CHAPTER 2

THE QUEUE LENGTH DISTRIBUTION IN A SYSTEM OF *M* HYPO(2)/HYPO(2)/1 DEPENDENT QUEUES

2.1 INTRODUCTION

Consider a system of M parallel Hypo(2)/Hypo(2)/1 queues each of which has a server with a 2-phase hypoexponential distributed service time, an arrival stream of which the interarrival time is also 2-phase hypoexponentially distributed, a first come first served queue discipline, and an infinite system capacity. Assume that the customer in a given queue m may cross over to the other queue m' ($m' \neq m$) with probability $q_{mm'}$ or stay in queue m with probability q_{mm}, $m = 1, 2, \ldots, M$.

As the interarrival time distribution is 2-phase hypoexponential, the arrival of a customer may be regarded as the result of the completion of 2 phases in a sequential manner. We may classify the arrival process in queue m to be in state j_m , if (j_m -1) phases of the process have been completed and the process is currently going through the j_m-th phase, $j_m \in \{1, 2\}$. Similarly, the completion of the service of a customer may be regarded as the completion of 2 phases in a sequential manner. We may classify the service process in queue m to be in state i_m, if (i_m -1) phases of the process have been completed and the service process is currently going through the i_m-th phase, $i_m \in \{1, 2\}$. If queue m is empty, then we may define the state of the service process to be zero.

Let $P_{i_1 j_1 i_2 j_2 \ldots i_M j_M}[n_1][n_2]\ldots[n_M](t)$ be the probability that at time t there are n_m customers (including the customer that is being served) in the m-th queue, the service process in the m-th queue is in state i_m, and the arrival process in the m-th queue is in state j_m, where $m = 1,2,\ldots, M$, $i_m \in \{0, 1, 2\}$, $j_m \in \{1, 2\}$.

Suppose

$$\lim_{t\to\infty} P_{i_1 j_1 i_2 j_2 \ldots i_M j_M}[n_1][n_2]\ldots[n_M](t) = P_{i_1 j_1 i_2 j_2 \ldots i_M j_M}[n_1][n_2]\ldots[n_M]$$

exists. We may refer to $P_{i_1 j_1 i_2 j_2 \ldots i_M j_M}[n_1][n_2]\ldots[n_M]$ as the stationary probability that there are n_m customers (including the customer that is being served) in the m-th queue, the service process in the m-th queue is in state i_m, and the arrival process in the m-th queue is in state j_m, where $m = 1,2,\ldots, M$, $i_m \in \{0, 1, 2\}$ and $j_m \in \{1, 2\}$.

A method is proposed in Section 2.2 to derive the stationary joint queue length distribution in a system of 2 dependent Hypo(2)/Hypo(2)/1 queues. We show in Section 2.6 and Section 2.10 that the method in Section 2.2 can be adapted to find the stationary probability in the system of 3 dependent Hypo(2)/Hypo(2)/1 queues and a system of M dependent Hypo(2)/Hypo(2)/1 queues.

2.2 DERIVATION OF THE FORWARD EQUATIONS IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES

Suppose in the system of M parallel queues described in Section 2.1, the value of M is 2. The crossing of a customer to another queue is illustrated in Figure 2.2.1.

Suppose that the hypoexponential distribution of the interarrival time T in queue m has the parameters λ_{m1}, λ_{m2}. The probability density function (pdf) will then be given by

$$f_m(t) = \lambda_{m1} e^{-\lambda_{m1} t} * \lambda_{m2} e^{-\lambda_{m2} t} \qquad \text{(" * " denotes convolution)}$$

$$f_m(t) = \frac{\lambda_{m1}\lambda_{m2}}{\lambda_{m1} - \lambda_{m2}}\left(e^{-\lambda_{m2}t} - e^{-\lambda_{m1}t}\right), \ t \geq 0.$$

or

Furthermore, assume that the hypoexponential distribution of the service time S in queue m has the parameters μ_{m1}, μ_{m2}. The pdf of the service time S in queue m will then be given by

$$g_m(t) = \mu_{m1}e^{-\mu_{m1}t} * \mu_{m2}e^{-\mu_{m2}t}$$

or

$$g_m(t) = \frac{\mu_{m1}\mu_{m2}}{\mu_{m1} - \mu_{m2}}\left(e^{-\mu_{m2}t} - e^{-\mu_{m1}t}\right), \ t \geq 0.$$

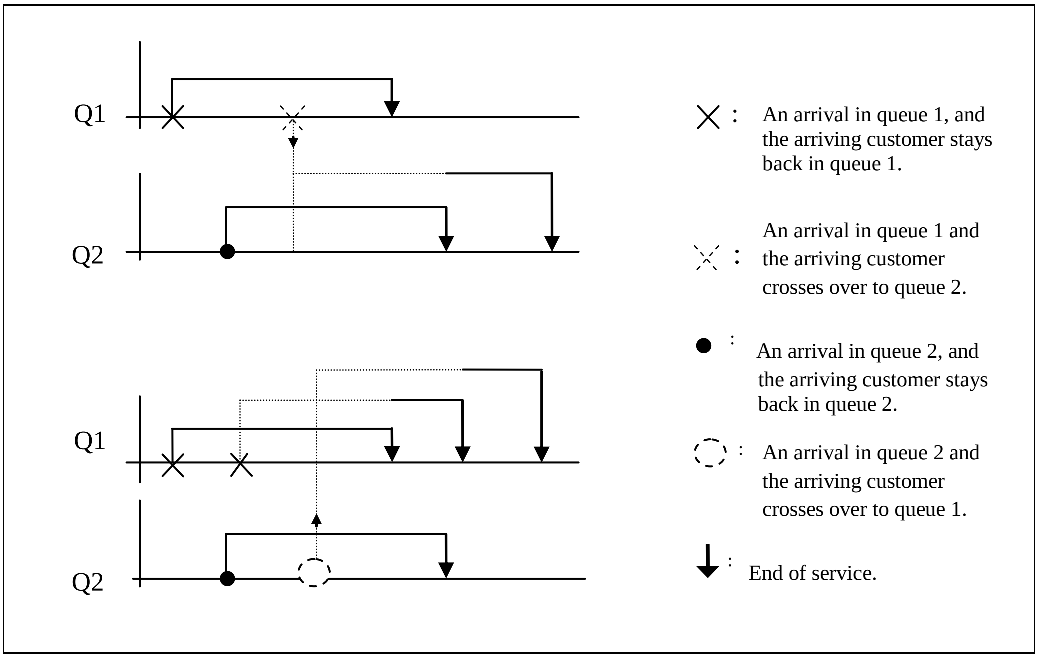

Figure 2.2.1 : Crossing over of a customer to another queue in a system of two one-server queues.

Let $\Delta t > 0$ be a small increment in time and $\tau_k = ((k-1)\Delta t, k\Delta t]$ a time interval, $k = 1, \ 2, \ \ldots$. Next let $P^{(k)}_{i_1 j_1 i_2 j_2}[n_1][n_2]$ be the probability that at the end of the interval τ_k, the number of customers in the system is n_1 in queue 1 and n_2 in queue 2 (including the customers that is being served), the service process in queue m is in the state i_m and the arrival process in queue m is in the state j_m, $i_1, i_2 \in \{0, 1, 2\}$ and $j_1, j_2 \in \{1, 2\}$. If queue m is empty, then we may define the state of the service process to be zero. Assume that

$$P_{i_1 j_1 i_2 j_2}[n_1][n_2] = \lim_{k\to\infty} P^{(k)}_{i_1 j_1 i_2 j_2}[n_1][n_2]$$

exists.

Let $\mathbf{h}^{(k)}$ be the vector

$$\mathbf{h}^{(k)} = \left(i_1^{(k)},\ j_1^{(k)},\ i_2^{(k)},\ j_2^{(k)},\ n_1^{(k)},\ n_2^{(k)} \right)$$

of which the components are respectively the values of i_1, j_1, i_2, j_2, n_1, n_2 at the end of τ_k. We may refer to $\mathbf{h}^{(k)}$ as the vector of characteristics of the queueing system at the end of τ_k.

The value $\mathbf{h}^{(k)}$ may be developed from $\mathbf{h}^{(k-1)}$ after some appropriate activities in the interval τ_k. The set of possible activities may be denoted by the set A = {A_1, A_2, …, A_{32} }. The events in A are shown below.

(i) For the case when $n_1^{(k-1)} \geq 1$ and $n_2^{(k-1)} \geq 1$:

A_1 = (1, 0, 0, 0, –1, –1)

A_2 = (0, 1, 0, 0, –1, –1)

A_3 = (0, 1, 0, 0, 0, –1)

A_4 = (0, 1, 0, 0, 1, –1)

A_5 = (0, 0, 1, 0, –1, –1)

A_6 = (0, 0, 0, 1, –1, –1)

A_7 = (0, 0, 0, 1, –1, 0)

A_8 = (0, 0, 0, 1, –1, 1)

A_9 = (0, 0, 0, 0, –1, –1)

(ii) For the case when $n_1^{(k-1)} = 0$ and $n_2^{(k-1)} \geq 1$:

A_{10} = (–1, 0, 0, 0, –1, –1)

$A_{11} = (-1, 1, 0, 0, -1, -1)$

$A_{12} = (-1, 1, 0, 0, \; 0, -1)$

$A_{13} = (-1, 1, 0, 0, \; 1, -1)$

$A_{14} = (-1, 0, 1, 0, -1, -1)$

$A_{15} = (-1, 0, 0, 1, -1, -1)$

$A_{16} = (-1, 0, 0, 1, -1, \; 0)$

$A_{17} = (-1, 0, 0, 1, -1, \; 1)$

(iiii) For the case when $n_1^{(k-1)} \geq 1$ and $n_2^{(k-1)} = 0$:

$A_{18} = (1, 0, -1, 0, -1, -1)$

$A_{19} = (0, 1, -1, 0, -1, -1)$

$A_{20} = (0, 1, -1, 0, \; 0, -1)$

$A_{21} = (0, 1, -1, 0, \; 1, -1)$

$A_{22} = (0, 0, -1, 0, -1, -1)$

$A_{23} = (0, 0, -1, 1, -1, -1)$

$A_{24} = (0, 0, -1, 1, -1, \; 0)$

$A_{25} = (0, 0, -1, 1, -1, \; 1)$

(iv) For the case when $n_1^{(k-1)} = 0$ and $n_2^{(k-1)} = 0$:

$A_{26} = (-1, 1, -1, 0, -1, -1)$

$A_{27} = (-1, 1, -1, 0, \; 0, -1)$

$A_{28} = (-1, 1, -1, 0, \; 1, -1)$

$A_{29} = (-1, 0, -1, 1, -1, -1)$

$A_{30} = (-1, 0, -1, 1, -1, \; 0)$

$A_{31} = (-1, 0, -1, 1, -1, \; 1)$

$A_{32} = (-1, 0, -1, 0, -1, -1)$

The positions, values and meanings of the first four components of A_w are explained in Table 2.2.1.

Table 2.2.1 : The positions, values and meanings of the components in A_w.

Position of component	Value of component	Meaning
1	1	A transition in the state of the service process in queue 1 occurs in τ_k.
1	0	A transition in the state of the service process in queue 1 does not occur in τ_k.
1	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of the service process in queue 1 has occurred in τ_k is not relevant.
2	1	A transition in the state of the arrival process in queue 1 occurs in τ_k.
2	0	A transition in the state of the arrival process in queue 1 does not occur in τ_k.
3	1	A transition in the state of the service process in queue 2 occurs in τ_k.
3	0	A transition in the state of the service process in queue 2 does not occur in τ_k.
3	–1	Queue 2 is empty at the end of τ_{k-1} and whether a transition in the state of the service process in queue 2 has occurred in τ_k is not relevant.
4	1	A transition in the state of the arrival process in queue 2 occurs in τ_k.
4	0	A transition in the state of the arrival process in queue 2 does not occur in τ_k.

The meanings of the fifth and sixth components (A_{w5} and A_{w6}) of A_w are explained below:

$$A_{w5} = \begin{cases} 0, & \text{if the arriving customer in queue 1 stays back in queue 1.} \\ 1, & \text{if the arriving customer in queue 1 goes to queue 2.} \\ -1, & \text{no customers arrive in queue 1 and it is not relevant to find out whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w6} = \begin{cases} 0, & \text{if the arriving customer in queue 2 stays back in queue 2.} \\ 1, & \text{if the arriving customer in queue 2 goes to queue 1.} \\ -1, & \text{no customers arrive in queue 2 and it is not relevant to find out whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search for all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form. An example of the codes is as follows:

Table 2.2.2 : An example of the codes of $\mathbf{h}^{(k)}, \mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k.

$\mathbf{h}^{(k)}$	$\mathbf{h}^{(k-1)}$	**Power**		
2 1 1 1 2 2	2 1 1 1 2 2	0 1 1 0 1 0 1 0	0 0 0 0 0 0 0 0	0 0 0 0
2 1 1 1 2 2	1 1 1 1 2 2	0 0 1 0 1 0 1 0	1 0 0 0 0 0 0 0	0 0 0 0
2 1 1 1 2 2	2 2 1 1 1 2	0 1 0 0 1 0 1 0	0 0 0 1 0 0 0 0	1 0 0 0
2 1 1 1 2 2	2 2 1 1 2 1	0 1 0 0 1 0 1 0	0 0 0 1 0 0 0 0	0 1 0 0
2 1 1 1 2 2	2 1 2 1 2 3	0 1 1 0 0 0 1 0	0 0 0 0 0 1 0 0	0 0 0 0
2 1 1 1 2 2	2 1 1 2 2 1	0 1 1 0 1 0 0 0	0 0 0 0 0 0 0 1	0 0 1 0
2 1 1 1 2 2	2 1 1 2 1 2	0 1 1 0 1 0 0 0	0 0 0 0 0 0 0 1	0 0 0 1

In Table 2.2.2,

Columns 1 – 6 give the components of $\mathbf{h}^{(k)}$.

Columns 7 – 12 give the components of $\mathbf{h}^{(k-1)}$.

Columns 13 – 32 give respectively the powers of $(1-\mu_{11}\Delta t)$, $(1-\mu_{12}\Delta t)$, $(1-\lambda_{11}\Delta t)$, $(1-\lambda_{12}\Delta t)$, $(1-\mu_{21}\Delta t)$, $(1-\mu_{22}\Delta t)$, $(1-\lambda_{21}\Delta t)$, $(1-\lambda_{22}\Delta t)$, $(\mu_{11}\Delta t)$, $(\mu_{12}\Delta t)$, $(\lambda_{11}\Delta t)$, $(\lambda_{12}\Delta t)$, $(\mu_{21}\Delta t)$, $(\mu_{22}\Delta t)$, $(\lambda_{21}\Delta t)$, $(\lambda_{22}\Delta t)$, (q_{11}), (q_{12}), (q_{22}) and (q_{21}).

The multiplication of $(1-\mu_{11}\Delta t)$, $(1-\mu_{12}\Delta t)$, $(1-\lambda_{11}\Delta t)$, $(1-\lambda_{12}\Delta t)$, $(1-\mu_{21}\Delta t)$, $(1-\mu_{22}\Delta t)$, $(1-\lambda_{21}\Delta t)$, $(1-\lambda_{22}\Delta t)$, $(\mu_{11}\Delta t)$, $(\mu_{12}\Delta t)$, $(\lambda_{11}\Delta t)$, $(\lambda_{12}\Delta t)$, $(\mu_{21}\Delta t)$, $(\mu_{22}\Delta t)$, $(\lambda_{21}\Delta t)$, $(\lambda_{22}\Delta t)$, (q_{11}), (q_{12}), (q_{22}) and (q_{21}) raised respectively to the corresponding powers will represent the probability of occurrence of the corresponding event which may be represented by an element in A.

The information represented by the above codes may be used to form the following equation:

$$
\begin{aligned}
P_{2111}^{(k)}[2][2] \cong{} & P_{2111}^{(k-1)}[2][2](1-\mu_{12}\Delta t-\lambda_{11}\Delta t-\mu_{21}\Delta t-\lambda_{21}\Delta t) \\
& + P_{1111}^{(k-1)}[2][2](\mu_{11}\Delta t)+P_{2211}^{(k-1)}[1][2](\lambda_{12}\Delta t)(q_{11}) \\
& + P_{2211}^{(k-1)}[2][1](\lambda_{12}\Delta t)(q_{12})+P_{2121}^{(k-1)}[2][3](\mu_{22}\Delta t) \\
& + P_{2112}^{(k-1)}[2][1](\lambda_{22}\Delta t)(q_{22})+P_{2112}^{(k-1)}[1][2](\lambda_{22}\Delta t)(q_{21})
\end{aligned}
\tag{2.2.1}
$$

The derivation of Equation (2.2.1) may also be illustrated by Figure 2.2.2.

Subtracting the term $P_{2111}^{(k-1)}[2][2]$ from both sides of (2.2.1), dividing both sides of the resulting equation by Δt, and letting $\Delta t \to 0$, and later letting $k \to \infty$, we get the balance equation

$$
\begin{aligned}
0 \cong{} & P_{2111}[2][2](-\mu_{12}-\lambda_{11}-\mu_{21}-\lambda_{21})+P_{1111}[2][2]\mu_{11} \\
& + P_{2211}[1][2]\lambda_{12}q_{11}+P_{2211}[2][1]\lambda_{12}q_{12} \\
& + P_{2121}[2][3]\mu_{22}+P_{2112}[2][1]\lambda_{22}q_{22} \\
& + P_{2112}[1][2]\lambda_{22}q_{21}
\end{aligned}
\tag{2.2.2}
$$

Equation (2.2.2) may be represented in a coded form as shown in Table 2.2.3.

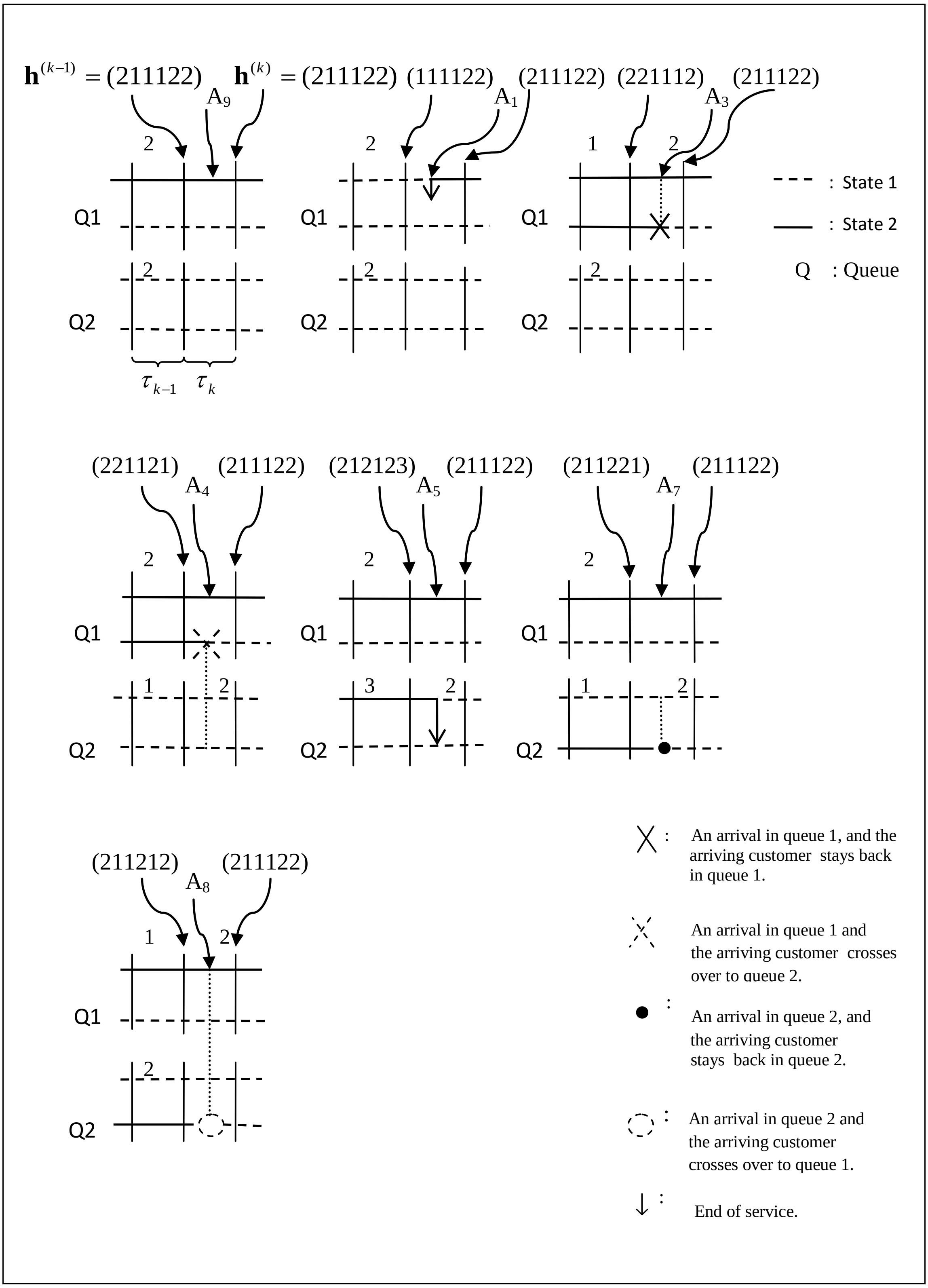

Figure 2.2.2 : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$.

Table 2.2.3 : Representation of balance equation in (2.2.2) by codes.

Constant	h	Power
-1	2 1 1 1 2 2	0 1 0 0 0 0 0 0 0 0 0 0
-1	2 1 1 1 2 2	0 0 1 0 0 0 0 0 0 0 0 0
-1	2 1 1 1 2 2	0 0 0 0 1 0 0 0 0 0 0 0
-1	2 1 1 1 2 2	0 0 0 0 0 0 1 0 0 0 0 0
1	1 1 1 1 2 2	1 0 0 0 0 0 0 0 0 0 0 0
1	2 2 1 1 1 2	0 0 0 1 0 0 0 0 1 0 0 0
1	2 2 1 1 2 1	0 0 0 1 0 0 0 0 0 1 0 0
1	2 1 2 1 2 3	0 0 0 0 0 1 0 0 0 0 0 0
1	2 1 1 2 2 1	0 0 0 0 0 0 0 1 0 0 1 0
1	2 1 1 2 1 2	0 0 0 0 0 0 0 1 0 0 0 1
1*		

In each row of Table 2.2.3,

Column 1 gives a coefficient value.

Columns 2 – 7 give the components of **h**.

Columns 8 – 19 give respectively the powers of (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (q_{11}), (q_{12}), (q_{22}) and (q_{21}).

The symbol " 1* " in the last row denotes the end of the equation.

For each row in Table 2.2.3, we form a product of

(i) the coefficient in column 1,

(ii) the term $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ of which the values i_1 , j_1 , i_2 , j_2 , n_1 , n_2 are given by **h**, and

(iii) the product of (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (q_{11}), (q_{12}), (q_{22}) and (q_{21}) raised respectively to the corresponding powers.

We then equate the sum of the products for all the rows in Table 2.2.3 to zero to form Equation (2.2.2).

Now if we change $\mathbf{h}^{(k)} = (2,\ 1,\ 1,\ 1,\ 2,\ 2)$ to other values and search for the values of $\mathbf{h}^{(k-1)}$ and the component of A_w which lead to the new value of $\mathbf{h}^{(k)}$, we can likewise obtain an equation similar to Equation (2.2.2). Again the resulting equation can be represented in a coded form. The resulting table of codes for $0 \leq n_1^{(k)} + n_2^{(k)} \leq 10$ can be found in the file *TwoQueueSystem_codes.txt* in the CD attached. A subset of the file *TwoQueueSystem_codes.txt* for $0 \leq n_1^{(k)} + n_2^{(k)} \leq 2$ is given in Appendix A. The first page of Appendix A is shown in Table 2.2.4. The equations for a given $n_1^{(k)}$ and $n_2^{(k)}$ of which $n_1^{(k)} + n_2^{(k)} \geq 11$ can be found from the file *TwoQueueSystem_codes.txt* in the CD by

(i) increasing the nonzero value of $n_1^{(k)}$ in the file *TwoQueueSystem_codes.txt* to $n_1^{(k)} + 1$, or

(ii) increasing the nonzero value of $n_2^{(k)}$ in the file *TwoQueueSystem_codes.txt* to $n_2^{(k)} + 1$.

Table 2.2.4 : Representation of some balance equations by codes.

Constant	h						Power											
-1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
-1	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	2	1	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	2	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0
1*																		
-1	0	1	0	2	0	0	0	0	1	0	0	0	0	0	0	0	0	0
-1	0	1	0	2	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	2	1	0	2	1	0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	2	2	0	1	0	0	0	0	0	1	0	0	0	0	0	0
1	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1*																		
-1	0	1	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
-1	0	1	1	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0
-1	0	1	1	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0
1	2	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0
1	0	2	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
1	0	1	2	1	0	2	0	0	0	0	0	1	0	0	0	0	0	0
1	0	1	0	2	0	0	0	0	0	0	0	0	0	1	0	0	1	0
1*																		
-1	0	1	1	2	0	1	0	0	1	0	0	0	0	0	0	0	0	0
-1	0	1	1	2	0	1	0	0	0	0	1	0	0	0	0	0	0	0
-1	0	1	1	2	0	1	0	0	0	0	0	0	0	1	0	0	0	0
1	2	1	1	2	1	1	0	1	0	0	0	0	0	0	0	0	0	0
1	0	2	0	2	0	0	0	0	0	1	0	0	0	0	0	1	0	0
1	0	1	2	2	0	2	0	0	0	0	0	1	0	0	0	0	0	0
1	0	1	1	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0
1*																		
-1	0	1	2	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
-1	0	1	2	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1	0	1	2	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0
1	2	1	2	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0
1*																		
-1	0	1	2	2	0	1	0	0	1	0	0	0	0	0	0	0	0	0
-1	0	1	2	2	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1	0	1	2	2	0	1	0	0	0	0	0	0	0	1	0	0	0	0
1	2	1	2	2	1	1	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	1	2	0	1	0	0	0	0	1	0	0	0	0	0	0	0
1	0	1	2	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0
1*																		
-1	0	2	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
-1	0	2	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	2	2	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
1	0	2	2	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0
1*																		
-1	0	2	0	2	0	0	0	0	0	1	0	0	0	0	0	0	0	0
-1	0	2	0	2	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	2	2	0	2	1	0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	2	0	0	0	0	1	0	0	0	0	0	0	0	0	0
1	0	2	2	2	0	1	0	0	0	0	0	1	0	0	0	0	0	0
1	0	2	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1*																		
-1	0	2	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0
-1	0	2	1	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0
-1	0	2	1	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0
1	2	2	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
1	0	2	2	1	0	2	0	0	0	0	0	1	0	0	0	0	0	0
1	0	2	0	2	0	0	0	0	0	0	0	0	0	1	0	0	1	0
1*																		

2.3 COMPUTATION OF THE VALUE OF $\boldsymbol{P_{i_1 j_1 i_2 j_2}[n_1][n_2]}$

Before solving the balance equations to obtain the stationary queue length distribution, we first introduce the following notations. Let

(a) $P_{i_1 j_1 i_2 j_2}[n]$ be a value of $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ of which $n_1 + n_2 = n$.

(b) $\{P\,[n_1][n_2]\}$ the set consisting of all the possible $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$.

(c) $\{P\,[n]\}$ a set formed by the $\{P\,[n_1][n_2]\}$ of which $n_1 + n_2 = n$.

(d) $\{P\,[n],\ \ P\,[n+1],\ \ P\,[n+2]\}$ the set of equations of the form

$$\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2} a_{i_1 j_1 i_2 j_2} P_{i_1 j_1 i_2 j_2}[n] + \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2} b_{i_1 j_1 i_2 j_2} P_{i_1 j_1 i_2 j_2}[n+1] + \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2} c_{i_1 j_1 i_2 j_2} P_{i_1 j_1 i_2 j_2}[n+2] = 0$$

where $a_{i_1 j_1 i_2 j_2}$, $b_{i_1 j_1 i_2 j_2}$, and $c_{i_1 j_1 i_2 j_2}$ are constants.

(e) $\left(P_{i_1 j_1 i_2 j_2}[n_1][n_2] \;\middle|\; \{P\,[0]\},\ \{P\,[n+1]\}\right)$ an equation of the form

$$P_{i_1 j_1 i_2 j_2}[n_1][n_2] = \sum_{i_1'=0}^{2}\sum_{j_1'=1}^{2}\sum_{i_2'=0}^{2}\sum_{j_2'=1}^{2} d_{i_1' j_1' i_2' j_2'} P_{i_1' j_1' i_2' j_2'}[0] + \sum_{i_1'=0}^{2}\sum_{j_1'=1}^{2}\sum_{i_2'=0}^{2}\sum_{j_2'=1}^{2} e_{i_1' j_1' i_2' j_2'} P_{i_1' j_1' i_2' j_2'}[n+1]$$

where $d_{i_1' j_1' i_2' j_2'}$ and $e_{i_1' j_1' i_2' j_2'}$ are constants.

With the above notations, the balance equations represented by the codes in the file *TwoQueueSystem_codes.txt* in the CD can be represented as

$$\{P\,[0],\ \ P\,[1]\} \qquad (2.3.1)$$

and

$$\{P\,[n-1],\ \ P\,[n],\ \ P\,[n+1]\} \quad ,n = 1,\ 2,\ \ldots \qquad (2.3.2)$$

For example the codes of the equations represented by $\{P[0], P[1]\}$ are shown in Table 2.3.1, and the codes of the equations represented by $\{P[0], P[1], P[2]\}$ are shown in Table 2.3.2.

Table 2.3.1 : The codes for the equations represented by $\{P[0], P[1]\}$.

Constant	**h**	**Power**
-1	0 1 0 1 0 0	0 0 1 0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 0	0 0 0 0 0 0 1 0 0 0 0 0
1	2 1 0 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 2 1 0 1	0 0 0 0 0 1 0 0 0 0 0 0
1*		
-1	0 1 0 2 0 0	0 0 1 0 0 0 0 0 0 0 0 0
-1	0 1 0 2 0 0	0 0 0 0 0 0 0 1 0 0 0 0
1	2 1 0 2 1 0	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 2 2 0 1	0 0 0 0 0 1 0 0 0 0 0 0
1	0 1 0 1 0 0	0 0 0 0 0 0 1 0 0 0 0 0
1*		
-1	0 2 0 1 0 0	0 0 0 1 0 0 0 0 0 0 0 0
-1	0 2 0 1 0 0	0 0 0 0 0 0 1 0 0 0 0 0
1	2 2 0 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 0	0 0 1 0 0 0 0 0 0 0 0 0
1	0 2 2 1 0 1	0 0 0 0 0 1 0 0 0 0 0 0
1*		
-1	0 2 0 2 0 0	0 0 0 1 0 0 0 0 0 0 0 0
-1	0 2 0 2 0 0	0 0 0 0 0 0 0 1 0 0 0 0
1	2 2 0 2 1 0	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 0	0 0 1 0 0 0 0 0 0 0 0 0
1	0 2 2 2 0 1	0 0 0 0 0 1 0 0 0 0 0 0
1	0 2 0 1 0 0	0 0 0 0 0 0 1 0 0 0 0 0
1*		

Table 2.3.2 : The codes for the equations represented by $\{P\,[0], P\,[1], P\,[2]\}$.

Constant	h	Power
-1	0 1 1 1 0 1	0 0 1 0 0 0 0 0 0 0 0 0
-1	0 1 1 1 0 1	0 0 0 0 1 0 0 0 0 0 0 0
-1	0 1 1 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1	2 1 1 1 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 0	0 0 0 1 0 0 0 0 0 1 0 0
1	0 1 2 1 0 2	0 0 0 0 0 1 0 0 0 0 0 0
1	0 1 0 2 0 0	0 0 0 0 0 0 0 1 0 0 1 0
1*		
-1	0 1 1 2 0 1	0 0 1 0 0 0 0 0 0 0 0 0
-1	0 1 1 2 0 1	0 0 0 0 1 0 0 0 0 0 0 0
-1	0 1 1 2 0 1	0 0 0 0 0 0 0 1 0 0 0 0
1	2 1 1 2 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 2 0 2 0 0	0 0 0 1 0 0 0 0 0 1 0 0
1	0 1 2 2 0 2	0 0 0 0 0 1 0 0 0 0 0 0
1	0 1 1 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1*		
-1	0 1 2 1 0 1	0 0 1 0 0 0 0 0 0 0 0 0
-1	0 1 2 1 0 1	0 0 0 0 0 1 0 0 0 0 0 0
-1	0 1 2 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1	2 1 2 1 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 1 1 0 1	0 0 0 0 1 0 0 0 0 0 0 0
1*		
-1	0 1 2 2 0 1	0 0 1 0 0 0 0 0 0 0 0 0
-1	0 1 2 2 0 1	0 0 0 0 0 1 0 0 0 0 0 0
-1	0 1 2 2 0 1	0 0 0 0 0 0 0 1 0 0 0 0
1	2 1 2 2 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 1 2 0 1	0 0 0 0 1 0 0 0 0 0 0 0
1	0 1 2 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1*		
-1	0 2 1 1 0 1	0 0 0 1 0 0 0 0 0 0 0 0
-1	0 2 1 1 0 1	0 0 0 0 1 0 0 0 0 0 0 0
-1	0 2 1 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1	2 2 1 1 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 1 1 0 1	0 0 1 0 0 0 0 0 0 0 0 0
1	0 2 2 1 0 2	0 0 0 0 0 1 0 0 0 0 0 0
1	0 2 0 2 0 0	0 0 0 0 0 0 0 1 0 0 1 0
1*		

Table 2.3.2, continued

Constant	h	Power
-1	0 2 1 2 0 1	0 0 0 1 0 0 0 0 0 0 0 0
-1	0 2 1 2 0 1	0 0 0 0 1 0 0 0 0 0 0 0
-1	0 2 1 2 0 1	0 0 0 0 0 0 0 1 0 0 0 0
1	2 2 1 2 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 1 2 0 1	0 0 1 0 0 0 0 0 0 0 0 0
1	0 2 2 2 0 2	0 0 0 0 0 1 0 0 0 0 0 0
1	0 2 1 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1*		
-1	0 2 2 1 0 1	0 0 0 1 0 0 0 0 0 0 0 0
-1	0 2 2 1 0 1	0 0 0 0 0 1 0 0 0 0 0 0
-1	0 2 2 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1	2 2 2 1 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 2 1 0 1	0 0 1 0 0 0 0 0 0 0 0 0
1	0 2 1 1 0 1	0 0 0 0 1 0 0 0 0 0 0 0
1*		
-1	0 2 2 2 0 1	0 0 0 1 0 0 0 0 0 0 0 0
-1	0 2 2 2 0 1	0 0 0 0 0 1 0 0 0 0 0 0
-1	0 2 2 2 0 1	0 0 0 0 0 0 0 1 0 0 0 0
1	2 2 2 2 1 1	0 1 0 0 0 0 0 0 0 0 0 0
1	0 1 2 2 0 1	0 0 1 0 0 0 0 0 0 0 0 0
1	0 2 1 2 0 1	0 0 0 0 1 0 0 0 0 0 0 0
1	0 2 2 1 0 1	0 0 0 0 0 0 1 0 0 0 0 0
1*		
-1	1 1 0 1 1 0	1 0 0 0 0 0 0 0 0 0 0 0
-1	1 1 0 1 1 0	0 0 1 0 0 0 0 0 0 0 0 0
-1	1 1 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1	2 1 0 1 2 0	0 1 0 0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 0	0 0 0 1 0 0 0 0 1 0 0 0
1	1 1 2 1 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1	0 1 0 2 0 0	0 0 0 0 0 0 0 1 0 0 0 1
1*		
-1	1 1 0 2 1 0	1 0 0 0 0 0 0 0 0 0 0 0
-1	1 1 0 2 1 0	0 0 1 0 0 0 0 0 0 0 0 0
-1	1 1 0 2 1 0	0 0 0 0 0 0 0 1 0 0 0 0
1	2 1 0 2 2 0	0 1 0 0 0 0 0 0 0 0 0 0
1	0 2 0 2 0 0	0 0 0 1 0 0 0 0 1 0 0 0
1	1 1 2 2 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1	1 1 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1*		

Table 2.3.2, continued

Constant	h	Power
-1	1 2 0 1 1 0	1 0 0 0 0 0 0 0 0 0 0 0
-1	1 2 0 1 1 0	0 0 0 1 0 0 0 0 0 0 0 0
-1	1 2 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1	2 2 0 1 2 0	0 1 0 0 0 0 0 0 0 0 0 0
1	1 1 0 1 1 0	0 0 1 0 0 0 0 0 0 0 0 0
1	1 2 2 1 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1	0 2 0 2 0 0	0 0 0 0 0 0 0 1 0 0 0 1
1*		
-1	1 2 0 2 1 0	1 0 0 0 0 0 0 0 0 0 0 0
-1	1 2 0 2 1 0	0 0 0 1 0 0 0 0 0 0 0 0
-1	1 2 0 2 1 0	0 0 0 0 0 0 0 1 0 0 0 0
1	2 2 0 2 2 0	0 1 0 0 0 0 0 0 0 0 0 0
1	1 1 0 2 1 0	0 0 1 0 0 0 0 0 0 0 0 0
1	1 2 2 2 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1	1 2 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1*		
-1	2 1 0 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0
-1	2 1 0 1 1 0	0 0 1 0 0 0 0 0 0 0 0 0
-1	2 1 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1	1 1 0 1 1 0	1 0 0 0 0 0 0 0 0 0 0 0
1	2 1 2 1 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1*		
-1	2 1 0 2 1 0	0 1 0 0 0 0 0 0 0 0 0 0
-1	2 1 0 2 1 0	0 0 1 0 0 0 0 0 0 0 0 0
-1	2 1 0 2 1 0	0 0 0 0 0 0 0 1 0 0 0 0
1	1 1 0 2 1 0	1 0 0 0 0 0 0 0 0 0 0 0
1	2 1 2 2 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1	2 1 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1*		
-1	2 2 0 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0
-1	2 2 0 1 1 0	0 0 0 1 0 0 0 0 0 0 0 0
-1	2 2 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1	1 2 0 1 1 0	1 0 0 0 0 0 0 0 0 0 0 0
1	2 1 0 1 1 0	0 0 1 0 0 0 0 0 0 0 0 0
1	2 2 2 1 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1*		
-1	2 2 0 2 1 0	0 1 0 0 0 0 0 0 0 0 0 0
-1	2 2 0 2 1 0	0 0 0 1 0 0 0 0 0 0 0 0
-1	2 2 0 2 1 0	0 0 0 0 0 0 0 1 0 0 0 0
1	1 2 0 2 1 0	1 0 0 0 0 0 0 0 0 0 0 0
1	2 1 0 2 1 0	0 0 1 0 0 0 0 0 0 0 0 0
1	2 2 2 2 1 1	0 0 0 0 0 1 0 0 0 0 0 0
1	2 2 0 1 1 0	0 0 0 0 0 0 1 0 0 0 0 0
1*		

To solve (2.3.1) and (2.3.2), we first combine the set of equations given by $\{P[0], P[1]\}$ and $\{P[0], P[1], P[2]\}$. We then solve for $P_{i_1 j_1 i_2 j_2}[n]$ in terms of the $P_{i_1 j_1 i_2 j_2}[2]$ for n = 0, 1 to get

$$\left(P_{i_1 j_1 i_2 j_2}[n] \;\middle|\; P[2]\right) \quad , \quad n = 0,\ 1 \qquad (2.3.3)$$

An example of the codes which represent (2.3.3) are given in the Table 2.3.3.

The first column of Table 2.3.3 gives the values of (n_1, n_2). The second column gives the values of i_1, j_1, i_2, j_2 appearing in (2.3.3). The first row in the Table 2.3.3 gives the values of (n_1', n_2'). The second row gives the various values of i_1', j_1', i_2', j_2'. Each subsequent row in Table 2.3.3 represents an equation formed by equating $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ to the sum of the products formed by multiplying the remaining entries in the row by $P_{i_1' j_1' i_2' j_2'}[n_1'][n_2']$. For example, row 3 represents the equation

$$\begin{aligned} P_{0101}[0][0] = {} & 44.7P_{0121}[0][2] + 28.3P_{0122}[0][2] + 28.3P_{0221}[0][2] + 16.9P_{0222}[0][2] \\ & + 44.7P_{2101}[2][0] + 28.3P_{2102}[2][0] + 28.3P_{2201}[2][0] + 16.9P_{2202}[2][0] \\ & + 44.7P_{1121}[1][1] + 28.3P_{1122}[1][1] + 28.3P_{1221}[1][1] + 16.9P_{1222}[1][1] \\ & + 44.7P_{2111}[1][1] + 28.3P_{2112}[1][1] + 95.9P_{2121}[1][1] + 70.3P_{2122}[1][1] \\ & + 28.3P_{2211}[1][1] + 16.9P_{2212}[1][1] + 70.3P_{2221}[1][1] + 47.2P_{2222}[1][1] \end{aligned}$$

Table 2.3.3 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2}[n] \mid P[2]\}$ for $n = 0, 1$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, q_{11}=0.90, q_{22}=0.90, Δt=0.01).

		02								20								11															
		0111	0112	0121	0122	0211	0212	0221	0222	1101	1102	1201	1202	2101	2102	2201	2202	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
	0101	0	0	44.67	28.32	0	0	28.32	16.86	0	0	0	0	44.67	28.32	28.32	16.86	0	0	44.67	28.32	0	0	28.32	16.86	44.67	28.32	95.87	70.25	28.32	16.86	70.25	47.2
00	0102	0	0	24.48	20.92	0	0	16.46	11.12	0	0	0	0	24.48	20.92	16.46	11.12	0	0	24.48	20.92	0	0	16.46	11.12	24.48	20.92	51.28	52.17	16.46	11.12	40.57	31.15
	0201	0	0	24.48	16.46	0	0	20.92	11.12	0	0	0	0	24.48	16.46	20.92	11.12	0	0	24.48	16.46	0	0	20.92	11.12	24.48	16.46	51.28	40.57	20.92	11.12	52.17	31.15
	0202	0	0	13.69	10.79	0	0	10.79	9.398	0	0	0	0	13.69	10.79	10.79	9.398	0	0	13.69	10.79	0	0	10.79	9.398	13.69	10.79	28.54	26.17	10.79	9.398	26.17	26.31
	0111	0	0	5.747	3.412	0	0	2.818	1.854	0	0	0	0	4.08	3.412	2.818	1.854	0	0	4.08	3.412	0	0	2.818	1.854	5.747	3.412	8.546	8.501	2.818	1.854	6.955	5.192
	0112	0	0	0.653	1.967	0	0	0.383	0.287	0	0	0	0	0.524	0.428	0.383	0.287	0	0	0.524	0.428	0	0	0.383	0.287	0.653	1.967	1.097	1.057	0.383	0.287	0.938	0.804
	0121	0	0	2.612	1.551	0	0	1.281	0.843	0	0	0	0	1.855	1.551	1.281	0.843	0	0	1.855	1.551	0	0	1.281	0.843	2.612	1.551	4.794	3.864	1.281	0.843	3.161	2.36
	0122	0	0	0.397	0.923	0	0	0.222	0.162	0	0	0	0	0.309	0.254	0.222	0.162	0	0	0.309	0.254	0	0	0.222	0.162	0.397	0.923	0.685	1.497	0.222	0.162	0.545	0.452
01	0211	0	0	2.338	1.756	0	0	3.249	1.444	0	0	0	0	2.21	1.756	1.711	1.444	0	0	2.21	1.756	0	0	1.711	1.444	2.338	1.756	4.61	4.278	3.249	1.444	4.159	4.043
	0212	0	0	0.214	0.266	0	0	0.259	1.552	0	0	0	0	0.195	0.156	0.15	0.124	0	0	0.195	0.156	0	0	0.15	0.124	0.214	0.266	0.408	0.381	0.259	1.552	0.364	0.346
	0221	0	0	1.13	0.831	0	0	1.468	0.664	0	0	0	0	1.041	0.831	0.799	0.664	0	0	1.041	0.831	0	0	0.799	0.664	1.13	0.831	2.213	2.028	1.468	0.664	2.815	1.86
	0222	0	0	0.153	0.184	0	0	0.179	0.681	0	0	0	0	0.138	0.11	0.105	0.086	0	0	0.138	0.11	0	0	0.105	0.086	0.153	0.184	0.291	0.306	0.179	0.681	0.292	1.074
	1101	0	0	4.08	2.818	0	0	3.412	1.854	0	0	0	0	5.747	2.818	3.412	1.854	0	0	5.747	2.818	0	0	3.412	1.854	4.08	2.818	8.546	6.955	3.412	1.854	8.501	5.192
	1102	0	0	2.21	1.711	0	0	1.756	1.444	0	0	0	0	2.338	3.249	1.756	1.444	0	0	2.338	3.249	0	0	1.756	1.444	2.21	1.711	4.61	4.159	1.756	1.444	4.278	4.043
	1201	0	0	0.524	0.383	0	0	0.428	0.287	0	0	0	0	0.653	0.383	1.967	0.287	0	0	0.653	0.383	0	0	1.967	0.287	0.524	0.383	1.097	0.938	0.428	0.287	1.057	0.804
10	1202	0	0	0.195	0.15	0	0	0.156	0.124	0	0	0	0	0.214	0.259	0.266	1.552	0	0	0.214	0.259	0	0	0.266	1.552	0.195	0.15	0.408	0.364	0.156	0.124	0.381	0.346
	2101	0	0	1.855	1.281	0	0	1.551	0.843	0	0	0	0	2.612	1.281	1.551	0.843	0	0	2.612	1.281	0	0	1.551	0.843	1.855	1.281	4.794	3.161	1.551	0.843	3.864	2.36
	2102	0	0	1.041	0.799	0	0	0.831	0.664	0	0	0	0	1.13	1.468	0.831	0.664	0	0	1.13	1.468	0	0	0.831	0.664	1.041	0.799	2.213	2.815	0.831	0.664	2.028	1.86
	2201	0	0	0.309	0.222	0	0	0.254	0.162	0	0	0	0	0.397	0.222	0.923	0.162	0	0	0.397	0.222	0	0	0.923	0.162	0.309	0.222	0.685	0.545	0.254	0.162	1.497	0.452
	2202	0	0	0.138	0.105	0	0	0.11	0.086	0	0	0	0	0.153	0.179	0.184	0.681	0	0	0.153	0.179	0	0	0.184	0.681	0.138	0.105	0.291	0.292	0.11	0.086	0.306	1.074

From the equations represented by rows 3-6 in Table 2.3.3, we get two sets of equations of the forms

$$\{P_{i_1j_1i_2j_2}[0][2] \mid \{P[0][0]\},\ \{P[1][1]\},\ \{P[2][0]\}\} \tag{2.3.4}$$

and $$\{P_{i_1j_1i_2j_2}[2][0] \mid \{P[0][0]\},\ \{P[1][1]\},\ \{P[0][2]\}\} \tag{2.3.5}$$

respectively (see for example Tables 2.3.4 - 2.3.5).

From (2.3.4) and rows 7-14 in Table (2.3.3), we get a set of equations of the form

$$\{P_{i_1j_1i_2j_2}[0][1] \mid \{P[0][0]\},\ \{P[1][1]\},\ \{P[2][0]\}\} \tag{2.3.6}$$

(see for example Table 2.3.6).

Similarly from (2.3.5) and rows 15-22 in Table (2.3.3), we get a set of equations of the form (see for example Table 2.3.7)

$$\{P_{i_1j_1i_2j_2}[1][0] \mid \{P[0][0]\},\ \{P[1][1]\},\ \{P[0][2]\}\}. \tag{2.3.7}$$

The set of equations given by (2.3.6) and (2.3.7) may be combined to form

$$\{P_{i_1j_1i_2j_2}[1] \mid \{P[0]\},\ \{P[2]\}\} \tag{2.3.8}$$

From (2.3.8) and the set of equations of the form

$$\{P[1],\ P[2],\ P[3]\}\ ,$$

we get

$$\{P_{i_1j_1i_2j_2}[2] \mid \{P[0]\},\ \{P[3]\}\}\ . \tag{2.3.9}$$

Table 2.3.4 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2}[0][2] \mid \{P[0][0]\}, \{P[1][1]\}, \{P[2][0]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, q_{11}=0.90, q_{22}=0.90, Δt=0.01.

The dotted lines indicate that the relevant tables are to be joined up).

		00			
		0101	0102	0201	0202
02	0111	0	0	0	0
	0112	0	0	0	0
	0121	0.132	-0.1	-0.1	-8.95E-17
	0122	-0.09225	0.22425	0	-0.1
	0211	0	0	0	0
	0212	0	0	0	0
	0221	-0.09225	1.27E-16	0.22425	-0.1
	0222	0.0195	-0.11175	-0.11175	0.336

		11															
		1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
02	0111	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0112	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0121	0	0	-1	-7.95E-17	0	0	-2.80E-16	-1.73E-16	-1	0	-2.4	-1.59E-15	-1.19E-16	-1.06E-16	-1.35E-15	3.37E-16
	0122	0	0	1.15E-15	-1	0	0	1.07E-15	3.14E-16	7.97E-16	-1	0.2	-2.6	6.17E-17	3.79E-17	2.63E-15	5.91E-17
	0211	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0212	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0221	0	0	1.15E-15	1.78E-16	0	0	-1	8.48E-17	7.97E-16	0	0.2	9.12E-16	-1	2.54E-16	-2.6	-3.39E-16
	0222	0	0	-1.99E-15	-5.97E-16	0	0	-1.34E-15	-1	-7.94E-16	0	-4.51E-15	0.2	-3.73E-16	-1	0.2	-2.8

		20							
		1101	1102	1201	1202	2101	2102	2201	2202
02	0111	0	0	0	0	0	0	0	0
	0112	0	0	0	0	0	0	0	0
	0121	0	0	0	0	-1	-7.95E-17	-2.80E-16	-1.73E-16
	0122	0	0	0	0	1.15E-15	-1	1.07E-15	3.14E-16
	0211	0	0	0	0	0	0	0	0
	0212	0	0	0	0	0	0	0	0
	0221	0	0	0	0	1.15E-15	1.78E-16	-1	8.48E-17
	0222	0	0	0	0	-1.99E-15	-5.97E-16	-1.34E-15	-1

Table 2.3.5 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2}[2][0] \mid \{P[0][0]\}, \{P[1][1]\}, \{P[0][2]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, q_{11}=0.90, q_{22}=0.90, Δt=0.01. The dotted lines indicate that the relevant tables are to be joined up).

		00			
		0101	0102	0201	0202
20	1101	0	0	0	0
	1102	0	0	0	0
	1201	0	0	0	0
	1202	0	0	0	0
	2101	0.132	-0.1	-0.1	-1.09E-16
	2102	-0.09225	0.22425	1.29E-16	-0.1
	2201	-0.09225	1.17E-16	0.22425	-0.1
	2202	0.0195	-0.11175	-0.11175	0.336

		11															
		1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
20	1101	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1102	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1201	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1202	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	2101	0	0	-1	0	0	0	2.74E-17	-6.14E-17	-1	1.59E-16	-2.4	-1.27E-15	1.09E-16	2.32E-17	-2.61E-16	5.95E-16
	2102	0	0	6.19E-16	-1	0	0	8.22E-17	5.27E-17	2.64E-16	-1	0.2	-2.6	-9.25E-16	-2.91E-16	4.11E-16	-5.99E-16
	2201	0	0	6.19E-16	0	0	0	-1	1.70E-16	2.64E-16	-1.78E-16	0.2	3.50E-16	-1	2.54E-16	-2.6	-3.39E-16
	2202	0	0	-1.97E-16	0	0	0	-2.24E-16	-1	9.97E-16	5.97E-16	-8.02E-16	0.2	7.46E-16	-1	0.2	-2.8

		02							
		0111	0112	0121	0122	0211	0212	0221	0222
20	1101	0	0	0	0	0	0	0	0
	1102	0	0	0	0	0	0	0	0
	1201	0	0	0	0	0	0	0	0
	1202	0	0	0	0	0	0	0	0
	2101	0	0	-1	1.59E-16	0	0	1.09E-16	2.32E-17
	2102	0	0	2.64E-16	-1	0	0	-9.25E-16	-2.91E-16
	2201	0	0	2.64E-16	-1.78E-16	0	0	-1	2.54E-16
	2202	0	0	9.97E-16	5.97E-16	0	0	7.46E-16	-1

Table 2.3.6 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2}[0][1] \mid \{P[0][0]\}, \{P[1][1]\}, \{P[2][0]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, q_{11}=0.90, q_{22}=0.90, Δt=0.01.

The dotted lines indicate that the relevant tables are to be joined up).

		00			
		0101	0102	0201	0202
01	0111	0.22	-0.016666667	-0.15	0
	0112	-0.125	0.343717949	-0.011538462	-0.138461538
	0121	0.1	-0.007575758	-0.068181818	0
	0122	-0.05	0.149113206	-0.007981149	-0.060200669
	0211	-0.125	-0.001282051	0.333461538	-0.015384615
	0212	0.01	-0.13518315	-0.136648352	0.469010989
	0221	-0.05	-8.87E-04	0.142018851	-0.006688963
	0222	9.54E-17	-0.050150212	-0.051351909	0.192634177

		11															
		1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
01	0111	0	0	-1.6666667	0	0	0	4.44E-16	-4.44E-16	0	0	-4	-3.55E-15	-4.44E-16	-2.22E-16	-8.88E-16	0
	0112	0	0	-0.1282051	-1.5384615	0	0	1.55E-15	5.55E-16	1.22E-15	0	4.22E-15	-4	5.55E-17	5.55E-17	4.00E-15	2.22E-16
	0121	0	0	-0.7575758	2.22E-16	0	0	2.22E-16	0	-4.44E-16	0	-0.9090909	-2.66E-15	0	-1.11E-16	-4.44E-16	0
	0122	0	0	-0.0886794	-0.6688963	0	0	6.11E-16	2.22E-16	5.00E-16	0	-0.0395257	-0.8695652	2.78E-17	2.78E-17	1.55E-15	1.11E-16
	0211	0	0	-0.1282051	0	0	0	-1.5384615	2.22E-16	8.88E-16	0	4.44E-15	-8.88E-16	4.44E-16	4.44E-16	-4	0
	0212	0	0	-0.018315	-0.1098901	0	0	-0.1098901	-1.4285714	-9.44E-16	0	-5.72E-15	-6.66E-16	-4.44E-16	-4.44E-16	-2.78E-16	-4
	0221	0	0	-0.0886794	-1.11E-16	0	0	-0.6688963	2.22E-16	6.66E-16	0	-0.0395257	0	2.22E-16	2.22E-16	-0.8695652	2.22E-16
	0222	0	0	-0.0150212	-0.0736582	0	0	-0.0736582	-0.5952381	-3.61E-16	0	-0.0032938	-0.0362319	-1.94E-16	-1.11E-16	-0.0362319	-0.8333333

		20							
		1101	1102	1201	1202	2101	2102	2201	2202
01	0111	0	0	0	0	-1.66666667	0	4.44E-16	-4.44E-16
	0112	0	0	0	0	-0.12820513	-1.53846154	1.55E-15	5.55E-16
	0121	0	0	0	0	-0.75757576	2.22E-16	2.22E-16	0
	0122	0	0	0	0	-0.08867944	-0.66889632	6.11E-16	2.22E-16
	0211	0	0	0	0	-0.12820513	0	-1.53846154	2.22E-16
	0212	0	0	0	0	-0.01831502	-0.10989011	-0.10989011	-1.42857143
	0221	0	0	0	0	-0.08867944	-1.11E-16	-0.66889632	2.22E-16
	0222	0	0	0	0	-0.01502121	-0.07365823	-0.07365823	-0.5952381

Table 2.3.7 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2}[1][0] \mid \{P\,[0][0]\},\ \{P\,[1][1]\},\ \{P\,[0][2]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, q_{11}=0.90, q_{22}=0.90, Δt=0.01.
The dotted lines indicate that the relevant tables are to be joined up).

		00			
		0101	0102	0201	0202
10	1101	0.22	-0.15	-0.016667	0
	1102	-0.125	0.3334615	-0.001282	-0.015385
	1201	-0.125	-0.011538	0.3437179	-0.138462
	1202	0.01	-0.136648	-0.135183	0.469011
	2101	0.1	-0.068182	-0.007576	0
	2102	-0.05	0.1420189	-8.87E-04	-0.006689
	2201	-0.05	-0.007981	0.1491132	-0.060201
	2202	4.16E-17	-0.051352	-0.05015	0.1926342

	11															
	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
1101	0	0	0	0	0	0	0	0	-1.666667	-4.44E-16	-4	-4.44E-15	4.44E-16	-2.22E-16	-1.78E-15	8.88E-16
1102	0	0	8.88E-16	0	0	0	0	0	-0.128205	-1.538462	4.44E-15	-4	-1.11E-15	-4.44E-16	1.78E-15	-8.88E-16
1201	0	0	8.88E-16	0	0	0	0	2.78E-16	-0.128205	-2.78E-16	3.55E-15	3.33E-16	-1.538462	3.33E-16	-4	-6.66E-16
1202	0	0	-1.11E-16	0	0	0	-3.33E-16	-4.44E-16	-0.018315	-0.10989	-6.66E-16	1.94E-15	-0.10989	-1.428571	2.83E-15	-4
2101	0	0	0	0	0	0	0	0	-0.757576	0	-0.909091	-8.88E-16	4.44E-16	0	4.44E-16	4.44E-16
2102	0	0	4.44E-16	0	0	0	0	0	-0.088679	-0.668896	-0.039526	-0.869565	-4.44E-16	-1.11E-16	8.88E-16	-2.22E-16
2201	0	0	3.33E-16	0	0	0	0	1.11E-16	-0.088679	-1.94E-16	-0.039526	0	-0.668896	1.67E-16	-0.869565	-1.67E-16
2202	0	0	-2.78E-17	0	0	0	-1.11E-16	-1.11E-16	-0.015021	-0.073658	-0.003294	-0.036232	-0.073658	-0.595238	-0.036232	-0.833333

	02							
	0111	0112	0121	0122	0211	0212	0221	0222
1101	0	0	-1.666667	-4.44E-16	0	0	4.44E-16	-2.22E-16
1102	0	0	-0.128205	-1.538462	0	0	-1.11E-15	-4.44E-16
1201	0	0	-0.128205	-2.78E-16	0	0	-1.538462	3.33E-16
1202	0	0	-0.018315	-0.10989	0	0	-0.10989	-1.428571
2101	0	0	-0.757576	0	0	0	4.44E-16	0
2102	0	0	-0.088679	-0.668896	0	0	-4.44E-16	-1.11E-16
2201	0	0	-0.088679	-1.94E-16	0	0	-0.668896	1.67E-16
2202	0	0	-0.015021	-0.073658	0	0	-0.073658	-0.595238

Similarly from the set of equations

$$\{P_{i_1 j_1 i_2 j_2}[n-1] \mid \{P\,[0]\},\ \{P\,[n+1]\}\} \tag{2.3.10}$$

and the set of equations

$$\{P\,[n-1],\ \ P\,[n],\ \ P\,[n+1]\} \tag{2.3.11}$$

we get

$$\{P_{i_1 j_1 i_2 j_2}[n] \mid \{P\,[0]\},\ \{P\,[n+1]\}\} \quad ,\ \ n=3,\ 4,\ \ldots \tag{2.3.12}$$

When n = N is large enough, we may set each member of $P\,[N+1]$ to be zero to obtain

$$\{P_{i_1 j_1 i_2 j_2}[N] \mid \{P\,[0]\},\ \{P\,[N+1]\}\} \cong \{P_{i_1 j_1 i_2 j_2}[N] \mid \{P\,[0]\}\} \tag{2.3.13}$$

for n = N–1, N–2, …,1.

Substituting the left side of (2.3.13) into (2.3.12) when n = N –1, we get

$$\{P_{i_1 j_1 i_2 j_2}[N-1] \mid \{P\,[0]\},\ \{P\,[N]\}\} \cong \{P_{i_1 j_1 i_2 j_2}[N-1] \mid \{P\,[0]\}\,. \tag{2.3.14}$$

Similarly for n = N–2, N–3, N–4, …,1 we may repeat the substitution of $P\,[n+1]$ into (2.3.12) and obtain

$$\{P_{i_1 j_1 i_2 j_2}[n] \mid \{P\,[0]\}\} \tag{2.3.15}$$

When n = 1, we get from (2.3.15)

$$\{P_{i_1 j_1 i_2 j_2}[1] \mid \{P\,[0]\}\}\,. \tag{2.3.16}$$

Substituting the left side of (2.3.16) into (2.3.1), we get

$$\{P_{i_1 j_1 i_2 j_2}[0] \mid \{P\,[0]\}\}\,. \tag{2.3.17}$$

An inspection of (2.3.17) reveals that among the 4 equations represented by (2.3.17), only 3 of them are linearly independent of each other. Hence, we need to include another linearly independent equation so that the resulting system of equations has a unique solution. Equating the sum of the left sides of (2.3.13) to (2.3.15) to the sum of the right sides of (2.3.13) to (2.3.15), we get an equation of the form

$$\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{n_1=0}^{N}\sum_{\substack{n_2=0\\ n_1+n_2\geq 1}}^{N} P_{i_1 j_1 i_2 j_2}[n_1][n_2] = \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2} C_{i_1 j_1 i_2 j_2} P_{i_1 j_1 i_2 j_2}[0][0]$$

where the $C_{i_1 j_1 i_2 j_2}$ are constants, or

$$1-\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2} P_{i_1 j_1 i_2 j_2}[0][0] = \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2} C_{i_1 j_1 i_2 j_2} P_{i_1 j_1 i_2 j_2}[0][0] \quad (2.3.18)$$

Equation (2.3.18) together with 3 equations chosen from (2.3.17) will form a set of equations in 4 unknowns. Solving the set of 4 equations, we get the numerical answers for the $P_{i_1 j_1 i_2 j_2}[0][0]$.

Then, from (2.3.15) and the values of the $P_{i_1 j_1 i_2 j_2}[0][0]$, we can get the numerical answers for $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$, for the case when $n_1 + n_2 \leq N$.

Some of the ideas involved in the above method for solving the balance equations are similar to those used by Koh. et al (2011).

2.4 SIMULATED VALUE OF $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$

The probability $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ may also be estimated by using a simulation procedure described below.

Let N_T >0 be an integer such that $N_T\Delta t$ corresponds to a time which is "very long" after t = 0. The approximate probability that an event from set A will occur in τ_k given the conditions of the system at the end of τ_{k-1} is summarized in Table 2.4.1.

Suppose that at the end of τ_1,

$$\mathbf{h}^{(1)} = (1,\ 1,\ 1,\ 1,\ 1,\ 1). \tag{2.4.1}$$

Then the events which may occur in τ_2 are A_1, A_2, A_5, A_6 and A_9. The probability ($P_1^{(1)}, P_2^{(1)}$, ..., or $P_5^{(1)}$) of each of these events and the resulting value $\mathbf{h}^{(2)}$ are as shown in Table 2.4.2.

To generate $\mathbf{h}^{(2)}$, we may first generate a random number $U^{(1)}$ from the U(0,1) distribution. Let $P_0^{(1)} = 0$. If $\sum_{i=0}^{j-1} P_i^{(1)} < U^{(1)} < \sum_{i=0}^{j} P_i^{(1)}$, then event $A^{(2)} = E_j^{(1)}$ is said to have occurred, j=1, 2, ..., 5 and the resulting $\mathbf{h}^{(2)}$ is as given in Table 2.4.2.

Similarly given a value of $\mathbf{h}^{(k-1)}$, we may first find out the set of possible events $E_1^{(k-1)}$, $E_2^{(k-1)}$, ..., $E_L^{(k-1)}$ which can occur in τ_k. Suppose $E_i^{(k-1)}$ occurs with probability $P_i^{(k-1)}$. To generate $\mathbf{h}^{(k)}$, we may first generate a random number $U^{(k-1)}$ from the U(0,1) distribution. Let $P_0^{(k-1)} = 0$. If $\sum_{i=0}^{j-1} P_i^{(k-1)} < U^{(k-1)} < \sum_{i=0}^{j} P_i^{(k-1)}$, then event $A^{(k)} = E_j^{(k-1)}$ is said to have occurred and the resulting $\mathbf{h}^{(k)}$ can be determined.

In short, we generate $\left(A^{(2)}, A^{(3)}, \Lambda\ , A^{(N_T)}\right)$ starting from $\mathbf{h}^{(1)}$ given by (2.4.1). We repeat the generation of $\left(A^{(2)}, A^{(3)}, \Lambda\ , A^{(N_T)}\right)$ for N_s number of times. The probability

$P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ is then given approximately by the proportion of times the vector of characteristics given by $\mathbf{h} = (i_1, \; j_1, \; i_2, \; j_2, \; n_1, \; n_2)$ is obtained at $t = N_T \Delta t$.

Table 2.4.1 : The approximate probability that event A will occur in τ_k given the conditions of system at the end of τ_{k-1} [v_i =1 or 2, 1 ≤ i ≤ 4].

Conditions of system at the end of τ_{k-1}	Event occurring in τ_k	Approximate probability that event in Column 2 will occur given the conditions in Column 1.
$i_1^{(k-1)} = 1$	A_1	$\mu_{11}\Delta t$
$i_1^{(k-1)} = 2$	A_1	$\mu_{12}\Delta t$
$j_1^{(k-1)} = 1$	A_2	$\lambda_{11}\Delta t$
$j_1^{(k-1)} = 2$	A_3	$\lambda_{12}\Delta t q_{11}$
$j_1^{(k-1)} = 2$	A_4	$\lambda_{12}\Delta t q_{12}$
$i_2^{(k-1)} = 1$	A_5	$\mu_{21}\Delta t$
$i_2^{(k-1)} = 2$	A_5	$\mu_{22}\Delta t$
$j_2^{(k-1)} = 1$	A_6	$\lambda_{21}\Delta t$
$j_2^{(k-1)} = 2$	A_7	$\lambda_{22}\Delta t q_{22}$
$j_2^{(k-1)} = 2$	A_8	$\lambda_{22}\Delta t q_{21}$
$i_1^{(k-1)} = v_1, j_1^{(k-1)} = v_2,$ $i_2^{(k-1)} = v_3, j_2^{(k-1)} = v_4.$	A_9	$1-(\mu_{1v_1} + \lambda_{1v_2} + \mu_{2v_3} + \lambda_{2v_4})\Delta t$
$i_1^{(k-1)} = 0, j_1^{(k-1)} = v_2,$ $i_2^{(k-1)} = v_3, j_2^{(k-1)} = v_4.$	A_{10}	$1-(\lambda_{1v_2} + \mu_{2v_3} + \lambda_{2v_4})\Delta t$
$i_1^{(k-1)} = 0, j_1^{(k-1)} = 1$	A_{11}	$\lambda_{11}\Delta t$
$i_1^{(k-1)} = 0, j_1^{(k-1)} = 2$	A_{12}	$\lambda_{12}\Delta t q_{11}$
$i_1^{(k-1)} = 0, j_1^{(k-1)} = 2$	A_{13}	$\lambda_{12}\Delta t q_{12}$
$i_1^{(k-1)} = 0, i_2^{(k-1)} = 1$	A_{14}	$\mu_{21}\Delta t$
$i_1^{(k-1)} = 0, i_2^{(k-1)} = 2$	A_{14}	$\mu_{22}\Delta t$
$i_1^{(k-1)} = 0, j_2^{(k-1)} = 1$	A_{15}	$\lambda_{21}\Delta t$
$i_1^{(k-1)} = 0, j_2^{(k-1)} = 2$	A_{16}	$\lambda_{22}\Delta t q_{22}$
$i_1^{(k-1)} = 0, j_2^{(k-1)} = 2$	A_{17}	$\lambda_{22}\Delta t q_{21}$

Table 2.4.1, continued

Conditions of system at the end of τ_{k-1}	Event occurring in τ_k	Approximate probability that event in Column 2 occur will given the conditions in Column 1.
$i_1^{(k-1)}=1, i_2^{(k-1)}=0$	A_{18}	$\mu_{11}\Delta t$
$i_1^{(k-1)}=2, i_2^{(k-1)}=0$	A_{18}	$\mu_{12}\Delta t$
$j_1^{(k-1)}=1, i_2^{(k-1)}=0$	A_{19}	$\lambda_{11}\Delta t$
$j_1^{(k-1)}=2, i_2^{(k-1)}=0$	A_{20}	$\lambda_{12}\Delta t q_{11}$
$j_1^{(k-1)}=2, i_2^{(k-1)}=0$	A_{21}	$\lambda_{12}\Delta t q_{12}$
$i_1^{(k-1)}=v_1, j_1^{(k-1)}=v_2,$ $i_2^{(k-1)}=0, j_2^{(k-1)}=v_4.$	A_{22}	$1-(\mu_{1v_1}+\lambda_{1v_2}+\lambda_{2v_4})\Delta t$
$i_2^{(k-1)}=0, j_2^{(k-1)}=1$	A_{23}	$\lambda_{21}\Delta t$
$i_2^{(k-1)}=0, j_2^{(k-1)}=2$	A_{24}	$\lambda_{22}\Delta t q_{22}$
$i_2^{(k-1)}=0, j_2^{(k-1)}=2$	A_{25}	$\lambda_{22}\Delta t q_{21}$
$i_1^{(k-1)}=0, j_1^{(k-1)}=1,$ $i_2^{(k-1)}=0.$	A_{26}	$\lambda_{11}\Delta t$
$i_1^{(k-1)}=0, j_1^{(k-1)}=2,$ $i_2^{(k-1)}=0.$	A_{27}	$\lambda_{12}\Delta t q_{11}$
$i_1^{(k-1)}=0, j_1^{(k-1)}=2,$ $i_2^{(k-1)}=0.$	A_{28}	$\lambda_{12}\Delta t q_{12}$
$i_1^{(k-1)}=0, i_2^{(k-1)}=0,$ $j_2^{(k-1)}=1.$	A_{29}	$\lambda_{21}\Delta t$
$i_1^{(k-1)}=0, i_2^{(k-1)}=0,$ $j_2^{(k-1)}=2.$	A_{30}	$\lambda_{22}\Delta t q_{22}$
$i_1^{(k-1)}=0, i_2^{(k-1)}=0,$ $j_2^{(k-1)}=2.$	A_{31}	$\lambda_{22}\Delta t q_{21}$
$i_1^{(k-1)}=0, j_1^{(k-1)}=v_2,$ $i_2^{(k-1)}=0, j_2^{(k-1)}=v_4.$	A_{32}	$1-(\lambda_{1v_2}+\lambda_{2v_4})\Delta t$

Table 2.4.2 : The approximate probability that event A will occur in τ_2 given the conditions of system at the end of τ_1.

i	Event, $E_i^{(1)}$	Probability, $P_i^{(1)}$	$\mathbf{h}^{(2)}$
1	$E_1^{(1)} = \mathrm{A}_1$	$P_1^{(1)} = \mu_{11}\Delta t$	(2, 1, 1, 1, 1, 1)
2	$E_2^{(1)} = \mathrm{A}_2$	$P_2^{(1)} = \lambda_{11}\Delta t$	(1, 2, 1, 1, 1, 1)
3	$E_3^{(1)} = \mathrm{A}_5$	$P_3^{(1)} = \mu_{21}\Delta t$	(1, 1, 2, 1, 1, 1)
4	$E_4^{(1)} = \mathrm{A}_6$	$P_4^{(1)} = \lambda_{21}\Delta t$	(1, 1, 1, 2, 1, 1)
5	$E_5^{(1)} = \mathrm{A}_9$	$P_5^{(1)} = 1-(\mu_{11}+\lambda_{11}+\mu_{21}+\lambda_{21})\Delta t$	(1, 1, 1, 1, 1, 1)

2.5 NUMERICAL RESULTS FOR DISTRIBUTION OF QUEUE LENGTH AND STATES OF ARRIVAL AND SERVICE PROCESSES IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES

Suppose $(\mu_{11},\mu_{12},\mu_{21},\mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22})$ = (1, 2, 1, 2), and q_{11} = q_{22} = 0.9. The traffic intensities (ρ_1,ρ_2) in the two queues are then given respectively by

$$\rho_1 = (\mu_{11}{}^{-1} + \mu_{12}{}^{-1}) / (\lambda_{11}{}^{-1} + \lambda_{12}{}^{-1}) = 0.1 \ ,$$

$$\rho_2 = (\mu_{21}{}^{-1} + \mu_{22}{}^{-1}) / (\lambda_{21}{}^{-1} + \lambda_{22}{}^{-1}) = 0.1 \ .$$

By setting $P_{i_1 j_1 i_2 j_2}[n_1][n_2] = 0$ when $n_1 + n_2 = 10$, the probabilities $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ for $n_1 + n_2 \le 2$ computed by using the numerical method in Section 2.3 and the simulation procedure in Section 2.4 are presented in Table 2.5.1 and Figure 2.5.1.

Next, Table 2.5.2 and Figure 2.5.2 show the results for the case when $(\mu_{11},\mu_{12},\mu_{21},\mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22})$ = (5, 12, 5, 12), q_{11} = q_{22} = 0.9, N = 9, and the traffic intensities are $\rho_1 = 0.5294$, $\rho_2 = 0.5294$.

The tables and figures in this section show that the results for $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ found by the proposed method agree well with those found by the simulation procedure.

Table 2.5.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ based on the proposed method and simulation procedure [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (1, 2, 1, 2), $q_{11} = q_{22} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $N = 9$ and $N_s = 5000$].

$n_1 n_2$	$i_1 j_1 i_2 j_2$	Proposed method	Simulation procedure
00	0101	0.330503	0.328800
	0102	0.186519	0.185400
	0201	0.186519	0.187000
	0202	0.105157	0.105200
01	0111	0.034938	0.039000
	0112	0.004675	0.004600
	0121	0.016525	0.012000
	0122	0.002834	0.002800
	0211	0.017978	0.018800
	0212	0.001686	0.001600
	0221	0.008618	0.009200
	0222	0.001190	0.001400
10	1101	0.034938	0.034600
	1102	0.017978	0.019800
	1201	0.004675	0.003000
	1202	0.001686	0.001200
	2101	0.016525	0.020600
	2102	0.008618	0.006200
	2201	0.002834	0.002000
	2201	0.001190	0.001000
02	0111	0.001083	0.001200
	0112	0.000117	0.000000
	0121	0.000823	0.001000
	0122	0.000099	0.000200
	0211	0.000331	0.001000
	0212	0.000033	0.000200
	0221	0.000275	0.000000
	0222	0.000030	0.000000
20	1101	0.001083	0.000800
	1102	0.000331	0.000000
	1201	0.000117	0.000000
	1202	0.000033	0.000000
	2101	0.000823	0.000600
	2102	0.000275	0.000600
	2201	0.000099	0.000200
	2201	0.000030	0.000200
11	1111	0.003111	0.002600
	1112	0.000289	0.000600
	1121	0.001488	0.002200
	1122	0.000199	0.000000
	1211	0.000289	0.000600
	1212	0.000025	0.000000
	1221	0.000141	0.000200
	1222	0.000017	0.000000
	2111	0.001488	0.001800
	2112	0.000141	0.000200
	2121	0.000709	0.000800
	2122	0.000096	0.000000
	2211	0.000199	0.000200
	2212	0.000017	0.000000
	2221	0.000096	0.000200
	2222	0.000012	0.000000
	Total	**0.999487**	**0.999600**

Table 2.5.2 : Comparison of results for $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ based on the proposed method and simulation procedure [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (5, 12, 5, 12), $q_{11} = q_{22} = 0.9$, $\rho_1 = 0.5294$, $\rho_2 = 0.5294$, $N = 9$ and $N_s = 5000$].

$n_1 n_2$	$i_1 j_1 i_2 j_2$	Proposed method	Simulation procedure
00	0101	0.085722	0.089600
	0102	0.049541	0.052700
	0201	0.049541	0.052600
	0202	0.028507	0.034100
01	0111	0.053393	0.064350
	0112	0.017359	0.018200
	0121	0.021215	0.024900
	0122	0.008626	0.009150
	0211	0.030041	0.034400
	0212	0.008927	0.008950
	0221	0.012053	0.012450
	0222	0.004719	0.004750
10	1101	0.053393	0.062050
	1102	0.030041	0.033400
	1201	0.017359	0.018450
	1202	0.008927	0.010000
	2101	0.021215	0.025700
	2102	0.012053	0.013950
	2201	0.008626	0.011400
	2201	0.004719	0.005500
02	0111	0.020311	0.019350
	0112	0.006221	0.006700
	0121	0.012143	0.010600
	0122	0.003932	0.002700
	0211	0.010307	0.010300
	0212	0.003149	0.002950
	0221	0.006341	0.007050
	0222	0.002023	0.001600
20	1101	0.020311	0.018100
	1102	0.010307	0.008450
	1201	0.006221	0.004850
	1202	0.003149	0.002800
	2101	0.012143	0.011850
	2102	0.006341	0.005250
	2201	0.003932	0.002800
	2201	0.002023	0.001950
11	1111	0.031858	0.035550
	1112	0.009348	0.009000
	1121	0.012816	0.015000
	1122	0.004859	0.004500
	1211	0.009348	0.009550
	1212	0.002651	0.002800
	1221	0.003832	0.003900
	1222	0.001410	0.001900
	2111	0.012816	0.015900
	2112	0.003832	0.003850
	2121	0.005127	0.005750
	2122	0.001974	0.001800
	2211	0.004859	0.004650
	2212	0.001410	0.001300
	2221	0.001974	0.002500
	2222	0.000749	0.000600
	Total	**0.994070**	**0.997100**

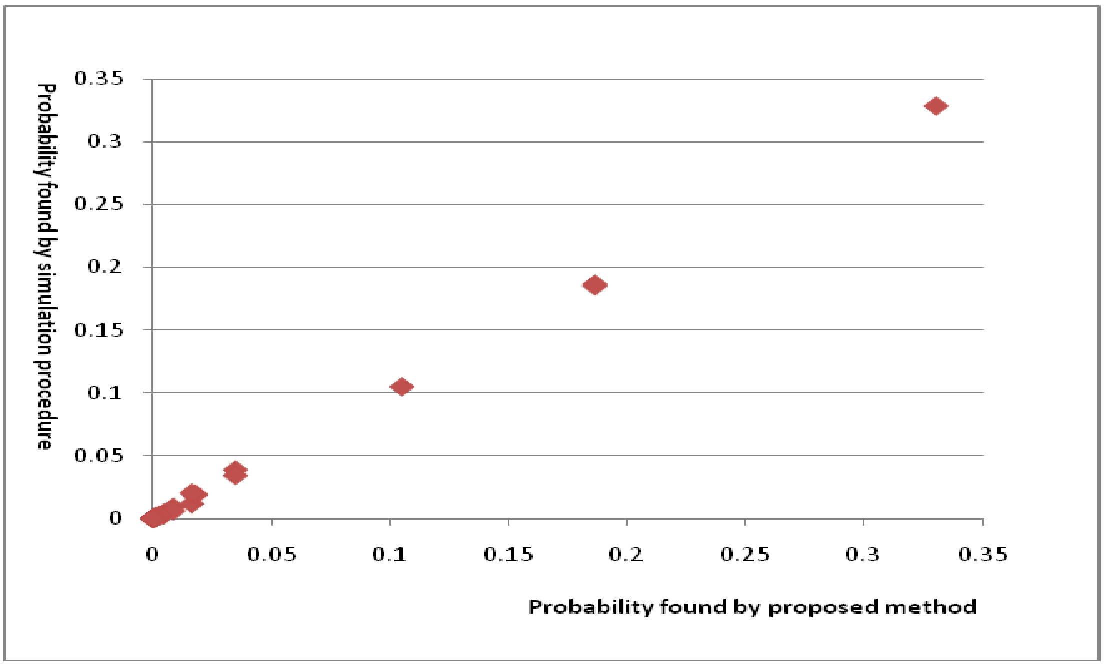

Figure 2.5.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ based on the proposed method and simulation procedure [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (1, 2, 1, 2), $q_{11} = q_{22} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $N = 9$ and $N_s = 5000$].

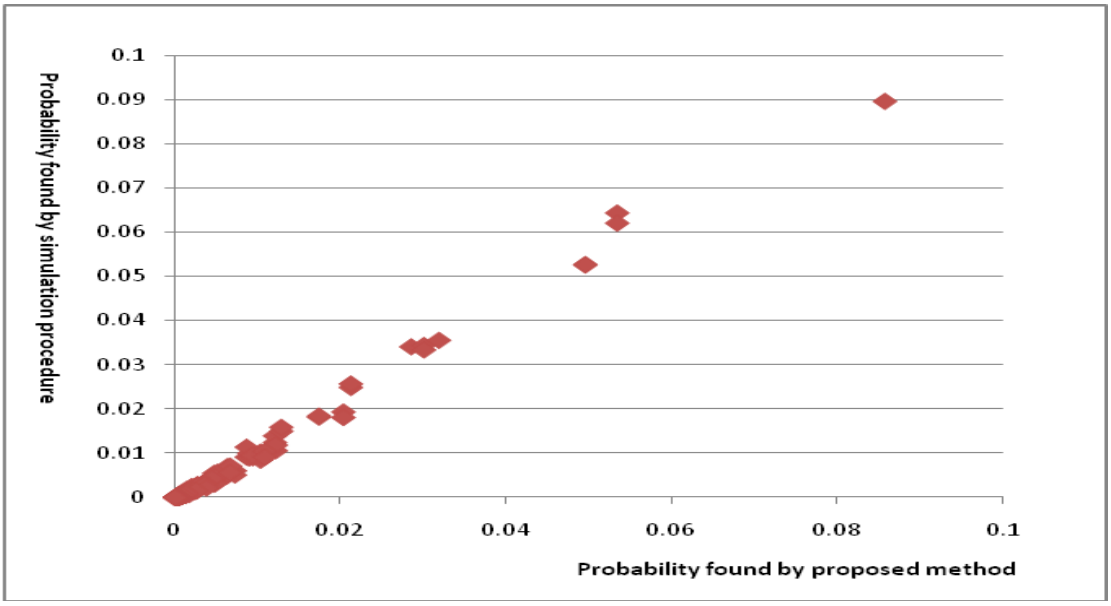

Figure 2.5.2 : Comparison of results for $P_{i_1 j_1 i_2 j_2}[n_1][n_2]$ based on the proposed method and simulation procedure [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (5, 12, 5, 12), $q_{11} = q_{22} = 0.9$, $\rho_1 = 0.5294$, $\rho_2 = 0.5294$, $N = 9$ and $N_s = 5000$].