CHAPTER 1

INTRODUCTION

1.1 Background

An option is a derivative security that gives the buyer of the option the right to buy or sell the underlying asset, at or before some maturity date T, for an agreed price K, called the strike price or exercise price. A call (or put) option is a right to buy (or sell). Obviously the profit of the buyer of a call (or put) option will depend on the price S(t) of the underlying asset at time t. As exercise is a right and not an obligation, the exercise payoff at time t is

$$(S(t)-K)^{+}=\max\{S(t)-K,0\} \tag{1.1.1}$$

for a call option and

$$(K-S(t))^{+}=\max\{K-S(t),0\} \tag{1.1.2}$$

for a put option. The payoff function of the call option is shown in Figure 1.1.1 while that of the put option is shown in Figure 1.1.2. There are two kinds of options: European options and American styled options. European options can only be exercised at the maturity date T, whereas American options can be exercised at any time at or before the maturity date. Thus American options allow the buyer to have extra flexibility and thus are never worth less than European options. As the date that may be chosen to exercise the options by the buyer, is completely random, the pricing of American options is difficult.

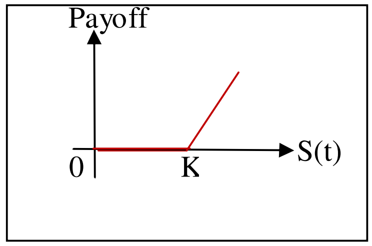

Figure 1.1.1: The payoff function of a call option

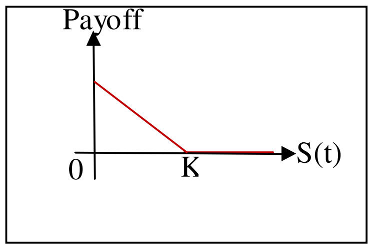

Figure 1.1.2: The payoff function of a put option

1.2 Overview of methods for pricing American options

The important methods for evaluating American option prices include the simulation procedure and the stochastic mesh approach:

(A) Simulation

Let $S_i(t)$ be the time-t price of the i-th asset and $\mathbf{S(t)} = [S_1(t), S_2(t), ..., S_N(t)]^T$ the vector of asset prices. In the simulation procedure, a large number of paths of asset prices of the form $\boldsymbol{S} = (\mathbf{S}(t_0), \mathbf{S}(t_1), \ldots, \mathbf{S}(t_{k^*}))$ at the times $0 = t_0 < t_1 < \ldots < t_{k^*} = T$ are generated. The following are some approaches for pricing American options based on simulation.

(I) The Regression Approach

At each point in time, the value of the American option is determined by the maximum of the value from immediate exercise and the conditional-expectation or continuation value. The conditional-expectation or continuation-value function is estimated by means of regression. A complete specification of the optimal exercise strategy along each path can be obtained by estimating the conditional-expectation function for each exercise date. American options can then be valued approximately by simulation (see for example Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999, 2001), and Carriére (1996)). The above approach tends to underprice American options. An upper bound to the price of an American option may be found by using a procedure proposed by Andersen and Broadie (2004). The above lower and upper bounds enable us to get a better idea of the true value of an American option.

(II) The Parametric Approach

The early decision rule is represented by a finite number of parameters and an approximation to the American option price is maximized over the parameter space to get an estimate of the American option price (see Bossaerts (1989), Li and Zhang (1996), Grant et al. (1997), Andersen (2000), and Garcia (2003)).

(III) The Stratification Approach

The space of the underlying assets is partitioned appropriately into a number of cells (or strata) such that the early exercise strategy is constant over these cells. Monte Carlo simulation is combined with the above stratification to compute the set of conditional probabilities corresponding to changes in the payoff value over time. An approximate value of the American price can then be computed backwards in time using the conditional probabilities (see Barraquand and Martineau (1995) and Raymar and Zwecher (1997)). Tilley (1993) used an idea similar to that of stratification for options based on one asset.

(IV) The Simulated Tree Approach

From the distribution of $\mathbf{S}(t_1)$ conditioned on $\mathbf{S}(t_0)$, *b* values (branches) of $\mathbf{S}(t_1)$ are generated. For k = 2, 3, …, k*, *b* values of $\mathbf{S}(t_k)$ are generated from the distribution of $\mathbf{S}(t_k)$ conditioned on the generated value of $\mathbf{S}(t_{k-1})$. Then from the set of generated paths of the form ($\mathbf{S}(t_0)$, $\mathbf{S}(t_1)$, …, $\mathbf{S}(t_{k^*})$), the option value (i.e. maximum (immediate exercise value, discounted expected option value)) at t_k are found for k = k*, k*-1, …, 0. The option value at t_0 will be an estimate of the American option price (see Broadie and Glasserman (1997) and Broadie et al. (1997b)).

(V) Neural Networks

Let $\mathbf{x}_{t_k}^{(j)}$ be the j-th chosen value of $\mathbf{S}(t_k)$ and $\hat{y}_{t_k}^{(j)}$ the continuation value evaluated at $\mathbf{x}_{t_k}^{(j)}$, j = 1, 2, …, n. Then $\{(\mathbf{x}_{t_k}^{(1)}, \hat{y}_{t_k}^{(1)}), (\mathbf{x}_{t_k}^{(2)}, \hat{y}_{t_k}^{(2)}), \ldots, (\mathbf{x}_{t_k}^{(n)}, \hat{y}_{t_k}^{(n)})\}$ may be treated as a training sample. There are various neural networks classes. An example is the Multi-layer perceptrons (MLP) networks which make use of

$$\hat{f}(\mathbf{x}_{t_k}^{(j)}) = h\{\sum_{i=1}^{L} c_i h(\beta_{0i} + \boldsymbol{\beta}_{1i}^{'} \mathbf{x}_{t_k}^{(j)}) + c_0\}$$

where $h(x)$ is a smooth, monotonic, increasing function of the form $1/(1+e^{-x})$, the c_i, β_{0i} and $\boldsymbol{\beta}_{1i}$ are parameters and L is the number of hidden units or neurons. The values of the c_i, β_{0i} and $\boldsymbol{\beta}_{1i}$ are chosen to minimize

$$H(\hat{f}) = \sum_{t=1}^{T} (\| \hat{y}_{t_k}^{(j)} - \hat{f}(\mathbf{x}_{t_k}^{(j)}) \|^2 + \lambda \| \nabla f(\mathbf{x}_{t_k}^{(j)}) \|^2)$$

where ∇ is the gradient operator and λ is a parameter accounting for the smoothness of the required solution. The optimal values of the c_i, β_{0i} and $\boldsymbol{\beta}_{1i}$ are then used to compute $\hat{f}(\mathbf{S}(t_k))$ when the value of $\mathbf{S}(t_k)$ is given. The corresponding option value at $\mathbf{S}(t_k)$ is next obtained. In this way, we may obtain the option values at the selected values of $\mathbf{S}(t_k)$ for k = k*-1, k*-2, …, 0. The option value at time t_0 is then an estimate of the American option price (see Hunt et al. (1992), Sanner et al. (1992), Kelly (1994), Morelli et al. (2004), Kohler et al. (2006) and Kohler and Krzyzak (2009)).

(B) The Stochastic Mesh Approach

The paths ($\mathbf{S}(t_0)$, $\mathbf{S}(t_1)$, …, $\mathbf{S}(t_{k^*})$) are generated using mesh density instead of the distribution of $\mathbf{S}(t_k)$ conditioned on the value of $\mathbf{S}(t_{k-1})$, k = 1, 2, …, k*. An example of the mesh density is the average density of the distributions of $\mathbf{S}(t_k)$ conditioned respectively on the *b* values of $\mathbf{S}(t_{k-1})$ chosen at time t_{k-1}. For k = k*-1, k*-2, …, 0, the continuation value at time t_k is obtained as the average value of the product of the option value at time t_{k+1} and a weight given by the ratio of the conditional density and the mesh density evaluated at the *b* chosen values of $S(t_{k+1})$ at time t_{k+1}. The resulting option value at time t_0 is then an estimate of the American option price (see for example, Broadie, Glasserman, and Ha (2000), Broadie and Glasserman (2004), Avramidis and Hyden (1999), Avramidis and Matzinger (2004), Liu and Hong (2009), Avramidis et al. (2000), Boyle et al. (2000, 2002), Broadie, Glasserman, and Jain (1997)).

1.3 Introduction to the Thesis

The thesis aims to estimate the American option price when there are N underlying assets, the possible exercise times prior to maturity are $0 = t_0, t_1, t_2, \ldots, t_{k^*} =$

T where $t_k = k\Delta t$ and Δt is a small increment in time, and the vector of asset prices $\mathbf{S(t)} = [S_1(t), S_2(t), ..., S_N(t)]^T$ is modeled as a Levy process. We use a backward procedure to find the option values at $\mathbf{S}(t_{k*})$, $\mathbf{S}(t_{k*-1})$, ..., $\mathbf{S}(0)$ with $\mathbf{S}(0)$ representing the American option price.

To find the option value at $\mathbf{S}(t_k)$, $0 \le k < k*$, we express the vector $\mathbf{S}(t_{k+1})$ of prices at time t_{k+1} given the value $\mathbf{S}(t_k)$ as a function of the vector $\mathbf{e}^{(k+1)}$ = $(e_1^{(k+1)}, e_2^{(k+1)}, ..., e_N^{(k+1)})$ of a set of uncorrelated random variables having respectively the standard normal distributions. The space formed by $\mathbf{e}^{(k+1)}$ is next transformed to the N-dimensional polar coordinate system. The continuation value at $\mathbf{S}(t_k)$ is computed by performing numerical integration along the radial direction and over the polar angles. The option value at $\mathbf{S}(t_k)$ is then given by the larger value of the immediate exercise value and the continuation value at $\mathbf{S}(t_k)$.

To find the option value as a function of $\mathbf{S}(t_k)$, we first derive the distribution of $\mathbf{S}(t_k)$ given $\mathbf{S}(t_0)$. It turns out the random vector $\mathbf{S}(t_k)$ can be expressed as a function of the vector $(\tilde{e}_1^{(k)}, \tilde{e}_2^{(k)}, ..., \tilde{e}_N^{(k)})$ of another set of uncorrelated random variables having respectively the standard normal distributions. The space formed by the $\tilde{\mathbf{e}}^{(k)}$ is next transformed to the N-dimensional polar coordinate system. We approximate the option values for the points along the radial direction by a low degree polynomial. By using a regression procedure, each of the coefficients of the polynomial in terms of the radial distance is next expressed as a low degree polynomial of the polar angles. In this way, we obtain a representation of the option value as a function of $\mathbf{S}(t_k)$. This function can then be used to find the option value at $\mathbf{S}(t_{k-1})$.

As the option values and continuation values are approximated by polynomials obtained by regression procedure, the computed American option price would not be exact. We estimate the standard error of the option value at time t_k for $k = k^*-1, k^*-2, \ldots, 0$ in the indicated order. The estimated standard error at t_0 will then be an estimate of the standard error of the American option price.

When N is large, instead of paying the high cost of estimating the standard errors for all the values of k in $\{k^*-1, k^*-2, \ldots, 0\}$, we may compute the initial few standard errors and use an extrapolation procedure to get an idea of the size of the standard error of American option price.

1.4 Layout of the Thesis

In Chapter 2, we introduce a numerical procedure to find the joint distribution of the vector of time-t asset prices of which the randomness in the underlying stochastic model is described via a Levy process. We then propose a method based on regression and numerical integration for pricing a two-dimensional American basket call option in Chapter 3. In Chapter 4, we use a procedure adapted from that in Chapter 3 to price high-dimensional American basket call options. Chapter 5 is devoted to the estimation of standard error of the computed American call option price. The thesis is concluded in Chapter 6.