the weak potency of certain one-relator groups with torsion.

Notation:

For the rest of this chapter, we let A and B be two groups such that $A \cap B = H$.

The following lemma will be used to prove most of the results in this chapter.

Lemma 5.1. *[5] Let $G = A \ast_H B$ where A and B are finite. Then G is weakly potent.*

5.2 Filters and weak potency

In this section we introduce w-filters (Definition 5.2) and then prove the main criterion (Theorem 5.3) for generalised free products to be weakly potent. We then apply our criterion to generalised free products amalgamating a finite subgroup.

Definition 5.2. *Let $G = A \ast_H B$. Let $\Delta = \{(M_i, N_i) | i \in I\}$ be a non-empty family of pairs (M_i, N_i), where $M_i \lhd A$ and $N_i \lhd B$, satisfying the following:*

(W1) $M_i \cap H = N_i \cap H$ for each $i \in I$;

(W2) For each $i \in I$, $A/M_i \ast_{\bar{H}} B/N_i$ is weakly potent, where $\bar{H} = HM_i/M_i \cong HN_i/N_i$;

(W3) For each $i_1, i_2, \ldots, i_n \in I$ *and* $n \in \mathbb{Z}^+$, $(\bigcap_{k=1}^n M_{i_k}, \bigcap_{k=1}^n N_{i_k}) \in \Delta$;

(W4) $\bigcap_{i\in I} HM_i = H = \bigcap_{i\in I} HN_i$.

Such Δ *is called a w-filter of the generalised free product* $G = A \ast_H B$.

We now prove our main criterion.

Theorem 5.3. *Let* $G = A \ast_H B$. *Suppose*

(1) Δ *is a w-filter of* G*; and*

(2) *For each* $x \in H$ *of infinite order, we can find a positive integer* r *such that for each positive integer* n*, there exists* $(M, N) \in \Delta$ *such that* $M \cap \langle x \rangle = \langle x^{rn} \rangle$.

Then G *is weakly potent.*

Proof. Let $g \in G$ be an element of infinite order.

Case 1. $g \in H$. We can find a positive integer r such that for every positive integer n, there exists $(M, N) \in \Delta$ such that $M \cap \langle g \rangle = \langle g^{rn} \rangle$. Let $\bar{G} = A/M \ast_{\bar{H}} B/N$ where $\bar{H} = HM/M \cong HN/N$. By $(W2)$, $\bar{G}$ is weakly potent and hence residually finite. Then there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{g}, \bar{g}^2, \ldots, \bar{g}^{rn-1} \notin \bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and gL has order rn in G/L.

Case 2. $g \notin H$. WLOG, we let $x = a_0 b_0 a_1 b_1 \ldots a_n b_n$, where $a_i \in A \backslash H$ and $b_i \in B \backslash H$. By $(W4)$, there exists $(M_{i_s}, N_{i_s}) \in \Delta$ such that $a_s \notin HM_{i_s}$ for each $1 \le s \le n$. Similarly, there exists $(M'_{i_s}, N'_{i_s}) \in \Delta$ such that $b_s \notin HN'_{i_s}$ for each $1 \le s \le n$. Let $M = \bigcap_{s=1}^n (M_{i_s} \cap M'_{i_s})$ and $N = \bigcap_{s=1}^n (N_{i_s} \cap N'_{i_s})$. Then $(M, N) \in \Delta$ by $(W3)$ and $M \cap H = N \cap H$ by $(W1)$. Let $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$, where $\bar{A} = A/M, \bar{B} = B/N$ and $\bar{H} = HM/M \cong HN/N$. Let $\bar{x}$ denote the image of $x \in G$ in $\bar{G}$. Note that $||\bar{x}|| = ||x||$ and $\bar{x}$ is nontrivial in $\bar{G}$. By $(W2)$, $\bar{G}$ is weakly potent. Then we can find a positive integer r such that for each positive integer n, there exists $\bar{L} \lhd_f \bar{G}$ such that $\bar{x}\bar{L}$ has order exactly rn in $\bar{G}/\bar{L}$. Let L be the preimage of $\bar{L}$ in G. Then $L \lhd_f G$ and xL has order of rn. Therefore, G is weakly potent. □

Theorem 5.4. *Let* $G = A \ast_H B$ *where* A, B *are weakly potent and* H *is finite. Let* $\Delta = \{(M, N) | M \lhd_f A, N \lhd_f B$ *such that* $M \cap H = 1 = N \cap H\}$. *Then* G *is weakly potent.*

Proof. Since H is finite, we need only show that Δ is a w-filter of G and the result follows from Theorem 5.3.

Let $H = \{1, h_1, \ldots, h_k\}$. Since A, B are weakly potent and hence residually finite, for each nontrivial element $h_i \in H$, there exist $M_i \lhd_f A, N_i \lhd_f B$ such that $h_i \notin M_i$ and $h_i \notin N_i$ for $1 \leq i \leq k$. Let $M = \bigcap_{i=1}^k M_i$ and $N = \bigcap_{i=1}^k N_i$. Then $M \lhd_f A$, $N \lhd_f B$ and $M \cap H = 1 = N \cap H$. Thus $(M, N) \in \Delta$. This implies $\Delta \neq \emptyset$. By Lemma 5.1, $A/M \underset{H}{*} B/N$ is weakly potent for each $(M, N) \in \Delta$. Hence $(W2)$ holds.

To show $(W4)$, let $a \in A \backslash H$. Then $ah_i^{-1} \neq 1$ for each $h_i \in H, i = 1, 2, \ldots, k$. Since A is residually finite, there exists $M \lhd_f A$ such that $ah_i^{-1} \notin M$ for all i and $M \cap H = 1$. Then $a \notin HM$. Similarly, there exists $N \lhd_f B$ such that $N \cap H = 1$. Then $(M, N) \in \Delta$ and $a \notin \bigcap_{(M,N)\in\Delta} HM$. This implies $\bigcap_{(M,N)\in\Delta} HM \subseteq H$. Hence $\bigcap_{(M,N)\in\Delta} HM = H$. Similarly, $\bigcap_{(M,N)\in\Delta} HN = H$. Thus, $(W4)$ holds. Since $(W1)$ and $(W3)$ hold trivially, Δ is a w-filter of G. Hence G is weakly potent by Theorem 5.3. □

The two conditions in Theorem 5.3 are sufficient for a generalised free product to be weakly potent. We will also examine the converse, that is, if a generalised free product is weakly potent then these two conditions are necessary. For ease of exposition, we state the two conditions given in Theorem 5.3 as Condition 5.3 below.

Condition 5.3:

Let $\Delta = \{(M, N) | M \lhd_f A, N \lhd_f B$ such that $M \cap H = N \cap H\}$ where

(1) Δ is a w-filter of $G = A \underset{H}{*} B$ and

(2) For each $x \in H$ of infinite order, we can find a positive integer r such that for each positive integer n, there exists $(M, N) \in \Delta$ such that $M \cap \langle x \rangle = \langle x^{rn} \rangle$.

5.3 Various criterions

We begin this section with a criterion for the weak potency of certain generalised free products with cyclic amalgamated subgroups.

Theorem 5.5. *Let $G = A \ast_H B$ where A and B are weakly potent and $H = \langle c \rangle$. Suppose A and B are $\langle c \rangle$-separable. Then G is weakly potent.*

Proof. If $\langle c \rangle$ is finite, the result follows from Theorem 5.4. Let $\langle c \rangle$ be infinite cyclic. Note that $\Delta \neq \emptyset$ since $(A, B) \in \Delta$. For each $(M, N) \in \Delta$, $\bar{G} = A/M \ast_{\langle \bar{c} \rangle} B/N$ is weakly potent by Lemma 5.1. Thus $(W2)$ holds.

By the weak potency of A and B, we can find positive integers r_1, r_2 with the property that for each positive integer n, there exist $P_1 \lhd_f A$, $P_2 \lhd_f B$ such that $P_1 \cap \langle c \rangle = \langle c^{r_1 n} \rangle$ and $P_2 \cap \langle c \rangle = \langle c^{r_2 n} \rangle$.

Let $x \in A \backslash \langle c \rangle$. Since A is $\langle c \rangle$-separable, there exists $S \lhd_f A$ such that $x \notin \langle c \rangle S$. Suppose $S \cap \langle c \rangle = \langle c^s \rangle$ for some positive integer s. Then from above. we can find $M^* \lhd_f A$ and $N \lhd_f B$ such that $M^* \cap \langle c \rangle = \langle c^{r_1 r_2 s} \rangle = N \cap \langle c \rangle$. Let $M = S \cap M^*$. Then $M \lhd_f A$ is such that $x \notin \langle c \rangle M$ and $M \cap \langle c \rangle = \langle c^{r_1 r_2 s} \rangle = N \cap \langle c \rangle$. Therefore $(M, N) \in \Delta$ and $x \notin \bigcap_{(M,N) \in \Delta} \langle c \rangle M$. Hence $\bigcap_{(M,N) \in \Delta} \langle c \rangle M \subseteq \langle c \rangle$. Thus $\bigcap_{(M,N) \in \Delta} \langle c \rangle M = \langle c \rangle$. Similarly, $\bigcap_{(M,N) \in \Delta} \langle c \rangle N = \langle c \rangle$. Thus $(W4)$ holds. Since $(W1)$ and $(W3)$ hold trivially, Δ is a w-filter of G.

Let $x \in H$ has infinite order. As above we can find $P \lhd_f A$ such that $P \cap \langle c \rangle = \langle c^{r_1 r_2} \rangle$. Suppose $P \cap \langle x \rangle = \langle x^s \rangle$ for some positive integer s. By the weak potency of A, we can find a positive integer r such that for every positive integer n, there exists $Q \lhd_f A$ such that $Q \cap \langle x \rangle = \langle x^{rsn} \rangle$. Let $M = P \cap Q$. Then $M \lhd_f A$ is such that $M \cap \langle x \rangle = \langle x^{rsn} \rangle$ and $M \cap \langle c \rangle = \langle c^{r_1 r_2 t} \rangle$ for some positive integer t. As above, we can find $N \lhd_f B$ be such that $N \cap \langle c \rangle = \langle c^{r_1 r_2 t} \rangle$. Then $(M, N) \in \Delta$ and $M \cap \langle x \rangle = \langle x^{rsn} \rangle$. By Theorem 5.3, G is weakly potent. □

Corollary 5.6. *Let $G = A \ast_{\langle c^s \rangle} B$ where A and B are weakly potent and s is any positive integer. If A and B are $\langle c \rangle$-separable, then G is weakly potent.*

Proof. Follows from Lemma 3.14 and Theorem 5.5. □

We now show a criterion for the weak potency of certain generalised free products amalgamating a retract.

Theorem 5.7. *Let $G = A *_H B$, where A and B are weakly potent. Suppose, for each $R \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = R = N_B \cap H$. If A and B are H-separable, then G is weakly potent.*

Proof. We note that $\Delta \neq \emptyset$ since $(A, B) \in \Delta$. For each $(M, N) \in \Delta$, $\bar{G} = A/M *_{\bar{H}} B/N$ is weakly potent by Lemma 5.1. Thus $(W2)$ holds.

Let $x \in A \backslash H$. Since A is H-separable, there exists $M \lhd_f A$ such that $x \notin HM$. Suppose $M \cap H = R$. Then $R \lhd_f H$. By assumption, there exists $N \lhd_f B$ such that $N \cap H = R$. Therefore $(M, N) \in \Delta$. Since $x \notin HM$, $x \notin \bigcap_{(M,N)\in\Delta} HM$. Therefore $\bigcap_{(M,N)\in\Delta} HM \subseteq H$ and thus $\bigcap_{(M,N)\in\Delta} HM = H$. Similarly, $\bigcap_{(M,N)\in\Delta} HN = H$. Thus $(W4)$ holds.

Since $(W1)$ and $(W3)$ hold trivially, Δ is a w-filter of G.

Let $x \in H$ has infinite order. Since A is weakly potent, we can find a positive integer r with the property that for each positive integer n, there exists $M \lhd_f A$ such that xM has order rn. Note that $M \cap H = R \lhd_f H$. By assumption, there exists $N \lhd_f B$ such that $N \cap H = R$. Hence $(M, N) \in \Delta$ and $M \cap \langle x \rangle = \langle x^{rn} \rangle$. Hence G is weakly potent by Theorem 5.3. □

Corollary 5.8. *Let $G = A *_H B$ where A and B are weakly potent groups. Suppose H is a retract in both A and B. Then G is weakly potent.*

Proof. This follows from Lemmas 4.7 and 4.8, and Theorem 5.7. □

5.4 Generalised free products amalgamating central subgroups

In this section we study the weak potency of certain generalised free products amalgamating a central subgroup. We begin with a result which shows that Condition 5.3 is necessary.

Theorem 5.9. *Let $G = A *_H B$ where $H \leq Z(A) \cap Z(B)$. If G is weakly potent, then Condition 5.3 holds.*

Proof. We note that $\Delta \neq \emptyset$ since $(A, B) \in \Delta$. For each $(M, N) \in \Delta$, $\bar{G} = A/M *_{\bar{H}} B/N$ is weakly potent by Lemma 5.1. Thus $(W2)$ holds.

Let $x \in A \backslash H$. Suppose $x \in HM$, whenever $(M, N) \in \Delta$. Let $y \in B \backslash H$. Then $g = [x, y] \neq 1$. Since G is residually finite, there exists $L \lhd_f G$ such that $g \notin L$. Let $M = L \cap A$ and $N = L \cap B$. Then $M \lhd_f A$, $N \lhd_f B$ and $M \cap H = L \cap A \cap H = L \cap H = L \cap B \cap H = N \cap B$. Therefore, $(M, N) \in \Delta$. Now, we can form $\bar{G} = A/M *_{\bar{H}} B/N$ where $\bar{H} = HM/M = HN/N$. Since $x \in HM$, we can find some $h \in H$ such that $\bar{x} = \bar{h}$. Then $\bar{g} = [\bar{x}, \bar{y}] = [\bar{h}, \bar{y}] = \bar{1}$ since $H \subseteq Z(B)$, a contradiction. Therefore, $x \notin HM$ for some $(M, N) \in \Delta$. Thus, $x \notin \bigcap_{(M,N) \in \Delta} HM$ and $\bigcap_{(M,N) \in \Delta} HM \subseteq H$. Hence $\bigcap_{(M,N) \in \Delta} HM = H$. Similarly, $\bigcap_{(M,N) \in \Delta} HN = H$. We have $(W4)$.

Since $(W1)$ and $(W3)$ hold trivially, Δ is a w-filter of G.

Let $x \in H$ has infinite order. Since G is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $L \lhd_f G$ such that $L \cap \langle x \rangle = \langle x^{rn} \rangle$. As above, we have $(M, N) \in \Delta$ such that $M = L \cap A$ and $N = L \cap B$. Then $M \cap \langle x \rangle = L \cap A \cap \langle x \rangle = L \cap \langle x \rangle = \langle x^{rn} \rangle$. We are done. □

It is clear that if $G = A *_H B$ satisfy Condition 5.3, then A and B are H-separable. Therefore we can obtain the following two corollaries from Theorem 5.5 and Corollary 5.8.

Corollary 5.10. *Let $G = A *_H B$ where A and B are weakly potent, $H = \langle c \rangle$ and $H \leq Z(A) \cap Z(B)$. Then G is weakly potent if and only if A and B are H-separable.*

Corollary 5.11. *Let $G = A *_H B$, where A and B are weakly potent and $H \leq Z(A) \cap Z(B)$. Suppose H is a retract in both A and B. Then G is weakly potent if and only if A and B are H-separable.*

We now show a criterion for the weak potency of certain generalised free products of subgroup separable groups and then apply it to polycyclic-by-finite or free-by-finite groups.

Lemma 5.12. *Let $H \lhd A$. If A is H-separable, then A/H is residually finite.*

Proof. Let $x \in A \backslash H$. Then there exists $N \lhd_f A$ such that $x \notin HN$. Note that $xH \neq H$ in A/H. It follows that $xH \notin HN/H$ and $HN/H \lhd_f A/H$. Thus, A/H is residually finite. $\square$

Theorem 5.13. *Let $G = A *_H B$ where A and B are weakly potent. Suppose $H \leq Z(A) \cap Z(B)$. If A and B are M-separable for every subgroup $M \lhd_f H$, then G is weakly potent.*

Proof. Note that $\Delta \neq \emptyset$ since $(A, B) \in \Delta$. For each $(M, N) \in \Delta$, $\bar{G} = A/M *_{\bar{H}} B/N$ is weakly potent by Lemma 5.1. Thus $(W2)$ holds.

Let $x \in A \backslash H$. Since A is H-separable, there exists $M \lhd_f A$ such that $x \notin HM$. Then $M \cap H \lhd_f H$ and $M \cap H \lhd B$. Let $\bar{B} = B/(M \cap H)$ and $\bar{H} = H/M \cap H$. Then $\bar{H}$ is finite and $\bar{B}$ is residually finite by Lemma 5.12. Hence, there exists $\bar{N} \lhd_f \bar{B}$ such that $\bar{N} \cap \bar{H} = \bar{1}$. Let N be the preimage of $\bar{N}$ in B. Then $N \lhd_f B$ such that $M \cap H = N \cap H$. Therefore, $(M, N) \in \Delta$. Thus, $x \notin \bigcap_{(M,N) \in \Delta} HM$. Hence, $\bigcap_{(M,N) \in \Delta} HM \subseteq H$ and $\bigcap_{(M,N) \in \Delta} HM = H$. Similarly, we have $\bigcap_{(M,N) \in \Delta} HN = H$ and $(W4)$ holds.

Since $(W1)$ and $(W3)$ hold trivially, Δ is a w-filter of G.

Let $x \in H$ has infinite order. Since A is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $M \lhd_f A$ such that $M \cap \langle x \rangle = \langle x^{rn} \rangle$. As above, we can find $N \lhd_f B$ such that $(M, N) \in \Delta$. Thus G is weakly potent by Theorem 5.3. $\square$

Corollary 5.14. *Let $G = A *_H B$ where A and B are weakly potent and H is a finitely generated central subgroup of A and B. If A and B are subgroup separable then G is weakly potent.*

Corollary 5.15. *Let $G = A *_H B$ where A and B are polycyclic-by-finite or free-by-finite and H is a finitely generated central subgroup of A and B. Then G is weakly potent.*

5.5 Tree products

In this section, we extend our results in the last section to tree products of finitely many weakly potent groups. We begin with the tree products amalgamating finite subgroups.

Theorem 5.16. *Let $A_1, A_2, ..., A_n$ be weakly potent groups. Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating finite subgroups H_{ij} of A_i and H_{ji} of A_j. Then G is weakly potent.*

Proof. We prove this theorem by induction on n. When $n = 2$, the result follows from Theorem 5.4. The tree product G has an extremal vertex, say A_n, which is joined to a unique vertex, say A_{n-1}. We write $G = \langle \bar{A}, A_n | H_{(n-1)n} = H_{n(n-1)} \rangle$, where $\bar{A}$ is the tree product generated by $A_1, A_2, \ldots, A_{n-1}$. By our inductive hypothesis, $\bar{A}$ is weakly potent. Then G is weakly potent by Theorem 5.4. □

We need the following lemma to prove the next result.

Lemma 5.17. *Let $G = \langle A_1, A_2, \ldots, A_n | c_{ij}^{s_{ij}} = c_{ji}^{s_{ji}} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating infinite cyclic subgroups $\langle c_{ij}^{s_{ij}} \rangle$ of A_i and $\langle c_{ji}^{s_{ji}} \rangle$ of A_j where s_{ij}, s_{ji} are any positive integers. Suppose G is weakly potent and A_i are $\langle c_{ij} \rangle$-separable for each i. Let K be a subgroup of A_r where $1 \leq r \leq n$ and A_r is K-separable. Then G is K-separable.*

Proof. By Lemma 3.14, A_i is $\langle c_{ij}^{s_{ij}} \rangle$-separable for each i. The result follows from Lemma 4.4 of [20]. □

Theorem 5.18. *Let $A_1, A_2, ..., A_n$ be weakly potent groups. Let $G = \langle A_1, A_2, \ldots, A_n | c_{ij}^{s_{ij}} = c_{ji}^{s_{ji}} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating infinite cyclic subgroups $\langle c_{ij}^{s_{ij}} \rangle$ of A_i and $\langle c_{ji}^{s_{ji}} \rangle$ of A_j where s_{ij}, s_{ji} are any positive integers. If A_i is $\langle c_{ij} \rangle$-separable for each i, then G is weakly potent.*

Proof. As in Theorem 5.16, we prove this by induction. When $n = 2$, the theorem follows from Corollary 5.6. We denote $G = \langle \bar{A}, A_n | c_{(n-1)n}^{s_{(n-1)n}} = c_{n(n-1)}^{s_{n(n-1)}} \rangle$, where $\bar{A}$ is the tree product generated by $A_1, A_2, \ldots, A_{n-1}$. By inductive hypothesis, $\bar{A}$ is weakly potent. Note that $\bar{A}$ is $\langle c_{(n-1)n}^{s_{(n-1)n}}\rangle$-separable by Lemma 5.17. Therefore G is weakly potent by Corollary 5.6. □

Corollary 5.19. *Let $A_1, A_2, ..., A_n$ be polycyclic-by-finite or free-by-finite groups. Let $G = \langle A_1, A_2, \ldots, A_n | c_{ij}^{s_{ij}} = c_{ji}^{s_{ji}}\rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating cyclic subgroups $\langle c_{ij}^{s_{ij}}\rangle$ of A_i and $\langle c_{ji}^{s_{ji}}\rangle$ of A_j where s_{ij}, s_{ji} are any positive integers. Then G is again weakly potent.*

Next, we show that tree products amalgamating along retracts and tree products amalgamating central subgroups are weakly potent.

Lemma 5.20. *[19] Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji}\rangle$ be a tree product of A_1, $A_2, \ldots, A_n$ amalgamating subgroups H_{ij} of A_i and H_{ji} of A_j. Suppose*

(i) *A_i is H_{ij}-separable for each i;*

(ii) *for each $R_i \lhd_f H_{ij}$, there exists $N_i \lhd_f A_i$ such that $N_i \cap H_{ij} = R_i$ for each i.*

Let K be a subgroup of A_r where $1 \leq r \leq n$ and A_r is K-separable. Then G is K-separable.

Lemma 5.21. *[19] Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji}\rangle$ be a tree product of A_1, A_2, $\ldots, A_n$ amalgamating the subgroups H_{ij} of A_i and H_{ji} of A_j. Suppose, for each $R_i \lhd_f H_{ij}$ there exists $N_i \lhd_f A_i$ such that $N_i \cap H_{ij} = R_i$. Let K be a subgroup of A_r where $1 \leq r \leq n$ and suppose, for each $S \lhd_f K$, there exists $N_r \lhd_f A_r$ such that $N_r \cap K = S$. Then there exists $P \lhd_f G$ such that $P \cap K = S$.*

Theorem 5.22. *Let $A_1, A_2, ..., A_n$ be weakly potent groups. Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji}\rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating subgroups H_{ij} of A_i and H_{ji} of A_j. Suppose*

(1) *A_i is H_{ij}-separable for each i;*

(2) *for each $R_i \lhd_f H_{ij}$, there exists $N_i \lhd_f A_i$ such that $N_i \cap H_{ij} = R_i$ for each i.*

Then G is weakly potent.

Proof. We prove by induction. When $n = 2$, the result follows from Theorem 5.7. As in Theorem 5.16, we write $G = \langle \bar{A}, A_n | H_{(n-1)n} = H_{n(n-1)} \rangle$, where $\bar{A}$ is the tree product generated by $A_1, A_2, \ldots, A_{n-1}$. By our inductive hypothesis, $\bar{A}$ is weakly potent. By Lemma 5.20, $\bar{A}$ is $H_{(n-1)n}$-separable. On the other hand, for each $R_{n-1} \lhd_f H_{(n-1)n}$, there exists $\bar{N} \lhd_f \bar{A}$ such that $\bar{N} \cap H_{(n-1)n} = R_{n-1}$ by Lemma 5.21. Thus the result follows from Theorem 5.7. $\square$

Corollary 5.23. *Let $A_1, A_2, ..., A_n$ be weakly potent groups. Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating subgroups H_{ij} of A_i and H_{ji} of A_j. Suppose H_{ij} is a retract in A_i for each i. Then G is weakly potent.*

Proof. Follows from Lemma 4.7, Lemma 4.8 and Theorem 5.22. $\square$

Theorem 5.24. *Let $A_1, A_2, ..., A_n$ be weakly potent groups. Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating central subgroups H_{ij} of A_i and H_{ji} of A_j. If A_i is M_i-separable for every subgroup $M_i \lhd_f H_{ij}$ for each i, then G is weakly potent.*

Proof. When $n = 2$, the result follows from Theorem 5.13. As in Theorem 5.16, we write $G = \langle \bar{A}, A_n | H_{(n-1)n} = H_{n(n-1)} \rangle$, where $\bar{A}$ is the tree product generated by $A_1, A_2, \ldots, A_{n-1}$. By inductive hypothesis, $\bar{A}$ is weakly potent.

Suppose $R_i \lhd_f H_{ij}$ be given. Let $\hat{A}_i = A_i/R_i$ and $\hat{H}_{ij} = H_{ij}/R_{ij}$. Since $\hat{A}_i$ is residually finite by Lemma 5.12 and $\hat{H}_{ij}$ is finite, there exists $\hat{N}_i \lhd_f \hat{A}_i$ such that $\hat{N}_i \cap \hat{H}_{ij} = \hat{1}$. Let N_i be the preimage of $\hat{N}_i$ in A_i. Then $N_i \lhd_f A_i$ such that $N_i \cap H_{ij} = R_i$. Thus, by Lemma 5.20 $\bar{A}$ is M_r-separable, for each $M_r \lhd_f H_{r(r+1)}$ for $1 \leq r \leq n-1$. Hence G is weakly potent by Theorem 5.13. $\square$

Corollary 5.25. *Let $A_1, A_2, ..., A_n$ be weakly potent groups. Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating finitely generated central subgroups H_{ij} of A_i and H_{ji} of A_j. If A_i is subgroup separable for each i, then G is weakly potent.*

Corollary 5.26. *Let $A_1, A_2, ..., A_n$ be polycyclic-by-finite or free-by-finite. Let $G = \langle A_1, A_2, \ldots, A_n | H_{ij} = H_{ji} \rangle$ be a tree product of $A_1, A_2, \ldots, A_n$ amalgamating finitely* *generated central subgroups H_{ij} of A_i and H_{ji} of A_j. If A_i is subgroup separable for each i, then G is weakly potent.*

5.6 One-relator groups with torsion

For our final result, we show that the one-relator groups with torsion $\langle a, b | (a^l b^m)^t \rangle$, $t > 1$ is weakly potent. First, we have the following definition.

Definition 5.27. *[1] Let x be an element of a group G. G is said to be $\langle x \rangle$-potent if and only if, for every positive integer n, there exists $N \lhd_f G$ such that xN has order exactly n in G/N. If G is $\langle x \rangle$-potent for every nontrivial element x, then G is said to be potent.*

Lemma 5.28. *[1] The group $G = \langle a, b | (ab)^t \rangle, t > 1$ is $\langle a \rangle$-separable, $\langle b \rangle$-separable, and also $\langle a \rangle$-Pot and $\langle b \rangle$-Pot.*

Theorem 5.29. *The group $G = \langle a, b | (a^l b^m)^t \rangle, t > 1$ is weakly potent.*

Proof. Let $S = \langle b, c | c^t \rangle$ and $d = cb^{-m}$ for some positive integer m. Then $S = \langle d, b | (db^m)^t \rangle$. Now, we can write $G = S \underset{d=a^l}{*} \langle a \rangle$. It is clear that $\langle d \rangle$ is of infinite order in S. Since S is $\langle d \rangle$-Pot by Lemma 5.28, there exists $N \lhd_f S$ such that $N \cap \langle d \rangle = \langle d^m \rangle$. Similarly, there exists $M \lhd_f \langle a \rangle$ such that $M \cap \langle a^l \rangle = \langle a^{lm} \rangle$ since infinite cyclic groups are potent. Since S is $\langle d \rangle$-separable and $\langle a \rangle$ is $\langle a^l \rangle$-separable, G is weakly potent by Theorem 5.7. $\square$