CHAPTER 3

PRICING OF AMERICAN CALL OPTIONS ON TWO ASSETS

3.1 Introduction

A major characteristic of the American options is that they can be exercised prior to the maturity date. Pricing of American options is known to be a difficult problem especially when the number of underlying assets is large.

Consider an American basket call option on two assets of which the distribution of the vector $(S_1(t), S_2(t))$ of asset prices at time t is described via a two dimensional Levy process. Pricing the American call option will entail calculating the expected discounted value of its payoff. Presently we propose a method based on numerical integration and regression for pricing a two-dimensional American basket call option. In modeling for the prices of the assets underlying the option, we use a non-normal distribution which is derived from the standard normal distribution. This type of non-normal distribution allows us to speed up the numerical integration as we can make use of the predetermined moments related to the standard normal distributions. A major feature of the method proposed is the use of the distribution for the vector of asset prices to determine the important region over which the required functions of the option values should be determined. In determining the required functions over the important region, we make use of polar coordinate system. It is found that when the polar angle is given, the required functions turn out to be of order two only when viewed as functions of the radial distance. These functions of low order have simplified the numerical integration process to a great extent.

The numerical results for the American call option prices show that the variation of the prices is not negligible as we vary the non-normality of the underlying distributions.

The layout of this chapter is as follows. In Section 3.2, we discuss the arbitrage-free pricing method and the risk-neutral pricing method for pricing the European call option written on N assets. In Section 3.3, we discuss the method based on numerical integration and regression for pricing the American call option written on two assets. In implementing the method in Section 3.3, we use the risk-neutral pricing formula discussed in Section 3.2. We then discuss the method based on simulation for pricing the American call options in Section 3.4. In Section 3.5 we present the results on the American call option price obtained by using the proposed method and simulation. Section 3.6 concludes the chapter.

3.2 Pricing of European call options on N assets

Suppose we are at time $t = (k-1)\Delta t$ and the price of the i-th asset at time $t = k\Delta t$ is as given by Eq. (2.4.1):

$$S_i(t_k) = S_i^{(k)} \cong S_i^{(k-1)} + S_i^{(k-1)}\mu_i \Delta t + S_i^{(k-1)}\sigma_i w_i^{(k)}\sqrt{\Delta t},\ \ i = 1, 2, \ldots,\ N.$$

Consider an European call option with purchase time t_{k-1} and maturity time t_k, strike price K and a payoff at time t_k given by $\left[\sum_{i=1}^{N} a_i S_i^{(k)} - K\right]^+$ where $a_i \geq 0$, $\sum_{i=1}^{N} a_i = 1$ and $[y]^+ = \max(y, 0)$. Suppose we replicate the option by a replicating portfolio consisting of one unit of bond of which the price is B and n_p units of the portfolio of N assets of which the weight of the i-th asset is a_i.

When the value $F_k = \sum_{i=1}^{N} a_i S_i^{(k)}$ of the portfolio of N assets at time t_k is larger than the strike price K, the payoff of the option will be $h\left(t_k, \mathbf{S}^{(k)}\right) = F_k - K$. When the value of the portfolio of N assets at time t_k is smaller or equal to K, the payoff of the option will be $h\left(t_k, \mathbf{S}^{(k)}\right) = 0$.

Assume that there are no arbitrage opportunities. Then

(A) when $F_k > K$, the time-t_k expected return from the replicating portfolio is equal to the time-t_k expected payoff of the option:

$$E^{(+)}(F_k)n_p + e^{r\Delta t}B = E^{(+)}[h(t_k, \mathbf{S}^{(k)})] \tag{3.2.1}$$

where r is the risk free interest rate and $E^{(+)}$ denotes the expectation obtained by using the conditional joint distribution of $\mathbf{S}^{(k)}$ given that $F_k > K$, and

(B) when $F_k \leq K$, the time-t_k expected return from the replicating portfolio is equal to the time-t_k expected payoff of the option:

$$E^{(-)}(F_k)n_p + e^{r\Delta t}B = 0 \tag{3.2.2}$$

where $E^{(-)}$ denotes the expectation obtained by using the conditional joint distribution of $\mathbf{S}^{(k)}$ given that $F_k \leq K$.

Let α and γ be respectively the expected return of the portfolio of N assets and the discounting rate for the option. The discounting rate for the option must be the same as the discounting rate for the replicating portfolio, which is a weighted average of the discounting rates of the two components of the replicating portfolio:

$$e^{\gamma\Delta t} = \frac{F_{k-1}n_p}{F_{k-1}n_p + B}e^{\alpha\Delta t} + \frac{B}{F_{k-1}n_p + B}e^{r\Delta t} \tag{3.2.3}$$

Let C be the option price at time t_{k-1}. By equating the return of the option to that of the replicating portfolio, we have

$$Ce^{\gamma\Delta t} = F_{k-1} n_p e^{\alpha\Delta t} + Be^{r\Delta t} \tag{3.2.4}$$

As Eq.(3.2.4) represents the situation of the no arbitrage, we may refer to C as the arbitrage-free price of the option. By solving Eq.(3.2.1) – (3.2.4), we get

$$C = \frac{E^{(+)}[h(t_k, S^{(k)})][F_{k-1} - E^{(-)}(F_k)e^{-r\Delta t}]}{E^{(+)}(F_k) - E^{(-)}(F_k)} \tag{3.2.5}$$

We next proceed to price the option using the risk-free neutral pricing method. First we may rewrite Eq.(2.4.1) as

$$S_i^{(k)} \cong S_i^{(k-1)} + S_i^{(k-1)} r\Delta t + S_i^{(k-1)} \sigma_i w_i^{*(k)} \sqrt{\Delta t}, \quad i = 1,2,\ldots,N \tag{3.2.6}$$

where $w_i^{*(k)} = w_i^{(k)} + \dfrac{\mu_i - r}{\sigma_i}\sqrt{\Delta t}$.

We note that corr($w_i^{(k)}$, $w_j^{(k)}$) = corr($w_i^{*(k)}$, $w_j^{*(k)}$). Thus the correlation matrix of $\left(w_1^{*(k)}, w_2^{*(k)}, \ldots, w_N^{*(k)}\right)$ is still equal to $\mathbf{P}$ (see Section 2.4). Let

$$\mathbf{v}^{*(k)} = \mathbf{B}^T \mathbf{w}^{*(k)} \tag{3.2.7}$$

where **B** is as defined in Section 2.4.

Let

$$\mathcal{E}_i = e^{-v_i^{*(k)}\vartheta_i} e^{-\frac{1}{2}\vartheta_i^2 + \Theta_3^{(i)}\vartheta_i^3 + \Theta_4^{(i)}\vartheta_i^4}, \quad i = 1, 2, \ldots, N \tag{3.2.8}$$

where ϑ_i is a constant; $\Theta_3^{(i)} = \overline{m}_3^{(i)}/6$, $\Theta_4^{(i)} = (\overline{m}_4^{(i)} - 3)/24$; $\overline{m}_3^{(i)} = m_3^{(i)}/(m_2^{(i)})^{3/2}$, $\overline{m}_4^{(i)} = m_4^{(i)}/(m_2^{(i)})^2$ and $m_j^{(i)} = E\left(v_i^{(k)}\right)^j$, $j = 2, 3, 4$.

Let $f_{v_i^{(k)}}(v_i^{(k)})$ be the probability density function (pdf) of the random variable $v_i^{(k)}$. We next assign $v_i^{*(k)}$ to a pdf given by

$$f_{v_i^{*(k)}}(v_i^{*(k)}) = \mathcal{E}_i \, f_{v_i^{(k)}}(v_i^{*(k)}) \tag{3.2.9}$$

The change of the pdf from $f_{v_i^{(k)}}(v_i^{*(k)})$ to $f_{v_i^{*(k)}}(v_i^{*(k)})$ is essentially the result of applying the Esscher transformation (see Gerber and Goovaerts (1981), Gerber and Shiu (1994), Goovaerts et al. (2004) and Badescu et al. (2009)) to transform the original probability measure P to a new probability measure P*.

We now choose $\vartheta_i = \vartheta_i^*$, i = 1, 2 ,…, N to be ones such that the expected value of $S_i^{(k)}$ under the distribution of $v_i^{*(k)}$ given by Eq.(3.2.9) is equal to $e^{r\Delta t}S_i^{(k-1)}$, i = 1, 2 ,…, N. With these chosen ϑ_i, and sufficiently small Δt, the price C of the option , according to Girsanov's Theorem, is approximately equal to the discounted expected value of $h\left(t_k, \mathbf{S}^{(k)}\right)$ under the distribution of $v_i^{*(k)}$ given by Eq.(3.2.9), i. e.

$$C = E^*[e^{-r\Delta t}h(t_k, \mathbf{S}^{(k)})] \tag{3.2.10}$$

where E^* is an abbreviation of E^{P^*}.

The probability measure P* is called the risk-neutral measure. It turns out that when Δt is sufficiently small,

1. ϑ_i^* will be small.
2. $E^*[v_i^{*(k)}] \cong E^P[v_i^{(k)}] = 0$.
3. $E^*[w_i^{*(k)}] \cong 0$.
4. $E^*[v_i^{*(k)}]^j \cong E^P[v_i^{(k)}]^j$, j = 2, 3, 4.

Thus when Δt is sufficiently small, we may price the option using Eq.(3.2.10) with $S_i^{(k)}$ (see Eq.(3.2.6)) given by

$$S_i^{(k)} \cong S_i^{(k-1)} + S_i^{(k-1)}r\Delta t + S_i^{(k-1)}\sigma_i w_i^{*(k)}\sqrt{\Delta t}$$

where $\mathbf{w}^{*(k)} = \mathbf{B}\mathbf{v}^{*(k)}$, and $f_{v_i^{*(k)}}(v_i^{*(k)}) = f_{v_i^{(k)}}(v_i^{*(k)})$.

Consider the case when N = 2, t_k = 10, r = 0.05, K = 55, ρ = 0.01 and $a_1 = a_2 = 0.5$. When $\overline{m}_3^{(i)}$ = 0 and $\overline{m}_4^{(i)}$ = 3, the random variable $v_i^{*(k)}$ will have the standard normal distribution (or equivalently the quadratic-normal distribution with parameter 0 and $(1, 0, -1)^T$). The values of $\mathbf{S}^{(0)}, \mu_i, \sigma_i, \overline{m}_3^{(i)}, \overline{m}_4^{(i)}$ and $\boldsymbol{\lambda}_i$ are given by Table 3.2.1. The results for the European call option prices when $\overline{m}_3^{(i)}$ and $\overline{m}_4^{(i)}$ vary are as shown in Table 3.2.2. We see from Table 3.2.2 that when Δt is close to zero, the European call option prices found by using Eq.(3.2.10) are close to the arbitrage-free prices found by Eq.(3.2.5).

Table 3.2.1: Values of $\mathbf{S}^{(0)}, \mu_i, \sigma_i, \overline{m}_3^{(i)}, \overline{m}_4^{(i)}$ and $\boldsymbol{\lambda}_i$

Case	i	$S^{(0)}$	μ_i	σ_i	$\overline{m}_3^{(i)}$	$\overline{m}_4^{(i)}$	λ_{i1}	λ_{i2}	λ_{i3}
1	1	50	0.05	0.15	0	3.0	1	0	-1
	2	60	0.05	0.10	0	3.0	1	0	-1
2	1	50	0.05	0.15	0.1	3.6	0.8495	0.1041	-0.7174
	2	60	0.05	0.10	0.1	3.2	0.9737	0.0352	-0.0915
3	1	50	0.05	0.15	0	7.0	0.4103	0.3509	-1
	2	60	0.05	0.10	0	7.0	0.4103	0.3509	-1

Table 3.2.2: European call option prices

Δt	Case 1		Case 2		Case 3	
	Arbitrage-free Price	Risk-neutral Price	Arbitrage-free Price	Risk-neutral Price	Arbitrage-free Price	Risk-neutral Price
0.01	0.24046	0.23940	0.23951	0.23844	0.21580	0.21508
0.05	0.53587	0.53755	0.53374	0.53538	0.48191	0.48404
0.1	0.78812	0.79465	0.78485	0.79132	0.71650	0.72316
0.2	1.19148	1.20883	1.18631	1.20363	1.10236	1.11898
0.3	1.53925	1.56773	1.53247	1.56097	1.44305	1.46982
0.4	1.85891	1.89839	1.85075	1.89028	1.76103	1.79778
0.5	2.16096	2.21106	2.15157	2.20178	2.06468	2.11107
0.6	2.45066	2.51091	2.44023	2.50058	2.35819	2.41374

3.3 Pricing of American call options using numerical integration

Consider an American basket call option on the two assets with time T to expiration and a strike price of K. Suppose the distribution of the vector of asset prices $\mathbf{S}(t) = (S_1(t), S_2(t))$ is described via a Levy process. As in Chapter 2, let Δt be a small increment in time, $t_k = k\Delta t$, $k = 0, 1, \ldots, k^*$, and $k^*\Delta t = T$. The i-th component of the time t_k value of the vector of asset prices $\mathbf{S}(t_k) = [S_1(t_k), S_2(t_k)]^T$ is then given approximately by Eq.(2.4.1).

The payoff from exercise of the basket call option at time t_k when $\mathbf{S}(t_k) = \mathbf{x}^{(k)}$ is given by

$$h(t_k, \mathbf{x}^{(k)}) = [a_1 S_1(t_k) + a_2 S_2(t_k) - K]^+ \quad \text{for } k = 0, 1, 2, \ldots, k^*. \tag{3.3.1}$$

The conditional-expectation of the option value when $\mathbf{S}(t_{k^*-1}) = \mathbf{x}^{(k^*-1)}$ is given by $E^*[h(t_{k^*}, \mathbf{x}^{(k^*)}) \mid \mathbf{S}(t_{k^*-1}) = \mathbf{x}^{(k^*-1)}]$ where E^* is as defined in Section 3.2. Let

$$Q(t_k, \mathbf{x}^{(k)}) = \max(\, h(t_k, \mathbf{x}^{(k)}), e^{(-r\Delta t)} E^*[Q(t_{k+1}, \mathbf{S}(t_{k+1})) \mid \mathbf{S}(t_k) = \mathbf{x}^{(k)}]\,) \text{ for } k < k^* \tag{3.3.2}$$

and

$$Q(t_{k^*}, \mathbf{x}^{(k^*)}) = h(t_{k^*}, \mathbf{x}^{(k^*)}). \tag{3.3.3}$$

The value $Q = Q(0, \mathbf{S}(0))$ will then represent the price of the American basket call option.

When $\mathbf{S}(t_{k^*}) = \mathbf{x}^{(k^*)}$, we have

$$Q(t_{k^*}, \mathbf{x}^{(k^*)}) = h(t_{k^*}, \mathbf{x}^{(k^*)}) = (a_1 S_1(t_{k^*}) + a_2 S_2(t_{k^*}) - K)^+ \tag{3.3.4}$$

The function $Q(t_{k^*}, \mathbf{x}^{(k^*)})$ of $\mathbf{x}^{(k^*)}$ may be summarized as follows.

First we note that the distribution of $\mathbf{S}^{(k^*)}$ (see Section 2.4) is specified by

$$\mathbf{S}^{(k^*)} = \begin{pmatrix} \tilde{\mu}_1^{(k^*)} \\ \tilde{\mu}_2^{(k^*)} \end{pmatrix} + \tilde{\mathbf{B}}^{(k^*)} \begin{pmatrix} \tilde{v}_1^{(k^*)} \\ \tilde{v}_2^{(k^*)} \end{pmatrix} \tag{3.3.5}$$

where

$$\tilde{v}_i^{(k^*)} = \begin{cases} \tilde{\lambda}_{i1}^{(k^*)} \tilde{e}_i^{(k^*)} + \tilde{\lambda}_{i2}^{(k^*)} \left([\tilde{e}_i^{(k^*)}]^2 - \left(\dfrac{1 + \tilde{\lambda}_{i3}^{(k^*)}}{2} \right) \right), & \tilde{e}_i^{(k^*)} \geq 0 \\ \tilde{\lambda}_{i1}^{(k^*)} \tilde{e}_i^{(k^*)} + \tilde{\lambda}_{i2}^{(k^*)} \left(\tilde{\lambda}_{i3}^{(k^*)} [\tilde{e}_i^{(k^*)}]^2 - \left(\dfrac{1 + \tilde{\lambda}_{i3}^{(k^*)}}{2} \right) \right), & \tilde{e}_i^{(k^*)} < 0 \end{cases} \tag{3.3.6}$$

and $\tilde{e}_i^{(k*)} \sim N(0,1)$, i = 1, 2.

We introduce a polar coordinate system given by

$$\tilde{e}_1^{(k*)} = \tilde{\rho}^{(k*)} \cos\tilde{\theta}^{(k*)} \tag{3.3.7}$$

and

$$\tilde{e}_2^{(k*)} = \tilde{\rho}^{(k*)} \sin\tilde{\theta}^{(k*)}. \tag{3.3.8}$$

For a given value of $\tilde{\theta}^{(k*)}$ in a set $\{\tilde{\theta}_1^{(k*)}, \tilde{\theta}_2^{(k*)}, ..., \tilde{\theta}_M^{(k*)}\}$ of M equally spaced angles with $\tilde{\theta}_1^{(k*)} = 0°$ and $\tilde{\theta}_M^{(k*)} = 360°$, we use Eq. (3.3.5), (3.3.6), (3.3.7) and (3.3.8) to find $Q(t_{k*}, \mathbf{x}^{(k*)})$ as a function of $\tilde{\rho}^{(k*)}$. This function of $\tilde{\rho}^{(k*)}$ turns out to be the form

$$Q(t_{k*}, \mathbf{x}^{(k*)}) = \begin{cases} Q_1(t_{k*}, \mathbf{x}^{(k*)}), & \text{for} \quad 0 \le \tilde{\rho}^{(k*)} \le \tilde{\xi}^{(k*)} \\ 0, \quad \text{for} & \tilde{\rho}^{(k*)} > \tilde{\xi}^{(k*)} \end{cases} \tag{3.3.9}$$

where $\tilde{\xi}^{(k*)}$ is a constant which depends on $\tilde{\theta}^{(k*)}$.

We may use a regression procedure to approximate $Q_1(t_{k*}, \mathbf{x}^{(k*)})$ by a quadratic function and express $Q(t_{k*}, \mathbf{x}^{(k*)})$ as

$$Q(t_{k*}, \mathbf{x}^{(k*)}) = \begin{cases} \tilde{c}_0^{(k*)} + \tilde{c}_1^{(k*)}\tilde{\rho}^{(k*)} + \tilde{c}_2^{(k*)}[\tilde{\rho}^{(k*)}]^2, & 0 \le \tilde{\rho}^{(k*)} \le \tilde{\xi}^{(k*)} \\ 0, & \tilde{\rho}^{(k*)} > \tilde{\xi}^{(k*)} \end{cases} \tag{3.3.10}$$

where $\tilde{c}_0^{(k*)}, \tilde{c}_1^{(k*)}, \tilde{c}_2^{(k*)}$, and $\tilde{\xi}^{(k*)}$ are constants which depend on $\tilde{\theta}^{(k*)}$.

For example, consider the case when N = 2, T = 10/365, r = 0.05, K = 54, ρ = 0.01 and $a_1 = a_2 = 0.5$. Let $\overline{m}_3^{(i)} = E[v_i^{(k)}]^3 = 0$ and $\overline{m}_4^{(i)} = E[v_i^{(k)}]^4 = 3$ be respectively the measures of skewness and kurtosis of the random variable $v_i^{(k)}$ for i = 1, 2. The values of $\mathbf{S}^{(0)}, \mu_i, \sigma_i$, $\overline{m}_3^{(i)}, \overline{m}_4^{(i)}$ and $\boldsymbol{\lambda}_i$ are given by Table 3.2.1. Examples of the fitted quadratic function of $Q(t_{k*}, \mathbf{x}^{(k*)})$ are shown in Figures 3.3.1 – 3.3.4.

Table 3.3.1: Values of $\mathbf{S}^{(0)}, \mu_i, \sigma_i, \overline{m}_3^{(i)}, \overline{m}_4^{(i)}$ and $\boldsymbol{\lambda}_i$

i	$S^{(0)}$	μ_i	σ_i	$\overline{m}_3^{(i)}$	$\overline{m}_4^{(i)}$	λ_{i1}	λ_{i2}	λ_{i3}
1	50	0.05	0.15	0	3	1	0	-1
2	60	0.05	0.10	0	3	1	0	-1

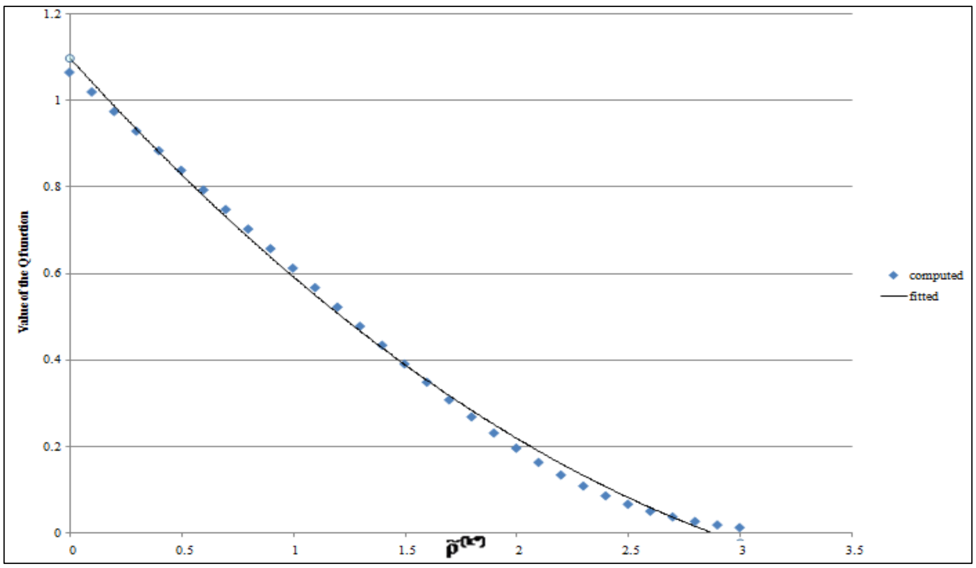

Figure 3.3.1: Computed and fitted quadratic function of $Q(t_{k*}, \mathbf{x}^{(k*)})$ at $\tilde{\theta}^{(k*)} = 0°$.

[Exercise dates are 1/365, 2/365,…, 10/365, r = 0.05, K = 54, ρ = 0.01 and $a_1 = a_2 = 0.5$, the fitted function is $y=0.0661\ x^2-0.5719x+1.0976$, other parameters are as given in Table 3.3.1]

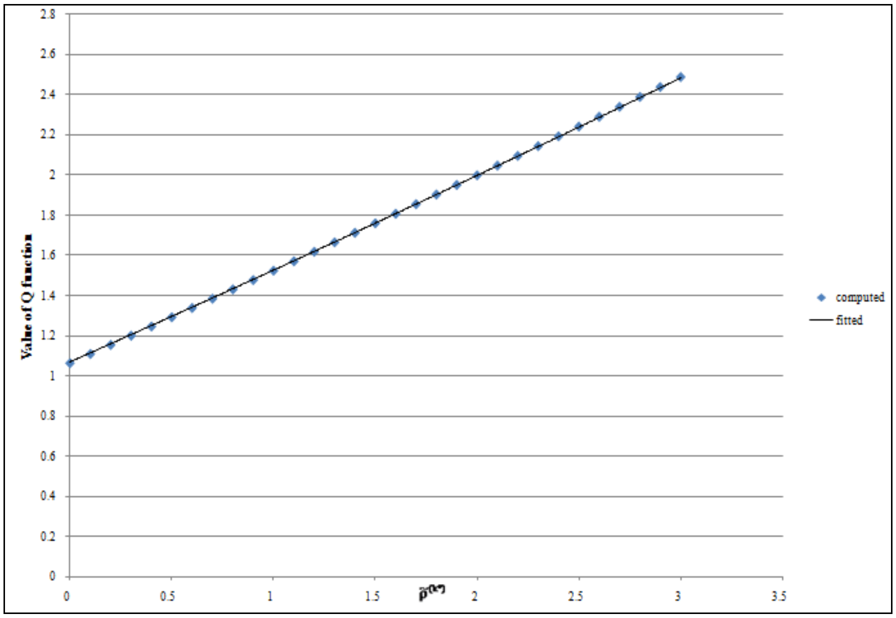

Figure 3.3.2: Computed and fitted quadratic function of Q(t_{k*}, $\mathbf{x}^{(k*)}$) at $\tilde{\theta}^{(k*)} = 180°$.

[Exercise dates are 1/365, 2/365,…, 10/365, r = 0.05, K = 54, ρ = 0.01 and $a_1 = a_2 = 0.5$, the fitted function is y=0.0071 x^2+0.4518x+1.0646, other parameters are as given in Table 3.3.1]

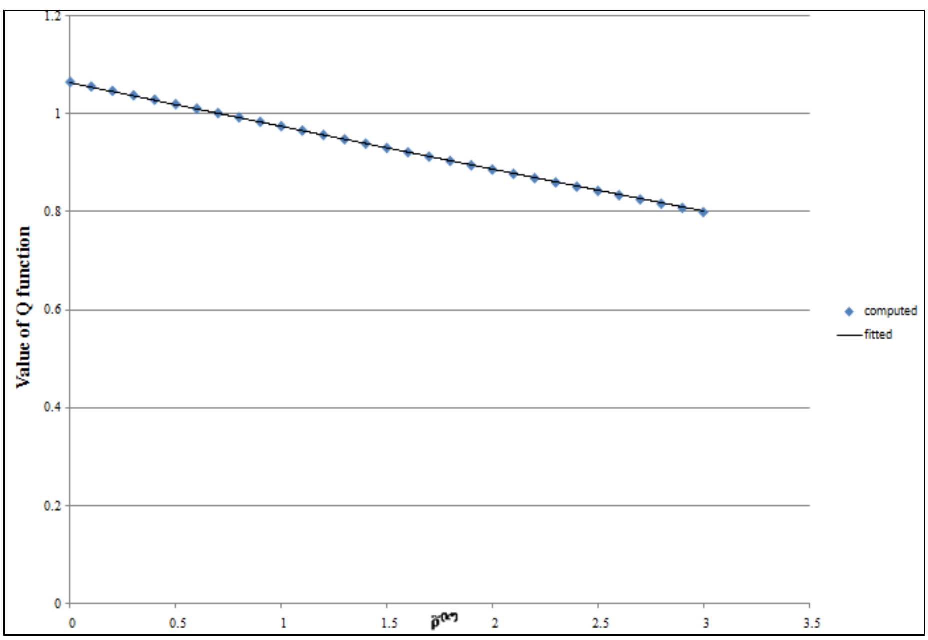

Figure 3.3.3: Computed and fitted quadratic function of Q(t_{k*}, $\mathbf{x}^{(k*)}$) at $\tilde{\theta}^{(k*)} = 30°$.

[Exercise dates are 1/365, 2/365,…, 10/365, r = 0.05, K = 54, ρ = 0.01 and $a_1 = a_2 = 0.5$, the fitted function is y = 0.0008 x^2-0.0905x+1.0646, other parameters are as given in Table 3.3.1]

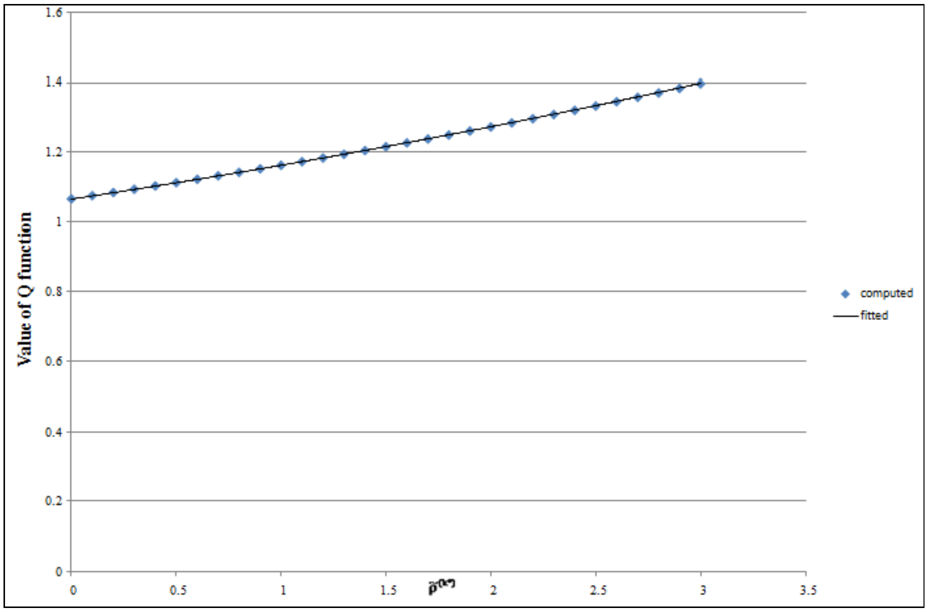

Figure 3.3.4: Computed and fitted quadratic function of $Q(t_{k*}, \mathbf{x}^{(k*)})$ at $\tilde{\theta}^{(k*)} = 210°$.

[Exercise dates are 1/365, 2/365,…, 10/365, r = 0.05, K = 54, ρ = 0.01 and $a_1 = a_2 = 0.5$, the fitted function is $y = 0.0069x^2+0.0905x+1.0646$, other parameters are as given in Table 3.3.1]

The simulation results in Section 3.5 also indicate that the above quadratic approximation is adequate. This function of $\tilde{\rho}^{(k*)}$ may be summarized by the vector ($\tilde{c}_0^{(k*)}, \tilde{c}_1^{(k*)}, \tilde{c}_2^{(k*)}, \tilde{\xi}^{(k*)}$), and an approximation for the function $Q(t_{k*}, \mathbf{x}^{(k*)})$ may be represented by a table containing five columns for $\tilde{\theta}^{(k*)}$, $\tilde{c}_0^{(k*)}$, $\tilde{c}_1^{(k*)}$, $\tilde{c}_2^{(k*)}$ and $\tilde{\xi}^{(k*)}$ respectively.

For $k = k^*, k^*-1, ..., 2, 1$, we next find $Q(t_{k-1}, \mathbf{x}^{(k-1)})$. To achieve this, we first note that

$$\mathbf{S}^{(k-1)} = \begin{pmatrix} \tilde{\mu}_1^{(k-1)} \\ \tilde{\mu}_2^{(k-1)} \end{pmatrix} + \tilde{\mathbf{B}}^{(k-1)} \begin{pmatrix} \tilde{v}_1^{(k-1)} \\ \tilde{v}_2^{(k-1)} \end{pmatrix} \tag{3.3.11}$$

where

$$\tilde{v}_i^{(k-1)} = \begin{cases} \tilde{\lambda}_{i1}^{(k-1)} \tilde{e}_i^{(k-1)} + \tilde{\lambda}_{i2}^{(k-1)} ([\tilde{e}_i^{(k-1)}]^2 - (\frac{1+\tilde{\lambda}_{i3}^{(k-1)}}{2})), & \tilde{e}_i^{(k-1)} \geq 0 \\ \tilde{\lambda}_{i1}^{(k-1)} \tilde{e}_i^{(k-1)} + \tilde{\lambda}_{i2}^{(k-1)} (\tilde{\lambda}_{i3}^{(k-1)} [\tilde{e}_i^{(k-1)}]^2 - (\frac{1+\tilde{\lambda}_{i3}^{(k-1)}}{2})), & \tilde{e}_i^{(k-1)} < 0 \end{cases} \tag{3.3.12}$$

We again introduce a polar coordinate system given by

$$\tilde{e}_1^{(k-1)} = \tilde{\rho}^{(k-1)} \cos \tilde{\theta}^{(k-1)} \tag{3.3.13}$$

and

$$\tilde{e}_2^{(k-1)} = \tilde{\rho}^{(k-1)} \sin \tilde{\theta}^{(k-1)}. \tag{3.3.14}$$

For a given value of $\tilde{\theta}^{(k-1)}$ in a set $\{\tilde{\theta}_1^{(k-1)}, \tilde{\theta}_2^{(k-1)}, ..., \tilde{\theta}_M^{(k-1)}\}$ of M equally spaced angles with $\tilde{\theta}_1^{(k-1)} = 0°$ and $\tilde{\theta}_M^{(k-1)} = 360°$, and $\tilde{\rho}_j^{(k-1)}$=jh where j=0, 1, 2, ..., ℓ and ℓh = ϕ where $\phi^2 = \chi^2_{N,0.01}$ is the 99% point of the chi square distribution with N degrees of freedom, we compute $\tilde{e}_1^{(k-1)} = \tilde{\rho}_j^{(k-1)} \cos \tilde{\theta}^{(k-1)}$ and $\tilde{e}_2^{(k-1)} = \tilde{\rho}_j^{(k-1)} \sin \tilde{\theta}^{(k-1)}$, and then use Eq.(3.3.11) to compute $\mathbf{S}(t_{k-1}) = \mathbf{x}^{(k-1)}$. We next need to find $h(t_{k-1}, \mathbf{x}^{(k-1)})$ and $E^*[Q(t_k, \mathbf{S}^{(k)} | \mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})]$ in order to determine $Q(t_{k-1}, \mathbf{x}^{(k-1)})$.

To find $E^*[Q(t_k, \mathbf{S}^{(k)} | \mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})]$, we may perform a two-dimensional numerical integration. The relevant procedure is as follows. First we introduce a polar coordinate system given by

$$e_1^{(k)} = \rho^{(k)} \cos\theta^{(k)} \tag{3.3.15}$$

and

$$e_2^{(k)} = \rho^{(k)} \sin\theta^{(k)}. \tag{3.3.16}$$

For a given value of $\theta^{(k)}$ in a set $\{\theta_1^{(k)}, \theta_2^{(k)}, ..., \theta_M^{(k)}\}$ of M equally spaced angles with $\theta_1^{(k)} = 0°$ and $\theta_M^{(k)} = 360°$, and a given value of $\rho_j^{(k)}$=jh where j=0, 1, 2, ..., ℓ and ℓh = ϕ where $\phi^2 = \chi^2_{N,0.01}$ is the 99% point of the chi square distribution with N degrees of freedom, we compute $e_1^{(k)} = \rho_j^{(k)} \cos\theta^{(k)}$ and $e_2^{(k)} = \rho_j^{(k)} \sin\theta^{(k)}$. We next use Eq.(2.4.3) to get

$$v_i^{*(k)} = \begin{cases} \lambda_{i1} e_i^{(k)} + \lambda_{i2}([e_i^{(k)}]^2 - (\frac{1+\lambda_{i3}}{2})), & e_i^{(k)} \geq 0 \\ \lambda_{i1} e_i^{(k)} + \lambda_{i2}(\lambda_{i3}[e_i^{(k)}]^2 - (\frac{1+\lambda_{i3}}{2})), & e_i^{(k)} < 0 \end{cases} \tag{3.3.17}$$

We now compute

$$\mathbf{w}^{*(k)} = \mathbf{B}\mathbf{v}^{*(k)}, \tag{3.3.18}$$

and

$x_i^{(k)} = S_i^{(k)}$(conditioned on $S_i^{(k-1)}$) $= S_i^{(k-1)}(1 + r\Delta t + \sigma_i w_i^{*(k)} \sqrt{\Delta t})$, for i = 1,2. (3.3.19)

Then we find $\tilde{\mathbf{v}}^{(k)} = \tilde{\mathbf{B}}^{(k)T}(\mathbf{x}^{(k)} - \tilde{\boldsymbol{\mu}}^{(k)})$ (see Eq. (2.4.4)), and $\tilde{e}_i^{(k)}$ by using

$$\tilde{v}_i^{(k)} = \begin{cases} \tilde{\lambda}_{i1}\tilde{e}_i^{(k)} + \tilde{\lambda}_{i2}([\tilde{e}_i^{(k)}]^2 - (\dfrac{1+\tilde{\lambda}_{i3}}{2})), & \tilde{e}_i^{(k)} \geq 0 \\ \tilde{\lambda}_{i1}\tilde{e}_i^{(k)} + \tilde{\lambda}_{i2}(\tilde{\lambda}_{i3}[\tilde{e}_i^{(k)}]^2 - (\dfrac{1+\tilde{\lambda}_{i3}}{2})), & \tilde{e}_i^{(k)} < 0 \end{cases} \tag{3.3.20}$$

and obtain

$$\tilde{\rho}^{(k)} = \sqrt{[\tilde{e}_1^{(k)}]^2 + [\tilde{e}_2^{(k)}]^2} \tag{3.3.21}$$

and
$$\tilde{\theta}^{(k)} = \tan^{-1}\left(\frac{\tilde{e}_2^{(k)}}{\tilde{e}_1^{(k)}}\right). \tag{3.3.22}$$

From the value of $\tilde{\theta}^{(k)}$, we find $\tilde{c}_0^{(k)}$, $\tilde{c}_1^{(k)}$, $\tilde{c}_2^{(k)}$, and $\tilde{\xi}^{(k)}$ from the previously found table which contains five columns for $\tilde{\theta}^{(k)}$, $\tilde{c}_0^{(k^*)}$, $\tilde{c}_1^{(k^*)}$, $\tilde{c}_2^{(k^*)}$, and $\tilde{\xi}^{(k^*)}$ respectively, and compute

$$Q(t_k, \mathbf{x}^{(k)}) = \begin{cases} \tilde{c}_0^{(k)} + \tilde{c}_1^{(k)}\tilde{\rho}^{(k)} + \tilde{c}_2^{(k)}[\tilde{\rho}^{(k)}]^2, & 0 \leq \tilde{\rho}^{(k)} \leq \tilde{\xi}^{(k)} \\ 0, & \tilde{\rho}^{(k)} > \tilde{\xi}^{(k)} \end{cases} \tag{3.3.23}$$

In short, for a given value of $\theta^{(k)}$ and the values $\rho_j^{(k)}$, j = 0,1, 2, …, ℓ of $\rho^{(k)}$, we find $\ell+1$ corresponding values of $Q(t_k, \mathbf{x}^{(k)})$. From these $\ell+1$ values of $Q(t_k, \mathbf{x}^{(k)})$, we use a regression procedure to express $Q(t_k, \mathbf{x}^{(k)})$ as

$$Q(t_k, \mathbf{x}^{(k)}) = \begin{cases} c_0^{(k)} + c_1^{(k)}\rho^{(k)} + c_2^{(k)}[\rho^{(k)}]^2, & 0 \leq \rho^{(k)} \leq \xi^{(k)} \\ 0, & \rho^{(k)} > \xi^{(k)} \end{cases} \tag{3.3.24}$$

The value of

$$E^*[Q(t_k, \mathbf{S}^{(k)} \mid \mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})] = \sum_{i=1}^{M} \frac{1}{M} \int_{\rho^{(k)}=0}^{\xi^{(k)}} (c_0^{(k)} + c_1^{(k)}\rho^{(k)} + c_2^{(k)}[\rho^{(k)}]^2)\rho^{(k)} e^{-[\rho^{(k)}]^2/2} d\rho^{(k)}$$

(3.3.25)

can then be computed using numerical integration. To speed up the numerical integration, we may first compute

$$F(\xi^{(k)}, n) = \int_0^{\xi^{(k)}} [\rho^{(k)}]^n e^{-[\rho^{(k)}]^2/2} d\rho^{(k)}, \qquad n = 1,2,... \tag{3.3.26}$$

for selected values of $\xi^{(k)}$ and n, and use interpolation to find the value of F(v, n) based on the values of F($\xi_1^{(k)}$, n) and F($\xi_2^{(k)}$, n) of which $\xi_1^{(k)} < v < \xi_2^{(k)}$.

Then

$$Q(t_{k-1}, \mathbf{x}^{(k-1)}) = \max(h(t_{k-1}, \mathbf{x}^{(k-1)}), e^{(-r\Delta t)}E^*[Q(t_k, \mathbf{S}^{(k)}|\mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})]). \tag{3.3.27}$$

For a given $\tilde{\theta}^{(k-1)}$, we may use a regression procedure to approximate $Q(t_{k-1}, \mathbf{x}^{(k-1)})$ by a quadratic function :

$$Q(t_{k-1}, \mathbf{x}^{(k-1)}) = \begin{cases} \tilde{c}_0^{(k-1)} + \tilde{c}_1^{(k-1)}\tilde{\rho}^{(k-1)} + \tilde{c}_2^{(k-1)}[\tilde{\rho}^{(k-1)}]^2, & 0 \le \tilde{\rho}^{(k-1)} \le \tilde{\xi}^{(k-1)} \\ 0, & \tilde{\rho}^{(k-1)} > \tilde{\xi}^{(k-1)} \end{cases} \tag{3.3.28}$$

where $\tilde{c}_0^{(k-1)}$, $\tilde{c}_1^{(k-1)}$, $\tilde{c}_2^{(k-1)}$, and $\tilde{\xi}^{(k-1)}$ are constants which depend on $\tilde{\theta}^{(k-1)}$.

An approximation of $Q(t_{k-1}, \mathbf{x}^{(k-1)})$ again may be represented by a table containing five columns for $\tilde{\theta}^{(k-1)}$, $\tilde{c}_0^{(k-1)}$, $\tilde{c}_1^{(k-1)}$, $\tilde{c}_2^{(k-1)}$ and $\tilde{\xi}^{(k-1)}$ respectively.

By finding $Q(t_{k^*}, \mathbf{x}^{(k^*)})$, $Q(t_{k^*-1}, \mathbf{x}^{(k^*-1)})$, ..., $Q(t_1, \mathbf{x}^{(1)})$, $Q(t_0, \mathbf{x}^{(0)})$ in the indicated order, we can finally obtain the price of the American basket call option

$$Q = Q(t_0, \mathbf{x}^{(0)}) = Q(0, \mathbf{S}(0)). \tag{3.3.29}$$

3.4 Pricing of American call options using simulation

For a given value of $\tilde{\theta}^{(k^*)}$ in a set $\{\tilde{\theta}_1^{(k^*)}, \tilde{\theta}_2^{(k^*)}, ..., \tilde{\theta}_M^{(k^*)}\}$ of M equally spaced angles with $\tilde{\theta}_1^{(k^*)} = 0°$ and $\tilde{\theta}_M^{(k^*)} = 360°$, and a given value of $\tilde{\rho}_j^{(k^*)} = jh$ where j = 0, 1, 2, ..., ℓ and $\ell h = \phi$ where $\phi^2 = \chi^2_{N,0.01}$ is the 99% point of the chi square distribution with N degrees of freedom, we

(i) find $\tilde{e}_1^{(k^*)}$ and $\tilde{e}_2^{(k^*)}$ using Eq.(3.3.7) and (3.3.8) with $\tilde{\rho}^{(k^*)} = \tilde{\rho}_j^{(k^*)}$,

(ii) find $\tilde{v}_1^{(k^*)}$ and $\tilde{v}_2^{(k^*)}$ using Eq.(3.3.6) ,

(iii) find $\tilde{S}_1^{(k^*)}$ and $\tilde{S}_2^{(k^*)}$ using Eq.(3.3.5),

and

(iv) find $Q(t_{k^*}, \mathbf{x}^{(k^*)})$ using Eq.(3.3.4).

For a given $\tilde{\theta}^{(k^*)}$, we may then use a regression procedure to approximate $Q(t_{k^*}, \mathbf{x}^{(k^*)})$ by a quadratic function of $\tilde{\rho}^{(k^*)}$:

$$Q(t_k, \mathbf{x}^{(k^*)}) = \begin{cases} c_0^{(k^*)} + c_1^{(k^*)}\tilde{\rho}^{(k^*)} + c_2^{(k^*)}[\tilde{\rho}^{(k^*)}]^2, & 0 \le \tilde{\rho}^{(k^*)} \le \tilde{\xi}^{(k^*)} \\ 0, & \tilde{\rho}^{(k^*)} > \tilde{\xi}^{(k^*)} \end{cases} \tag{3.4.1}$$

where $\tilde{c}_0^{(k^*)}$, $\tilde{c}_1^{(k^*)}$, $\tilde{c}_2^{(k^*)}$ and $\tilde{\xi}^{(k^*)}$ are constants which depend on $\tilde{\theta}^{(k^*)}$.

This function of $\tilde{\rho}^{(k^*)}$ may be summarized by the vector ($\tilde{c}_0^{(k^*)}$, $\tilde{c}_1^{(k^*)}$, $\tilde{c}_2^{(k^*)}$, $\tilde{\xi}^{(k^*)}$) and an approximation for the function $Q(t_{k^*}, \mathbf{x}^{(k^*)})$ may be represented by a table containing five columns for $\tilde{\theta}^{(k^*)}, \tilde{c}_0^{(k^*)}, \tilde{c}_1^{(k^*)}, \tilde{c}_2^{(k^*)}$ and $\tilde{\xi}^{(k^*)}$ respectively.

We next find $Q(t_{k-1}, \mathbf{x}^{(k-1)})$ for $k = k^*, k^*-1, \ldots, 2, 1$. For a given value of $\tilde{\theta}^{(k-1)}$ in a set $\{\tilde{\theta}_1^{(k-1)}, \tilde{\theta}_2^{(k-1)}, \ldots, \tilde{\theta}_M^{(k-1)}\}$ of M equally spaced angles with $\tilde{\theta}_1^{(k-1)} = 0°$ and $\tilde{\theta}_M^{(k-1)} = 360°$, we consider the value of $\tilde{\rho}_j^{(k-1)} = jh$ where $j = 0, 1, 2, \ldots, \ell$ and $\ell h = \phi$ where $\phi^2 = \chi^2_{N,0.01}$ is the 99% point of the chi square distribution with N degrees of freedom. From ($\tilde{\theta}^{(k-1)}, \tilde{\rho}_j^{(k-1)}$), we first find $\tilde{e}_1^{(k-1)}$ and $\tilde{e}_2^{(k-1)}$ using Eq.(3.3.13) and (3.3.14) with $\tilde{\rho}^{(k-1)} = \tilde{\rho}_j^{(k-1)}$. We next find $\mathbf{S}(t_{k-1}) = \mathbf{x}^{(k-1)}$ using Eq.(3.3.12) and then Eq.(3.3.11). We then need to find $h(t_{k-1}, \mathbf{x}^{(k-1)})$ and $E^*[Q(t_k, \mathbf{S}^{(k)} | \mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})]$ so that we can determine $Q(t_{k-1}, \mathbf{x}^{(k-1)})$.

To find $E^*[Q(t_k, \mathbf{S}^{(k)} | \mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})]$ using simulation, we first generate n_s values of $(e_1^{(k)}, e_2^{(k)})$ where $e_i^{(k)} \sim N(0,1)$ and $e_1^{(k)}$, $e_2^{(k)}$ are uncorrelated. For each generated $(e_1^{(k)}, e_2^{(k)})$ we compute the corresponding $(v_1^{*(k)}, v_2^{*(k)})$, $(w_1^{*(k)}, w_2^{*(k)})$ and $(x_1^{(k)}, x_2^{(k)})$ (see Eq.(3.3.17) – (3.3.19)). We then

(i) find $\tilde{v}_1^{(k)}$ and $\tilde{v}_2^{(k)}$ using Eq.(2.4.4),

(ii) find $\tilde{e}_1^{(k)}$ and $\tilde{e}_2^{(k)}$ using Eq.(3.3.20),

(iii) find $(\tilde{\theta}^{(k)}, \tilde{\rho}^{(k)})$ using Eq.(3.3.21) and (3.3.22),

and

(iv) find $Q(t_k, \mathbf{x}^{(k)})$ using Eq.(3.3.23).

Based on the resulting n_s values of $Q(t_k, \mathbf{x}^{(k)})$, we find the average value of $Q(t_k, \mathbf{x}^{(k)})$ and use the average value to estimate $E^*[Q(t_k, \mathbf{S}^{(k)}|\mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})]$.

Then

$$Q(t_{k-1}, \mathbf{x}^{(k-1)}) = \max(h(t_{k-1}, \mathbf{x}^{(k-1)}), e^{(-r\Delta t)}E^*[Q(t_k, \mathbf{S}^{(k)}|\mathbf{S}^{(k-1)} = \mathbf{x}^{(k-1)})]). \tag{3.4.2}$$

For a given value of $\tilde{\theta}^{(k-1)}$, we may use a regression procedure to approximate $Q(t_{k-1}, \mathbf{x}^{(k-1)})$ by a quadratic function :

$$Q(t_{k-1}, \mathbf{x}^{(k-1)}) = \begin{cases} \tilde{c}_0^{(k-1)} + \tilde{c}_1^{(k-1)}\tilde{\rho}^{(k-1)} + \tilde{c}_2^{(k-1)}[\tilde{\rho}^{(k-1)}]^2, & 0 \le \tilde{\rho}^{(k-1)} \le \tilde{\xi}^{(k-1)} \\ 0, & \tilde{\rho}^{(k-1)} > \tilde{\xi}^{(k-1)} \end{cases} \tag{3.4.3}$$

where $\tilde{c}_0^{(k-1)}$, $\tilde{c}_1^{(k-1)}$, $\tilde{c}_2^{(k-1)}$, and $\tilde{\xi}^{(k-1)}$ are constants which depend on $\tilde{\theta}^{(k-1)}$.

An approximation of $Q(t_{k-1}, \mathbf{x}^{(k-1)})$ again may be represented by a table containing five columns for $\tilde{\theta}^{(k-1)}, \tilde{c}_0^{(k-1)}$, $\tilde{c}_1^{(k-1)}$, $\tilde{c}_2^{(k-1)}$ and $\tilde{\xi}^{(k-1)}$ respectively.

By finding $Q(t_{k^*}, \mathbf{x}^{(k^*)})$, $Q(t_{k^*-1}, \mathbf{x}^{(k^*-1)})$, ..., $Q(t_1, \mathbf{x}^{(1)})$, $Q(t_0, \mathbf{x}^{(0)})$ in the indicated order, we can finally obtain the price of the American basket call option

$$Q = Q(t_0, \mathbf{x}^{(0)}) = Q(0, \mathbf{S}(0)). \tag{3.4.4}$$

3.5 Numerical results

Let T = 10/365, r = 0.05, K = 54, $\rho = 0.01$ and $a_1 = a_2 = 0.5$. Table 3.5.1 displays some chosen values of the parameters λ_{i1}, λ_{i2} and λ_{i3} of the distribution of $v_i^{(k)}$ (see Eq.(2.4.2)). The corresponding measure of skewness $\overline{m}_3^{(i)}$ and measure of kurtosis $\overline{m}_4^{(i)}$ are also given in the table. When $\mathbf{S}^{(0)}$, μ_i, σ_i, $\boldsymbol{\lambda}_i$ are given by Table 3.5.1, the results for the American call option prices are as shown in Table 3.5.2.

Table 3.5.1: Values of $\mathbf{S}^{(0)}$, μ_i, σ_i, and $\boldsymbol{\lambda}_i$

Example	i	$S^{(0)}$	μ_i	σ_i	$\overline{m}_3^{(i)}$	$\overline{m}_4^{(i)}$	λ_{i1}	λ_{i2}	λ_{i3}
1	1	50	0.05	0.15	0	3	1	0	-1
	2	60	0.05	0.10	0	3	1	0	-1
2	1	50	0.05	0.15	0.1	3.6	0.8495	0.1041	-0.7174
	2	60	0.05	0.10	0.1	3.2	0.9737	0.0352	-0.0915
3	1	50	0.05	0.15	1.0	3.4	0.9369	0.1852	0.8003
	2	60	0.05	0.10	1.1	5.0	0.8859	0.2257	0.5082

Table 3.5.2: Results for American call option prices

Example	Numerical Method	Simulation
1	1.088243	1.087655
2	1.086198	1.087103
3	1.041091	1.048527

Table 3.5.2 shows non-negligible decreases of the option prices as we increase the non-normality of the underlying distributions in the Levy process.

3.6 Concluding remarks

From the distribution of the vector of asset prices at $t = t_k$, we are able to determine the values of the vector $\mathbf{x}^{(k)}$ of asset prices of which the probability density of $\mathbf{x}^{(k)}$ is not negligible. The simple nature of the function $Q(t_k, \mathbf{x}^{(k)})$ for $\mathbf{x}^{(k)}$ with non-negligible probability density enables $Q(t_k, \mathbf{x}^{(k)})$ to be approximated by a quadratic function of the radial distance in the polar coordinate system for each value of the polar angle.

By expressing the non-normal random variable underlying the Levy process as a function of the standard normal random variable, we can use the predetermined values of the moments related to standard normal distribution to speed up the numerical integration involved in computing the price of the American call option.