2.6 DERIVATION OF THE FORWARD EQUATIONS IN A SYSTEM OF THREE DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES

Consider a system of three Hypo(2)/Hypo(2)/1 queues in which the customer who arrives at queue m has a probability of $q_{mm'} \geq 0$ of joining queue m', for $m \in \{1, 2, 3\}$, $m' \in \{1, 2, 3\}$ and $\sum_{m'=1}^{3} q_{mm'} = 1$. An illustration of the possible crossing over to other queues is given in Figure 2.6.1.

As in Section 2.2, let the parameters of the hypoexponential distribution in two phases for the interarrival time T in queue m be λ_{m1}, λ_{m2}. Furthermore, assume that the parameters of the hypoexponential distribution in two phases for the service time S in queue m be μ_{m1}, μ_{m2}.

Similarly, let $\Delta t > 0$ be a small increment in time and $\tau_k = ((k-1)\Delta t, k\Delta t]$ a time interval, $k = 1, 2, \ldots$. Next let $P^{(k)}_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ be the probability that at the end of the interval τ_k, the number of customers in the system is n_m in queue m, $m \in \{1, 2, 3\}$ (In determining the queue size in a given queue, we include the customer that is currently being served), the service process in queue m is in the state i_m and the arrival process in queue m is in the state j_m, $m \in \{1, 2, 3\}$, $i_m \in \{0, 1, 2\}$ and $j_m \in \{1, 2\}$. If queue m is empty, then we may define the state of the service process to be zero. Assume that

$$P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3] = \lim_{k \to \infty} P^{(k)}_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$$

exists.

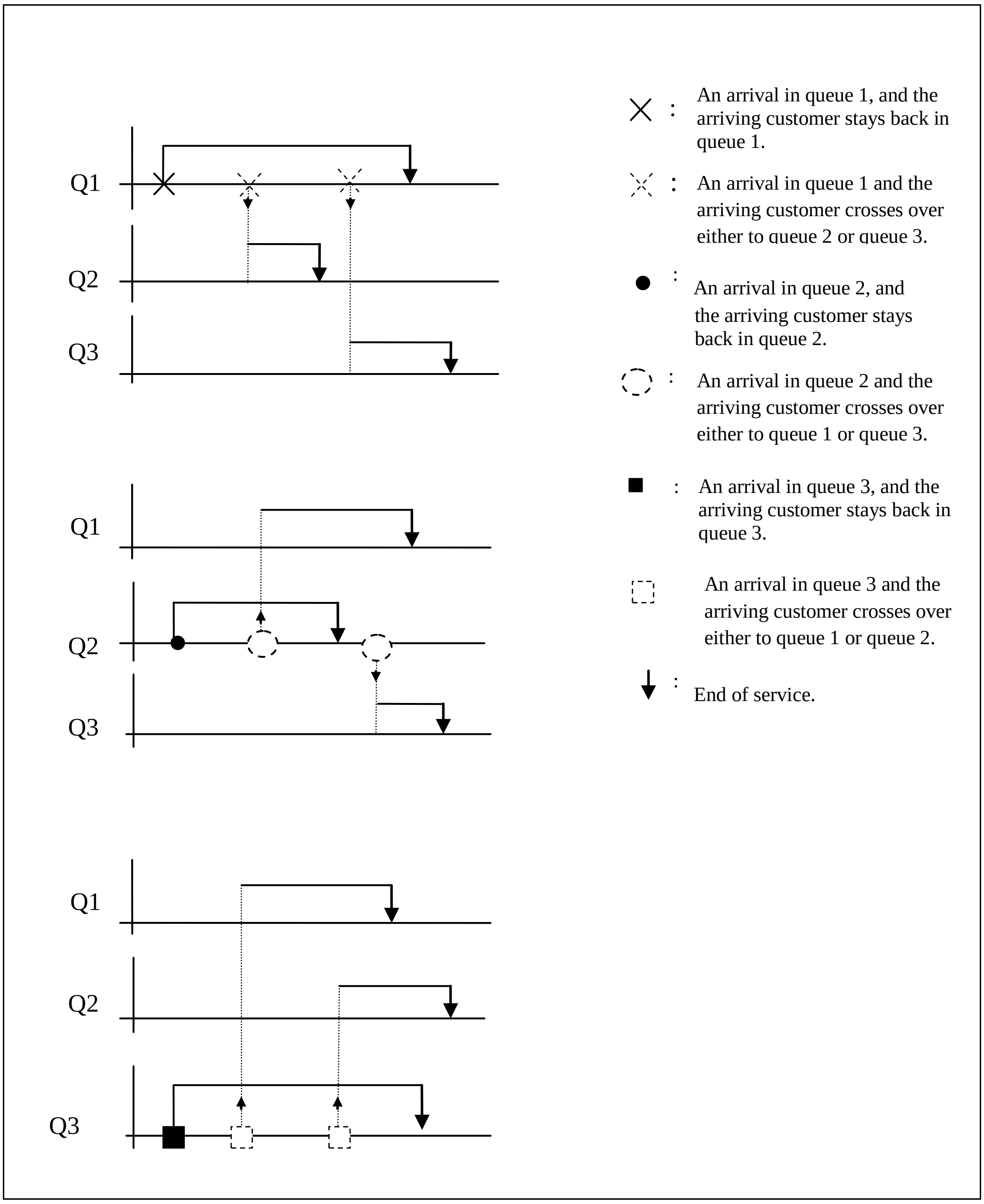

Figure 2.6.1 : Cross over probabilities in a system of three one-server queues.

Let $\mathbf{h}^{(k)}$ be the vector

$$\mathbf{h}^{(k)} = \left(i_1^{(k)},\ j_1^{(k)},\ i_2^{(k)},\ j_2^{(k)},\ i_3^{(k)},\ j_3^{(k)},\ n_1^{(k)},\ n_2^{(k)},\ n_3^{(k)} \right)$$

of which the components are respectively the values of $i_1, j_1, i_2, j_2, i_3, j_3, n_1, n_2, n_3$ at the end of τ_k. Again we refer to $\mathbf{h}^{(k)}$ as the vector of characteristics of the queueing system at the end of τ_k.

The value $\mathbf{h}^{(k)}$ may be developed from $\mathbf{h}^{(k-1)}$ after some appropriate activities in the interval τ_k. The set of possible activities may be denoted by the set A = {A_1, A_2, …, A_{116}}. Some feasible events in A are shown below.

A_1 = (1, 0, 0, 0, 0, 0, –1, –1, –1)

A_2 = (0, 1, 0, 0, 0, 0, –1, –1, –1)

A_3 = (0, 1, 0, 0, 0, 0, 11, –1, –1)

A_4 = (0, 1, 0, 0, 0, 0, 12, –1, –1)

A_5 = (0, 1, 0, 0, 0, 0, 13, –1, –1)

A_6 = (0, 0, 1, 0, 0, 0, –1, –1, –1)

A_7 = (0, 0, 0, 1, 0, 0, –1, –1, –1)

A_8 = (0, 0, 0, 1, 0, 0, –1, 21, –1)

A_9 = (0, 0, 0, 1, 0, 0, –1, 22, –1)

A_{10} = (0, 0, 0, 1, 0, 0, –1, 23, –1)

A_{11} = (0, 0, 0, 0, 1, 0, –1, –1, –1)

A_{12} = (0, 0, 0, 0, 0, 1, –1, –1, –1)

A_{13} = (0, 0, 0, 0, 0, 1, –1, –1, 31)

A_{14} = (0, 0, 0, 0, 0, 1, –1, –1, 32)

A_{15} = (0, 0, 0, 0, 0, 1, –1, –1, 33)

A_{16} = (0, 0, 0, 0, 0, 0, –1, –1, –1)

The positions, values and meanings of the components in A are explained in Table 2.6.1.

Table 2.6.1 : The positions, values and meanings of the components in A_w.

Position of component	**Value of Component**	**Meaning**
1	1	A transition in the state of the service process in queue 1 occurs in τ_k.
1	0	A transition in the state of the service process in queue 1 does not occur in τ_k.
1	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of the service process in queue 1 has occurred in τ_k is not relevant.
2	1	A transition in the state of the arrival process in queue 1 occurs in τ_k.
2	0	A transition in the state of the arrival process in queue 1 does not occur in τ_k.
3	1	A transition in the state of the service process in queue 2 occurs in τ_k.
3	0	A transition in the state of the service process in queue 2 does not occur in τ_k.
3	–1	Queue 2 is empty at the end of τ_{k-1} and whether a transition in the state of the service process in queue 2 has occurred in τ_k is not relevant.
4	1	A transition in the state of the arrival process in queue 2 occurs in τ_k.
4	0	A transition in the state of the arrival process in queue 2 does not occur in τ_k.
5	1	A transition in the state of the service process in queue 3 occurs in τ_k.
5	0	A transition in the state of the service process in queue 3 does not occur in τ_k.
5	–1	Queue 3 is empty at the end of τ_{k-1} and whether a transition in the state of the service process in queue 3 has occurred in τ_k is not relevant.
6	1	A transition in the state of the arrival process in queue 3 occurs in τ_k.
6	0	A transition in the state of the arrival process in queue 3 does not occur in τ_k.

The meanings of the seventh, eighth and ninth components (A_{w7} and A_{w8} and A_{w9}) of A_w are explained below:

$$A_{w7} = \begin{cases} 11, & \text{if the arriving customer in queue 1 stays back in queue 1.} \\ 12, & \text{if the arriving customer in queue 1 goes to queue 2.} \\ 13, & \text{if the arriving customer in queue 1 goes to queue 3.} \\ -1, & \text{no customers arrive in queue 1 and it is not relevant to find out whether} \\ & \text{the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w8} = \begin{cases} 21, & \text{if the arriving customer in queue 2 goes to queue 1.} \\ 22, & \text{if the arriving customer in queue 2 stays back in queue 2.} \\ 23, & \text{if the arriving customer in queue 2 goes to queue 3.} \\ -1, & \text{no customers arrive in queue 2 and it is not relevant to find out whether} \\ & \text{the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w9} = \begin{cases} 31, & \text{if the arriving customer in queue 3 goes to queue 1.} \\ 32, & \text{if the arriving customer in queue 3 goes to queue 2.} \\ 33, & \text{if the arriving customer in queue 3 stays back in queue 3.} \\ -1, & \text{no customers arrive in queue 3 and it is not relevant to find out whether} \\ & \text{the arriving customer is staying back or going elsewhere.} \end{cases}$$

The complete set of feasible events of A is shown in the Appendix B.

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search for all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form. An example of the codes is as follows:

Table 2.6.2 : An example of the codes of $\mathbf{h}^{(k)}, \mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k .

$\mathbf{h}^{(k)}$	$\mathbf{h}^{(k-1)}$	**Power**		
1 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1	1 0 1 0 1 0 1 0 1 0 1 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1	2 1 1 1 1 1 2 1 1	0 0 1 0 1 0 1 0 1 0 1 0	1 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1	0 2 1 1 1 1 0 1 1	0 0 0 1 1 0 1 0 1 0 1 0	0 0 0 1 0 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1	1 2 0 1 1 1 1 0 1	1 0 0 1 0 0 1 0 1 0 1 0	0 0 0 1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1	1 2 1 1 0 1 1 1 0	1 0 0 1 1 0 1 0 0 0 1 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1	1 1 2 1 1 1 1 2 1	1 0 1 0 0 0 1 0 1 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1	0 1 1 2 1 1 0 1 1	0 0 1 0 1 0 0 0 1 0 1 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 1 0 0 0 0 0
1 1 1 1 1 1 1 1 1	1 1 0 2 1 1 1 0 1	1 0 1 0 0 0 0 0 1 0 1 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0 0
1 1 1 1 1 1 1 1 1	1 1 1 2 0 1 1 1 0	1 0 1 0 1 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0
1 1 1 1 1 1 1 1 1	1 1 1 1 2 1 1 1 2	1 0 1 0 1 0 1 0 0 0 1 0	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1	0 1 1 1 1 2 0 1 1	0 0 1 0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 1 0 0
1 1 1 1 1 1 1 1 1	1 1 0 1 1 2 1 0 1	1 0 1 0 0 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 1 0
1 1 1 1 1 1 1 1 1	1 1 1 1 0 2 1 1 0	1 0 1 0 1 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1

In Table 2.6.2,

Columns 1 – 9 give the components of $\mathbf{h}^{(k)}$.

Columns 10 – 18 give the components of $\mathbf{h}^{(k-1)}$.

Columns 19 – 51 give respectively the powers of (1-$\mu_{11}\Delta t$), (1-$\mu_{12}\Delta t$), (1-$\lambda_{11}\Delta t$), (1-$\lambda_{12}\Delta t$), (1-$\mu_{21}\Delta t$), (1-$\mu_{22}\Delta t$), (1-$\lambda_{21}\Delta t$), (1-$\lambda_{22}\Delta t$), (1-$\mu_{31}\Delta t$), (1-$\mu_{32}\Delta t$), (1-$\lambda_{31}\Delta t$), (1-$\lambda_{32}\Delta t$), ($\mu_{11}\Delta t$), ($\mu_{12}\Delta t$), ($\lambda_{11}\Delta t$), ($\lambda_{12}\Delta t$), ($\mu_{21}\Delta t$), ($\mu_{22}\Delta t$), ($\lambda_{21}\Delta t$), ($\lambda_{22}\Delta t$), ($\mu_{31}\Delta t$), ($\mu_{32}\Delta t$), ($\lambda_{31}\Delta t$), ($\lambda_{32}\Delta t$), (q_{11}), (q_{12}), (q_{13}), (q_{21}), (q_{22}), (q_{23}), (q_{31}), (q_{32}) and (q_{33}).

The multiplication of (1-$\mu_{11}\Delta t$), (1-$\mu_{12}\Delta t$), (1-$\lambda_{11}\Delta t$), (1-$\lambda_{12}\Delta t$), (1-$\mu_{21}\Delta t$), (1-$\mu_{22}\Delta t$), (1-$\lambda_{21}\Delta t$), (1-$\lambda_{22}\Delta t$), (1-$\mu_{31}\Delta t$), (1-$\mu_{32}\Delta t$), (1-$\lambda_{31}\Delta t$), (1-$\lambda_{32}\Delta t$), ($\mu_{11}\Delta t$), ($\mu_{12}\Delta t$), ($\lambda_{11}\Delta t$), ($\lambda_{12}\Delta t$), ($\mu_{21}\Delta t$), ($\mu_{22}\Delta t$), ($\lambda_{21}\Delta t$), ($\lambda_{22}\Delta t$), ($\mu_{31}\Delta t$), ($\mu_{32}\Delta t$), ($\lambda_{31}\Delta t$), ($\lambda_{32}\Delta t$), (q_{11}), (q_{12}), (q_{13}), (q_{21}), (q_{22}), (q_{23}), (q_{31}), (q_{32}) , (q_{33}) raised respectively to the corresponding powers will represent the probability of occurrence of the corresponding event which may be represented by an element in A.

The information represented by the above codes may be used to form the following equation:

$$
\begin{aligned}
P^{(k)}_{111111}[1][1][1] \cong\ & P^{(k-1)}_{111111}[1][1][1](1-\mu_{11}\Delta t-\lambda_{11}\Delta t-\mu_{21}\Delta t-\lambda_{21}\Delta t-\mu_{31}\Delta t-\lambda_{31}\Delta t) \\
& + P^{(k-1)}_{211111}[1][1][1](\mu_{12}\Delta t) + P^{(k-1)}_{021111}[0][1][1](\lambda_{12}\Delta t)q_{11} \\
& + P^{(k-1)}_{120111}[1][0][1](\lambda_{12}\Delta t)q_{12} + P^{(k-1)}_{121101}[1][1][0](\lambda_{12}\Delta t)q_{13} \\
& + P^{(k-1)}_{112111}[1][2][1](\mu_{22}\Delta t) + P^{(k-1)}_{011211}[0][1][1](\lambda_{22}\Delta t)q_{21} \\
& + P^{(k-1)}_{110211}[1][0][1](\lambda_{22}\Delta t)q_{22} + P^{(k-1)}_{111201}[1][1][0](\lambda_{22}\Delta t)q_{23} \\
& + P^{(k-1)}_{111121}[1][1][2](\mu_{32}\Delta t) + P^{(k-1)}_{011112}[0][1][1](\lambda_{32}\Delta t)q_{31} \\
& + P^{(k-1)}_{110112}[1][0][1](\lambda_{32}\Delta t)q_{32} + P^{(k-1)}_{111102}[1][1][0](\lambda_{32}\Delta t)q_{33}
\end{aligned}
\tag{2.6.1}
$$

The derivation of Equation (2.6.1) may also be illustrated by Figure 2.6.2.

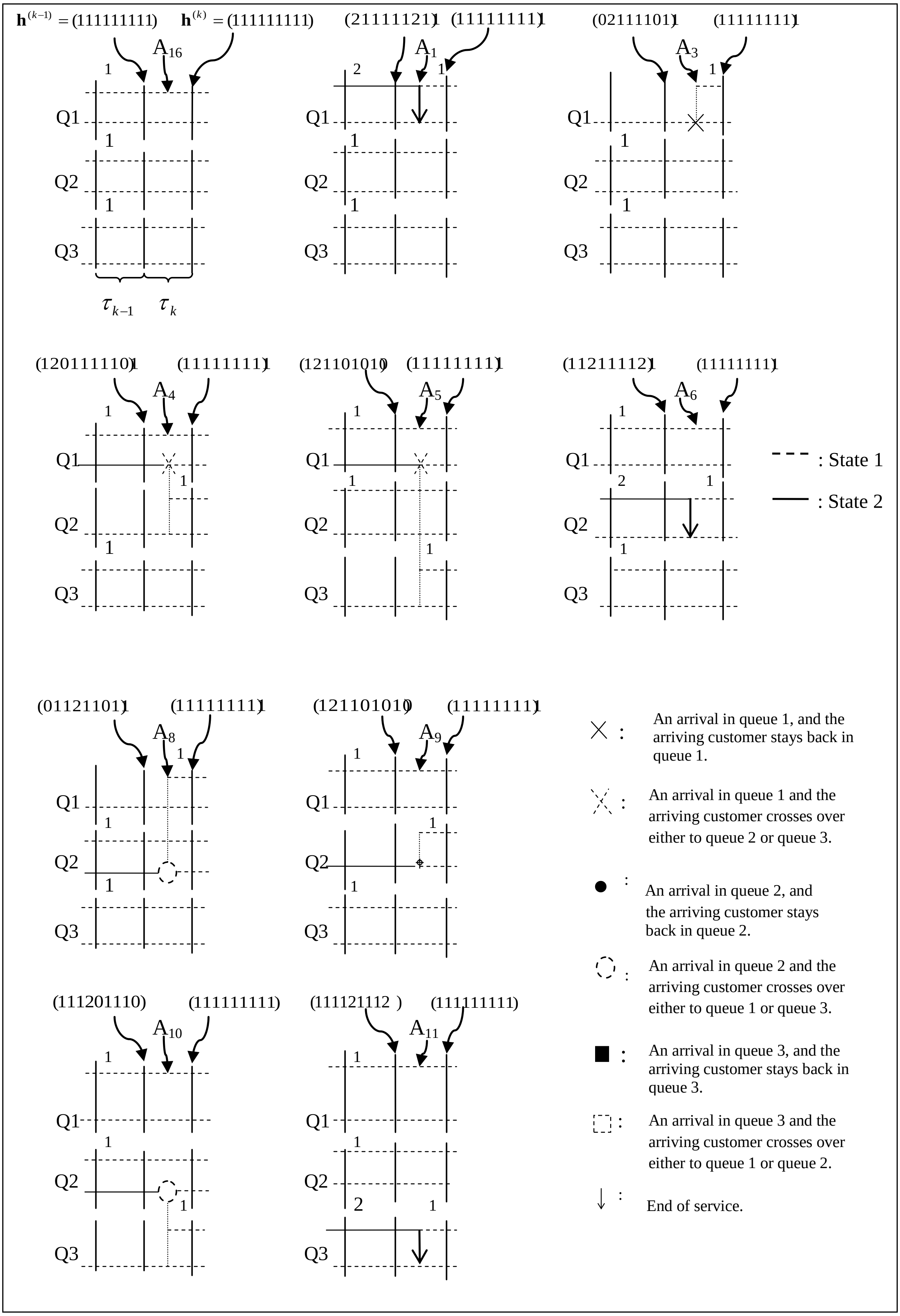

Figure 2.6.2 : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$.

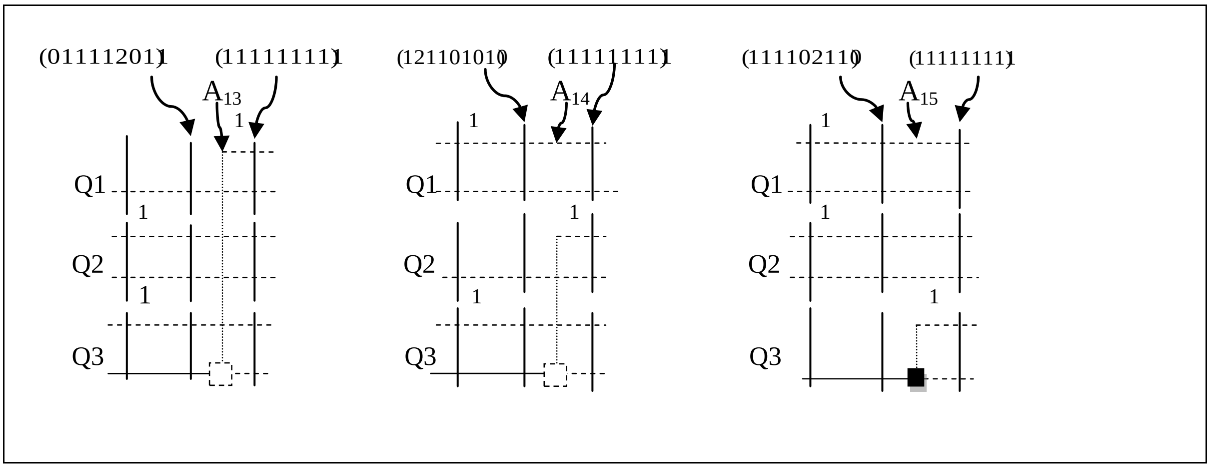

Figure 2.6.2, continued

Subtracting the term $P^{(k-1)}_{111111}[1][1][1]$ from both sides of (2.6.1), dividing both sides of the resulting equation by Δt, and letting $\Delta t \to 0$, and later letting $k \to \infty$, we get the balance equation

$$
\begin{aligned}
0 \cong\ & P_{111111}[1][1][1](-\mu_{11} - \lambda_{11} - \mu_{21} - \lambda_{21} - \mu_{31} - \lambda_{31}) \\
& + P_{211111}[1][1][1]\mu_{12} + P_{021111}[0][1][1]\lambda_{12}q_{11} \\
& + P_{120111}[1][0][1]\lambda_{12}q_{12} + P_{121101}[1][1][0]\lambda_{12}q_{13} \\
& + P_{112111}[1][2][1]\mu_{22} + P_{011211}[0][1][1]\lambda_{22}q_{21} \\
& + P_{110211}[1][0][1]\lambda_{22}q_{22} + P_{111201}[1][1][0]\lambda_{22}q_{23} \\
& + P_{110112}[1][0][1]\lambda_{32}q_{32} + P_{111102}[1][1][0]\lambda_{32}q_{33}
\end{aligned}
$$

(2.6.2)

Equation (2.6.2) may be represented in a coded form as shown in Table 2.6.3.

Table 2.6.3 : Representation of Equation in (2.6.2) by codes.

Constant	**h**									**Power**																				
-1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	1	1	1	1	1	1	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
-1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
1	2	1	1	1	1	1	2	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	1	1	1	1	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
1	1	2	0	1	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	1	2	1	1	0	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	1	1	2	1	1	1	1	2	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	2	1	1	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
1	1	1	0	2	1	1	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
1	1	1	1	2	0	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
1	1	1	1	1	2	1	1	1	2	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	2	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
1	1	1	0	1	1	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
1	1	1	1	1	0	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
1*																														

In each row of Table 2.6.3,

Column 1 gives a coefficient value.

Columns 2 – 10 give the components of $\mathbf{h}$.

Columns 11 – 31 give respectively the powers of (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (μ_{31}), (μ_{32}), (λ_{31}), (λ_{32}), (q_{11}), (q_{12}), (q_{13}), (q_{21}), (q_{22}), (q_{23}), (q_{31}), (q_{32}) and (q_{33}).

The symbol " 1* " in the last row denotes the end of the equation.

For each row in Table 2.6.3, we form a product of

(i) the coefficient in column 1,

(ii) the term $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ of which the values i_1, j_1, i_2, j_2, i_3, j_3, n_1, n_2, n_3 are given by $\mathbf{h}$, and

(iii) the product of (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (μ_{31}), (μ_{32}), (λ_{31}), (λ_{32}), (q_{11}), (q_{12}), (q_{13}), (q_{21}), (q_{22}), (q_{23}), (q_{31}), (q_{32}) and (q_{33}) raised respectively to the corresponding powers.

We then equate the sum of the products for all the rows in Table 2.6.3 to zero to form (2.6.2).

Now if we change $\mathbf{h}^{(k)} = (1, 1, 1, 1, 1, 1, 1, 1, 1)$ to other values and search for the values of $\mathbf{h}^{(k-1)}$ and the component of A which lead to the new value of $\mathbf{h}^{(k)}$, we can similarly obtain an equation alike to (2.6.2). Again the resulting equation can be represented in a coded form. The resulting table of codes for $0 \leq n_1^{(k)} + n_2^{(k)} + n_3^{(k)} \leq 15$ can be found in the file *ThreeQueueSystem_codes.txt* in the CD attached. The first page of *ThreeQueueSystem_codes.txt* is shown in Table 2.6.4. The equations for a given $n_1^{(k)}$, $n_2^{(k)}$ and $n_3^{(k)}$ of which $n_1^{(k)} + n_2^{(k)} + n_3^{(k)} \geq 16$ can be found from the file *ThreeQueueSystem_codes.txt* in the CD by

(i) increasing the nonzero value of $n_1^{(k)}$ in the file *ThreeQueueSystem_codes.txt* to $n_1^{(k)} + 1$, or

(ii) increasing the nonzero value of $n_2^{(k)}$ in the file *ThreeQueueSystem_codes.txt* to $n_2^{(k)} + 1$, or

(iii) increasing the nonzero value of $n_3^{(k)}$ in the file *ThreeQueueSystem_codes.txt* to $n_3^{(k)} + 1$.

Table 2.6.4 : Representation of some balance equations by codes.

Constant	h	Power	
-1	0 1 0 1 0 1 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 0 1 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 0 1 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 2 1 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 1 0 2 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 0 1 0 2 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 0 2 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 2 2 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 1 1 1 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 1 1 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 1 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 1 1 1 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 1 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0
1	0 1 0 1 2 1 0 0 2	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
1*			
-1	0 1 0 1 1 1 0 0 2	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 2	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 2	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 2	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 1 1 1 0 2	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 1 1 0 0 1	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 1 1 1 0 1 2	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 1 1 0 0 1	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0
1	0 1 0 1 2 1 0 0 3	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 2 0 0 1	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
1*			
-1	0 1 0 1 1 1 0 0 3	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 3	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 3	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 3	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 1 1 1 0 3	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 1 1 0 0 2	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 1 1 1 0 1 3	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 1 1 0 0 2	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0
1	0 1 0 1 2 1 0 0 4	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 2 0 0 2	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
1*			
-1	0 1 0 1 1 1 0 0 4	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 4	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 4	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 4	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 1 1 1 0 4	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 1 1 0 0 3	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 1 1 1 0 1 4	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 1 1 0 0 3	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0
1	0 1 0 1 2 1 0 0 5	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 2 0 0 3	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
1*			

2.7 COMPUTATION OF THE VALUE OF $\boldsymbol{P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]}$

Before solving the balance equations to obtain the stationary queue length distribution, we first introduce the following notations. Let

(a) $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n]$ be a value of $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ of which $n_1 + n_2 + n_3 = n$.

(b) $\{P\,[n_1][n_2][n_3]\}$ the set consisting of all the possible $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$.

(c) $\{P\,[n]\}$ a set formed by the $\{P\,[n_1][n_2][n_3]\}$ of which $n_1 + n_2 + n_3 = n$.

(d) $\{P\,[n],\ P\,[n+1],\ P\,[n+2]\}$ the set of equations of the form

$$\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{i_3=0}^{2}\sum_{j_3=1}^{2} a_{i_1 j_1 i_2 j_2 i_3 j_3} P_{i_1 j_1 i_2 j_2 i_3 j_3}[n]$$
$$+\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{i_3=0}^{2}\sum_{j_3=1}^{2} b_{i_1 j_1 i_2 j_2 i_3 j_3} P_{i_1 j_1 i_2 j_2 i_3 j_3}[n+1]$$
$$+\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{i_3=0}^{2}\sum_{j_3=1}^{2} c_{i_1 j_1 i_2 j_2 i_3 j_3} P_{i_1 j_1 i_2 j_2 i_3 j_3}[n+2] = 0$$

where $a_{i_1 j_1 i_2 j_2 i_3 j_3}, b_{i_1 j_1 i_2 j_2 i_3 j_3}$ and $c_{i_1 j_1 i_2 j_2 i_3 j_3}$ are constants.

(e) $\left(P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3] \,\middle|\, \{P\,[0]\},\ \{P\,[n+1]\}\right)$ an equation of the form

$$P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3] = \sum_{i'_1=0}^{2}\sum_{j'_1=1}^{2}\sum_{i'_2=0}^{2}\sum_{j'_2=1}^{2}\sum_{i'_3=0}^{2}\sum_{j'_3=1}^{2} d_{i'_1 j'_1 i'_2 j'_2 i'_3 j'_3} P_{i'_1 j'_1 i'_2 j'_2 i'_3 j'_3}[0]$$
$$+\sum_{i'_1=0}^{2}\sum_{j'_1=1}^{2}\sum_{i'_2=0}^{2}\sum_{j'_2=1}^{2}\sum_{i'_3=0}^{2}\sum_{j'_3=1}^{2} e_{i'_1 j'_1 i'_2 j'_2 i'_3 j'_3} P_{i'_1 j'_1 i'_2 j'_2 i'_3 j'_3}[n+1]$$

where $d_{i'_1 j'_1 i'_2 j'_2 i'_3 j'_3}$ and $e_{i'_1 j'_1 i'_2 j'_2 i'_3 j'_3}$ are constants.

With the above notations, the balance equations given by codes in the file *ThreeQueueSystem_codes.txt* in the CD can be represented as

$$\{P\,[0],\ P\,[1]\} \tag{2.7.1}$$

and

$$\{P\,[n-1],\ P\,[n],\ P\,[n+1]\} \quad ,n = 1,\ 2,\ \ldots \tag{2.7.2}$$

For example the codes of the equations represented by $\{P[0], P[1]\}$ are shown in Table 2.7.1 and the codes of some of the equations in the set of equations represented by $\{P[0], P[1], P[2]\}$ are shown in Table 2.7.2. The complete set of equations given by $\{P[0], P[1], P[2]\}$ is shown in Appendix C.

Table 2.7.1 : The codes for the equations represented by $\{P[0], P[1]\}$.

Constant	h	Power	
-1	0 1 0 1 0 1 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 0 1 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 0 1 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 2 1 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 1 0 2 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 0 1 0 2 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 0 2 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 2 2 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 2 0 1 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 0 1 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 2 0 1 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 2 0 1 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 2 1 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 2 0 2 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 0 2 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 0 2 0 2 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 2 0 2 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 2 2 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			

Table 2.7.1, continued

Constant	h	Power	
-1	0 2 0 1 0 1 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 1 0 1 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 2 0 1 0 1 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 1 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 2 1 0 1 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 2 1 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 2 0 1 0 2 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 1 0 2 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 1 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 2 0 1 0 2 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 2 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 2 1 0 2 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 2 2 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 2 0 2 0 1 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 2 0 1 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 2 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 2 0 2 0 1 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 1 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 2 2 0 1 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 1 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 2 2 1 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 2 0 2 0 2 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 2 0 2 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 2 0 2 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 2 0 2 0 2 1 0 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 2 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 2 2 0 2 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 2 0 0 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 2 2 2 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 2 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			

Table 2.7.2 : The codes for some equations represented by $\{P[0], P[1], P[2]\}$.

Constant	h	Power	
-1	0 1 0 1 1 1 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 1 1 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 1 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 1 1 1 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 1 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0
1	0 1 0 1 2 1 0 0 2	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
1*			
-1	0 1 0 1 1 2 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 2 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 2 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 1 2 0 0 1	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 0 1 1 2 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 2 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 1 1 2 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 2 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 1 0 0 0
1	0 1 0 1 2 2 0 0 2	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 1 2 1 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 2 1 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 2 1 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 2 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 1 2 1 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 2 1 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 1 2 2 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 2 2 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 2 2 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 1 2 2 0 0 1	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 0 1 2 2 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 2 2 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 2 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 2 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 2 1 1 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 1 1 0 0 1	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 1 1 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 1 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 2 1 1 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 2 0 1 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 2 1 1 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 1 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 2 1 0 0 2	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1
1*			
-1	0 1 0 2 1 2 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 1 2 0 0 1	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 1 2 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 1 2 0 0 1	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 0 2 1 2 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 2 0 2 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0
1	0 1 2 2 1 2 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 1 2 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 2 2 0 0 2	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 1 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 0 2 2 1 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 2 1 0 0 1	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 2 1 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 2 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 0 2 2 1 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 2 2 1 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 2 1 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 1 1 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
1*			

Table 2.7.2, continued

Constant	h	Power	
-1	0 1 0 2 2 2 0 0 1	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 2 2 0 0 1	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 2 2 0 0 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 0 2 2 2 0 0 1	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 0 2 2 2 1 0 1	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 2 2 2 0 1 1	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 2 2 0 0 1	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 1 2 0 0 1	0 0 0 0 0 0 0 0 1 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 2 1 0 0 1	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 1 1 0 1 0 1 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 1 0 1 0 1 0	0 0 0 0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 1 0 1 0 1 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 1 0 1 0 1 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 1 1 0 1 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 1 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0
1	0 1 2 1 0 1 0 2 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 1 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0 0
1	0 1 1 1 2 1 0 1 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 1 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 1 0
1*			
-1	0 1 1 1 0 2 0 1 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 1 0 2 0 1 0	0 0 0 0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 1 0 2 0 1 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 1 0 2 0 1 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 1 1 0 2 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 1 0 2 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0
1	0 1 2 1 0 2 0 2 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 2 0 0 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 1 0 0 0 0
1	0 1 1 1 2 2 0 1 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 1 0 1 0 1 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 1 2 0 1 0 1 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 2 0 1 0 1 0	0 0 0 0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 2 0 1 0 1 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 2 0 1 0 1 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 1 2 0 1 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 2 0 1 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0
1	0 1 2 2 0 1 0 2 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 1 0 1 0 1 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 2 2 1 0 1 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 0 2 0 2 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 1 0
1*			
-1	0 1 1 2 0 2 0 1 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 2 0 2 0 1 0	0 0 0 0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 2 0 2 0 1 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 1 2 0 2 0 1 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 1 2 0 2 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 2 0 2 0 2 0 0 0	0 0 0 1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0
1	0 1 2 2 0 2 0 2 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 1 0 2 0 1 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 2 2 2 0 1 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 2 0 1 0 1 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 2 1 0 1 0 1 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 2 1 0 1 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 2 1 0 1 0 1 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 2 1 0 1 0 1 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1	2 1 2 1 0 1 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 1 0 1 0 1 0	0 0 0 0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 2 1 0 1 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1*			
-1	0 1 2 1 0 2 0 1 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 2 1 0 2 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 2 1 0 2 0 1 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0
-1	0 1 2 1 0 2 0 1 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 0
1	2 1 2 1 0 2 1 1 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 1 1 0 2 0 1 0	0 0 0 0 1 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 2 2 0 1 1	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
1	0 1 2 1 0 1 0 1 0	0 0 0 0 0 0 0 0 0 0 1 0	0 0 0 0 0 0 0 0 0
1*			

To solve (2.7.1) and (2.7.2), we first combine the set of equations given by $\{P\,[0], P\,[1]\}$ and $\{P\,[0], P\,[1], P\,[2]\}$. We then solve for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n]$ in terms of the $P_{i_1 j_1 i_2 j_2 i_3 j_3}[2]$ for n = 0, 1 to get

$$\left(P_{i_1 j_1 i_2 j_2 i_3 j_3}[n] \;\middle|\; P\,[2]\right) \quad , \quad n = 0,\ 1. \tag{2.7.3}$$

An example of the codes which represent (2.7.3) are given in the Table 2.7.3.

The first column of Table 2.7.3 gives the values of (n_1, n_2, n_3). The second column gives the values of i_1, j_1, i_2, j_2, i_2, j_2 appearing in (2.7.3). The first row in the Table 2.7.3 gives the values of (n_1', n_2', n_3'). The second row gives the various values of $i_1', j_1', i_2', j_2', i_3', j_3'$. Each subsequent row in Table 2.7.3 represents an equation formed by equating $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ to the sum of the products formed by multiplying the remaining entries in the row by $P_{i_1' j_1' i_2' j_2' i_3' j_3'}[n_1'][n_2'][n_3']$. For example, row 3 represents the equation

$$\begin{aligned}
P_{010101}[0][0][0] = {} & 12.92P_{010121}[0][0][2] + 6.25P_{010222}[0][0][2] + 6.26P_{010221}[0][0][2] \\
& + 3.172P_{010222}[0][0][2] + 6.25P_{020121}[0][0][2] + 3.172P_{020122}[0][0][2] \\
& + 3.172P_{020221}[0][0][2] + 1.525P_{020222}[0][0][2] \\
& + \quad \Lambda \quad + \\
& + 12.92P_{011112}[0][1][1] + 6.25P_{011112}[0][1][1] + 6.25P_{011221}[0][1][1] \\
& + 3.172P_{011222}[0][1][1] + 29.85P_{012121}[0][1][1] + 16.23P_{012122}[0][1][1] \\
& + 3.172P_{012212}[0][1][1] + 16.23P_{012221}[0][1][1] + 9.21P_{012222}[0][1][1] \\
& + 6.25P_{021121}[0][1][1] + 3.172P_{021221}[0][1][1] + 1.525P_{021222}[0][1][1] \\
& + 6.25P_{022111}[0][1][1] + 3.172P_{022112}[0][1][1] + 16.23P_{022121}[0][1][1] \\
& + 9.21P_{022122}[0][1][1] + 3.172P_{022211}[0][1][1] + 1.525P_{022212}[0][1][1] \\
& + 9.21P_{022221}[0][1][1] + 4.879P_{022222}[0][1][1]
\end{aligned}$$

Table 2.7.3 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[n] \mid P[2]\}$ for n = 0, 1

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, μ_{31}=10, μ_{32}=20, λ_{31}=1, λ_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, Δt=0.01.

The dotted lines indicate that the present table is to be joined up with the table in the next page).

		002																020		
		010111	010112	010121	010122	010211	010212	010221	010222	020111	020112	020121	020122	020211	020212	020221	020222	011101	011102	…
000	010101	0	0	12.92187254	6.249825957	0	0	6.249825957	3.171610367	0	0	6.249825957	3.171610367	0	0	3.171610367	1.524812676	0	0	
	010102	0	0	6.325766393	6.83849824	0	0	3.588368234	2.601899105	0	0	3.588368234	2.601899105	0	0	1.908538037	1.139797476	0	0	
	010201	0	0	6.325766393	3.588368234	0	0	6.83849824	2.601899105	0	0	3.588368234	1.908538037	0	0	2.601899105	1.139797476	0	0	
	010202	0	0	3.407191257	2.996586789	0	0	2.996586789	3.763899798	0	0	2.076149972	1.43420661	0	0	1.43420661	1.062870676	0	0	
	020101	0	0	6.325766393	3.588368234	0	0	3.588368234	1.908538037	0	0	6.83849824	2.601899105	0	0	2.601899105	1.139797476	0	0	…
	020102	0	0	3.407191257	2.996586789	0	0	2.076149972	1.43420661	0	0	2.996586789	3.763899798	0	0	1.43420661	1.062870676	0	0	
	020201	0	0	3.407191257	2.076149972	0	0	2.996586789	1.43420661	0	0	2.996586789	1.43420661	0	0	3.763899798	1.062870676	0	0	
	020202	0	0	1.889892476	1.517298781	0	0	1.517298781	1.401276356	0	0	1.517298781	1.401276356	0	0	1.401276356	2.276254658	0	0	
001	010111	0	0	2.511656368	1.002074652	0	0	0.577057651	0.394958623	0	0	0.577057651	0.394958623	0	0	0.30428833	0.175353458	0	0	
	010112	0	0	0.228078187	1.542956572	0	0	0.077452238	0.065340662	0	0	0.077452238	0.065340662	0	0	0.042223547	0.027709114	0	0	
	010121	0	0	1.092024508	0.435684631	0	0	0.250894631	0.171721141	0	0	0.250894631	0.171721141	0	0	0.132299274	0.076240634	0	0	
	010122	0	0	0.140533599	0.661052098	0	0	0.042725709	0.034380323	0	0	0.042725709	0.034380323	0	0	0.023105614	0.014722157	0	0	
	010211	0	0	0.641808554	0.471681848	0	0	1.876469468	0.522385637	0	0	0.329556163	0.222853656	0	0	0.233017872	0.156771982	0	0	
	010212	0	0	0.070591733	0.144424553	0	0	0.140376772	1.381856929	0	0	0.037249219	0.028554797	0	0	0.02769127	0.027473771	0	0	…
	010221	0	0	0.312921252	0.21468763	0	0	0.792316221	0.22481573	0	0	0.147769011	0.100010737	0	0	0.10260325	0.068498352	0	0	
	010222	0	0	0.046374887	0.09279941	0	0	0.089552386	0.563110614	0	0	0.022519476	0.016797561	0	0	0.016104863	0.014318329	0	0	
	020111	0	0	0.641808554	0.471681848	0	0	0.329556163	0.222853656	0	0	1.876469468	0.522385637	0	0	0.233017872	0.156771982	0	0	
	020112	0	0	0.070591733	0.144424553	0	0	0.037249219	0.028554797	0	0	0.140376772	1.381856929	0	0	0.02769127	0.027473771	0	0	
	020121	0	0	0.312921252	0.21468763	0	0	0.147769011	0.100010737	0	0	0.792316221	0.22481573	0	0	0.10260325	0.068498352	0	0	
	020122	0	0	0.046374887	0.09279941	0	0	0.022519476	0.016797561	0	0	0.089552386	0.563110614	0	0	0.016104863	0.014318329	0	0	
	020211	0	0	0.312361571	0.244966767	0	0	0.329144229	0.217835782	0	0	0.329144229	0.217835782	0	0	1.532555546	0.29405349	0	0	
	020212	0	0	0.028346565	0.033363492	0	0	0.031673139	0.101765469	0	0	0.031673139	0.101765469	0	0	0.09924613	1.271812564	0	0	
	020221	0	0	0.149978329	0.115161717	0	0	0.169261101	0.100127372	0	0	0.169261101	0.100127372	0	0	0.621230478	0.123101264	0	0	
	020222	0	0	0.020238221	0.024399825	0	0	0.023002475	0.065295778	0	0	0.023002475	0.065295778	0	0	0.063303904	0.494994753	0	0	
010	011101	0	0	0.97319483	0.577057651	0	0	1.002074652	0.394958623	0	0	0.577057651	0.30428833	0	0	0.394958623	0.175353458	0	0	
	011102	0	0	0.531918444	0.447898039	0	0	0.471681848	0.522385637	0	0	0.329556163	0.233017872	0	0	0.222853656	0.156771982	0	0	
	011201	0	0	0.118188077	0.077452238	0	0	0.114385144	0.065340662	0	0	0.077452238	0.042223547	0	0	0.065340662	0.027709114	0	0	
	011202	0	0	0.055939718	0.045138677	0	0	0.049186458	0.048523596	0	0	0.037249219	0.02769127	0	0	0.028554797	0.027473771	0	0	…
	012101	0	0	0.423128187	0.250894631	0	0	0.435684631	0.171721141	0	0	0.250894631	0.132299274	0	0	0.171721141	0.076240634	0	0	
	012102	0	0	0.239263026	0.197078126	0	0	0.21468763	0.22481573	0	0	0.147769011	0.10260325	0	0	0.100010737	0.068498352	0	0	
	012201	0	0	0.066875373	0.042725709	0	0	0.065814003	0.034380323	0	0	0.042725709	0.023105614	0	0	0.034380323	0.014722157	0	0	
	012202	0	0	0.034621423	0.027647624	0	0	0.030894648	0.02977728	0	0	0.022519476	0.016104863	0	0	0.016797561	0.014318329	0	0	
	021101	0	0	0.531918444	0.329556163	0	0	0.471681848	0.222853656	0	0	0.447898039	0.233017872	0	0	0.522385637	0.156771982	0	0	
	021102	0	0	0.297709556	0.233906134	0	0	0.244966767	0.217835782	0	0	0.233906134	0.199222212	0	0	0.217835782	0.29405349	0	0	
	021201	0	0	0.055939718	0.037249219	0	0	0.049186458	0.028554797	0	0	0.045138677	0.02769127	0	0	0.048523596	0.027473771	0	0	
	021202	0	0	0.025599312	0.019768377	0	0	0.02145873	0.018432136	0	0	0.019768377	0.015912797	0	0	0.018432136	0.021812564	0	0	
	022101	0	0	0.239263026	0.147769011	0	0	0.21468763	0.100010737	0	0	0.197078126	0.10260325	0	0	0.22481573	0.068498352	0	0	
	022102	0	0	0.138224865	0.107356339	0	0	0.115161717	0.100127372	0	0	0.107356339	0.087897145	0	0	0.100127372	0.123101264	0	0	
	022201	0	0	0.034621423	0.022519476	0	0	0.030894648	0.016797561	0	0	0.027647624	0.016104863	0	0	0.02977728	0.014318329	0	0	
	022202	0	0	0.017825417	0.013661816	0	0	0.015059166	0.012731676	0	0	0.013661816	0.010739802	0	0	0.012731676	0.014225522	0	0	
100	110101	0	0	0.97319483	0.577057651	0	0	0.577057651	0.30428833	0	0	1.002074652	0.394958623	0	0	0.394958623	0.175353458	0	0	
	110102	0	0	0.531918444	0.447898039	0	0	0.329556163	0.233017872	0	0	0.471681848	0.522385637	0	0	0.222853656	0.156771982	0	0	
	110201	0	0	0.531918444	0.329556163	0	0	0.447898039	0.233017872	0	0	0.471681848	0.222853656	0	0	0.522385637	0.156771982	0	0	
	110202	0	0	0.297709556	0.233906134	0	0	0.233906134	0.199222212	0	0	0.244966767	0.217835782	0	0	0.217835782	0.29405349	0	0	…
	120101	0	0	0.118188077	0.077452238	0	0	0.077452238	0.042223547	0	0	0.114385144	0.065340662	0	0	0.065340662	0.027709114	0	0	
	120102	0	0	0.055939718	0.045138677	0	0	0.037249219	0.02769127	0	0	0.049186458	0.048523596	0	0	0.028554797	0.027473771	0	0	
	120201	0	0	0.055939718	0.037249219	0	0	0.045138677	0.02769127	0	0	0.049186458	0.028554797	0	0	0.048523596	0.027473771	0	0	
	120202	0	0	0.025599312	0.019768377	0	0	0.019768377	0.015912797	0	0	0.02145873	0.018432136	0	0	0.018432136	0.021812564	0	0	
	210101	0	0	0.423128187	0.250894631	0	0	0.250894631	0.132299274	0	0	0.435684631	0.171721141	0	0	0.171721141	0.076240634	0	0	
	210102	0	0	0.239263026	0.197078126	0	0	0.147769011	0.10260325	0	0	0.21468763	0.22481573	0	0	0.100010737	0.068498352	0	0	
	210201	0	0	0.239263026	0.147769011	0	0	0.197078126	0.10260325	0	0	0.21468763	0.100010737	0	0	0.22481573	0.068498352	0	0	
	210202	0	0	0.138224865	0.107356339	0	0	0.107356339	0.087897145	0	0	0.115161717	0.100127372	0	0	0.100127372	0.123101264	0	0	
	220101	0	0	0.066875373	0.042725709	0	0	0.042725709	0.023105614	0	0	0.065814003	0.034380323	0	0	0.034380323	0.014722157	0	0	
	220102	0	0	0.034621423	0.027647624	0	0	0.022519476	0.016104863	0	0	0.030894648	0.02977728	0	0	0.016797561	0.014318329	0	0	
	220201	0	0	0.034621423	0.022519476	0	0	0.027647624	0.016104863	0	0	0.030894648	0.016797561	0	0	0.02977728	0.014318329	0	0	
	220202	0	0	0.017825417	0.013661816	0	0	0.013661816	0.010739802	0	0	0.015059166	0.012731676	0	0	0.012731676	0.014225522	0	0	

Table 2.7.3, continued

(The dotted lines indicate that some of the columns after the previous page have been omitted,

and the remaining columns are listed in the present page.)

011111	011112	011 0111121	0111122	011211	011212	011221	011222	012111	012112	012121	012122	012211	012212	012221	012222	021111	021112	021121	021122	021211	021212	021221	021222	022111	022112	022121	022122	022211	022212	022221	022222
0	0	12.922	6.24983	0	0	6.2498	3.17161	12.922	6.2498	29.847	16.2309	6.24983	3.17161	16.23087	9.20987	0	0	6.2498	3.1716	0	0	3.17161	1.52481	6.2498	3.1716	16.2309	9.20987	3.17161	1.52481	9.209869	4.8794006
0	0	6.3258	6.8385	0	0	3.5884	2.6019	6.3258	6.8385	13.644	18.107	3.58837	2.6019	9.145344	7.57774	0	0	3.5884	2.6019	0	0	1.908538	1.1398	3.5884	2.6019	9.14534	7.57774	1.908538	1.1398	5.497655	3.6473519
0	0	6.3258	3.58837	0	0	6.8385	2.6019	6.3258	3.5884	13.644	9.14534	6.8385	2.6019	18.10704	7.57774	0	0	3.5884	1.9085	0	0	2.601899	1.1398	3.5884	1.9085	9.14534	5.49765	2.601899	1.1398	7.577738	3.6473519
0	0	3.4072	2.99659	0	0	2.9966	3.7639	3.4072	2.9966	7.2448	7.35082	2.99659	3.7639	7.350822	11.0791	0	0	2.0761	1.4342	0	0	1.434207	1.06287	2.0761	1.4342	5.23954	4.09005	1.434207	1.06287	4.090046	3.4011862
0	0	6.3258	3.58837	0	0	3.5884	1.90854	6.3258	3.5884	13.644	9.14534	3.58837	1.90854	9.145344	5.49765	0	0	6.8385	2.6019	0	0	2.601899	1.1398	6.8385	2.6019	18.107	7.57774	2.601899	1.1398	7.577738	3.6473519
0	0	3.4072	2.99659	0	0	2.0761	1.43421	3.4072	2.9966	7.2448	7.35082	2.07615	1.43421	5.239537	4.09005	0	0	2.9966	3.7639	0	0	1.434207	1.06287	2.9966	3.7639	7.35082	11.0791	1.434207	1.06287	4.090046	3.4011862
0	0	3.4072	2.07615	0	0	2.9966	1.43421	3.4072	2.0761	7.2448	5.23954	2.99659	1.43421	7.350822	4.09005	0	0	2.9966	1.4342	0	0	3.7639	1.06287	2.9966	1.4342	7.35082	4.09005	3.7639	1.06287	11.07913	3.4011862
0	0	1.8899	1.5173	0	0	1.5173	1.40128	1.8899	1.5173	4.0033	3.68793	1.5173	1.40128	3.687926	3.74858	0	0	1.5173	1.4013	0	0	1.401276	2.27625	1.5173	1.4013	3.68793	3.74858	1.401276	2.27625	3.748578	7.2840149
0	0	0.9732	1.00207	0	0	0.5771	0.39496	2.5117	1.0021	2.0991	2.64783	0.57706	0.39496	1.475912	1.14981	0	0	0.5771	0.395	0	0	0.304288	0.17535	0.5771	0.395	1.47591	1.14981	0.304288	0.17535	0.877794	0.5611311
0	0	0.1182	0.11439	0	0	0.0775	0.06534	0.2281	1.543	0.2534	0.29414	0.07745	0.06534	0.195353	0.19048	0	0	0.0775	0.0653	0	0	0.042224	0.02771	0.0775	0.0653	0.19535	0.19048	0.042224	0.02771	0.121129	0.0886692
0	0	0.4231	0.43568	0	0	0.2509	0.17172	1.092	0.4357	1.7822	1.15123	0.25089	0.17172	0.641701	0.49992	0	0	0.2509	0.1717	0	0	0.132299	0.07624	0.2509	0.1717	0.6417	0.49992	0.132299	0.07624	0.38165	0.24397
0	0	0.0669	0.06581	0	0	0.0427	0.03438	0.1405	0.6611	0.1799	1.00386	0.04273	0.03438	0.108135	0.1002	0	0	0.0427	0.0344	0	0	0.023106	0.01472	0.0427	0.0344	0.10813	0.1002	0.023106	0.01472	0.066372	0.0471109
0	0	0.5319	0.47168	0	0	0.4479	0.52239	0.6418	0.4717	1.1332	1.17166	1.87647	0.52239	1.103034	1.5358	0	0	0.3296	0.2229	0	0	0.233018	0.15677	0.3296	0.2229	0.83158	0.63721	0.233018	0.15677	0.667699	0.5016703
0	0	0.0559	0.04919	0	0	0.0451	0.04852	0.0706	0.1444	0.1191	0.12231	0.14038	1.38186	0.111145	0.14008	0	0	0.0372	0.0286	0	0	0.027691	0.02747	0.0372	0.0286	0.09305	0.08017	0.027691	0.02747	0.077579	0.0879161
0	0	0.2393	0.21469	0	0	0.1971	0.22482	0.3129	0.2147	0.5464	0.53616	0.79232	0.22482	1.319668	0.66075	0	0	0.1478	0.1	0	0	0.102603	0.0685	0.1478	0.1	0.37323	0.28633	0.102603	0.0685	0.29411	0.2191947
0	0	0.0346	0.03089	0	0	0.0276	0.02978	0.0464	0.0928	0.0767	0.11052	0.08955	0.56311	0.10157	0.88647	0	0	0.0225	0.0168	0	0	0.016105	0.01432	0.0225	0.0168	0.05647	0.04753	0.016105	0.01432	0.045451	0.0458187
0	0	0.5319	0.47168	0	0	0.3296	0.22285	0.6418	0.4717	1.1332	1.17166	0.32956	0.22285	0.831583	0.63721	0	0	0.4479	0.5224	0	0	0.233018	0.15677	1.8765	0.5224	1.10303	1.5358	0.233018	0.15677	0.667699	0.5016703
0	0	0.0559	0.04919	0	0	0.0372	0.02855	0.0706	0.1444	0.1191	0.12231	0.03725	0.02855	0.093049	0.08017	0	0	0.0451	0.0485	0	0	0.027691	0.02747	0.1404	1.3819	0.11115	0.14008	0.027691	0.02747	0.077579	0.0879161
0	0	0.2393	0.21469	0	0	0.1478	0.10001	0.3129	0.2147	0.5464	0.53616	0.14777	0.10001	0.37323	0.28633	0	0	0.1971	0.2248	0	0	0.102603	0.0685	0.7923	0.2248	1.31967	0.66075	0.102603	0.0685	0.29411	0.2191947
0	0	0.0346	0.03089	0	0	0.0225	0.0168	0.0464	0.0928	0.0767	0.11052	0.02252	0.0168	0.056474	0.04753	0	0	0.0276	0.0298	0	0	0.016105	0.01432	0.0896	0.5631	0.10157	0.88647	0.016105	0.01432	0.045451	0.0458187
0	0	0.2977	0.24497	0	0	0.2339	0.21784	0.3124	0.245	0.6315	0.59877	0.32914	0.21784	0.571526	0.5947	0	0	0.2339	0.2178	0	0	0.199222	0.29405	0.3291	0.2178	0.57153	0.5947	1.532556	0.29405	0.538856	0.9409712
0	0	0.0256	0.02146	0	0	0.0198	0.01843	0.0283	0.0334	0.0544	0.05271	0.03167	0.10177	0.048482	0.05093	0	0	0.0198	0.0184	0	0	0.015913	0.02181	0.0317	0.1018	0.04848	0.05093	0.099246	1.27181	0.043376	0.0698002
0	0	0.1382	0.11516	0	0	0.1074	0.10013	0.15	0.1152	0.2963	0.2824	0.16926	0.10013	0.296326	0.27576	0	0	0.1074	0.1001	0	0	0.087897	0.1231	0.1693	0.1001	0.29633	0.27576	0.62123	0.1231	1.039071	0.393924
0	0	0.0178	0.01506	0	0	0.0137	0.01273	0.0202	0.0244	0.0382	0.03964	0.023	0.0653	0.036123	0.06612	0	0	0.0137	0.0127	0	0	0.01074	0.01423	0.023	0.0653	0.03612	0.06612	0.063304	0.49499	0.060144	0.8147524
0	0	2.5117	0.57706	0	0	1.0021	0.39496	0.9732	0.5771	2.0991	1.47591	1.00207	0.39496	2.647826	1.14981	0	0	0.5771	0.3043	0	0	0.394959	0.17535	0.5771	0.3043	1.47591	0.87779	0.394959	0.17535	1.149805	0.5611311
0	0	0.6418	1.87647	0	0	0.4717	0.52239	0.5319	0.4479	1.1332	1.10303	0.47168	0.52239	1.171661	1.5358	0	0	0.3296	0.233	0	0	0.222854	0.15677	0.3296	0.233	0.83158	0.6677	0.222854	0.15677	0.637207	0.5016703
0	0	0.2281	0.07745	0	0	1.543	0.06534	0.1182	0.0775	0.2534	0.19535	0.11439	0.06534	0.294142	0.19048	0	0	0.0775	0.0422	0	0	0.065341	0.02771	0.0775	0.0422	0.19535	0.12113	0.065341	0.02771	0.19048	0.0886692
0	0	0.0706	0.14038	0	0	0.1444	1.38186	0.0559	0.0451	0.1191	0.11115	0.04919	0.04852	0.122306	0.14008	0	0	0.0372	0.0277	0	0	0.028555	0.02747	0.0372	0.0277	0.09305	0.07758	0.028555	0.02747	0.08017	0.0879161
0	0	1.092	0.25089	0	0	0.4357	0.17172	0.4231	0.2509	1.7822	0.6417	0.43568	0.17172	1.151229	0.49992	0	0	0.2509	0.1323	0	0	0.171721	0.07624	0.2509	0.1323	0.6417	0.38165	0.171721	0.07624	0.499915	0.24397
0	0	0.3129	0.79232	0	0	0.2147	0.22482	0.2393	0.1971	0.5464	1.31967	0.21469	0.22482	0.53616	0.66075	0	0	0.1478	0.1026	0	0	0.100011	0.0685	0.1478	0.1026	0.37323	0.29411	0.100011	0.0685	0.286333	0.2191947
0	0	0.1405	0.04273	0	0	0.6611	0.03438	0.0669	0.0427	0.1799	0.10813	0.06581	0.03438	1.00386	0.1002	0	0	0.0427	0.0231	0	0	0.03438	0.01472	0.0427	0.0231	0.10813	0.06637	0.03438	0.01472	0.100197	0.0471109
0	0	0.0464	0.08955	0	0	0.0928	0.56311	0.0346	0.0276	0.0767	0.10157	0.03089	0.02978	0.110523	0.88647	0	0	0.0225	0.0161	0	0	0.016798	0.01432	0.0225	0.0161	0.05647	0.04545	0.016798	0.01432	0.047529	0.0458187
0	0	0.6418	0.32956	0	0	0.4717	0.22285	0.5319	0.3296	1.1332	0.83158	0.47168	0.22285	1.171661	0.63721	0	0	1.8765	0.233	0	0	0.522386	0.15677	0.4479	0.233	1.10303	0.6677	0.522386	0.15677	1.535803	0.5016703
0	0	0.3124	0.32914	0	0	0.245	0.21784	0.2977	0.2339	0.6315	0.57153	0.24497	0.21784	0.598773	0.5947	0	0	0.3291	1.5326	0	0	0.217836	0.29405	0.2339	0.1992	0.57153	0.53886	0.217836	0.29405	0.594697	0.9409712
0	0	0.0706	0.03725	0	0	0.1444	0.02855	0.0559	0.0372	0.1191	0.09305	0.04919	0.02855	0.122306	0.08017	0	0	0.1404	0.0277	0	0	1.381857	0.02747	0.0451	0.0277	0.11115	0.07758	0.048524	0.02747	0.140076	0.0879161
0	0	0.0283	0.03167	0	0	0.0334	0.10177	0.0256	0.0198	0.0544	0.04848	0.02146	0.01843	0.052712	0.05093	0	0	0.0317	0.0992	0	0	0.101765	1.27181	0.0198	0.0159	0.04848	0.04338	0.018432	0.02181	0.050934	0.0698002
0	0	0.3129	0.14777	0	0	0.2147	0.10001	0.2393	0.1478	0.5464	0.37323	0.21469	0.10001	0.53616	0.28633	0	0	0.7923	0.1026	0	0	0.224816	0.0685	0.1971	0.1026	1.31967	0.29411	0.224816	0.0685	0.660748	0.2191947
0	0	0.15	0.16926	0	0	0.1152	0.10013	0.1382	0.1074	0.2963	0.29633	0.11516	0.10013	0.282402	0.27576	0	0	0.1693	0.6212	0	0	0.100127	0.1231	0.1074	0.0879	0.29633	1.03907	0.100127	0.1231	0.275762	0.393924
0	0	0.0464	0.02252	0	0	0.0928	0.0168	0.0346	0.0225	0.0767	0.05647	0.03089	0.0168	0.110523	0.04753	0	0	0.0896	0.0161	0	0	0.563111	0.01432	0.0276	0.0161	0.10157	0.04545	0.029777	0.01432	0.886468	0.0458187
0	0	0.0202	0.023	0	0	0.0244	0.0653	0.0178	0.0137	0.0382	0.03612	0.01506	0.01273	0.039637	0.06612	0	0	0.023	0.0633	0	0	0.065296	0.49499	0.0137	0.0107	0.03612	0.06014	0.012732	0.01423	0.066119	0.8147524
0	0	0.9732	0.57706	0	0	0.5771	0.30429	0.9732	0.5771	2.0991	1.47591	0.57706	0.30429	1.475912	0.87779	0	0	1.0021	0.395	0	0	0.394959	0.17535	1.0021	0.395	2.64783	1.14981	0.394959	0.17535	1.149805	0.5611311
0	0	0.5319	0.4479	0	0	0.3296	0.23302	0.5319	0.4479	1.1332	1.10303	0.32956	0.23302	0.831583	0.6677	0	0	0.4717	0.5224	0	0	0.222854	0.15677	0.4717	0.5224	1.17166	1.5358	0.222854	0.15677	0.637207	0.5016703
0	0	0.5319	0.32956	0	0	0.4479	0.23302	0.5319	0.3296	1.1332	0.83158	0.4479	0.23302	1.103034	0.6677	0	0	0.4717	0.2229	0	0	0.522386	0.15677	0.4717	0.2229	1.17166	0.63721	0.522386	0.15677	1.535803	0.5016703
0	0	0.2977	0.23391	0	0	0.2339	0.19922	0.2977	0.2339	0.6315	0.57153	0.23391	0.19922	0.571526	0.53886	0	0	0.245	0.2178	0	0	0.217836	0.29405	0.245	0.2178	0.59877	0.5947	0.217836	0.29405	0.594697	0.9409712
0	0	0.1182	0.07745	0	0	0.0775	0.04222	0.1182	0.0775	0.2534	0.19535	0.07745	0.04222	0.195353	0.12113	0	0	0.1144	0.0653	0	0	0.065341	0.02771	0.1144	0.0653	0.29414	0.19048	0.065341	0.02771	0.19048	0.0886692
0	0	0.0559	0.04514	0	0	0.0372	0.02769	0.0559	0.0451	0.1191	0.11115	0.03725	0.02769	0.093049	0.07758	0	0	0.0492	0.0485	0	0	0.028555	0.02747	0.0492	0.0485	0.12231	0.14008	0.028555	0.02747	0.08017	0.0879161
0	0	0.0559	0.03725	0	0	0.0451	0.02769	0.0559	0.0372	0.1191	0.09305	0.04514	0.02769	0.111145	0.07758	0	0	0.0492	0.0286	0	0	0.048524	0.02747	0.0492	0.0286	0.12231	0.08017	0.048524	0.02747	0.140076	0.0879161
0	0	0.0256	0.01977	0	0	0.0198	0.01591	0.0256	0.0198	0.0544	0.04848	0.01977	0.01591	0.048482	0.04338	0	0	0.0215	0.0184	0	0	0.018432	0.02181	0.0215	0.0184	0.05271	0.05093	0.018432	0.02181	0.050934	0.0698002
0	0	0.4231	0.25089	0	0	0.2509	0.1323	0.4231	0.2509	0.9126	0.6417	0.25089	0.1323	0.641701	0.38165	0	0	0.4357	0.1717	0	0	0.171721	0.07624	0.4357	0.1717	1.15123	0.49992	0.171721	0.07624	0.499915	0.24397
0	0	0.2393	0.19708	0	0	0.1478	0.1026	0.2393	0.1971	0.5102	0.48633	0.14777	0.1026	0.37323	0.29411	0	0	0.2147	0.2248	0	0	0.100011	0.0685	0.2147	0.2248	0.53616	0.66075	0.100011	0.0685	0.286333	0.2191947
0	0	0.2393	0.14777	0	0	0.1971	0.1026	0.2393	0.1478	0.5102	0.37323	0.19708	0.1026	0.486335	0.29411	0	0	0.2147	0.1	0	0	0.224816	0.0685	0.2147	0.1	0.53616	0.28633	0.224816	0.0685	0.660748	0.2191947
0	0	0.1382	0.10736	0	0	0.1074	0.0879	0.1382	0.1074	0.2934	0.26299	0.10736	0.0879	0.262993	0.23907	0	0	0.1152	0.1001	0	0	0.100127	0.1231	0.1152	0.1001	0.2824	0.27576	0.100127	0.1231	0.275762	0.393924
0	0	0.0669	0.04273	0	0	0.0427	0.02311	0.0669	0.0427	0.1436	0.10813	0.04273	0.02311	0.108135	0.06637	0	0	0.0658	0.0344	0	0	0.03438	0.01472	0.0658	0.0344	0.17053	0.1002	0.03438	0.01472	0.100197	0.0471109
0	0	0.0346	0.02765	0	0	0.0225	0.0161	0.0346	0.0276	0.0738	0.06824	0.02252	0.0161	0.056474	0.04545	0	0	0.0309	0.0298	0	0	0.016798	0.01432	0.0309	0.0298	0.07719	0.08647	0.016798	0.01432	0.047529	0.0458187
0	0	0.0346	0.02252	0	0	0.0276	0.0161	0.0346	0.0225	0.0738	0.05647	0.02765	0.0161	0.068237	0.04545	0	0	0.0309	0.0168	0	0	0.029777	0.01432	0.0309	0.0168	0.07719	0.04753	0.029777	0.01432	0.086468	0.0458187
0	0	0.0178	0.01366	0	0	0.0137	0.01074	0.0178	0.0137	0.0379	0.03356	0.01366	0.01074	0.033559	0.02937	0	0	0.0151	0.0127	0	0	0.012732	0.01423	0.0151	0.0127	0.03707	0.03535	0.012732	0.01423	0.03535	0.0455217

From the equations represented by Rows 3-10 in Table 2.7.3, we get three sets of equations of the forms

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][2] \mid \{P[0][0][0]\}, \{P[0][2][0]\}, \{P[2][0][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\ \} \tag{2.7.4}$$

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][2][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[2][0][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\ \} \tag{2.7.5}$$

and

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[2][0][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[0][2][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\ \} \tag{2.7.6}$$

respectively (see for example Tables 2.7.4 - 2.7.6).

From (2.7.4) and rows 11-26 in Table 2.7.3, we get a set of equations of the form

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][1] \mid \{P[0][0][0]\}, \{P[0][2][0]\}, \{P[2][0][0]\} \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\ \} \tag{2.7.7}$$

(see for example Table 2.7.7).

Next from (2.7.5) and rows 27-42 in Table 2.7.3, we get a set of equations of the form (see for example Table 2.7.8)

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][1][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[2][0][0]\} \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\ \}. \tag{2.7.8}$$

Similarly from (2.7.6) and rows 43-58 in Table 2.7.3, we get a set of equations of the form (see for example Table 2.7.9)

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[1][0][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[0][2][0]\} \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\ \}\ . \tag{2.7.9}$$

Table 2.7.4 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][2] \mid \{P[0][0][0]\}, \{P[0][2][0]\}, \{P[2][0][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, μ_{31}=10, μ_{32}=20, λ_{31}=1, λ_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, Δt=0.01.

The dotted lines indicate that the present table is to be joined up with the table in the next page).

		000								020		
		010101	010102	010201	020102	020101	020102	020201	020201	011101	011102	...
	010111	0	0	0	0	0	0	0	0	0	0	
	010112	0	0	0	0	0	0	0	0	0	0	
002	010121	0	0	0	0	-1	-6.87E-17	5.22E-17	2.19E-17	0	0	
	010122	0	0	0	0	2.98E-16	-1	1.76E-16	-4.19E-18	0	0	
	010211	0	0	0	0	0	0	0	0	0	0	
	010212	0	0	0	0	0	0	0	0	0	0	...
	010221	0	0	0	0	1.65E-16	2.95E-17	-1	3.00E-17	0	0	
	010222	0	0	0	0	-4.98E-17	3.06E-16	-7.77E-17	-1	0	0	
	020111	0	0	0	0	0	0	0	0	0	0	
	020112	0	0	0	0	0	0	0	0	0	0	
	020121	0	0	0	0	2.54E-16	-1.49E-17	9.53E-17	-1.23E-16	0	0	
	020122	0	0	0	0	-4.66E-16	5.14E-16	-1.48E-16	1.71E-16	0	0	
	020211	0	0	0	0	0	0	0	0	0	0	
	020212	0	0	0	0	0	0	0	0	0	0	
	020221	0	0	0	0	1.58E-16	2.95E-16	-1.78E-17	2.49E-16	0	0	
	020222	0	0	0	0	6.06E-17	-3.40E-16	4.67E-16	5.11E-17	0	0	

Table 2.7.4, continued

(The dotted lines indicate that some of the columns after the previous page have been omitted, and the remaining columns are listed in the present page.)

	011													
021112	021121	021122	021211	021212	021221	021222	022111	022112	022121	022122	022211	022212	022221	022222
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4.75E-18	-2.29E-18	1.15E-15	-4.81E-19	7.89E-18	-1.94E-16	3.19E-16	0.22425	-0.1	-0.1	-5.16E-17	-0.1	-6.87E-17	3.85E-17	-1.43E-17
-1.65E-18	9.69E-17	-1.86E-16	-8.07E-19	8.11E-18	-5.06E-16	-3.52E-16	-0.11175	0.336	-5.14E-17	-0.1	-5.34E-18	-0.1	6.24E-17	1.36E-17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.54E-17	-3.02E-17	-5.96E-16	-1.13E-17	-4.50E-18	3.65E-16	-1.67E-16	-0.11175	-1.87E-17	0.336	-0.1	3.00E-17	1.21E-17	-0.1	3.37E-17
1.35E-17	-1.22E-16	1.03E-15	5.18E-17	4.54E-17	1.26E-17	6.52E-16	0.021	-0.13275	-0.13275	0.46875	3.06E-17	-5.73E-17	-6.11E-17	-0.1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-3.05E-17	-2.8	-1.22E-16	1.17E-17	3.82E-18	-4.31E-16	-2.02E-16	-0.11175	-1.14E-17	2.47E-17	4.44E-17	0.336	-0.1	-0.1	-7.61E-18
-1	0.2	-3	-4.91E-17	2.32E-17	7.86E-16	5.89E-16	0.021	-0.13275	3.99E-17	-4.48E-17	-0.13275	0.46875	-4.91E-17	-0.1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-5.20E-17	0.2	6.65E-16	-1	2.46E-17	-3	0	0.021	-3.70E-17	-0.13275	-2.16E-17	-0.13275	-6.16E-18	0.46875	-0.1
1.72E-17	-1.82E-16	0.2	0	-1	0.2	-3.2	-0.0015	0.0225	0.0225	-0.15525	0.0225	-0.15525	-0.15525	0.624

Table 2.7.5 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][2][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[2][0][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, μ_{31}=10, μ_{32}=20, λ_{31}=1, λ_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, Δt=0.01.

The dotted lines indicate that the present table is to be joined up with the table in the next page).

			000								002	
		010101	010102	010201	020102	020101	020102	020201	020201	010111	010112	...
	011101	0	0	0	0	0	0	0	0	0	0	
	011102	0	0	0	0	0	0	0	0	0	0	
020	011201	0	0	0	0	0	0	0	0	0	0	
	011202	0	0	0	0	0	0	0	0	0	0	
	012101	0	0	-1	2.75E-16	0	0	4.49E-17	-9.44E-18	0	0	
	012102	0	0	-3.43E-16	-1	0	0	-2.35E-16	8.39E-18	0	0	…
	012201	0	0	-2.10E-16	-7.39E-17	0	0	-1	-5.99E-17	0	0	
	012202	0	0	2.76E-17	-2.02E-16	0	0	2.86E-16	-1	0	0	
	021101	0	0	0	0	0	0	0	0	0	0	
	021102	0	0	0	0	0	0	0	0	0	0	
	021201	0	0	0	0	0	0	0	0	0	0	
	021202	0	0	0	0	0	0	0	0	0	0	
	022101	0	0	-3.43E-16	-2.95E-17	0	0	-2.29E-16	3.66E-17	0	0	
	022102	0	0	6.52E-16	-4.10E-16	0	0	1.92E-16	-2.07E-16	0	0	
	022201	0	0	2.76E-17	-1.91E-16	0	0	1.66E-16	-1.02E-16	0	0	
	022202	0	0	-6.00E-17	2.37E-16	0	0	-5.77E-16	-1.24E-17	0	0	

Table 2.7.5, continued

(The dotted lines indicate that some of the columns after the previous page have been omitted, and the remaining columns are listed in the present page.)

	011														
021111	021112	021121	021122	021211	021212	021221	021222	022111	022112	022121	022122	022211	022212	022221	022222
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4.50E-16	-1.93E-16	9.06E-16	4.64E-16	-2.09E-17	-5.45E-17	-5.70E-16	1.28E-16	0.22425	-0.1	-0.1	-6.87E-17	-0.1	3.44E-17	7.20E-17	-2.54E-18
-1.05E-16	1.94E-16	2.95E-17	6.04E-16	1.33E-16	-7.01E-19	7.61E-17	-2.77E-16	-0.11175	0.336	-1.47E-17	-0.1	2.36E-17	-0.1	-4.22E-17	4.30E-17
-1.24E-16	-2.48E-17	-4.84E-17	-5.46E-16	1.14E-17	-5.87E-17	7.56E-16	-2.71E-16	-0.11175	-5.24E-17	0.336	-0.1	2.10E-17	6.92E-18	-0.1	5.99E-17
1.83E-17	-1.29E-16	-2.93E-16	3.95E-16	1.19E-16	-3.19E-17	-4.87E-17	4.39E-16	0.021	-0.13275	-0.13275	0.46875	3.67E-17	-5.35E-18	-4.81E-17	-0.1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	1.52E-16	-2.8	4.87E-16	-3.45E-17	-1.22E-17	-2.85E-16	-2.37E-16	-0.11175	-3.43E-17	1.92E-17	-8.06E-18	0.336	-0.1	-0.1	2.67E-17
2.96E-16	-1	0.2	-3	-1.11E-16	-3.28E-17	1.72E-16	4.69E-16	0.021	-0.13275	1.54E-17	1.51E-17	-0.13275	0.46875	4.30E-17	-0.1
2.96E-16	-5.01E-17	0.2	4.00E-16	-1	1.48E-16	-3	3.94E-16	0.021	3.31E-17	-0.13275	5.85E-17	-0.13275	3.08E-17	0.46875	-0.1
-1.03E-16	8.22E-17	-3.55E-16	0.2	-2.08E-16	-1	0.2	-3.2	-0.0015	0.0225	0.0225	-0.15525	0.0225	-0.15525	-0.15525	0.624

Table 2.7.6 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[2][0][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[0][2][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, μ_{31}=10, μ_{32}=20, λ_{31}=1, λ_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, Δt=0.01.

The dotted lines indicate that the present table is to be joined up with the table in the next page).

		000								002		
		010101	010102	010201	020102	020101	020102	020201	020201	010111	010112	…
	110101	0	0	0	0	0	0	0	0	0	0	
	110102	0	0	0	0	0	0	0	0	0	0	
200	110201	0	0	0	0	0	0	0	0	0	0	
	110202	0	0	0	0	0	0	0	0	0	0	
	120101	0	0	0	0	0	0	0	0	0	0	…
	120102	0	0	0	0	0	0	0	0	0	0	
	120201	0	0	0	0	0	0	0	0	0	0	
	120202	0	0	0	0	0	0	0	0	0	0	
	210101	0	0	-1	2.75E-16	0	0	-4.03E-17	2.12E-17	0	0	
	210102	0	0	1.21E-16	-1	0	0	-5.88E-17	1.43E-16	0	0	
	210201	0	0	1.65E-16	-1.43E-16	0	0	-1	2.40E-16	0	0	
	210202	0	0	1.36E-16	-2.05E-16	0	0	2.86E-16	-1	0	0	
	220101	0	0	1.65E-16	1.55E-16	0	0	5.27E-17	1.71E-17	0	0	
	220102	0	0	1.36E-16	-3.23E-16	0	0	3.87E-17	-2.21E-16	0	0	
	220201	0	0	-7.21E-17	-3.10E-16	0	0	-1.30E-16	-1.17E-16	0	0	
	220202	0	0	-7.66E-18	3.94E-16	0	0	-4.72E-16	1.86E-16	0	0	

Table 2.7.6, continued

(The dotted lines indicate that some of the columns after the previous page have been omitted, and the remaining columns are listed in the present page.)

	011															
012222	021111	021112	021121	021122	021211	021212	021221	021222	022111	022112	022121	022122	022211	022212	022221	022222
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3.63E-17	-3.83E-17	-1.62E-16	4.33E-17	4.87E-16	1.50E-16	-2.65E-17	2.15E-16	2.88E-16	0.22425	-0.1	-0.1	-5.16E-17	-0.1	1.72E-17	-5.21E-17	2.52E-18
-1.08E-17	2.46E-16	1.44E-16	6.70E-16	5.52E-16	-1.28E-16	1.39E-17	-7.64E-16	-3.16E-16	-0.11175	0.336	-1.47E-17	-0.1	-4.55E-17	-0.1	1.43E-16	3.67E-18
4.12E-17	-6.05E-18	-2.50E-17	-5.44E-17	-6.65E-16	-1.39E-16	2.85E-17	3.38E-17	-7.08E-17	-0.11175	8.62E-17	0.336	-0.1	1.81E-17	3.27E-17	-0.1	0
-1.09E-16	-3.67E-17	-1.13E-16	-2.69E-16	6.39E-16	1.37E-16	-1.64E-16	2.74E-17	9.49E-17	0.021	-0.13275	-0.13275	0.46875	4.89E-17	-4.20E-17	-8.79E-17	-0.1
-3.60E-17	-1	6.09E-17	-2.8	4.87E-16	-1.31E-16	-4.21E-17	-6.57E-16	-4.04E-16	-0.11175	-3.81E-18	-3.66E-19	4.10E-17	0.336	-0.1	-0.1	7.61E-18
-9.41E-17	1.11E-17	-1	0.2	-3	3.68E-17	-1.77E-16	7.12E-16	1.91E-16	0.021	-0.13275	1.54E-18	1.41E-17	-0.13275	0.46875	-1.01E-16	-0.1
-7.39E-17	1.15E-16	-3.82E-18	0.2	5.85E-16	-1	9.86E-17	-3	9.86E-17	0.021	1.08E-17	-0.13275	2.16E-17	-0.13275	-4.31E-17	0.46875	-0.1
-1	-1.04E-16	-1.30E-16	-2.77E-16	0.2	6.93E-17	-1	0.2	-3.2	-0.0015	0.0225	0.0225	-0.15525	0.0225	-0.15525	-0.15525	0.624

Table 2.7.7 : The set of equations given by $\{P_{i_1j_1i_2j_2i_3j_3}[0][0][1] \mid \{P[0][0][0]\}, \{P[0][2][0]\}, \{P[2][0][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, μ_{31}=10, μ_{32}=20, λ_{31}=1, λ_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, Δt=0.01.

The dotted lines indicate that the present table is to be joined up with the table in the next page).

		000								020		
		010101	010102	010201	020102	020101	020102	020201	020201	011101	011102	…
	010111	0.345	-0.015384615	-0.146153846	1.39E-17	-0.146153846	-3.82E-17	5.55E-17	-1.39E-17	0	0	
	010112	-0.135	0.478901099	-0.01043956	-0.135714286	-0.01043956	-0.135714286	9.63E-17	1.04E-17	0	0	
001	010121	0.15	-0.006688963	-0.063545151	-2.60E-17	-0.063545151	-3.47E-17	8.67E-18	0	0	0	
	010122	-0.05	0.199263418	-0.006997531	-0.056547619	-0.006997531	-0.056547619	3.90E-17	6.94E-18	0	0	
	010211	-0.135	-0.001098901	0.46956044	-0.014285714	-0.01043956	3.82E-17	-0.135714286	2.78E-17	0	0	…
	010212	0.01	-0.14514652	-0.146391941	0.615	-0.001391941	-0.009047619	-0.009047619	-0.126666667	0	0	
	010221	-0.05	-7.37E-04	0.193002469	-0.005952381	-0.006997531	1.21E-17	-0.056547619	6.94E-18	0	0	
	010222	1.50E-17	-0.050117535	-0.051116579	0.2435	-0.001116579	-0.005880952	-0.005880952	-0.050666667	0	0	
	020111	-0.135	-0.001098901	-0.01043956	2.78E-17	0.46956044	-0.014285714	-0.135714286	-2.78E-17	0	0	
	020112	0.01	-0.14514652	-0.001391941	-0.009047619	-0.146391941	0.615	-0.009047619	-0.126666667	0	0	
	020121	-0.05	-7.37E-04	-0.006997531	2.43E-17	0.193002469	-0.005952381	-0.056547619	-6.94E-18	0	0	
	020122	3.30E-17	-0.050117535	-0.001116579	-0.005880952	-0.051116579	0.2435	-0.005880952	-0.050666667	0	0	
	020211	0.01	-1.47E-04	-0.146391941	-9.52E-04	-0.146391941	-9.52E-04	0.606904762	-0.013333333	0	0	
	020212	-7.13E-17	0.009972527	0.009739011	-0.15625	0.009739011	-0.15625	-0.157261905	0.763333333	0	0	
	020221	3.59E-17	-1.18E-04	-0.051116579	-6.19E-04	-0.051116579	-6.19E-04	0.238238095	-0.005333333	0	0	
	020222	-2.39E-17	-2.41E-05	-2.29E-04	-0.050980769	-2.29E-04	-0.050980769	-0.051774725	0.289487179	0	0	

Table 2.7.7, continued

(The dotted lines indicate that some of the columns after the previous page have been omitted, and the remaining columns are listed in the present page.)

	011														
021111	021112	021121	021122	021211	021212	021221	021222	022111	022112	022121	022122	022211	022212	022221	022222
0	0	-4.44E-16	1.67E-16	0	0	5.55E-17	5.55E-17	0	0	-8.88E-16	1.33E-15	0	2.78E-17	-2.22E-16	5.55E-16
0	0	1.39E-16	-2.22E-16	0	0	-2.01E-16	-4.86E-17	-1.39E-17	0	1.11E-16	-1.94E-16	0	1.39E-17	-7.63E-16	-4.58E-16
0	0	-1.11E-16	8.33E-17	0	0	0	1.39E-17	0	0	-1.11E-16	8.33E-16	0	1.39E-17	-5.55E-17	2.22E-16
0	0	4.86E-17	-9.02E-17	0	0	-8.67E-17	-1.91E-17	-6.94E-18	0	1.39E-17	-2.78E-17	0	5.20E-18	-3.05E-16	-1.87E-16
0	0	0	8.33E-17	0	0	-8.33E-17	2.78E-17	-1.11E-16	2.78E-17	-3.33E-16	-6.66E-16	0	0	4.44E-16	-1.11E-16
0	0	6.94E-17	5.20E-17	0	0	-3.47E-18	2.19E-16	9.02E-17	2.08E-17	-1.53E-16	1.28E-15	6.94E-17	5.90E-17	1.39E-17	8.33E-16
0	0	-2.78E-17	2.78E-17	0	0	-1.39E-17	2.78E-17	-2.78E-17	1.39E-17	-1.11E-16	-2.78E-16	0	0	1.67E-16	-2.78E-17
0	0	3.12E-17	2.08E-17	0	0	-3.47E-18	8.67E-17	3.47E-17	6.94E-18	-7.63E-17	5.27E-16	2.78E-17	2.43E-17	-6.94E-18	3.19E-16
0	0	-1.4285714	-2.22E-16	0	0	1.67E-16	2.78E-17	0	0	-4	-4.44E-16	0	2.78E-17	-4.44E-16	-1.11E-16
0	0	-0.0952381	-1.3333333	0	0	5.20E-17	1.98E-16	2.78E-17	0	4.30E-16	-4	-6.59E-17	2.78E-17	9.71E-16	7.63E-16
0	0	-0.5952381	-2.78E-17	0	0	6.94E-17	2.78E-17	0	-2.78E-17	-0.8333333	-1.11E-16	0	0	-2.22E-16	-1.11E-16
0	0	-0.0619048	-0.5333333	0	0	2.08E-17	7.98E-17	0	0	-0.0333333	-0.8	-2.78E-17	1.21E-17	3.75E-16	2.84E-16
0	0	-0.0952381	0	0	0	-1.3333333	-1.67E-16	0	-8.33E-17	-3.33E-16	7.77E-16	0	0	-4	-1.11E-16
0	0	-0.0119048	-0.0833333	0	0	-0.0833333	-1.25	-8.33E-17	2.78E-17	-2.29E-16	7.63E-17	0	-2.22E-16	-3.75E-16	-4
0	0	-0.0619048	-2.78E-17	0	0	-0.5333333	-4.16E-17	0	-2.78E-17	-0.0333333	1.11E-16	0	0	-0.8	-5.55E-17
0	0	-0.0093407	-0.0525641	0	0	-0.0525641	-0.4807692	-3.12E-17	1.39E-17	-0.0025641	-0.0307692	0	-1.11E-16	-0.0307692	-0.7692308

Table 2.7.8 : The set of equations given by $\{P_{i_1j_1i_2j_2i_3j_3}[0][1][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[2][0][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, μ_{31}=10, μ_{32}=20, λ_{31}=1, λ_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, Δt=0.01.

The dotted lines indicate that the present table is to be joined up with the table in the next page).

		000								002		
		010101	010102	010201	020102	020101	020102	020201	020201	010111	010112	…
010	011101	0.345	-0.14615385	-0.01538462	-8.33E-17	-0.14615385	7.29E-17	2.78E-17	0	0	0	
	011102	-0.135	0.46956044	-0.0010989	-0.01428571	-0.01043956	-0.13571429	-6.59E-17	5.55E-17	0	0	
	011201	-0.135	-0.01043956	0.478901099	-0.13571429	-0.01043956	1.21E-17	-0.13571429	8.67E-17	0	0	
	011202	0.01	-0.14639194	-0.14514652	0.615	-0.00139194	-0.00904762	-0.00904762	-0.12666667	0	0	
	012101	0.15	-0.06354515	-0.00668896	-3.64E-17	-0.06354515	8.67E-18	3.30E-17	6.94E-18	0	0	
	012102	-0.05	0.193002469	-7.37E-04	-0.00595238	-0.00699753	-0.05654762	-2.43E-17	2.08E-17	0	0	…
	012201	-0.05	-0.00699753	0.199263418	-0.05654762	-0.00699753	6.51E-18	-0.05654762	3.64E-17	0	0	
	012202	2.62E-17	-0.05111658	-0.05011753	0.2435	-0.00111658	-0.00588095	-0.00588095	-0.05066667	0	0	
	021101	-0.135	-0.01043956	-0.0010989	2.08E-17	0.46956044	-0.13571429	-0.01428571	0	0	0	
	021102	0.01	-0.14639194	-1.47E-04	-9.52E-04	-0.14639194	0.606904762	-9.52E-04	-0.01333333	0	0	
	021201	0.01	-0.00139194	-0.14514652	-0.00904762	-0.14639194	-0.00904762	0.615	-0.12666667	0	0	
	021202	-1.34E-16	0.009739011	0.009972527	-0.15625	0.009739011	-0.1572619	-0.15625	0.763333333	0	0	
	022101	-0.05	-0.00699753	-7.37E-04	8.67E-18	0.193002469	-0.05654762	-0.00595238	6.94E-18	0	0	
	022102	-3.80E-17	-0.05111658	-1.18E-04	-6.19E-04	-0.05111658	0.238238095	-6.19E-04	-0.00533333	0	0	
	022201	4.53E-17	-0.00111658	-0.05011753	-0.00588095	-0.05111658	-0.00588095	0.2435	-0.05066667	0	0	
	022202	-5.02E-17	-2.29E-04	-2.41E-05	-0.05098077	-2.29E-04	-0.05177473	-0.05098077	0.289487179	0	0	

Table 2.7.8, continued

(The dotted lines indicate that some of the columns after the previous page have been omitted, and the remaining columns are listed in the present page.)

	011														
021111	021112	021121	021122	021211	021212	021221	021222	022111	022112	022121	022122	022211	022212	022221	022222
0	0	0	0	0	0	0	0	6.66E-16	-3.33E-16	1.33E-15	3.33E-16	0	-8.33E-17	-4.44E-16	2.22E-16
0	0	0	0	0	0	0	0	-1.67E-16	2.50E-16	0	9.99E-16	1.67E-16	2.78E-17	0	-2.22E-16
0	0	-2.78E-17	2.08E-17	0	0	1.11E-16	-1.73E-17	-1.25E-16	-5.55E-17	5.55E-17	-7.49E-16	1.39E-17	-8.33E-17	1.05E-15	-3.47E-16
0	0	-6.25E-17	-1.11E-16	0	0	1.04E-17	-3.47E-18	6.94E-18	-1.60E-16	-4.30E-16	5.27E-16	1.67E-16	-3.82E-17	0	5.41E-16
0	0	0	0	0	0	0	0	2.78E-16	-1.39E-16	4.44E-16	2.78E-16	-5.55E-17	-2.78E-17	-1.67E-16	8.33E-17
0	0	0	0	0	0	0	0	-5.55E-17	1.11E-16	0	4.44E-16	6.94E-17	0	5.55E-17	-8.33E-17
0	0	-1.39E-17	6.94E-18	0	0	4.86E-17	-6.94E-18	-4.86E-17	-3.12E-17	0	-2.91E-16	6.94E-18	-3.47E-17	4.30E-16	-1.60E-16
0	0	-2.43E-17	-4.51E-17	0	0	3.47E-18	-1.73E-18	3.47E-18	-6.25E-17	-1.46E-16	2.22E-16	6.59E-17	-1.39E-17	1.39E-17	2.08E-16
0	0	0	0	0	0	1.11E-16	2.78E-17	-1.4285714	2.22E-16	-4	7.77E-16	2.22E-16	8.33E-17	2.22E-16	-1.11E-16
0	0	0	0	0	0	-8.33E-17	5.55E-17	-0.0952381	-1.3333333	1.33E-15	-4	-1.11E-16	1.11E-16	1.11E-16	4.44E-16
0	0	0	-1.04E-17	0	0	-2.22E-16	-3.47E-17	-0.0952381	-4.86E-17	8.60E-16	5.27E-16	-1.3333333	1.98E-16	-4	4.86E-16
0	0	3.47E-17	6.94E-17	0	0	5.55E-17	0	-0.0119048	-0.0833333	-3.47E-16	1.11E-15	-0.0833333	-1.25	-6.25E-16	-4
0	0	0	0	0	0	5.55E-17	0	-0.5952381	6.94E-17	-0.8333333	3.33E-16	0	1.39E-17	0	-8.33E-17
0	0	0	0	0	0	-2.78E-17	1.39E-17	-0.0619048	-0.5333333	-0.0333333	-0.8	-5.55E-17	2.78E-17	0	2.78E-16
0	0	0	-6.94E-18	0	0	-1.11E-16	-1.39E-17	-0.0619048	-2.43E-17	-0.0333333	2.01E-16	-0.5333333	7.98E-17	-0.8	1.87E-16
0	0	1.39E-17	2.78E-17	0	0	1.39E-17	0	-0.0093407	-0.0525641	-0.0025641	-0.0307692	-0.0525641	-0.4807692	-0.0307692	-0.7692308

Table 2.7.9 : The set of equations given by $\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[1][0][0] \mid \{P[0][0][0]\}, \{P[0][0][2]\}, \{P[0][2][0]\}, \{P[1][1][0]\}, \{P[1][0][1]\}, \{P[0][1][1]\}\}$

(μ_{11}=10, μ_{12}=20, λ_{11}=1, λ_{12}=2, μ_{21}=10, μ_{22}=20, λ_{21}=1, λ_{22}=2, μ_{31}=10, μ_{32}=20, λ_{31}=1, λ_{32}=2, q_{11}=0.9, q_{22}=0.9, q_{33}=0.9, Δt=0.01.

The dotted lines indicate that the present table is to be joined up with the table in the next page).

		000								002		
		010101	010102	010201	020102	020101	020102	020201	020201	010111	010112	···
	110101	0.345	-0.14615385	-0.14615385	-5.90E-17	-0.01538462	6.94E-18	-6.94E-17	1.39E-17	0	0	
	110102	-0.135	0.46956044	-0.01043956	-0.13571429	-0.0010989	-0.01428571	1.91E-16	0	0	0	
100	110201	-0.135	-0.01043956	0.46956044	-0.13571429	-0.0010989	3.12E-17	-0.01428571	-1.39E-17	0	0	
	110202	0.01	-0.14639194	-0.14639194	0.60690476	-1.47E-04	-9.52E-04	-9.52E-04	-0.01333333	0	0	
	120101	-0.135	-0.01043956	-0.01043956	5.81E-17	0.4789011	-0.13571429	-0.13571429	1.04E-17	0	0	···
	120102	0.01	-0.14639194	-0.00139194	-0.00904762	-0.14514652	0.615	-0.00904762	-0.12666667	0	0	
	120201	0.01	-0.00139194	-0.14639194	-0.00904762	-0.14514652	-0.00904762	0.615	-0.12666667	0	0	
	120202	-1.35E-16	0.00973901	0.00973901	-0.1572619	0.00997253	-0.15625	-0.15625	0.76333333	0	0	
	210101	0.15	-0.06354515	-0.06354515	-2.60E-17	-0.00668896	1.39E-17	-3.99E-17	6.94E-18	0	0	
	210102	-0.05	0.19300247	-0.00699753	-0.05654762	-7.37E-04	-0.00595238	7.46E-17	-6.94E-18	0	0	
	210201	-0.05	-0.00699753	0.19300247	-0.05654762	-7.37E-04	2.43E-17	-0.00595238	-6.94E-18	0	0	
	210202	3.00E-17	-0.05111658	-0.05111658	0.2382381	-1.18E-04	-6.19E-04	-6.19E-04	-0.00533333	0	0	
	220101	-0.05	-0.00699753	-0.00699753	2.13E-17	0.19926342	-0.05654762	-0.05654762	3.47E-18	0	0	
	220102	1.48E-17	-0.05111658	-0.00111658	-0.00588095	-0.05011753	0.2435	-0.00588095	-0.05066667	0	0	
	220201	5.21E-17	-0.00111658	-0.05111658	-0.00588095	-0.05011753	-0.00588095	0.2435	-0.05066667	0	0	
	220202	-4.73E-17	-2.29E-04	-2.29E-04	-0.05177473	-2.41E-05	-0.05098077	-0.05098077	0.28948718	0	0	

Table 2.7.9, continued

(The dotted lines indicate that some of the columns after the previous page have been omitted, and the remaining columns are listed in the present page.)

021111	021112	021121	021122	021211	021212	021221	021222	022111	022112	022121	022122	022211	022212	022221	022222
0	0	0	0	0	0	2.78E-16	2.78E-17	-2.22E-16	-1.67E-16	-8.88E-16	6.66E-16	2.78E-16	-8.33E-17	2.22E-16	4.44E-16
0	0	3.89E-16	1.11E-16	0	0	-3.33E-16	-2.78E-17	4.44E-16	2.22E-16	1.11E-15	0	-1.39E-16	-2.78E-17	-9.99E-16	-2.22E-16
0	0	-1.67E-16	0	0	0	-2.22E-16	8.33E-17	0	-2.78E-17	-2.22E-16	-6.66E-16	-2.22E-16	1.11E-16	-2.22E-16	0
0	0	1.11E-16	-8.33E-17	0	0	1.39E-16	-5.55E-17	-2.78E-17	-1.11E-16	-2.22E-16	6.66E-16	2.50E-16	-5.55E-17	2.22E-16	3.33E-16
0	0	-1.4285714	-1.80E-16	0	0	-4.16E-17	-4.86E-17	-1.4285714	6.94E-17	-4	6.66E-16	-1.80E-16	-6.59E-17	-8.88E-16	-5.41E-16
0	0	-0.0952381	-1.3333333	0	0	1.73E-16	-1.28E-16	-0.0952381	-1.3333333	7.63E-16	-4	3.82E-17	-2.39E-16	8.47E-16	1.67E-16
0	0	-0.0952381	6.94E-18	0	0	-1.3333333	-9.71E-17	-0.0952381	-6.94E-18	5.00E-16	7.63E-16	-1.3333333	1.32E-16	-4	6.94E-17
0	0	-0.0119048	-0.0833333	0	0	-0.0833333	-1.25	-0.0119048	-0.0833333	-2.84E-16	1.39E-16	-0.0833333	-1.25	1.46E-16	-4
0	0	-1.11E-16	-2.78E-17	0	0	1.11E-16	1.39E-17	-5.55E-17	-8.33E-17	0	1.67E-16	1.11E-16	0	1.11E-16	1.67E-16
0	0	1.39E-16	0	0	0	-1.25E-16	-1.39E-17	1.39E-16	1.39E-16	4.44E-16	1.11E-16	-5.55E-17	-1.39E-17	-3.33E-16	-1.39E-16
0	0	-5.55E-17	0	0	0	0	2.78E-17	5.55E-17	-4.16E-17	-1.11E-16	-3.89E-16	-2.78E-17	4.16E-17	3.33E-16	-2.78E-17
0	0	5.55E-17	-2.78E-17	0	0	2.78E-17	-5.55E-17	1.39E-17	-4.16E-17	-1.11E-16	2.22E-16	4.16E-17	-5.55E-17	0	5.55E-17
0	0	-0.5952381	-7.63E-17	0	0	-1.39E-17	-1.91E-17	-0.5952381	3.47E-17	-1.6666667	3.05E-16	-6.25E-17	-2.60E-17	-3.89E-16	-2.15E-16
0	0	-0.0619048	-0.5333333	0	0	6.25E-17	-5.55E-17	-0.0619048	-0.5333333	-0.0666667	-1.6	6.94E-18	-1.01E-16	2.98E-16	5.55E-17
0	0	-0.0619048	-3.47E-18	0	0	-0.5333333	-4.16E-17	-0.0619048	-6.94E-18	-0.0666667	2.84E-16	-0.5333333	4.86E-17	-1.6	4.16E-17
0	0	-0.0093407	-0.0525641	0	0	-0.0525641	-0.4807692	-0.0093407	-0.0525641	-0.0051282	-0.0615385	-0.0525641	-0.4807692	-0.0615385	-1.5384615

The set of equations given by (2.7.7), (2.7.8) and (2.7.9) may be combined to form

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[1] \mid \{P[0]\},\ \{P[2]\}\}. \tag{2.7.10}$$

From (2.7.10) and the set of equations of the form

$$\{P[1],\ P[2],\ P[3]\}\ ,$$

we get

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[2] \mid \{P[0]\},\ \{P[3]\}\}\ . \tag{2.7.11}$$

Similarly from the set of equations

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[n-1] \mid \{P[0]\},\ \{P[n+1]\}\} \tag{2.7.12}$$

and the set of equations

$$\{P[n-1],\ P[n],\ P[n+1]\} \tag{2.7.13}$$

we get

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[n] \mid \{P[0]\},\ \{P[n+1]\}\} \quad ,\ n = 3,\ 4,\ \dots. \tag{2.7.14}$$

When n=N is large enough, we may set each member of $P[N+1]$ to be zero to obtain

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[N] \mid \{P[0]\},\ \{P[N+1]\}\} \cong \{P_{i_1 j_1 i_2 j_2 i_3 j_3}[N] \mid \{P[0]\}\} \tag{2.7.15}$$

for $n = N-1,\ N-2,\ \dots, 1$.

Substituting the left side of (2.7.15) into (2.7.14) when $n = N-1$, we get

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[N-1] \mid \{P[0]\},\ \{P[N]\}\} \cong \{P_{i_1 j_1 i_2 j_2 i_3 j_3}[N-1] \mid \{P[0]\} \tag{2.7.16}$$

Similarly for n = N–2, N–3, N–4, …,1 we may repeat the substitution of $P[n+1]$ into (2.7.14) and obtain

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[n] \,\big|\, \{P[0]\}\}. \tag{2.7.17}$$

When n =1, we get from (2.7.17)

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[1] \,\big|\, \{P[0]\}\}. \tag{2.7.18}$$

Substituting the left side of (2.7.18) into (2.7.1), we get

$$\{P_{i_1 j_1 i_2 j_2 i_3 j_3}[0] \,\big|\, \{P[0]\}\} \tag{2.7.19}$$

An inspection of (2.7.19) reveals that among the 8 equations represented by (2.7.19), only 7 of them are linearly independent of each other. Hence, we need to include another linearly independent equation so that the resulting system of equations has a unique solution. Equating the sum of the left sides of (2.7.15) to (2.7.17) to the sum of the right sides of (2.7.15) to (2.7.17), we get an equation of the form

$$\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{i_3=0}^{2}\sum_{j_3=1}^{2}\sum_{\substack{n_1=0 \\ n_1+n_2+n_3\geq 1}}^{N}\sum_{n_2=0}^{N}\sum_{n_3=0}^{N} P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$$
$$= \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{i_3=0}^{2}\sum_{j_3=1}^{2} C_{i_1 j_1 i_2 j_2 i_3 j_3} P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][0]$$

where the $C_{i_1 j_1 i_2 j_2 i_3 j_3}$ are constants, or

$$1-\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{i_3=0}^{2}\sum_{j_3=1}^{2} P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][0]$$
$$= \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\sum_{i_3=0}^{2}\sum_{j_3=1}^{2} C_{i_1 j_1 i_2 j_2 i_3 j_3} P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][0] \tag{2.7.20}$$

Equation (2.7.20) together with 7 equations chosen from (2.7.19) will form a set of equations in 8 unknowns. Solving the set of 8 equations, we get the numerical answers for the $P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][0]$.

Then, from (2.7.17) and the values of the $P_{i_1 j_1 i_2 j_2 i_3 j_3}[0][0][0]$, we can get the numerical answers for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$, for the case when $n_1 + n_2 + n_3 \leq N$.

2.8 SIMULATED VALUE OF $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$

The probability $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ may also be estimated by using a simulation procedure similar to that described in Section 2.4.

In the simulation procedure, we need to know the approximate probability that an event from set A will occur in τ_k given the conditions of the system at the end of τ_{k-1} . Some examples of the above conditional probabilities are given in Table 2.8.1.

Suppose that at the end of τ_1,

$$\mathbf{h}^{(1)} = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1). \tag{2.8.1}$$

To generate $\mathbf{h}^{(k)}$ given the conditions specified by $\mathbf{h}^{(k-1)}$ for $k = 2,\ 3,\ ..., N_T$, we may first find out the set of possible events $E_1^{(k-1)},\ E_2^{(k-1)},\ ..., E_L^{(k-1)}$ which can occur in τ_k. Suppose $E_i^{(k-1)}$ occurs with probability $P_i^{(k-1)}$. We next generate a random number $U^{(k-1)}$ from the U(0,1) distribution. Let $P_0^{(k-1)} = 0$. If $\sum_{i=0}^{j-1} P_i^{(k-1)} < U^{(k-1)} < \sum_{i=0}^{j} P_i^{(k-1)}$, then event $A^{(k)} = E_j^{(k-1)}$ is said to have occurred and the resulting $\mathbf{h}^{(k)}$ can be determined.

In short, we generate $\left(A^{(2)}, A^{(3)}, \Lambda\ , A^{(N_T)}\right)$ starting from $\mathbf{h}^{(1)}$ given by (2.8.1). We repeat the generation of $\left(A^{(2)}, A^{(3)}, \Lambda\ , A^{(N_T)}\right)$ for N_s number of times. The probability $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ is then given approximately by the proportion of times the vector of characteristics given by $\mathbf{h} = \left(i_1,\ j_1,\ i_2,\ j_2,\ i_3,\ j_3,\ n_1,\ n_2,\ n_3\right)$ is obtained at $t = N_T \Delta t$.

Table 2.8.1 : Some examples of the approximate probability that event A will occur in τ_k given the conditions of system at the end of τ_{k-1} [v_i =1 2, or 3, $1 \le i \le 6$].

Conditions of system at the end of τ_{k-1}	Event occurring in τ_k	Approximate probability that event in Column 2 will occur given the conditions in Column 1.
$i_1^{(k-1)} = 1$	A_1	$\mu_{11}\Delta t$
$i_1^{(k-1)} = 2$	A_1	$\mu_{12}\Delta t$
$j_1^{(k-1)} = 1$	A_2	$\lambda_{11}\Delta t$
$j_1^{(k-1)} = 2$	A_3	$\lambda_{12}\Delta t q_{11}$
$j_1^{(k-1)} = 2$	A_4	$\lambda_{12}\Delta t q_{12}$
$j_1^{(k-1)} = 2$	A_5	$\lambda_{12}\Delta t q_{13}$
$i_2^{(k-1)} = 1$	A_6	$\mu_{21}\Delta t$
$i_2^{(k-1)} = 2$	A_6	$\mu_{22}\Delta t$
$j_2^{(k-1)} = 1$	A_7	$\lambda_{21}\Delta t$
$j_2^{(k-1)} = 2$	A_8	$\lambda_{22}\Delta t q_{21}$
$j_2^{(k-1)} = 2$	A_9	$\lambda_{22}\Delta t q_{22}$
$j_2^{(k-1)} = 2$	A_{10}	$\lambda_{22}\Delta t q_{23}$
$i_3^{(k-1)} = 1$	A_{11}	$\mu_{31}\Delta t$
$i_3^{(k-1)} = 2$	A_{11}	$\mu_{32}\Delta t$
$j_3^{(k-1)} = 1$	A_{12}	$\lambda_{31}\Delta t$
$j_3^{(k-1)} = 2$	A_{13}	$\lambda_{32}\Delta t q_{31}$
$j_2^{(k-1)} = 2$	A_{14}	$\lambda_{32}\Delta t q_{32}$
$j_2^{(k-1)} = 2$	A_{15}	$\lambda_{32}\Delta t q_{33}$
$i_1^{(k-1)} = v_1, j_1^{(k-1)} = v_2,$ $i_2^{(k-1)} = v_3, j_2^{(k-1)} = v_4,$ $i_3^{(k-1)} = v_5, j_3^{(k-1)} = v_6.$	A_{16}	$1-(\mu_{1v_1}+\lambda_{1v_2}+\mu_{2v_3}+\lambda_{2v_4}+\mu_{3v_5}+\lambda_{3v_6})\Delta t$

2.9 NUMERICAL RESULTS FOR DISTRIBUTION OF QUEUE LENGTH AND STATES OF ARRIVAL AND SERVICE PROCESSES IN A SYSTEM OF THREE DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES

Suppose $(\mu_{11},\mu_{12},\mu_{21},\mu_{22},\mu_{31},\mu_{32})$ = (10, 20, 10, 20, 10, 20), $(\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22},\lambda_{31},\lambda_{32})$ = (1, 2, 1, 2, 1, 2), and $q_{11} = q_{22} = q_{33} = 0.9$. The traffic intensities (ρ_1,ρ_2,ρ_3) in the three queues are then given respectively by

$$\rho_1 = (\mu_{11}{}^{-1} + \mu_{12}{}^{-1}) / (\lambda_{11}{}^{-1} + \lambda_{12}{}^{-1}) = 0.1 ,$$

$$\rho_2 = (\mu_{21}{}^{-1} + \mu_{22}{}^{-1}) / (\lambda_{21}{}^{-1} + \lambda_{22}{}^{-1}) = 0.1 ,$$

$$\rho_3 = (\mu_{31}{}^{-1} + \mu_{32}{}^{-1}) / (\lambda_{31}{}^{-1} + \lambda_{32}{}^{-1}) = 0.1.$$

By setting $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3] = 0$ when $n_1 + n_2 + n_3 = 4$, the probabilities $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ for $n_1 + n_2 + n_3 \le 4$ computed by using the proposed method in Section 2.7 and the simulation procedure in Section 2.8 are presented in Table (2.9.1) and Figure 2.9.1.

Next, Table 2.9.2 and Figure 2.9.2 show the results for the case when $(\mu_{11},\mu_{12},\mu_{21},\mu_{22},\mu_{31},\mu_{32})$= (10, 20, 10, 20, 10, 20), $(\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22},\lambda_{31},\lambda_{32})$ = (3, 4, 3, 4, 3, 4), $q_{11} = q_{22} = q_{33} = 0.9$, $N = 4$, and the traffic intensities are $\rho_1 = 0.2571$, $\rho_2 = 0.2571$, $\rho_3 = 0.2571$.

The tables and figures show that the results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ found by the proposed method agree well with those found by the simulation procedure.

Table 2.9.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) =$ (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (1, 2, 1, 2, 1, 2), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $\rho_3 = 0.1$, $N = 3$ and $N_s = 50000$].

$n_1n_2n_3$	$i_1 j_1 i_2 j_2 i_3 j_3$	Proposed method	Simulation procedure
000	010101	0.18761	0.19062
	010102	0.10666	0.10816
	010201	0.10666	0.10792
	010202	0.06033	0.05964
	020101	0.10907	0.1098
	020102	0.06121	0.05908
	020201	0.06120	0.06014
	020202	0.03441	0.03572
001	010111	0.01940	0.01992
	010112	0.00269	0.00234
	010121	0.00917	0.00948
	010122	0.00161	0.00134
	010211	0.01082	0.01026
	010212	0.00124	0.00118
	010221	0.00513	0.00520
	010222	0.00080	0.00050
	020111	0.01117	0.01080
	020112	0.00129	0.00102
	020121	0.00536	0.00504
	020122	0.00084	0.00078
	020211	0.00592	0.00616
	020212	0.00056	0.00058
	020221	0.00285	0.00300
	020222	0.00040	0.00032
010	011101	0.01939	0.02018
	011102	0.01082	0.01028
	011201	0.00269	0.00258
	011202	0.00124	0.00108
	012101	0.00916	0.00954
	012102	0.00513	0.00556
	012201	0.00161	0.00176
	012202	0.00080	0.00066
	021101	0.01116	0.01116
	021102	0.00592	0.00588
	021201	0.00129	0.00120
	021202	0.00056	0.00068
	022101	0.00536	0.00608
	022102	0.00285	0.00286
	022201	0.00084	0.00044
	022202	0.00040	0.00030
100	110101	0.01978	0.02024
	110102	0.01097	0.01134
	110201	0.01097	0.01040
	110202	0.00588	0.00572
	120101	0.00278	0.00238
	120102	0.00126	0.00108
	120201	0.00126	0.00082
	120202	0.00055	0.00058
	210101	0.00934	0.00954
	210102	0.00521	0.00502
	210201	0.00520	0.0048
	210202	0.00282	0.00278
	220101	0.00171	0.00178
	220102	0.00082	0.00066
	220201	0.00082	0.00064
	220202	0.00039	0.00028
	Total	**0.96542**	**0.96730**

Table 2.9.2 : Comparison of results for $P_{i_1j_1i_2j_2i_3j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32})$ = (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (3, 4, 3, 4, 3, 4), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.2571$, $\rho_2 = 0.2571$, $\rho_3 = 0.2571$, $N = 4$ and $N_s = 20000$].

$n_1n_2n_3$	$i_1j_1i_2j_2i_3j_3$	Proposed method	Simulation procedure
	010101	0.04988	0.04790
	010102	0.04964	0.04950
	010201	0.04956	0.05175
000	010202	0.05036	0.05480
	020101	0.05349	0.05165
	020102	0.05327	0.05230
	020201	0.05322	0.05085
	020202	0.05344	0.05310
	010111	0.01927	0.01735
	010112	0.00494	0.00495
001	010121	0.00800	0.00770
	010122	0.00296	0.00300
	010211	0.01607	0.01820
	010212	0.00466	0.00495
	010221	0.00704	0.00745
	010222	0.00279	0.00220
	020111	0.01772	0.01650
	020112	0.00524	0.00470
	020121	0.00797	0.00875
	020122	0.00323	0.00290
	020211	0.01707	0.01725
	020212	0.00443	0.00390
	020221	0.00757	0.00780
	020222	0.00288	0.00265
	011101	0.01912	0.01950
	011102	0.01602	0.01720
	011201	0.00487	0.00540
010	011202	0.00463	0.00415
	012101	0.00794	0.00875
	012102	0.00703	0.00840
	012201	0.00291	0.00305
	012202	0.00278	0.00320
	021101	0.01765	0.01730
	021102	0.01704	0.01680
	021201	0.00520	0.00440
	021202	0.00442	0.00350
	022101	0.00793	0.00785
	022102	0.00756	0.00800
	022201	0.00320	0.00255
	022202	0.00287	0.00250
	110101	0.02011	0.01835
	110102	0.01681	0.01800
	110201	0.01682	0.01660
100	110202	0.01664	0.01595
	120101	0.00524	0.00480
	120102	0.00488	0.00355
	120201	0.00487	0.00430
	120202	0.00424	0.00425
	210101	0.00831	0.00805
	210102	0.00734	0.00645
	210201	0.00733	0.00730
	210202	0.00724	0.00750
	220101	0.00335	0.00270
	220102	0.00304	0.00270
	220201	0.00304	0.00280
	220202	0.00278	0.00250
	Total	**0.80809**	**0.80045**

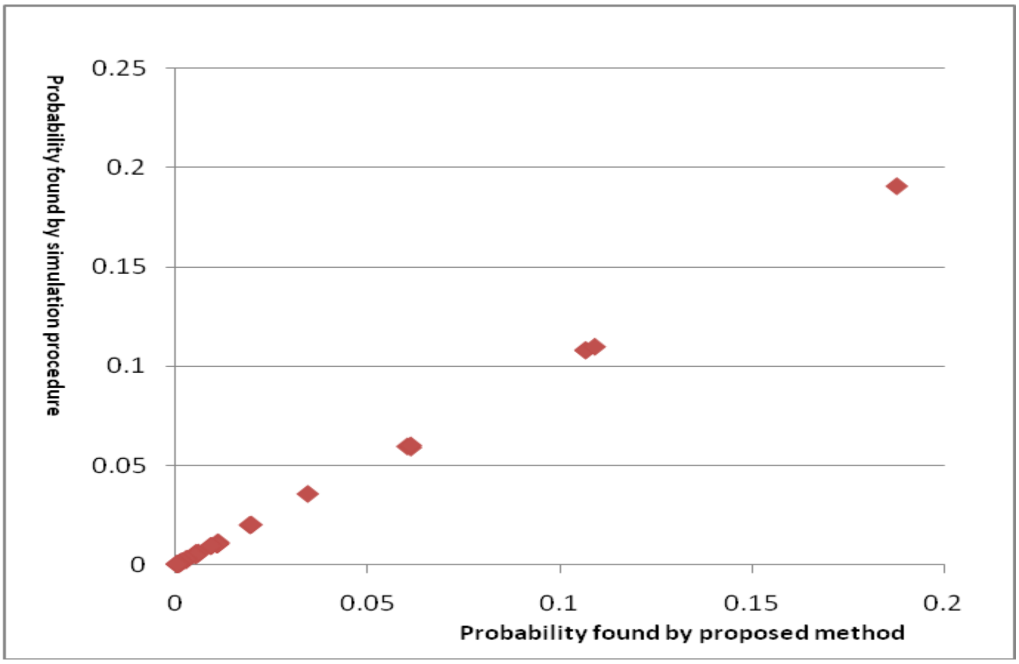

Figure 2.9.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) =$ (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (1, 2, 1, 2, 1, 2), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $\rho_3 = 0.1$, $N = 3$ and $N_s = 50000$].

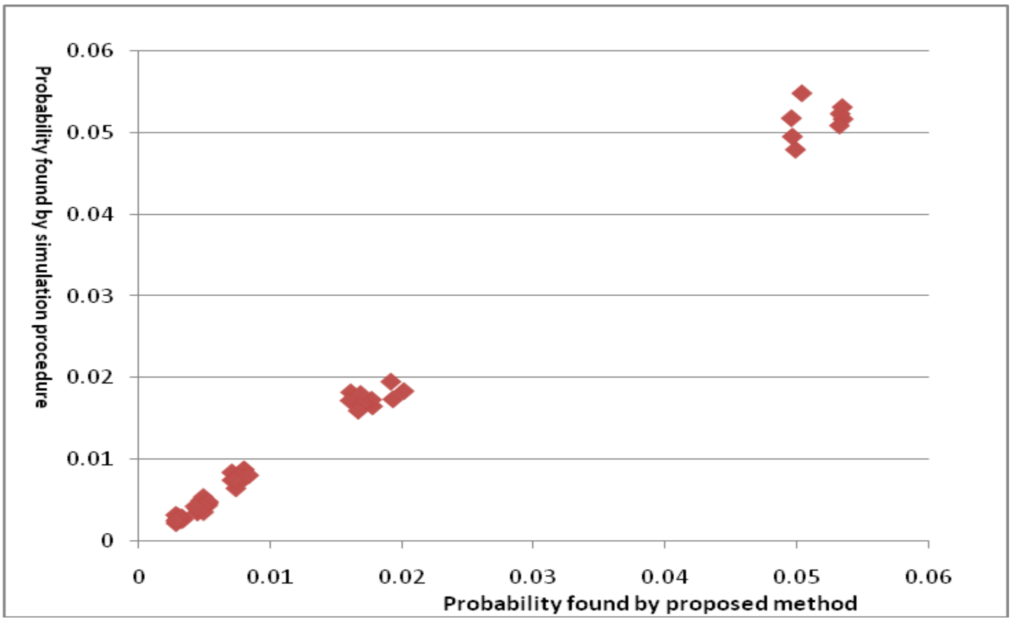

Figure 2.9.2 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) =$ (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (3, 4, 3, 4, 3, 4), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.2571$, $\rho_2 = 0.2571$, $\rho_3 = 0.2571$, $N = 4$ and $N_s = 20000$].

2.10 DERIVATION OF THE FORWARD EQUATIONS IN A SYSTEM OF M DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES

Consider a system of M Hypo(2)/Hypo(2)/1 queues in which the customer who arrives at queue m has a probability of $q_{mm'} \geq 0$ of joining queue m', where $m \in \{1, 2, \ldots, M\}$, $m' \in \{1, 2, \ldots, M\}$ and $\sum_{m'=1}^{M} q_{mm'} = 1$.

Assume that for $1 \leq m \leq M$, the interarrival time T in queue m has a hypoexponential distribution in two phases with parameters λ_{m1}, λ_{m2} , and the service time S in queue m has another hypoexponential distribution in two phases with parameters μ_{m1}, μ_{m2}.

As in Section 2.2, let $\Delta t > 0$ be a small increment in time and $\tau_k = ((k-1)\Delta t, k\Delta t]$ a time interval, $k = 1, 2, \ldots$. Next let $P^{(k)}_{i_1 j_1 i_2 j_2 .. i_M j_M}[n_1][n_2]...[n_M]$ be the probability that at the end of the interval τ_k , the number of customers in the system is n_m in queue m, the service process in queue m is in the state i_m and the arrival process in queue m is in the state j_m , $1 \leq m \leq M$, $i_m \in \{0, 1, 2\}$ and $j_m \in \{1, 2\}$. Assume that

$$P_{i_1 j_1 i_2 j_2 .. i_M j_M}[n_1][n_2]\ldots[n_M] = \lim_{k \to \infty} P^{(k)}_{i_1 j_1 i_2 j_2 .. i_M j_M}[n_1][n_2]\ldots[n_M]$$

exists.

Let $\mathbf{h}^{(k)}$ be the vector

$$\mathbf{h}^{(k)} = \left(\; i_1^{(k)}, j_1^{(k)}, \; i_2^{(k)}, j_2^{(k)}, \; ..., \; i_M^{(k)}, j_M^{(k)}, \; n_1^{(k)}, n_2^{(k)}, \; ..., n_M^{(k)} \right)$$

of which the components are respectively the values of $i_1, j_1, i_2, j_2, \ldots, i_M, j_M, n_1, n_2, \ldots, n_M$ at the end of τ_k . Again we refer to $\mathbf{h}^{(k)}$ as the vector of characteristics of the queueing system at the end of τ_k .

The value $\mathbf{h}^{(k)}$ may be developed from $\mathbf{h}^{(k-1)}$ after some appropriate activities in the interval τ_k. The set of possible activities may be denoted by a set A = {A_1, A_2, …, A_W} where A_w = (A_{w1}, $A_{w2, \ldots,}$ A_{w3M}), $1 \le w \le W$. The meanings of the components in A_w are explained in Table 2.10.1.

Table 2.10.1 : The meanings of the components A_{wj} in A_w, $1 \le j \le 2M$.

j	A_{wj}	**Meaning**
1	1	A transition in the state of the service process in queue 1 occurs in τ_k .
1	0	A transition in the state of the service process in queue 1 does not occur in τ_k .
1	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of the service process in queue 1 has occurred in τ_k is not relevant.
2	1	A transition in the state of the arrival process in queue 1 occurs in τ_k .
2	0	A transition in the state of the arrival process in queue 1 does not occur in τ_k .
Λ	Λ	Λ
$2M-1$	1	A transition in the state of the service process in queue $(M-1)$ occurs in τ_k .
$2M-1$	0	A transition in the state of the service process in queue $(M-1)$ does not occur in τ_k .
$2M-1$	–1	Queue $(M-1)$ is empty at the end of τ_{k-1} and whether a transition in the state of the service process in queue $(M-1)$ has occurred in τ_k is not relevant.
$2M$	1	A transition in the state of the arrival process in queue M occurs in τ_k .
$2M$	0	A transition in the state of the arrival process in queue M does not occur in τ_k .

The meanings of A_{wj} in for $(2M+1) \le j \le 3M$ are given below:

$$
A_{w(2M+1)} = \begin{cases}
11, & \text{if the arriving customer in queue 1 stays back in queue 1.} \\
12, & \text{if the arriving customer in queue 1 goes to queue 2.} \\
13, & \text{if the arriving customer in queue 1 goes to queue 3.} \\
\Lambda\ , & \qquad\qquad \Lambda \\
1(M-1), & \text{if the arriving customer in queue 1 goes to queue } (M-1). \\
1M, & \text{if the arriving customer in queue 1 goes to queue } M. \\
\text{-1}, & \text{no customers arrive in queue 1 and it is not relevant to find out} \\
 & \text{whether the arriving customer staying back or going elsewhere.}
\end{cases}
$$

$$
A_{w(2M+2)} = \begin{cases}
21, & \text{if the arriving customer in queue 2 goes to queue 1.} \\
22, & \text{if the arriving customer in queue 2 stays back in queue 2.} \\
23, & \text{if the arriving customer in queue 2 goes to queue 3.} \\
\Lambda\ , & \qquad\qquad \Lambda \\
2(M-1), & \text{if the arriving customer in queue 2 goes to queue } (M-1). \\
2M, & \text{if the arriving customer in queue 2 goes to queue } M. \\
\text{-1}, & \text{no customers arrive in queue 2 and it is not relevant to find out} \\
 & \text{whether the arriving customer staying back or going elsewhere.}
\end{cases}
$$

$\Lambda \qquad \Lambda \qquad\qquad \Lambda$

$$
A_{w(3M)} = \begin{cases}
M1, & \text{if the arriving customer in queue } M \text{ goes to queue 1.} \\
M2, & \text{if the arriving customer in queue } M \text{ goes to queue 2.} \\
M3, & \text{if the arriving customer in queue } M \text{ goes to queue 3.} \\
\Lambda\ , & \qquad\qquad \Lambda \\
M(M-1), & \text{if the arriving customer in queue } M \text{ goes to queue } (M-1). \\
MM, & \text{if the arriving customer in queue } M \text{ stays back in queue } M. \\
\text{-1}, & \text{no customers arrive in queue } M \text{ and it is not relevant to find out} \\
 & \text{whether the arriving customer staying back or going elsewhere.}
\end{cases}
$$

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form as has been done in Sections 2.2 and 2.6.

Next, the codes for the corresponding balance equations similar to (2.2.2) and (2.6.2) may be obtained.

2.11 COMPUTATION OF THE VALUE OF $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M]$

Before solving the balance equations to obtain the stationary queue length distribution, we first introduce the following notations. Let

(a) $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n]$ be a value of $P_{i_1 j_1 i_2 j_2 i \Lambda_3 j_3}[n_1][n_2]\Lambda\,[n_M]$ of which $n_1 + n_2 + ... + n_M = n$.

(b) $\{P\,[n_1][n_2]\Lambda\,[n_M]\}$ the set consisting of all the possible $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M]$.

(c) $\{P\,[n]\}$ a set formed by the $\{P\,[n_1][n_2]\Lambda\,[n_M]\}$ of which $n_1 + n_2 + ... + n_M = n$.

(d) $\{P\,[n],\ \ P\,[n+1],\ \ P\,[n+2]\}$ the set of equations of the form

$$\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\Lambda \sum_{i_M=0}^{2}\sum_{j_M=1}^{2} a_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M} P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n]$$
$$+\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\Lambda \sum_{i_M=0}^{2}\sum_{j_M=1}^{2} b_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M} P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n+1]$$
$$+\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\Lambda \sum_{i_M=0}^{2}\sum_{j_M=1}^{2} c_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M} P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n+2] = 0$$

where $a_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}$, $b_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}$, and $c_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}$ are constants.

(e) $\left(P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M] \;\middle|\; \{P\,[0]\},\ \{P\,[n+1]\}\right)$ an equation of the form

$$P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M] = \sum_{i'_1=0}^{2}\sum_{j'_1=1}^{2}\sum_{i'_2=0}^{2}\sum_{j'_2=1}^{2}\Lambda\ \sum_{i'_M=0}^{2}\sum_{j'_M=1}^{2} d_{i'_1 j'_1 i'_2 j'_2 \Lambda\, i'_M j'_M}\, P_{i'_1 j'_1 i'_2 j'_2 \Lambda\, i'_M j'_{M3}}[0]$$

$$+\sum_{i'_1=0}^{2}\sum_{j'_1=1}^{2}\sum_{i'_2=0}^{2}\sum_{j'_2=1}^{2}\Lambda\ \sum_{i'_M=0}^{2}\sum_{j'_M=1}^{2} e_{i'_1 j'_1 i'_2 j'_2 \Lambda\, i'_M j'_M}\, P_{i'_1 j'_1 i'_2 j'_2 \Lambda\, i'_M j'_{M3}}[n+1]$$

where $d_{i'_1 j'_1 i'_2 j'_2 \Lambda\, i'_M j'_{M3}}$ and $e_{i'_1 j'_1 i'_2 j'_2 \Lambda\, i'_M j'_{M3}}$ are constants.

With the above notations, the balance equations given in the coded form can be represented as

$$\{P\,[0],\ \ P\,[1]\} \tag{2.11.1}$$

and

$$\{P\,[n-1],\ \ P\,[n],\ \ P\,[n+1]\} \qquad ,n = 1,\ 2,\ \ldots \tag{2.11.2}$$

To solve (2.11.1) and (2.11.2), we first combined the set of equations given by $\{P\,[0], P\,[1]\}$ and $\{P\,[0], P\,[1], P\,[2]\}$. We then solve for $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n]$ in terms of the $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[2]$ for n = 0, 1 to get

$$\left(P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n] \;\middle|\; P\,[2]\right) \quad , \quad n = 0,\ 1. \tag{2.11.3}$$

From the equations represented by $\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0] \;|\; P\,[2]\}$, we get M sets of equations of the forms

$$\begin{aligned}\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0][0]\Lambda\,[0][2] \;\Big|\; & \{P\,[0][0][0]\Lambda\,[0][0]\},\ \{P\,[0][0][0]\Lambda\,[2][0]\},\\ & ,\Lambda\ ,\{P\,[2][0][0]\Lambda\,[0][0],\ \{P\,[1][1][0]\Lambda\,[0][0]\}\\ & ,\{P\,[1][0][1]\Lambda\,[0][0]\},\Lambda\ ,\ \{P\,[0][0][0]\Lambda\,[1][1]\}\ \}\end{aligned} \tag{2.11.4}$$

$$\begin{aligned}\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0][0]\Lambda\,[2][0] \;\Big|\; & \{P\,[0][0][0]\Lambda\,[0][0]\},\ \{P\,[0][0][0]\Lambda\,[0][2]\},\\ & ,\Lambda\ ,\{P\,[2][0][0]\Lambda\,[0][0],\ \{P\,[1][1][0]\Lambda\,[0][0]\}\\ & ,\{P\,[1][0][1]\Lambda\,[0][0]\},\Lambda\ ,\ \{P\,[0][0][0]\Lambda\,[1][1]\}\ \}\end{aligned} \tag{2.11.5}$$

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and

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[2][0][0]\Lambda\,[0][0] \;\Big|\; \{P\,[0][0][0]\Lambda\,[0][0]\},\ \{P\,[0][0][0]\Lambda\,[0][2]\},$$
$$,\Lambda\ ,\{P\,[0][2][0]\Lambda\,[0][0],\ \{P\,[1][1][0]\Lambda\,[0][0]\}$$
$$,\{P\,[1][0][1]\Lambda\,[0][0]\},\Lambda\ ,\ \{P\,[0][0][0]\Lambda\,[1][1]\}\ \}$$

(2.11.6)

respectively.

From (2.11.4) and $\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0][0]\Lambda\,[0][1] \;\Big|\; P\,[2]\}$ in (2.11.3), we get a set of equations of the form

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0][0]\Lambda\,[0][1] \;\Big|\; \{P\,[0][0][0]\Lambda\,[0][0]\},\ \{P\,[0][0][0]\Lambda\,[2][0]\}$$
$$,\Lambda\ ,\{P\,[2][0][0]\Lambda\,[0][0]\},\{P\,[1][1][0]\Lambda\,[0][0]\}$$
$$,\{P\,[1][0][1]\Lambda\,[0][0]\},\Lambda\ ,\ \{P\,[0][0][0]\Lambda\,[1][1]\}\ \}$$

(2.11.7)

Next from (2.11.5) and $\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0][0]\Lambda\,[1][0] \;\Big|\; P\,[2]\}$ in (2.11.3), we get a set of equations of the form

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0][0]\Lambda\,[1][0] \;\Big|\; \{P\,[0][0][0]\Lambda\,[0][0]\},\ \{P\,[0][0][0]\Lambda\,[0][2]\}$$
$$,\Lambda\ ,\{P\,[2][0][0]\Lambda\,[0][0]\},\{P\,[1][1][0]\Lambda\,[0][0]\}$$
$$,\{P\,[1][0][1]\Lambda\,[0][0]\},\Lambda\ ,\ \{P\,[0][0][0]\Lambda\,[1][1]\}\ \}$$

(2.11.8)

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Similarly from (2.11.6) and $\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[1][0][0]\Lambda\,[0][0] \;\Big|\; P\,[2]\}$ in (2.11.3), we get a set of equations of the form

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[1][0][0]\Lambda\,[0][0] \;\Big|\; \{P\,[0][0][0]\Lambda\,[0][0]\},\ \{P\,[0][0][0]\Lambda\,[0][2]\}$$
$$,\Lambda\ ,\{P\,[0][2][0]\Lambda\,[0][0]\},\{P\,[1][1][0]\Lambda\,[0][0]\}$$
$$,\{P\,[1][0][1]\Lambda\,[0][0]\},\Lambda\ ,\ \{P\,[0][0][0]\Lambda\,[1][1]\}\ \}\ .$$

(2.11.9)

The set of equations given by (2.11.7) to (2.11.9) and other similar equations may be combined to form

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[1] \;\big|\; \{P[0]\},\ \{P[2]\}\}. \tag{2.11.10}$$

From (2.11.10) and the set of equations of the form

$$\{P[1],\ \ P[2],\ \ P[3]\}\ \ ,$$

we get

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[2] \;\big|\; \{P[0]\},\ \{P[3]\}\}\ . \tag{2.11.11}$$

Similarly from the set of equations

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n-1] \;\big|\; \{P[0]\},\ \{P[n+1]\}\} \tag{2.11.12}$$

and the set of equations

$$\{P[n-1],\ \ P[n],\ \ P[n+1]\} \tag{2.11.13}$$

we get

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n] \;\big|\; \{P[0]\},\ \{P[n+1]\}\} \qquad ,\ \ n=3,\ 4,\ \ldots . \tag{2.11.14}$$

When *n*=*N* is large enough, we may set each member of $P[N+1]$ to be zero to obtain

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[N] \;\big|\; \{P[0]\},\ \{P[N+1]\}\} \cong \{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[N] \;\big|\; \{P[0]\}\} \tag{2.11.15}$$

for $n = N–1, N–2, \ldots, 1$.

Substituting the left side of (2.11.15) into (2.11.14) when $n = N-1$, we get

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[N-1] \;\big|\; \{P[0]\},\ \{P[N]\}\} \cong \{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[N-1] \;\big|\; \{P[0]\}. \tag{2.11.16}$$

Similarly for $n = N–2, N–3, N–4, \ldots, 1$ we may repeat the substitution of $P[n+1]$ into (2.11.14) and obtain

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n] \;\big|\; \{P[0]\}\}. \tag{2.11.17}$$

When $n = 1$, we get from (2.11.17)

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[1] \;\Big|\; \{P[0]\}\}. \qquad (2.11.18)$$

Substituting the left side of (2.11.18) into (2.11.1), we get

$$\{P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0] \;\Big|\; \{P[0]\}\}. \qquad (2.11.19)$$

An inspection of (2.11.19) reveals that among the 2^M equations represented by (2.11.19), only $(2^M - 1)$ of them are linearly independent of each other. Hence, we need to include another linearly independent equation so that the resulting system of equations has a unique solution. Equating the sum of the left sides of (2.11.15) to (2.11.17) to the sum of the right sides of (2.11.15) to (2.11.17), we get an equation of the form

$$\sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\Lambda\sum_{i_M=0}^{2}\sum_{j_M=1}^{2}\sum_{\substack{n_1=0 \\ n_1+n_2+\Lambda+n_M \geq 1}}^{N}\sum_{n_2=0}^{N}\Lambda\sum_{n_M=0}^{N} P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M]$$
$$= \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\Lambda\sum_{i_M=0}^{2}\sum_{j_M=1}^{2} C_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M} P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0]\Lambda\,[0]$$

where the $C_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}$ are constants, or

$$1 - \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\Lambda\sum_{i_M=0}^{2}\sum_{j_M=1}^{2} P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0]\Lambda\,[0]$$
$$= \sum_{i_1=0}^{2}\sum_{j_1=1}^{2}\sum_{i_2=0}^{2}\sum_{j_2=1}^{2}\Lambda\sum_{i_M=0}^{2}\sum_{j_M=1}^{2} C_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M} P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0]\Lambda\,[0]\,. \qquad (2.11.20)$$

Equation (2.11.20) together with $(2^M - 1)$ equations chosen from (2.11.19) will form a set of equations in 2^M unknowns. Solving the set of 2^M equations, we get the numerical answers for the $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0]\Lambda\,[0]$.

Then, from (2.11.17) and the values of the $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[0][0]\Lambda\,[0]$, we can get the numerical answers for $P_{i_1 j_1 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M]$, for the case when $n_1 + n_2 + \Lambda + n_M \leq N$.