# CHAPTER 2

# DISTRIBUTION FOR ASSET PRICES AT A GIVEN TIME

## 2.1 Introduction

The vector of asset prices $\mathbf{S(t)} = [S_1(t), S_2(t), ..., S_N(t)]^T$ is very often described via an N-dimensional Brownian motion. As the individual asset price usually has a fat tail distribution, the vector of asset prices is more appropriately described via an N-dimensional Levy process.

In this chapter, we find the time-t joint distribution of the vector of asset prices $\mathbf{S(t)}$ which is described via a Levy process. We shall see in Chapters 3 and 4 that this joint distribution plays an important role in the proposed numerical method for pricing American option on N assets.

## 2.2 N-dimensional Brownian motion for asset prices

Suppose the vector of asset prices $\mathbf{S(t)}$ is described by the stochastic differential equations

$$dS_i = \mu_i S_i dt + \sigma_i S_i dz_i, \qquad i = 1, 2, ..., N \tag{2.2.1}$$

where $\mu_i$ and $\sigma_i$ are respectively the mean rate and volatility of the price of asset i and $[z_1(t), z_2(t), ..., z_N(t)]^T$ is an N-dimensional Brownian motion that allows for a correlated structure specified by $\text{corr}(dz_i, dz_j) = \rho_{ij} dt$ of which $\rho_{ii} = 1$ for i=1, 2,…, N .

Let $s \geq 0$ and $t > 0$ . The Brownian random quantity $z_i(t+s) - z_i(t)$ has the following properties:

1. $z_i(t+s)-z_i(t)$ is a normal random variables with mean zero and variance s for all t>0.

2. for $s'>0$ and $t'>0$ such that $(t,t+s)\cap(t',t'+s')=\emptyset$, $z_i(t+s)-z_i(t)$ and $z_i(t'+s')-z_i(t')$ are independent.

Let $\Delta t$ be a small increment in time, $t_k = k\Delta t$, k=0, 1, 2,…, k* and $k^*\Delta t = T$. Furthermore let $S_i^{(k)} = S_i(t_k)$. The i-th component of the time-$t_k$ value of the vector of asset prices $\mathbf{S}(t_k)=[S_1(t_k),S_2(t_k),...,S_N(t_k)]^T$ is then given approximately by

$$S_i(t_k)=S_i^{(k)} \cong S_i^{(k-1)} + S_i^{(k-1)}\mu_i\Delta t + S_i^{(k-1)}\sigma_i z_i^{(k)}\sqrt{\Delta t}, \qquad i=1,2,...,N \tag{2.2.2}$$

where $z_i^{(k)} \sim N(0,1)$ for all i and k, $z_i^{(k_1)}$ and $z_i^{(k_2)}$ for $k_1 \neq k_2$ are independent, and $E\left(z_i^{(k)}z_j^{(k)}\right)=\rho_{ij}$.

## 2.3 Levy process

A real-valued process W(t), with W(0)=0, is called a Levy process if :

(i) it has independent increments; that is, for any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < t_2 < ... < t_n$, the random variables $W(t_0), W(t_1)-W(t_0),...,$ $W(t_n)-W(t_{n-1})$ are independent;

(ii) it is time-homogeneous; that is, the distribution of $W(t+s)-W(t), s\geq 0, t>0$ does not depend upon t;

(iii) it is stochastically continuous; that is, for any $\varepsilon>0$, $\Pr\{|W(t+s)-W(t)|>\varepsilon\}\to 0$ as $s\to 0$.

An example of the Levy process can be obtained by requiring dw(t) to have a type of non-normal distribution called the quadratic-normal distribution. The definition of the quadratic-normal distribution (see Pooi (2003)) is given below:

Let $\mu$ and $\boldsymbol{\lambda} = (\lambda_1\ \lambda_2\ \lambda_3)^T$ be constants and consider the following transformation of the standard normal variate e to the variate r:

$$r = \begin{cases} \mu + \lambda_1 e + \lambda_2 (e^2 - (\frac{1+\lambda_3}{2})), & e \geq 0 \\ \mu + \lambda_1 e + \lambda_2 (\lambda_3 e^2 - (\frac{1+\lambda_3}{2})), & e < 0 \end{cases} \tag{2.3.1}$$

The variate r is then said to have a quadratic-normal distribution with parameters $\mu$ and $\boldsymbol{\lambda}$, and we may write $r \sim QN(\mu, \boldsymbol{\lambda})$. By examining the extreme values of r, it can be shown that r is a one-to-one function of e for |e|<Zq provided that $-\lambda_1/(2\lambda_2\lambda_3) < -Z_q$ when $-\lambda_1/(2\lambda_2\lambda_3) < 0$, and $-\lambda_1/(2\lambda_2) > Z_q$ when $-\lambda_1/(2\lambda_2) \geq 0$. The mean of r is $\mu$. The shape of the distribution of r will vary as the value of $\boldsymbol{\lambda}$ varies. If $\lambda_2 = 0$, then r has the normal distribution with mean $\mu$ and variance $\lambda_1^2$. If $\lambda_2 \neq 0$ and $\lambda_3 = -1$, then the distribution of r will be symmetrical about r=$\mu$. When $\lambda_3 \neq -1$, the distribution of r is skewed.

As the standard normal distribution has a third moment which is equal to zero and a fourth moment given by three, the severity of departure from normality may be measured by the standardized third and fourth central moments of r given by $\overline{m}_3 = m_3/m_2^{3/2}$ and $\overline{m}_4 = m_4/m_2^2$ where $m_i = E(r-\mu)^i$, i= 2, 3, 4. As we vary the value of $\boldsymbol{\lambda}$, it is possible to get a fairly large deviation of the value of $(\overline{m}_3, \overline{m}_4)$ from the value (0, 3) which corresponds to the case when r is normal. Thus in a situation in which the random variable involved is non-normal but uni-modal, the quadratic-normal distribution would be a suitable distribution for the random variable.

## 2.4 Multivariate quadratic-normal distribution of asset prices

The prices of the assets are usually correlated and each of them has fatter tails and thinner waist than the normal distribution. The multivariate normal distribution is thus not suitable for approximating the joint distribution of asset prices. Presently we use the multivariate quadratic-normal distribution to approximate the distribution of the asset prices.

The discrete (approximate) version of the model for asset prices based on Levy process is given by

$$S_i(t_k) = S_i^{(k)} \cong S_i^{(k-1)} + S_i^{(k-1)}\mu_i \Delta t + S_i^{(k-1)}\sigma_i w_i^{(k)}\sqrt{\Delta t}, i = 1,2,...,N, k = 0,1,2,...,k^* \qquad (2.4.1)$$

where $\mu_i$ and $\sigma_i$ are respectively the mean rate and volatility of the price of asset i and $\mathbf{w}^{(k)} = (w_1^{(k)}, w_2^{(k)},..., w_N^{(k)})$ is a set of N random variables that has a correlation structure specified by the correlation matrix **P={** $\rho_{ij}$ **}** where $\rho_{ij} = \mathrm{corr}(w_i^{(k)}, w_j^{(k)})$, for $i \neq j$, $i, j = 1, 2,..., N$ and $\mathrm{var}(w_i^{(k)}) = 1$, for i=1, 2,…, N.

Let **B**={ $b_{ij}$ } be the (NxN) matrix formed by the eigenvectors of the (NxN) matrix **P=**{ $\rho_{ij}$ } and

$$\mathbf{v}^{(k)} = \mathbf{B}^T \mathbf{w}^{(k)} \qquad (2.4.2)$$

Suppose the distribution of $v_i^{(k)}$ is given by a quadratic-normal distribution with parameters 0 and $\boldsymbol{\lambda}_i$, i.e.

$$v_i^{(k)} = \begin{cases} \lambda_{i1}e_i^{(k)} + \lambda_{i2}([e_i^{(k)}]^2 - (\dfrac{1+\lambda_{i3}}{2})), & e_i^{(k)} \geq 0 \\ \lambda_{i1}e_i^{(k)} + \lambda_{i2}(\lambda_{i3}[e_i^{(k)}]^2 - (\dfrac{1+\lambda_{i3}}{2})), & e_i^{(k)} < 0 \end{cases} \qquad (2.4.3)$$

where $e_i^{(k)} \sim N(0,1)$.

Let $\mathbf{S}^{(k)*}$ be the value of $\mathbf{S}^{(k)}$ given the value of $\mathbf{S}^{(k-1)}$. By using Eq.(2.4.1) we find the moments $E([S_i^{(k)*}]^{m_1}[S_j^{(k)*}]^{m_2})$ for $m_1 \geq 0, m_2 \geq 0$ and $m_1 + m_2 \leq 4$; for i, j=1, 2,…, N and k=1, 2,…, k*. By using the moments $E([S_i^{(1)*}]^{m_1}[S_j^{(1)*}]^{m_2})$ and the value of $\mathbf{S}^{(0)}$ we can find the moments $E([S_i^{(1)}]^{m_1}[S_j^{(1)}]^{m_2})$, for $m_1 \geq 0, m_2 \geq 0$ and $m_1 + m_2 \leq 4$. Similarly by using the moments $E([S_i^{(k)*}]^{m_1}[S_j^{(k)*}]^{m_2})$ and the value of $E([S_i^{(k-1)}]^{m_1}[S_j^{(k-1)}]^{m_2})$ we can find the moments $E([S_i^{(k)}]^{m_1}[S_j^{(k)}]^{m_2})$ for k=2, 3,…, k*.

Let $\tilde{\mathbf{A}}^{(k)}$ be the (NxN) variance-covariance matrix of $\mathbf{S}^{(k)}$, and $\tilde{\mathbf{B}}^{(k)}$ the matrix formed by the eigenvectors of $\tilde{\mathbf{A}}^{(k)}$. Furthermore let $\tilde{\boldsymbol{\mu}}^{(k)}$ be a vector of which the i-th component is $E(S_i^{(k)})$. Then

$$\tilde{\mathbf{v}}^{(k)} = \tilde{\mathbf{B}}^{(k)T}(\mathbf{S}^{(k)} - \tilde{\boldsymbol{\mu}}^{(k)}), \text{ for k=1, 2,…, k*} \tag{2.4.4}$$

is a vector consisting of uncorrelated random variables.

Let $\overline{S}_i^{(k)m} = (S_i^{(k)} - \tilde{\mu}_i^{(k)})^m$, for m=1, 2, 3, 4. For i=1, 2,…, N, k=1, 2,…, k*, the first four moments of $\tilde{\mathbf{v}}_i^{(k)}$ given by Eq. (2.4.4) can be expressed as:

$$E(\tilde{v}_i^{(k)}) = \sum_{g=1}^{N} \tilde{b}_{gi} E(\overline{S}_g^{(k)}) \tag{2.4.5}$$

$$E([\tilde{v}_i^{(k}]^2) = \sum_{g=1}^{N} \tilde{b}_{gi}^2 E([\overline{S}_g^{(k}]^2) + 2\sum_{\substack{g=1\\g<h}}^{N}\sum_{h=1}^{N} \tilde{b}_{gi}\tilde{b}_{hi} E(\overline{S}_g^{(k)}\overline{S}_h^{(k)}) \tag{2.4.6}$$

$$E([\tilde{v}_i^{(k)}]^3) = \sum_{g=1}^{N} \tilde{b}_{gi}^3 E([\bar{S}_g^{(k)}]^3) + 3\sum_{\substack{g=1 \\ g\neq h}}^{N}\sum_{h=1}^{N} \tilde{b}_{gi}^2 \tilde{b}_{hi} E([\bar{S}_g^{(k)}]^2 \bar{S}_h^{(k)})$$

$$+ 6\sum_{\substack{g=1 \\ g\neq h}}^{N}\sum_{\substack{h=1 \\ h\neq w}}^{N}\sum_{\substack{w=1 \\ w\neq g}}^{N} \tilde{b}_{gi}\tilde{b}_{hi}\tilde{b}_{wi} E(\bar{S}_g^{(k)}\bar{S}_h^{(k)}\bar{S}_w^{(k)}) \tag{2.4.7}$$

$$E([\tilde{v}_i^{(k)}]^4) = \sum_{g=1}^{N} \tilde{b}_{gi}^4 E([\bar{S}_g^{(k}]^4) + 4\sum_{\substack{g=1 \\ g\neq h}}^{N}\sum_{h=1}^{N} \tilde{b}_{gi} \tilde{b}_{hi}^2 E([\bar{S}_g^{(k}]^3 \bar{S}_h^{(k)})$$

$$+ 6\sum_{\substack{g=1 \\ g<h}}^{N}\sum_{h=1}^{N} \tilde{b}_{gi}^2 \tilde{b}_{hi}^2 E([\bar{S}_g^{(k)}]^2 [\bar{S}_h^{(k)}]^2) + 12\sum_{\substack{g=1 \\ g\neq h}}^{N}\sum_{\substack{h=1 \\ h\neq w}}^{N}\sum_{\substack{w=1 \\ w\neq g}}^{N} \tilde{b}_{hi}\tilde{b}_{wi}\tilde{b}_{gi}^2 E([\bar{S}_g^{(k)}]^2 \bar{S}_h^{(k)}\bar{S}_w^{(k)})$$

$$+ 24\sum_{\substack{g=1 \\ g\neq h, g\neq w}}^{N}\sum_{\substack{h=1 \\ h\neq w}}^{N}\sum_{\substack{w=1 \\ w\neq v}}^{N}\sum_{\substack{v=1 \\ h\neq v}}^{N} \tilde{b}_{gi}\tilde{b}_{hi}\tilde{b}_{wi}\tilde{b}_{vi} E(\bar{S}_g^{(k)}\bar{S}_h^{(k)}\bar{S}_w^{(k)}\bar{S}_v^{(k)}) \tag{2.4.8}$$

Let $\tilde{\boldsymbol{\lambda}}_i^{(k)} = (\tilde{\lambda}_{i1}^{(k)}, \tilde{\lambda}_{i2}^{(k)}, \tilde{\lambda}_{i3}^{(k)})^T$ be the value of $\boldsymbol{\lambda}$ such that the moments of r (see Eq. (2.3.1)) match those of $\tilde{v}_i^{(k)}$:

$$E(r^m) = E([\tilde{v}_i^{(k)}]^m), \text{ m=1, 2, 3, 4.} \tag{2.4.9}$$

Then an approximate distribution for $\tilde{v}_i^{(k)}$ is $QN(0,(\tilde{\lambda}_{i1}^{(k)},\tilde{\lambda}_{i2}^{(k)},\tilde{\lambda}_{i3}^{(k)})^T)$, and an approximate distribution of $\mathbf{S}^{(k)}$ may be specified by using $(\tilde{\boldsymbol{\mu}}^{(k)}, \tilde{\mathbf{B}}^{(k)}, \tilde{\boldsymbol{\lambda}}_i^{(k)}, i = 1,2,..., N)$.

## 2.5 Numerical examples

Let N=3, **P={$\rho_{ij}$}**= $\begin{bmatrix} 1 & 0.1 & 0.15 \\ 0.1 & 1 & 0.05 \\ 0.15 & 0.05 & 1 \end{bmatrix}$, and $\mu_i, \sigma_i, \mathbf{S}^{(0)}$, together with the first four moments of $\mathbf{v}^{(k)}$ (see Eq. (2.4.2)) are given in Table 2.5.1.

**Table 2.5.1:** Values of $\mu_i, \sigma_i, \mathbf{S}^{(0)}$, and the first four moments of $\mathbf{v}^{(k)}$ for N=3

| i | $\mu_i$ | $\sigma_i$ | $S_i^{(0)}$ | $E(v_i^{(k)})$ | $E([v_i^{(k)}]^2)$ | $E([v_i^{(k)}]^3)$ | $E([v_i^{(k)}]^4)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.15 | 50.0 | 0 | 0.9375 | 0 | 2.6367 |
| 2 | 0.60 | 0.10 | 60.0 | 0 | 0.9900 | 0 | 2.9406 |
| 3 | 0.30 | 0.20 | 35.0 | 0 | 1.0724 | 0 | 3.4503 |

When $\Delta t = 1/365$ and k=100, the values of $[\tilde{\mathbf{B}}^{(100)}, \tilde{\mu}^{(100)}, \tilde{\lambda}_i^{(100)}, i = 1,2,3]$ are shown in Tables 2.5.2 – 2.5.3.

**Table 2.5.2:** The values of $\tilde{\mathbf{B}}^{(100)}$ obtained by using numerical procedure and simulation

| $\tilde{b}_{ij}^{(100)}$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0.0170<br>*(0.0215)* | -0.3720<br>*(-0.4054)* | 0.92807<br>*(0.9141)* |
| 2 | 0.9557<br>*(0.9496)* | 0.2787<br>*(0.2864)* | 0.0942<br>*(0.1066)* |
| 3 | -0.2937<br>*(-0.3131)* | 0.8853<br>*(0.8680)* | 0.3602<br>*(0.3851)* |

**Table 2.5.3:** The values of $\tilde{\boldsymbol{\mu}}_i^{(100)}$ and $\tilde{\boldsymbol{\lambda}}_i^{(100)T}$ obtained by using numerical procedure and simulation

| i | 1 | 2 | 3 |
|---|---|---|---|
| $\tilde{\boldsymbol{\mu}}_i^{(100)}$ | 53.5434 *(53.5414)* | 70.7104 *(70.7066)* | 37.9969 *(37.9889)* |
| $\tilde{\boldsymbol{\lambda}}_i^{(100)T}$ | [17.5656, 0.7549, 0.8185] *([17.6908, 0.7143, 0.9265])* | [13.6210, 0.3787, 0.8659] *([13.5901, 0.4282, 0.7093])* | [15.6623, 0.9302, 0.7628] *([15.7010, 0.9071, 0.7665])* |

In Tables 2.5.2 – 2.5.3, the values in italics are obtained by using simulation which involves the generation of 100,000 values of $(w_1^{(k)}, w_2^{(k)}, ..., w_N^{(k)})$ for each k. Figure 2.5.1 shows the cumulative probability functions of $\tilde{v}_i^{(100)}$ found by simulation and the numerical procedure for i=1, 2, 3. The simulated results are found to be very close to those based on the numerical procedure.

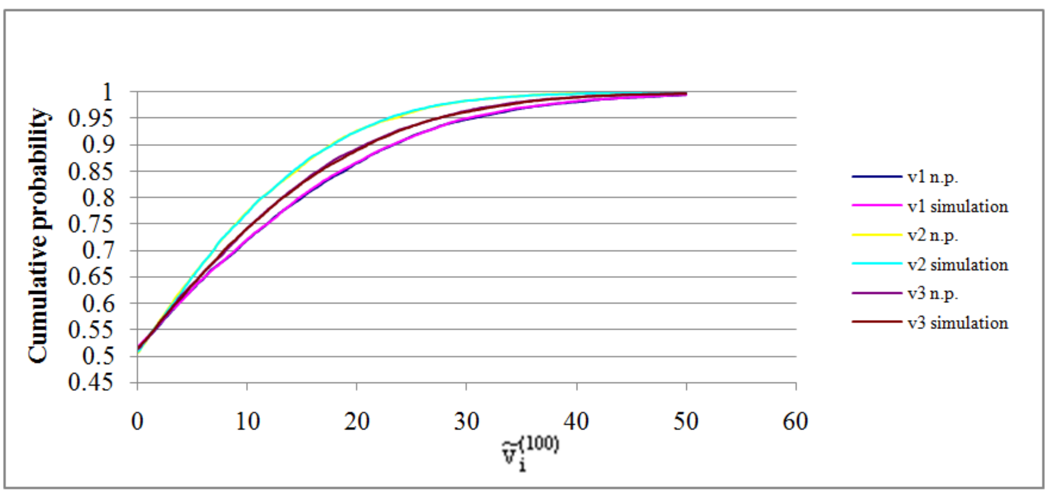


**Figure 2.5.1**: The comparison of the cumulative probability function of $\tilde{v}_i^{*(100)}$ found by simulation and the numerical procedure (n. p.) for i = 1, 2, 3.

Again when N=3 but **P={** $\rho_{ij}$ **}=** $\begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 1 & 0.8 \\ 0.7 & 0.8 & 1 \end{bmatrix}$. $\mu_i, \sigma_i, \mathbf{S^{(0)}}$, together with the first four moments of $\mathbf{v}^{(k)}$ (see Eq. (2.4.2)) are given in Table 2.5.4.

**Table 2.5.4:** Values of $\mu_i, \sigma_i, \mathbf{S^{(0)}}$, and the first four moments of $\mathbf{v}^{(k)}$ for N=3

| i | $\mu_i$ | $\sigma_i$ | $S_i^{(0)}$ | $E(v_i^{(k)})$ | $E([v_i^{(k)}]^2)$ | $E([v_i^{(k)}]^3)$ | $E([v_i^{(k)}]^4)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.15 | 50.0 | 0 | 0.15143 | 0 | 0.06879 |
| 2 | 0.60 | 0.10 | 60.0 | 0 | 0.50779 | 0 | 0.77356 |
| 3 | 0.30 | 0.20 | 35.0 | 0 | 2.34077 | 0 | 16.43764 |

When $\Delta t = 1/365$ and k=100, the values of $[\tilde{\mathbf{B}}^{(100)}, \tilde{\mu}^{(100)}, \tilde{\lambda}_i^{(100)}, i = 1,2,3]$ are shown in Tables 2.5.5 – 2.5.6.

**Table 2.5.5:** The values of $\tilde{\mathbf{B}}^{(100)}$ obtained by using numerical procedure and simulation

| $\tilde{b}_{ij}^{(100)}$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0.27458<br>*(0.27490)* | 0.75859<br>*(0.75750)* | 0.59088<br>*(0.59214)* |
| 2 | 0.59557<br>*(0.59600)* | -0.61662<br>*(-0.61752)* | 0.51486<br>*(0.51327)* |
| 3 | -0.75492<br>*(-0.75446)* | -0.21054<br>*(-0.21182)* | 0.62111<br>*(0.62123)* |

**Table 2.5.6:** The values of $\tilde{\boldsymbol{\mu}}_i^{(100)}$ and $\tilde{\boldsymbol{\lambda}}_i^{(100)T}$ obtained by using numerical procedure and simulation

| i | 1 | 2 | 3 |
|---|---|---|---|
| $\tilde{\boldsymbol{\mu}}_i^{(100)}$ | 53.54341 *(53.52024)* | 70.71044 *(70.69684)* | 37.99696 *(37.98373)* |
| $\tilde{\boldsymbol{\lambda}}_i^{(100)T}$ | [ 17.5639, 0.7557, 0.8166 ] *([17.5340, 0.7643, 0.7764 ])* | [13.6242, 0.3773, 0.8731 ] *([13.4676, 0.43764, 0.6807 ])* | [15.6599, 0.9314, 0.7607] *([15.5960, 0.9788, 0.7222 ])* |

In Tables 2.5.5 – 2.5.6, the values in italics are obtained by using simulation which involves the generation of 100,000 values of $(w_1^{(k)}, w_2^{(k)}, ..., w_N^{(k)})$ for each k. Figure 2.5.2 shows the cumulative probability functions of $\tilde{v}_i^{(100)}$ found by simulation and the numerical procedure for i=1, 2, 3. The simulated results are found to be very close to those based on the numerical procedure.

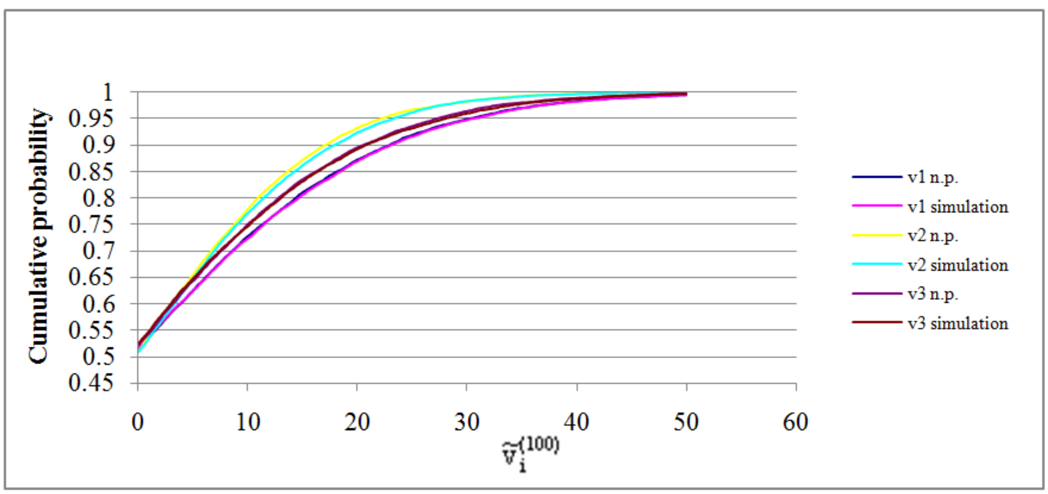


**Figure 2.5.2**: The comparison of the cumulative probability function of $\tilde{v}_i^{*(100)}$ found by simulation and the numerical procedure (n. p.) for i = 1, 2, 3.

Consider the example in which N=6,

$$\mathbf{P}=\{\rho_{ij}\} = \begin{bmatrix} 1 & 0.01 & 0.045 & 0.08 & 0.05 & 0.1 \\ 0.01 & 1 & 0.05 & 0.03 & 0.1 & 0.07 \\ 0.045 & 0.05 & 1 & 0.1 & 0.075 & 0.09 \\ 0.08 & 0.03 & 0.1 & 1 & 0.07 & 0.05 \\ 0.05 & 0.1 & 0.075 & 0.07 & 1 & 0.04 \\ 0.1 & 0.07 & 0.09 & 0.05 & 0.04 & 1 \end{bmatrix}$$

and $\mu_i$, $\sigma_i$, $\mathbf{S}^{(0)}$, together with the first four moments of $\mathbf{v}^{(k)}$ (see Eq. (2.4.2)) are given by

**Table 2.5.7:** Values of $\mu_i, \sigma_i, \mathbf{S}^{(0)}$, and the first four moments of $\mathbf{v}^{(k)}$ for N=6

| i | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\mu_i$ | 0.25 | 0.20 | 0.30 | 0.25 | 0.35 | 0.40 |
| $\sigma_i$ | 0.15 | 0.10 | 0.20 | 0.20 | 0.20 | 0.20 |
| $S_i^{(0)}$ | 50.0 | 60.0 | 35.0 | 40.0 | 45.0 | 52.0 |
| $E(v_i^{(k)})$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $E([v_i^{(k)}]^2)$ | 0.8498 | 0.8891 | 0.9406 | 0.9729 | 1.0240 | 1.3231 |
| $E([v_i^{(k)}]^3)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $E([v_i^{(k)}]^4)$ | 2.1669 | 2.3720 | 2.6543 | 2.8401 | 3.1462 | 5.2525 |

When $\Delta t = 1/365$ and k=100, the values of $[\tilde{\mathbf{B}}^{(100)}, \tilde{\mu}^{(100)}, \tilde{\lambda}_i^{(100)},\ i=1,2,...,6]$ are shown in Tables 2.5.8 – 2.5.9.

**Table 2.5.8:** The values of $\tilde{\mathbf{B}}^{(100)}$ obtained by using numerical procedure and simulation

| $\tilde{b}_{ij}^{(100)}$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0.0221 *(0.0154)* | 0.0015 *(0.0127)* | -0.9367 *(-0.9369)* | -0.3062 *(-0.3015)* | -0.0902 *(-0.0961)* | 0.1418 *(0.1469)* |
| 2 | 0.9911 *(0.9908)* | -0.0605 *(-0.0640)* | 0.0384 *(0.0353)* | 0.0107 *(0.0012)* | -0.0897 *(-0.0881)* | 0.0658 *(0.0716)* |
| 3 | -0.0789 *(-0.0844)* | -0.9488 *(-0.9480)* | 0.1019 *(0.0930)* | -0.2288 *(-0.2329)* | -0.1267 *(-0.1266)* | 0.1209 *(0.1231)* |
| 4 | -0.0129 *(-0.0196)* | 0.2885 *(0.2904)* | 0.3219 *(0.3224)* | -0.8670 *(-0.8651)* | -0.2198 *(-0.2247)* | 0.1132 *(0.1107)* |
| 5 | -0.0941 *(-0.0910)* | 0.0844 *(0.0826)* | 0.0167 *(0.0246)* | 0.2938 *(0.3023)* | -0.9327 *(-0.9290)* | 0.1653 *(0.1721)* |
| 6 | -0.0435 *(-0.0470)* | 0.0749 *(0.0763)* | 0.0820 *(0.0874)* | 0.1250 *(0.1218)* | 0.2221 *(0.2307)* | 0.9595 *(0.9571)* |

**Table 2.5.9:** The values of $\tilde{\boldsymbol{\mu}}_i^{(100)}$ and $\tilde{\boldsymbol{\lambda}}_i^{(100)T}$ obtained by using numerical procedure and simulation

| i | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\tilde{\boldsymbol{\mu}}^{(100)}$ | 53.5434 *(53.5637)* | 63.3784 *(63.3813)* | 37.9969 *(38.0152)* | 42.8347 *(42.8196)* | 49.5264 *(49.5292)* | 58.0191 *(58.0580)* |
| $\tilde{\boldsymbol{\lambda}}_i^{(100)T}$ | [10.6889, 0.2862, 0.8603] *([10.806, 0.3131, 0.8366])* | [13.5273, 0.1196, -10.5089] *([13.517, 0.1375, -9.7925])* | [16.8826, -0.4931, 1.0127] *([16.603, -0.3417, 1.7149])* | [18.5528, 0.2077, -7.6512] *([18.395, 0.2069, -7.5853])* | [24.1880, 0.4246, -6.3923] *([24.391, 0.3573, -7.0318])* | [37.6412, 2.0458, 0.7515] *([37.628, 2.0795, 0.6588])* |

In Tables 2.5.8 – 2.5.9, the values in italics are obtained by using simulation which involves the generation of 100,000 values of $(w_1^{(k)}, w_2^{(k)}, ..., w_N^{(k)})$ for each k. Figure 2.5.3 shows the cumulative probability function of $\tilde{v}_i^{(100)}$ for simulation and the numerical procedure for i=1, 2,…, 6. The simulated results are found to be very close to those based on the numerical procedure.

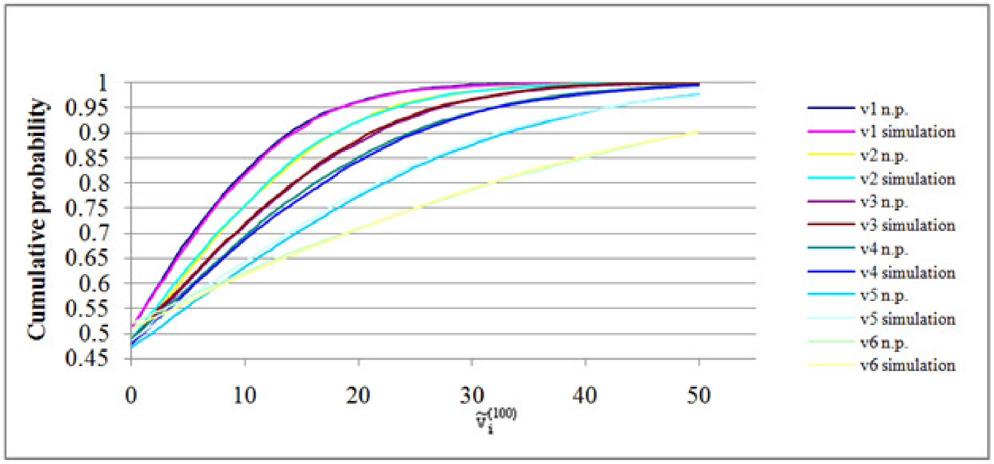


**Figure 2.5.3**: The comparison of the cumulative probability function of $\tilde{v}_i^{*(100)}$ found by simulation and the numerical procedure (n. p.) for i = 1, 2, 3, 4, 5, 6.