

**QUEUE LENGTH AND WAITING TIME DISTRIBUTIONS
IN A SYSTEM OF M DEPENDENT QUEUES**

SOO HUEI CHING

**FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR**

2011

**QUEUE LENGTH AND WAITING TIME DISTRIBUTIONS
IN A SYSTEM OF M DEPENDENT QUEUES**

SOO HUEI CHING

**THESIS SUBMITTED IN FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY**

**INSTITUTE OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR**

2011

ABSTRAK

Pertimbangkan sebuah sistem M giliran Hypo(2)/Hypo(2)/1 dengan hubungan kebersandaran ditetapkan melalui suatu skim interaksi. Dua jenis skim interaksi dikaji. Kita mencadangkan satu kaedah untuk menerbitkan taburan panjang giliran pegun tercantum, dan taburan masa tunggu pegun bagi pelanggan yang meminta perkhidmatan dalam suatu giliran yang tertentu. Kaedah yang dicadangkan juga telah diubahsuaikan untuk mencari taburan panjang giliran pegun tercantum dalam suatu sistem bersandar yang terdiri daripada dua giliran Hypo(2)/Hypo(2)/ c/c . Didapati bahawa keputusan berangka yang diperolehi dengan menggunakan kaedah yang dicadangkan adalah secocok dengan hasil berdasarkan simulasi.

ABSTRACT

Consider a system of M Hypo(2)/Hypo(2)/1 queues of which the dependence relation is specified via an interaction scheme. Two types of interaction scheme are considered. We proposed a method for deriving the stationary joint queue length distributions, and the stationary waiting time distribution, of a customer seeking service in a given queue. The proposed method has also been adapted to find the stationary joint queue length distribution in a system of two dependent Hypo(2)/Hypo(2)/ c/c queues. The numerical results found by using the proposed method agree well with those based on simulation.

ACKNOWLEDGEMENTS

Firstly, I would like to express my sincere appreciation and gratitude to my supervisor Professor Dr. Pooi Ah Hin for his patience guidance and for many discussion that I had with him. His assistance, advice, insight and helpful suggestions were invaluable to me throughout all these years.

Secondly, I wish to express my heartfelt thanks to my co-supervisors Dr. Tan Yi Fei and Associate Professor Dr. Nik Ahmad Kamal for their support and coaching all these years.

I would also like to thank University of Malaya for its financial support. Special thanks are extended to Faculty of Engineering, Multimedia University for employing me as a contract research officer during the beginning period of my study.

Lastly, my sincere thanks are also extended to Ms. Koh Siew Khew for her assistance, the staffs of the Institute of Mathematical Sciences, University of Malaya especially Miss Ng Lee Ling for her substantial technical advice and my family members for their continuous support and encouragement.

TABLE OF CONTENTS

ABSTRAK		iii
ABSTRACT		iv
ACKNOWLEDGEMENTS		v
TABLE OF CONTENTS		vi
LIST OF FIGURES		ix
LIST OF TABLES		xi
CHAPTER 1	INTRODUCTION	1
1.1	A Survey of Works on Queueing System	1
1.2	Introduction to the Thesis	4
1.3	Layout of the Dissertation	5
CHAPTER 2	THE QUEUE LENGTH DISTRIBUTION IN A SYSTEM OF M HYPO(2)/HYPO(2)/1 DEPENDENT QUEUES	6
2.1	Introduction	6
2.2	Derivation of the Forward Equations in a System of Two Dependent Hypo(2)/Hypo(2)/1 Queues	7
2.3	Computation of the Value of $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$	18
2.4	Simulated Value of $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$	32
2.5	Numerical Results for Distribution of Queue Length and States of Arrival and Service Processes in a System of Two Dependent Hypo(2)/Hypo(2)/1 Queues	35
2.6	Derivation of the Forward Equations in a System of Three Dependent Hypo(2)/Hypo(2)/1 Queues	39
2.7	Computation of the Value of $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$	51
2.8	Simulated Value of $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$	74

2.9	Numerical Results for Distribution of Queue Length and States of Arrival and Service Processes in a System of Three Dependent Hypo(2)/Hypo(2)/1 Queues	76
2.10	Derivation of the Forward Equations in a System of M Dependent Hypo(2)/Hypo(2)/1 Queues	80
2.11	Computation of the Value of $P_{i_1 j_1 i_2 j_2 \wedge i_M j_M} [n_1][n_2] \wedge [n_M]$	83
CHAPTER 3	A SYSTEM OF M HYPO(2)/HYPO(2)/1 QUEUES IN WHICH CUSTOMERS MAY CROSS OVER TO SHORTEST QUEUES	88
3.1	Introduction	88
3.2	Derivation of the Forward Equations in a System of Three Hypo(2)/Hypo(2)/1 Queues which Follow the Second Interaction Scheme	89
3.3	Numerical Results for Distribution of Queue Length and States of Arrival and Service Processes in a System of Three Hypo(2)/Hypo(2)/1 Queues with Interaction Scheme Joining the Shorter Queue	99
3.4	Derivation of the Forward Equations in a System of M Hypo(2)/Hypo(2)/1 Queues which Follow the Second Interaction Scheme	103
CHAPTER 4	WAITING TIME DISTRIBUTION IN A SYSTEM OF M HYPO(2)/HYPO(2)/1 DEPENDENT QUEUES	105
4.1	Introduction	105
4.2	Derivation of the Waiting Time Distribution in a System of Two Dependent Hypo(2)/Hypo(2)/1 Queues which Follow the First Interaction Scheme	106
4.3	Simulated Waiting Time Distribution in a System of Two Dependent Hypo(2)/Hypo(2)/1 Queues which Follow the First Interaction Scheme	109
4.4	Numerical Results for Waiting Time Distribution in a System of Two Dependent Hypo(2)/Hypo(2)/1 Queues which Follow the First Interaction Scheme	112
4.5	Derivation of the Waiting Time Distribution in a System of Three Dependent Hypo(2)/Hypo(2)/1 Follow the First Interaction Scheme	113

4.6	Numerical Results for Waiting Time Distribution in a System of Three Dependent Hypo(2)/Hypo(2)/1 Queues which Follow the First Interaction Scheme	119
4.7	Derivation of the Waiting Time Distribution in a System of M Dependent Hypo(2)/Hypo(2)/1 Queues which Follow the First Interaction Scheme	120
4.8	Derivation of the Waiting Time Distribution in a System of Three Dependent Hypo(2)/Hypo(2)/1 Queues which Follow the Second Interaction Scheme	122
4.9	Derivation of the Waiting Time Distribution in a System of M Dependent Hypo(2)/Hypo(2)/1 Queues which Follow the Second Interaction Scheme	128
CHAPTER 5	THE QUEUE LENGTH DISTRIBUTION IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/c/c QUEUES	131
5.1	Introduction	131
5.2	Derivation of the Forward Equations in a System of Two Dependent Hypo(2)/Hypo(2)/2/2 Queues	132
5.3	Computation of the Value of $P_{i_1 i_2 j_1 i_2 i_2 j_2} [n_1][n_2]$	141
5.4	Simulated Valued of $P_{i_1 i_2 j_1 i_2 i_2 j_2} [n_1][n_2]$	144
5.5	Numerical Results for Distribution of Queue Length and State of Arrival and Service Processes in a System of Two Dependent Hypo(2)/Hypo(2)/2/2 Queues	150
5.6	Derivation of the Forward Equations in a System of Two Dependent Hypo(2)/Hypo(2)/ c/c Queues	156
CONCLUDING REMARKS	160
APPENDIX A	161
APPENDIX B	166
APPENDIX C	167
REFERENCES	176

LIST OF FIGURES

- Figure 2.2.1** : Crossing over of a customer to another queue in a system of two one-server queues. 8
- Figure 2.2.2** : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$. 14
- Figure 2.5.1** : Comparison of results for $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure
 $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20)$, $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1, 2, 1, 2)$, $q_{11} = q_{22} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $N = 9$ and $N_s = 5000$]. 38
- Figure 2.5.2** : Comparison of results for $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure
 $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20)$, $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (5, 12, 5, 12)$, $q_{11} = q_{22} = 0.9$, $\rho_1 = 0.5294$, $\rho_2 = 0.5294$, $N = 9$ and $N_s = 5000$]. 38
- Figure 2.6.1** : Cross over probabilities in a system of three one-server queues. 40
- Figure 2.6.2** : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$. 46
- Figure 2.9.1** : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 20, 10, 20, 10, 20)$,
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (1, 2, 1, 2, 1, 2)$,
 $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$,
 $\rho_3 = 0.1$, $N = 3$ and $N_s = 50000$]. 79
- Figure 2.9.2** : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 20, 10, 20, 10, 20)$,
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (3, 4, 3, 4, 3, 4)$,
 $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.2571$, $\rho_2 = 0.2571$,
 $\rho_3 = 0.2571$, $N = 4$ and $N_s = 20000$]. 79
- Figure 3.2.1** : Crossing over of an arriving customer in a system of three one-server queues (Q_1, Q_2, Q_3) which follow the Second Interaction Scheme. 90

Figure 3.2.2 : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$. 96

Figure 3.3.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 20, 10, 20, 10, 20),$
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (1, 2, 1, 2, 1, 2),$
 $q_{11} = q_{22} = q_{33} = 0.9, \rho_1 = 0.1, \rho_2 = 0.1, \rho_3 = 0.1,$
 $N = 3$ and $N_s = 50000]$. 102

Figure 3.3.2 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = 10, 20, 10, 20, 10, 20),$
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (1, 2, 2, 4, 3, 6),$
 $q_{11} = q_{22} = q_{33} = 0.9, \rho_1 = 0.10, \rho_2 = 0.20, \rho_3 = 0.30,$
 $N = 4$ and $N_s = 50000]$. 102

Figure 4.3.1 : The waiting time in queue m for a system of 2 dependent queues which follow the First Interaction Scheme. 111

Figure 5.2.1 : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$. 139

Figure 5.5.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20), (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$
 $= (2, 6, 2, 6), q_{11} = q_{22} = 0.60, \rho_1 = 0.225, \rho_2 = 0.225$
and $N_s = 50000]$. 155

Figure 5.5.2 : Comparison of results for $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20), (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$
 $= (4, 5, 3, 8), q_{11} = q_{22} = 0.90, \rho_1 = 0.33333,$
 $\rho_2 = 0.32727$ and $N_s = 50000]$. 155

LIST OF TABLES

Table 2.2.1 : The positions, values and meanings of the components in A_w .	11
Table 2.2.2 : An example of the codes of $\mathbf{h}^{(k)}$, $\mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k .	12
Table 2.2.3 : Representation of balance equation in (2.2.2) by codes.	15
Table 2.2.4 : Representation of some balance equations by codes.	17
Table 2.3.1 : The codes for the equations represented by $\{P [0], P [1]\}$.	19
Table 2.3.2 : The codes for the equations represented by $\{P [0], P [1], P [2]\}$.	20
Table 2.3.3 : The set of equations given by $\{P_{i,j,i,j} [n] \mid P [2]\}$ for $n = 0, 1$ ($\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \lambda_{22}=2, q_{11}=0.9, q_{22}=0.9, \Delta t=0.01$).	24
Table 2.3.4 : The set of equations given by $\{P_{i,j,i,j} [0][2] \mid \{P [0][0]\}, \{P [1][1]\}, \{P [2][0]\}\}$ ($\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \lambda_{22}=2, q_{11}=0.9, q_{22}=0.9, \Delta t=0.01$. The dotted lines indicate that the relevant tables are to be joined up).	26
Table 2.3.5 : The set of equations given by $\{P_{i,j,i,j} [2][0] \mid \{P [0][0]\}, \{P [1][1]\}, \{P [0][2]\}\}$ ($\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \lambda_{22}=2, q_{11}=0.9, q_{22}=0.9, \Delta t=0.01$. The dotted lines indicate that the relevant tables are to be joined up).	27
Table 2.3.6 : The set of equations given by $\{P_{i,j,i,j} [0][1] \mid \{P [0][0]\}, \{P [1][1]\}, \{P [2][0]\}\}$ ($\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \lambda_{22}=2, q_{11}=0.9, q_{22}=0.9, \Delta t=0.01$. The dotted lines indicate that the relevant tables are to be joined up).	28
Table 2.3.7 : The set of equations given by $\{P_{i,j,i,j} [1][0] \mid \{P [0][0]\}, \{P [1][1]\}, \{P [0][2]\}\}$ ($\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \lambda_{22}=2, q_{11}=0.9, q_{22}=0.9, \Delta t=0.01$. The dotted lines indicate that the relevant tables are to be joined up).	29

Table 2.4.1 : The approximate probability that event A will occur in τ_k given the conditions of system at the end of τ_{k-1} [$v_i=1$ or $2, 1 \leq i \leq 4$].	33
Table 2.4.2 : The approximate probability that event A will occur in τ_2 given the conditions of system at the end of τ_1 .	35
Table 2.5.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20), (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (1, 2, 1, 2), q_{11} = q_{22} = 0.9, \rho_1 = 0.1, \rho_2 = 0.1,$ $N = 9$ and $N_s = 5000$].	36
Table 2.5.2 : Comparison of results for $P_{i_1 j_1 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20), (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (5, 12, 5, 12), q_{11} = q_{22} = 0.9, \rho_1 = 0.5294,$ $\rho_2 = 0.5294, N = 9$ and $N_s = 5000$].	37
Table 2.6.1 : The positions, values and meanings of the components in A_w .	42
Table 2.6.2 : An example of the codes of $\mathbf{h}^{(k)}, \mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k .	44
Table 2.6.3 : Representation of Equation in (2.6.2) by codes.	48
Table 2.6.4 : Representation of some balance equations by codes.	50
Table 2.7.1 : The codes for the equations represented by $\{P[0], P[1]\}$.	52
Table 2.7.2 : The codes for some equations represented by $\{P[0], P[1], P[2]\}$.	54
Table 2.7.3 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2 i_3 j_3} [n] \mid P[2]\}$ for $n = 0, 1 (\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20,$ $\lambda_{21}=1, \lambda_{22}=2, \mu_{31}=10, \mu_{32}=20, \lambda_{31}=1, \lambda_{32}=2, q_{11}=0.9,$ $q_{22}=0.9, q_{33}=0.9, \Delta t=0.01$. The dotted lines indicate of the present table is to be joined up with the table in the next page).	57
Table 2.7.4 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2 i_3 j_3} [0][0][2] \mid \{P[0][0][0]\}, \{P[0][2][0]\}, \{P[2][0][0]\},$ $\{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\} (\mu_{11}=10, \mu_{12}=20,$ $\lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \lambda_{22}=2, \mu_{31}=10, \mu_{32}=20,$ $\lambda_{31}=1, \lambda_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, \Delta t=0.01$. The dotted lines indicate of the present table is to be joined up with the table in the next page).	60

Table 2.7.5 : The set of equations given by

$$\{P_{i_1 i_2 j_2 i_3 j_3} [0][2][0] \mid \{P [0][0][0]\}, \{P [0][0][2]\}, \\ \{P [2][0][0]\} \{P [1][1][0]\}, \{P [1][0][1]\}, \{P [0][1][1]\} \\ (\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \\ \lambda_{22}=2, \mu_{31}=10, \mu_{32}=20, \lambda_{31}=1, \lambda_{32}=2, q_{11}=0.9, q_{22}=0.9, \\ q_{33}=0.9, \Delta t=0.01. \text{ The dotted lines indicate of the present} \\ \text{table is to be joined up with the table in the next page}).$$

62

Table 2.7.6 : The set of equations given by

$$\{P_{i_1 i_2 j_2 i_3 j_3} [2][0][0] \mid \{P [0][0][0]\}, \{P [0][0][2]\}, \\ \{P [0][2][0][0]\}, \{P [1][1][0]\}, \{P [1][0][1]\}, \{P [0][1][1]\} \\ (\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \\ \lambda_{22}=2, \mu_{31}=10, \mu_{32}=20, \lambda_{31}=1, \lambda_{32}=2, q_{11}=0.9, q_{22}=0.9, \\ q_{33}=0.9, \Delta t=0.01. \text{ The dotted lines indicate of the present} \\ \text{table is to be joined up with the table in the next page}).$$

64

Table 2.7.7 : The set of equations given by

$$\{P_{i_1 i_2 j_2 i_3 j_3} [0][0][1] \mid \{P [0][0][0]\}, \{P [0][2][0]\}, \\ \{P [2][0][0]\}, \{P [1][1][0]\}, \{P [1][0][1]\}, \{P [0][1][1]\} \\ (\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \\ \lambda_{22}=2, \mu_{31}=10, \mu_{32}=20, \lambda_{31}=1, \lambda_{32}=2, q_{11}=0.9, q_{22}=0.9, \\ q_{33}=0.9, \Delta t=0.01. \text{ The dotted lines indicate of the present} \\ \text{table is to be joined up with the table in the next page}).$$

66

Table 2.7.8 : The set of equations given by

$$\{P_{i_1 i_2 j_2 i_3 j_3} [0][1][0] \mid \{P [0][0][0]\}, \{P [0][0][2]\}, \\ \{P [2][0][0]\}, \{P [1][1][0]\}, \{P [1][0][1]\}, \{P [0][1][1]\} \\ (\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \\ \lambda_{22}=2, \mu_{31}=10, \mu_{32}=20, \lambda_{31}=1, \lambda_{32}=2, q_{11}=0.9, q_{22}=0.9, \\ q_{33}=0.9, \Delta t=0.01. \text{ The dotted lines indicate of the present} \\ \text{table is to be joined up with the table in the next page}).$$

68

Table 2.7.9 : The set of equations given by

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3} [1][0][0] \mid \{P [0][0][0]\}, \{P [0][0][2]\}, \\ \{P [0][2][0]\}, \{P [1][1][0]\}, \{P [1][0][1]\}, \{P [0][1][1]\} \\ (\mu_{11}=10, \mu_{12}=20, \lambda_{11}=1, \lambda_{12}=2, \mu_{21}=10, \mu_{22}=20, \lambda_{21}=1, \\ \lambda_{22}=2, \mu_{31}=10, \mu_{32}=20, \lambda_{31}=1, \lambda_{32}=2, q_{11}=0.9, q_{22}=0.9, \\ q_{33}=0.9, \Delta t=0.01. \text{ The dotted lines indicate of the present} \\ \text{table is to be joined up with the table in the next page}).$$

70

Table 2.8.1 : The approximate probability that event A will occur in τ_k given the conditions of system at the end of τ_{k-1}

$$[v_i=1, 2, \text{ or } 3, 1 \leq i \leq 6].$$

75

- Table 2.9.1 :** Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 20, 10, 20, 10, 20),$
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (1, 2, 1, 2, 1, 2), q_{11} = q_{22} =$
 $q_{33} = 0.9, \rho_1 = 0.1, \rho_2 = 0.1, \rho_3 = 0.1, N = 3$ and
 $N_s = 50000]$. 77
- Table 2.9.2 :** Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 20, 10, 20, 10, 20),$
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (3, 4, 3, 4, 3, 4), q_{11} = q_{22} =$
 $q_{33} = 0.9, \rho_1 = 0.2571, \rho_2 = 0.2571, \rho_3 = 0.2571,$
 $N = 4$ and $N_s = 20000]$. 78
- Table 2.10.1 :** The meanings of the components A_{wj} in $A_w, 1 \leq j \leq 2M$. 81
- Table 3.2.1 :** An example of the codes of $\mathbf{h}^{(k)}, \mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k . 94
- Table 3.2.2 :** Representation of balance equation in (3.2.2) by codes. 97
- Table 3.3.1 :** Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 20, 10, 20, 10, 20),$
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (1, 2, 1, 2, 1, 2),$
 $q_{11} = q_{22} = q_{33} = 0.9, \rho_1 = 0.1, \rho_2 = 0.1, \rho_3 = 0.1,$
 $N = 3$ and $N_s = 50000]$. 100
- Table 3.3.2 :** Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3} [n_1][n_2][n_3]$ based on the proposed method and simulation procedure
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 20, 10, 20, 10, 20),$
 $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (1, 2, 2, 4, 3, 6),$
 $q_{11} = q_{22} = q_{33} = 0.9, \rho_1 = 0.10, \rho_2 = 0.20, \rho_3 = 0.30,$
 $N = 4$ and $N_s = 50000]$. 101
- Table 4.4.1 :** Comparison of $W^{(m)}(t)$ obtained by the numerical method and simulation procedure for $m = 1, 2$
 $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 10, 10, 10), (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) =$
 $(1, 1, 1, 1), q_{11} = q_{22} = 0.9, \Delta t = 0.01, \rho_1 = 0.1, \rho_2 = 0.1,$
 $N = 13$ and $N_s = 10000000]$. 112

Table 4.6.1 : Comparison of $W^{(m)}(t)$ obtained by the numerical method and simulation procedure for $m = 1, 2, 3$ $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) = (10, 10, 10, 10, 10, 10),$ $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}) = (1, 1, 1, 1, 1, 1), q_{11} = q_{22} =$ $q_{33} = 0.9, \Delta t = 0.01, \rho_1 = 0.1, \rho_2 = 0.1, \rho_3 = 0.1,$ $N = 3$ and $N_s = 10000000]$.	119
Table 5.2.1 : The meanings of the components in A_w .	135
Table 5.2.2 : An example of the codes of $\mathbf{h}^{(k)}, \mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k .	137
Table 5.2.3 : Representation of balance equation in (5.2.2) by codes.	140
Table 5.4.1 : The approximate probability that event A will occur in τ_k given the conditions of system at the end of τ_{k-1} $[v_i = 1 \text{ or } 2, 1 \leq i \leq 6]$.	146
Table 5.4.2 : The approximate probability that event A will occur in τ_2 given the conditions of system at the end of τ_1 .	149
Table 5.5.1 : Comparison of results for $P_{i_1 i_2 j_1 i_2 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20), (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) =$ $(2, 6, 2, 6), q_{11} = q_{22} = 0.60, \rho_1 = 0.225, \rho_2 = 0.225$ and $N_s = 50000]$.	151
Table 5.5.2 : Comparison of results for $P_{i_1 i_2 j_1 i_2 i_2 j_2} [n_1][n_2]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (10, 20, 10, 20), (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) =$ $(4, 5, 3, 8), q_{11} = q_{22} = 0.90, \rho_1 = 0.33333, \rho_2 = 0.32727$ and $N_s = 50000]$.	153
Table 5.6.1 : The meanings of the components A_{wj} in $A_w, 1 \leq j \leq 2(c+1)$.	157