APPENDIX A

PARTIAL DERIVATIVES AND ELEMENTS OF INFORMATION

MATRIX FOR $GIT_{3,1}$ DISTRIBUTION

(A1) Partial Derivatives of Log likelihood function

$$\ln L = \sum_{s=0}^{\infty} \pi_s \ln f(s), \qquad \pi_s = \text{observed frequencies and } f(s) = \text{pmf}$$

$$\frac{\partial \ln L}{\partial p_2} = -\frac{\pi_0 \phi f(1)}{f(0)} + \sum_{s=1}^{\infty} \frac{\pi_s}{f(s)} \frac{\partial f(s)}{\partial p_2}$$

$$\frac{\partial \ln L}{\partial p_3} = -\frac{\pi_0 \phi f(1)}{f(0)} + \sum_{s=1}^{\infty} \frac{\pi_s}{f(s)} \frac{\partial f(s)}{\partial p_3}$$

$$\frac{\partial^2 \ln L}{\partial p_2^2} = \frac{\pi_0}{f(0)} \frac{\partial^2 f(0)}{\partial p_2^2} - \frac{\pi_0}{f^2(0)} \left(\frac{\partial f(0)}{\partial p_2} \right)^2 + \sum_{s=1}^{\infty} \left[\frac{\pi_s}{f(s)} \frac{\partial^2 f(s)}{\partial p_2^2} - \frac{\pi_s}{f^2(s)} \left(\frac{\partial f(s)}{\partial p_2} \right)^2 \right]$$

$$\frac{\partial^2 \ln L}{\partial p_3^2} = \frac{\pi_0}{f(0)} \frac{\partial^2 f(0)}{\partial p_3^2} - \frac{\pi_0}{f^2(0)} \left(\frac{\partial f(0)}{\partial p_3} \right)^2 + \sum_{s=1}^{\infty} \left[\frac{\pi_s}{f(s)} \frac{\partial^2 f(s)}{\partial p_3^2} - \frac{\pi_s}{f^2(s)} \left(\frac{\partial f(s)}{\partial p_3} \right)^2 \right]$$

$$\frac{\partial^2 \ln L}{\partial p_2 \partial p_3} = \frac{\pi_0}{f(0)} \frac{\partial^2 f(0)}{\partial p_2 \partial p_3} - \frac{\pi_0}{f^2(0)} \left(\frac{\partial f(0)}{\partial p_2} \frac{\partial f(0)}{\partial p_3} \right) + \sum_{s=1}^{\infty} \left[\frac{\pi_s}{f(s)} \frac{\partial^2 f(s)}{\partial p_2 \partial p_3} - \frac{\pi_s}{f^2(s)} \left(\frac{\partial f(s)}{\partial p_2} \frac{\partial f(s)}{\partial p_3} \right) \right]$$

$$\frac{\partial^2 \ln L}{\partial p_3 \partial p_2} = \frac{\pi_0}{f(0)} \frac{\partial^2 f_0}{\partial p_3 \partial p_2} - \frac{\pi_0}{f^2(0)} \left(\frac{\partial f(0)}{\partial p_3} \frac{\partial f(0)}{\partial p_3} \right) + \sum_{s=1}^{\infty} \left[\frac{\pi_s}{f(s)} \frac{\partial^2 f(s)}{\partial p_3 \partial p_2} - \frac{\pi_s}{f^2(s)} \left(\frac{\partial f(s)}{\partial p_3} \frac{\partial f(s)}{\partial p_2} \right) \right]$$

(A2) Elements of the Information Matrix

N=sample size

$$E\left[\frac{\partial^2 \ln L}{\partial p_2 \partial p_3}\right] = N\sum_{s=0}^{\infty} \frac{1}{\pi_s} \frac{\partial f(s)\partial f(s)}{\partial p_2 \partial p_3}$$

$$E\left[-\frac{\partial^2 \ln L}{\partial p_2^2}\right] = -N\left[\frac{\partial^2 f(0)}{\partial p_2^2} - \frac{1}{f(0)}\left(\frac{\partial f(0)}{\partial p_2}\right)^2\right] - N\sum_{s=1}^{\infty}\left[\frac{\partial^2 f(s)}{\partial p_2^2} - \frac{1}{f(s)}\left(\frac{\partial f(s)}{\partial p_2}\right)^2\right]$$

$$E\left[-\frac{\partial^2 \ln L}{\partial p_3^2}\right] = -N\left[\frac{\partial^2 f(0)}{\partial p_3^2} - \frac{1}{f(0)}\left(\frac{\partial f(0)}{\partial p_3}\right)^2\right] - N\sum_{s=1}^{\infty}\left[\frac{\partial^2 f(s)}{\partial p_3^2} - \frac{1}{f(s)}\left(\frac{\partial f(s)}{\partial p_3}\right)^2\right]$$

$$E\left[-\frac{\partial^2 \ln L}{\partial p_2 \partial p_3}\right] = -N\left[\frac{\partial^2 f(0)}{\partial p_2 \partial p_3} - \frac{1}{f(0)}\left(\frac{\partial f(0)}{\partial p_2}\frac{\partial f(0)}{\partial p_3}\right)\right] - N\sum_{s=1}^{\infty}\left[\frac{\partial^2 f(s)}{\partial p_2 \partial p_3} - \frac{1}{f(s)}\left(\frac{\partial f(s)}{\partial p_2}\frac{\partial f(s)}{\partial p_3}\right)\right]$$

$$E\left[-\frac{\partial^2 \ln L}{\partial p_3 \partial p_2}\right] = -N\left[\frac{\partial^2 f(0)}{\partial p_3 \partial p_2} - \frac{1}{f(0)}\left(\frac{\partial f(0)}{\partial p_3}\frac{\partial f(0)}{\partial p_2}\right)\right] - N\sum_{s=1}^{\infty}\left[\frac{\partial^2 f(s)}{\partial p_3 \partial p_2} - \frac{1}{f(s)}\left(\frac{\partial f(s)}{\partial p_3}\frac{\partial f(s)}{\partial p_2}\right)\right]$$

APPENDIX B

PARTIAL DERIVATIVES AND ELEMENTS OF INFORMATION

MATRIX FOR COMP-POISSON DISTRIBUTION

(B1) Partial Derivatives of Log likelihood function

$$\ln L = \sum_{s=0}^{\infty} \pi_s \ln f(s), \qquad \pi_s = \text{observed frequencies and } f(s) = \text{pmf}$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left[\sum_{s=0}^{\infty} \pi_s \ln f(s) \right] = \sum_{s=0}^{\infty} \pi_s \frac{\partial \ln f(s)}{\partial \lambda}$$

$$\frac{\partial \ln L}{\partial v} = \sum_{s=0}^{\infty} \pi_s \frac{\partial \ln f(s)}{\partial v}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \sum_{s=0}^{\infty} \pi_s \frac{\partial^2 \ln f(s)}{\partial \lambda^2}$$

$$\frac{\partial^2 \ln L}{\partial v^2} = \sum_{s=0}^{\infty} \pi_s \frac{\partial^2 \ln f(s)}{\partial v^2}$$

(B2) Elements of the Information Matrix

N=sample size

$$E\left[-\frac{\partial^2 \ln L}{\partial \lambda^2} \right] = -N \sum_{s=0}^{\infty} f(s) \frac{\partial^2 \ln f(s)}{\partial \lambda^2}$$

$$E\left[-\frac{\partial^2 \ln L}{\partial v^2} \right] = -N \sum_{s=0}^{\infty} f(s) \frac{\partial^2 \ln f(s)}{\partial v^2}$$

$$E\left[-\frac{\partial^2 \ln L}{\partial \lambda \partial v} \right] = -N \sum_{s=0}^{\infty} f(s) \frac{\partial^2 \ln f(s)}{\partial \lambda \partial v}$$

APPENDIX C

MIXTURE OF DISCRETE DISTRIBUTIONS

The mixture of the Neyman Type A (NTA) distribution, the $GIT_{3,1}$ distribution and the Poisson inverse Gaussian (PIG) distribution with POI distribution are examined.The pmf of the mixed distributions are

For $0 < \omega < 1$

1) A mixture of NTA distribution and POI distribution with pmf

$$\Pr(X = x) = \omega \frac{e^{-\lambda}\phi^x}{x!} \sum_{i=0}^{\infty} \frac{(\lambda e^{-\phi})^i i^x}{i!} + (1-\omega)\frac{e^{-\lambda}\lambda^x}{x!}, \quad \text{for } x = 0,\, 1,\, 2,\, ...$$

2) A mixture of $GIT_{3,1}$ distribution and POI distribution with pmf

$$\Pr(X = x) = \omega \binom{n+x-1}{x} p_1^n p_3^x \, {}_2F_1\left(-n, -x; -n-x+1; -\frac{p_2}{p_1 p_3}\right) + (1-\omega)\frac{e^{-\lambda}\lambda^x}{x!}$$

for $x = 0,\, 1,\, 2,\, ...$

3) A mixture of a PIG distribution and a POI distribution is performed. The pmf is

$$\Pr(X = x) = \omega \frac{\sqrt{\eta}}{x!}\left(\frac{\mu}{\eta}\right)^x \frac{K_{x-1/2}(\mu\eta/\tau)}{K_{-1/2}(\mu/\tau)} + (1-\omega)\frac{e^{-\lambda}\lambda^x}{x!}, \quad \text{for } x = 0,\, 1,\, 2,\, ...$$

Table C1: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(Without outliers, NTA ($\phi = 2, b = 4$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	-0.0427	0.2384	-0.3449	0.8840	-0.0891	0.7103	-0.0550	0.3565
	(0.1340)	(0.8854)	(0.2663)	(3.0361)	(0.3050)	(8.1646)	(0.2126)	(1.6349)
500	-0.0103	0.0459	-0.1137	0.2016	-0.0136	0.0732	-0.0092	0.0575
	(0.0263)	(0.1181)	(0.0399)	(0.1851)	(0.0537)	(0.2052)	(0.0421)	(0.1685)
1000	-0.0065	0.0247	-0.0671	0.1133	-0.0067	0.0350	-0.0044	0.0277
	(0.0131)	(0.0564)	(0.0180)	(0.0767)	(0.0264)	(0.0919)	(0.0210)	(0.0787)

Table C2: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(With 2.5% outliers, NTA ($\phi = 2, b = 4$)and POI ($\lambda = 30$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	0.4414	-0.3907	-0.2329	0.6443	-0.0289	0.6403	0.0486	0.2435
	(0.3925)	(0.6163)	(0.2287)	(2.4781)	(0.3101)	(6.7371)	(0.2172)	(1.2552)
500	0.4876	-0.5453	0.1146	-0.1316	0.0483	0.0486	0.0934	-0.0185
	(0.2770)	(0.3607)	(0.0496)	(0.1311)	(0.0580)	(0.1929)	(0.0519)	(0.1447)
1000	0.4926	-0.5619	0.1958	-0.2442	0.0551	0.0120	0.0979	-0.0450
	(0.2622)	(0.3462)	(0.0566)	(0.1076)	(0.0306)	(0.0871)	(0.0312)	(0.0708)

Table C3: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(With 2.5% outliers, NTA ($\phi = 2, b = 4$)and POI ($\lambda = 65$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	1.5388	-1.2102	-0.3505	0.9036	-0.0180	0.6198	0.0929	0.1767
	(3.1099)	(1.8728)	(0.2710)	(3.2126)	(0.3118)	(6.4970)	(0.2262)	(1.1004)
500	1.6616	-1.4094	-0.1184	0.2103	0.0590	0.0378	0.1359	-0.0674
	(2.9055)	(2.0206)	(0.0416)	(0.1941)	(0.0596)	(0.1894)	(0.0622)	(0.1384)
1000	1.6769	-1.4292	-0.0723	0.1220	0.0658	0.0015	0.1402	-0.0924
	(2.8823)	(2.0575)	(0.0188)	(0.0805)	(0.0321)	(0.0859)	(0.0415)	(0.0725)

Table C4: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(Without outliers, $\text{GIT}_{3,1}(p_1 = 0.7, p_2 = 0.2, p_3 = 0.1, n = 20)$)

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	0.0071	-0.0055	0.0577	-0.0451	0.0507	-0.0403	0.0125	-0.0096
	(0.0043)	(0.0024)	(0.0069)	(0.0041)	(0.0134)	(0.0079)	(0.0086)	(0.0050)
500	0.0005	-0.0004	0.0172	-0.0134	0.0172	-0.0140	-0.0002	0.0002
	(0.0009)	(0.0005)	(0.0013)	(0.0008)	(0.0089)	(0.0057)	(0.0023)	(0.0014)
1000	0.0006	-0.0005	0.0097	-0.0076	-0.0009	0.0007	-0.0009	0.0007
	(0.0004)	(0.0003)	(0.0006)	(0.0003)	(0.0012)	(0.0007)	(0.0012)	(0.0007)

Table C5: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(With 2.5% outliers, $\text{GIT}_{3,1}(p_1 = 0.7, p_2 = 0.2, p_3 = 0.1, n = 20)$ and POI ($\lambda = 30$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	-0.1764	0.1482	0.0586	-0.0458	-0.0538	0.0480	-0.0538	0.1480
	(0.0347)	(0.0246)	(0.0071)	(0.0041)	(0.0114)	(0.0075)	(0.0114)	(0.0271)
500	-0.1986	0.1651	0.0169	-0.0132	-0.0741	0.0639	-0.0741	0.0639
	(0.0394)	(0.0273)	(0.0013)	(0.0008)	(0.0076)	(0.0054)	(0.0076)	(0.0054)
1000	-0.1990	0.2654	0.0074	-0.0058	-0.0747	0.0643	-0.0747	0.0643
	(0.0396)	(0.0705)	(0.0006)	(0.0003)	(0.0067)	(0.0048)	(0.0067)	(0.0048)

Table C6: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(With 2.5% outliers, $\text{GIT}_{3,1}(p_1 = 0.7, p_2 = 0.2, p_3 = 0.1, n = 20)$ and POI ($\lambda = 65$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	-0.1839	0.1762	0.0599	-0.0468	-0.0805	0.0698	-0.0805	0.0698
	(0.0371)	(0.0343)	(0.0072)	(0.0041)	(0.0149)	(0.0102)	(0.0149)	(0.0102)
500	-0.1990	0.1884	0.0200	-0.0156	-0.1047	0.0888	-0.1047	0.0888
	(0.0396)	(0.0356)	(0.0014)	(0.0008)	(0.0131)	(0.0093)	(0.0131)	(0.0093)
1000	-0.1990	0.1881	0.0119	-0.0093	-0.1055	0.0894	-0.1055	0.0894
	(0.0396)	(0.0354)	(0.0007)	(0.0004)	(0.0123)	(0.0087)	(0.0123)	(0.0087)

Table C7: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(Without outliers, PIG ($\mu = 4, \theta = 1$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	-0.0040	-0.0099	-0.2172	-0.3365	0.0235	0.1380	0.0127	0.0806
	(0.0813)	(0.1248)	(0.1243)	(0.1941)	(0.1554)	(0.8885)	(0.1083)	(0.5147)
500	0.0014	0.0014	-0.0716	-0.1299	0.0125	0.0460	0.0096	0.0329
	(0.0155)	(0.0248)	(0.0204)	(0.0385)	(0.0297)	(0.1344)	(0.0207)	(0.0828)
1000	-0.0019	-0.0016	-0.0445	-0.0824	0.0072	0.0291	0.0045	0.0202
	(0.0078)	(0.0118)	(0.0098)	(0.0180)	(0.0153)	(0.0660)	(0.0106)	(0.0408)

Table C8: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(With 2.5% outliers, PIG ($\mu = 4, \theta = 1$) and POI ($\lambda = 30$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	0.6455	1.7078	-0.2048	-0.2986	0.2557	0.4532	0.3326	0.5676
	(0.6823)	(4.4107)	(0.1214)	(0.1889)	(0.2949)	(1.6219)	(0.3071)	(1.3709)
500	0.6543	1.7063	-0.0250	0.0013	0.2227	0.2912	0.2661	0.3689
	(0.4781)	(3.1941)	(0.0199)	(0.0477)	(0.0888)	(0.2714)	(0.1008)	(0.2646)
1000	0.6479	1.6879	0.0333	0.1271	0.2171	0.2713	0.2602	0.3525
	(0.4447)	(2.9869)	(0.0127)	(0.0515)	(0.0673)	(0.1661)	(0.0829)	(0.1880)

Table C9: MSE and bias (in bracket) for proposed estimators T_1 and T_2

(With 2.5% outliers, PIG ($\mu = 4, \theta = 1$) and POI ($\lambda = 65$))

(Number of iterations=1000, value written= mean of 1000 iterations)

Sample size	MLE		MHD		T_1		T_2	
	ϕ	b	ϕ	b	ϕ	b	ϕ	b
100	1.5257	5.1756	-0.2186	-0.3388	0.2493	0.4403	0.3149	0.5290
	(3.4015)	(41.6537)	(0.1245)	(0.1911)	(0.2869)	(1.5738)	(0.2845)	(1.2538)
500	1.5363	4.9559	-0.0769	-0.1404	0.2369	0.3146	0.3019	0.4375
	(2.5632)	(27.1662)	(0.0216)	(0.0409)	(0.0967)	(0.2934)	(0.1243)	(0.3356)
1000	1.5231	4.8777	-0.0478	-0.0904	0.2310	0.2938	0.2952	0.4188
	(2.4214)	(25.0634)	(0.0104)	(0.0194)	(0.0742)	(0.1825)	(0.1039)	(0.2467)