CHAPTER 3

A SYSTEM OF *M* HYPO(2)/HYPO(2)/1 QUEUES IN WHICH CUSTOMERS MAY CROSS OVER TO SHORTEST QUEUES

3.1 INTRODUCTION

Consider a system of M dependent Hypo(2)/Hypo(2)/1 queues which follow an interaction scheme below.

"The customer who arrives at queue m will stay back in queue m with probability q_{mm} or cross over to one of the I_s shortest queues (among the remaining $M-1$ queues) with probability $(1-q_{mm})/I_s$."

We may refer to the above interaction scheme as the Second Interaction Scheme, and call the interaction scheme in Chapter 2 the First Interaction Scheme.

A method is proposed in Section 3.2 to derive the stationary joint queue length distribution in a system of three Hypo(2)/Hypo(2)/1 queues which follow the Second Interaction Scheme.

In Section 3.3, we show some numerical results for the joint distribution of the queue length and states of arrival and service processes obtained by using the proposed method and simulation procedure.

In Section 3.4, we describe how the method in Section 3.2 may be adapted to find the joint queue length distribution in a system of M Hypo(2)/Hypo(2)/1 queues which follow the Second Interaction Scheme.

3.2 DERIVATION OF THE FORWARD EQUATIONS IN A SYSTEM OF THREE HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE SECOND INTERACTION SCHEME

Consider a system of three Hypo(2)/Hypo(2)/1 queues which follow the Second Interaction Scheme. An illustration of the possible crossing over to a shortest queue is given in Figure 3.2.1.

Let $P^{(k)}_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ be the probability that at the end of the interval τ_k, the number of customers in the system is n_m in queue m (including the customer that is being served), the service process in queue m is in the state i_m and the arrival process in queue m is in the state j_m, $m \in \{1, 2, 3\}$, $i_m \in \{0, 1, 2\}$ and $j_m \in \{1, 2\}$. If queue m is empty, then we may define the state of the service process to be zero.

Assume that

$$P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3] = \lim_{k\to\infty} P^{(k)}_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$$

exists.

Let $\mathbf{h}^{(k)}$ be the vector

$$\mathbf{h}^{(k)} = \left(i_1^{(k)},\ j_1^{(k)},\ i_2^{(k)},\ j_2^{(k)},\ i_3^{(k)},\ j_3^{(k)},\ n_1^{(k)},\ n_2^{(k)},\ n_3^{(k)} \right)$$

of which the components are respectively the values of i_1, j_1, i_2, j_2, i_3, j_3, n_1, n_2, n_3 at the end of τ_k. Again we refer to $\mathbf{h}^{(k)}$ as the vector of characteristics of the queueing system at the end of τ_k.

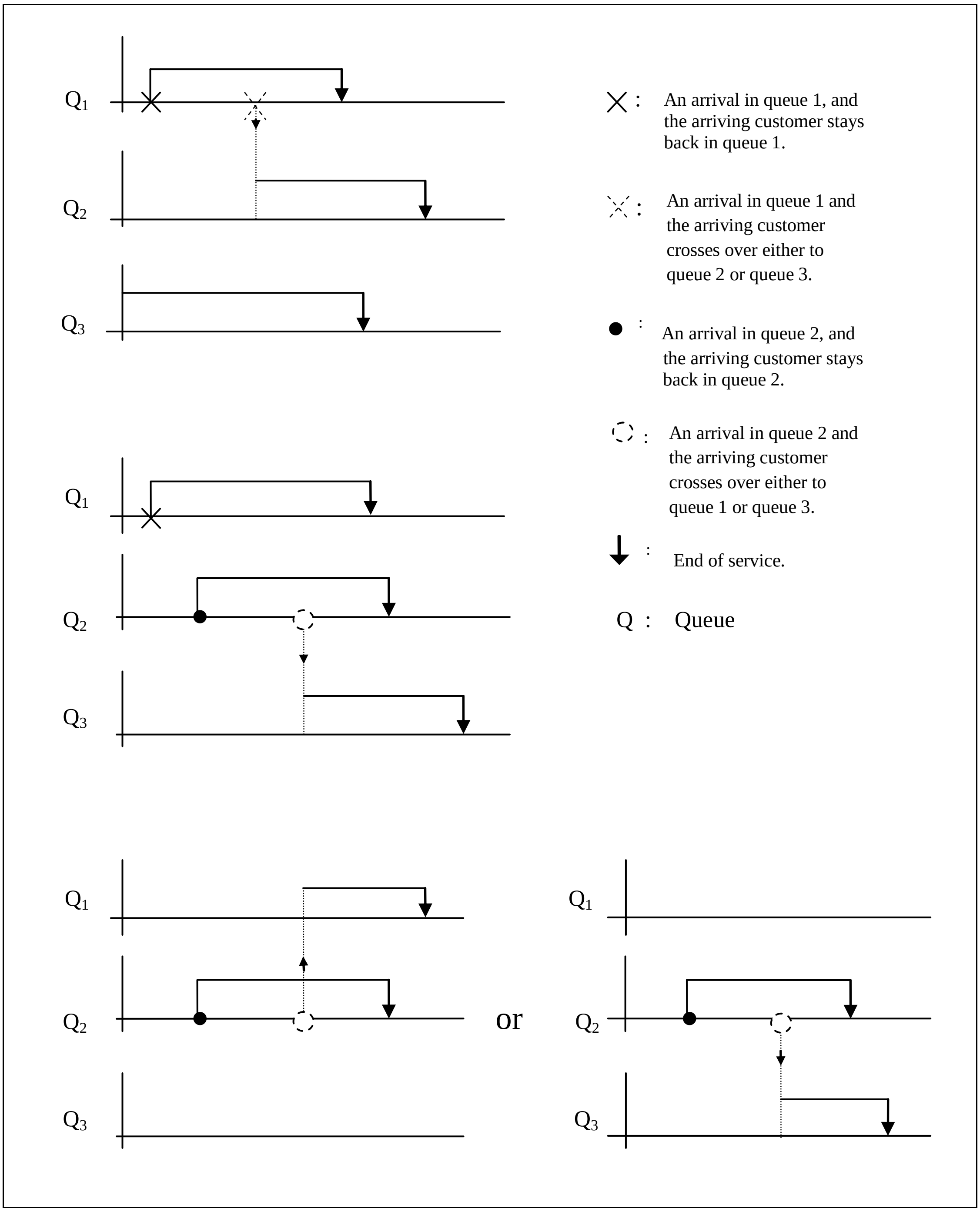

Figure 3.2.1 : Crossing over of an arriving customer in a system of three one-server queues (Q_1, Q_2, Q_3) which follow the Second Interaction Scheme.

The value $\mathbf{h}^{(k)}$ may be developed from $\mathbf{h}^{(k-1)}$ after some appropriate activities in the interval τ_k. The set of possible activities may be denoted by the set A = {A_1, A_2, …, A_{116}}. Some feasible events in A are shown below.

$$A_1 = (1, 0, 0, 0, 0, 0, -1, -1, -1)$$

$$A_2 = (0, 1, 0, 0, 0, 0, -1, -1, -1)$$

$$A_3 = (0, 1, 0, 0, 0, 0, 11, -1, -1)$$

$$A_4 = (0, 1, 0, 0, 0, 0, 12, -1, -1)$$

$$A_5 = (0, 1, 0, 0, 0, 0, 13, -1, -1)$$

$$A_6 = (0, 0, 1, 0, 0, 0, -1, -1, -1)$$

$$A_7 = (0, 0, 0, 1, 0, 0, -1, -1, -1)$$

$$A_8 = (0, 0, 0, 1, 0, 0, -1, 21, -1)$$

$$A_9 = (0, 0, 0, 1, 0, 0, -1, 22, -1)$$

$$A_{10} = (0, 0, 0, 1, 0, 0, -1, 23, -1)$$

$$A_{11} = (0, 0, 0, 0, 1, 0, -1, -1, -1)$$

$$A_{12} = (0, 0, 0, 0, 0, 1, -1, -1, -1)$$

$$A_{13} = (0, 0, 0, 0, 0, 1, -1, -1, 31)$$

$$A_{14} = (0, 0, 0, 0, 0, 1, -1, -1, 32)$$

$$A_{15} = (0, 0, 0, 0, 0, 1, -1, -1, 33)$$

$$A_{16} = (0, 0, 0, 0, 0, 0, -1, -1, -1)$$

The positions, values and meanings of the first six components in A_w are given in Table 2.6.1. The meanings of the seventh, eighth and ninth components (A_{w7} and A_{w8} and A_{w9}) of A_w are explained below:

$$A_{w7} = \begin{cases} 11, & \text{if the arriving customer in queue 1 stays back in queue 1.} \\ 1j, & \text{if the arriving customer in queue 1 goes to queue } j \text{ after noting that queue } j \text{ is one of the queues with the shortest queue size.} \\ -1, & \text{no customers arrive in queue 1 and it is not relevant to find out whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w8} = \begin{cases} 22, & \text{if the arriving customer in queue 2 stays back in queue 2.} \\ 2j, & \text{if the arriving customer in queue 2 goes to queue } j \text{ after noting that queue } j \text{ is one of the queues with the shortest queue size.} \\ -1, & \text{no customers arrive in queue 2 and it is not relevant to find out whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w9} = \begin{cases} 33, & \text{if the arriving customer in queue 3 stays back in queue 3.} \\ 3j, & \text{if the arriving customer in queue 3 goes to queue } j \text{ after noting that queue } j \text{ is one of the queues with the shortest queue size.} \\ -1, & \text{no customers arrive in queue 3 and it is not relevant to find out whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

The complete set of feasible events of A is shown by Appendix B.

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search for all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form. An example of the codes is given in Table 3.2.1.

In each row of Table 3.2.1,

Columns 1 – 9 give the components of $\mathbf{h}^{(k)}$.

Columns 10 – 18 give the components of $\mathbf{h}^{(k-1)}$.

Columns 19 – 51 give respectively the powers of $(1-\mu_{11}\Delta t)$, $(1-\mu_{12}\Delta t)$, $(1-\lambda_{11}\Delta t)$, $(1-\lambda_{12}\Delta t)$, $(1-\mu_{21}\Delta t)$, $(1-\mu_{22}\Delta t)$, $(1-\lambda_{21}\Delta t)$, $(1-\lambda_{22}\Delta t)$, $(1-\mu_{31}\Delta t)$, $(1-\mu_{32}\Delta t)$, $(1-\lambda_{31}\Delta t)$, $(1-\lambda_{32}\Delta t)$, $(\mu_{11}\Delta t)$, $(\mu_{12}\Delta t)$, $(\lambda_{11}\Delta t)$, $(\lambda_{12}\Delta t)$, $(\mu_{21}\Delta t)$, $(\mu_{22}\Delta t)$, $(\lambda_{21}\Delta t)$, $(\lambda_{22}\Delta t)$, $(\mu_{31}\Delta t)$, $(\mu_{32}\Delta t)$, $(\lambda_{31}\Delta t)$, $(\lambda_{32}\Delta t)$, (q_{11}), $(1-q_{11})$, $(1-q_{11})/2$, (q_{22}), $(1-q_{22})$, $(1-q_{22})/2$, (q_{33}), $(1-q_{33})$ and $(1-q_{33})/2$.

The multiplication of $(1-\mu_{11}\Delta t)$, $(1-\mu_{12}\Delta t)$, $(1-\lambda_{11}\Delta t)$, $(1-\lambda_{12}\Delta t)$, $(1-\mu_{21}\Delta t)$, $(1-\mu_{22}\Delta t)$, $(1-\lambda_{21}\Delta t)$, $(1-\lambda_{22}\Delta t)$, $(1-\mu_{31}\Delta t)$, $(1-\mu_{32}\Delta t)$, $(1-\lambda_{31}\Delta t)$, $(1-\lambda_{32}\Delta t)$, $(\mu_{11}\Delta t)$, $(\mu_{12}\Delta t)$, $(\lambda_{11}\Delta t)$,

($\lambda_{12}\Delta t$), ($\mu_{21}\Delta t$), ($\mu_{22}\Delta t$), ($\lambda_{21}\Delta t$), ($\lambda_{22}\Delta t$), ($\mu_{31}\Delta t$), ($\mu_{32}\Delta t$), ($\lambda_{31}\Delta t$), ($\lambda_{32}\Delta t$), (q_{11}), ($1-q_{11}$), ($1-q_{11}$)/2, (q_{22}), ($1-q_{22}$), ($1-q_{22}$)/2, (q_{33}), ($1-q_{33}$), ($1-q_{33}$)/2 raised respectively to the corresponding powers will represent the probability of occurrence of the corresponding event which may be represented by an element in A.

Table 3.2.1 : An example of the codes of $\mathbf{h}^{(k)}, \mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k .

$\mathbf{h}^{(k)}$	$\mathbf{h}^{(k-1)}$	**Power**		
0 1 1 1 1 1 0 2 2	0 1 1 1 1 1 0 2 2	0 0 1 0 1 0 1 0 1 0 1 0	0 0 1 0 1 0 1 0 1 0 1 0	0 0 0 0 0 0 0 0 0
0 1 1 1 1 1 0 2 2	2 1 1 1 1 1 1 2 2	0 0 1 0 1 0 1 0 1 0 1 0	0 1 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
0 1 1 1 1 1 0 2 2	0 2 1 1 1 1 0 1 2	0 0 0 0 1 0 1 0 1 0 1 0	0 0 0 1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0
0 1 1 1 1 1 0 2 2	0 2 1 1 1 1 0 2 1	0 0 0 0 1 0 1 0 1 0 1 0	0 0 0 1 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0
0 1 1 1 1 1 0 2 2	0 1 2 1 1 1 0 3 2	0 0 1 0 0 0 1 0 0 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0
0 1 1 1 1 1 0 2 2	0 1 1 2 1 1 0 1 2	0 0 1 0 1 0 0 0 0 0 1 0	0 0 0 0 0 0 0 1 0 0 0 0	0 0 0 1 0 0 0 0 0
0 1 1 1 1 1 0 2 2	0 1 1 1 2 1 0 2 3	0 0 1 0 1 0 1 0 0 0 1 0	0 0 0 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 0 0 0
0 1 1 1 1 1 0 2 2	0 1 1 1 1 2 0 2 1	0 0 1 0 1 0 1 0 1 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 1 0 0

The information represented by the above codes may be used to form the following equation:

$$
\begin{aligned}
P_{011111}^{(k)}[0][2][2] \cong\ & P_{011111}^{(k-1)}[0][2][2](1-\lambda_{11}\Delta t-\mu_{21}\Delta t-\lambda_{21}\Delta t-\mu_{31}\Delta t-\lambda_{31}\Delta t) \\
& +P_{211111}^{(k-1)}[1][2][2](\mu_{12}\Delta t)+P_{021111}^{(k-1)}[0][1][2](\lambda_{12}\Delta t)(1-q_{11}) \\
& +P_{021111}^{(k-1)}[0][2][1](\lambda_{12}\Delta t)(1-q_{11})+P_{012111}^{(k-1)}[0][3][2](\mu_{22}\Delta t) \\
& +P_{011211}^{(k-1)}[0][1][2](\lambda_{22}\Delta t)q_{22}+P_{011121}^{(k-1)}[0][2][3](\mu_{32}\Delta t) \\
& +P_{011112}^{(k-1)}[0][2][1](\lambda_{32}\Delta t)q_{33}
\end{aligned}
\tag{3.2.1}
$$

The derivation of Equation (3.2.1) may also be illustrated by Figure 3.2.2

Subtracting the term $P_{011111}^{(k-1)}[0][2][2]$ from both sides of (3.2.1), dividing both sides of the resulting equation by Δt, and letting $\Delta t \to 0$, and later letting $k \to \infty$, we get the balance equation

$$
\begin{aligned}
0 \cong\ & P_{011111}[0][2][2](-\lambda_{11}-\mu_{21}-\lambda_{21}-\mu_{31}-\lambda_{31}) \\
& +P_{211111}[1][2][2](\mu_{12})+P_{021111}[0][1][2]\lambda_{12}(1-q_{11}) \\
& +P_{021111}[0][2][1]\lambda_{12}(1-q_{11})+P_{012111}[0][3][2]\mu_{22} \\
& +P_{011211}[0][1][2]\lambda_{22}q_{22}+P_{011121}[0][2][3]\mu_{32} \\
& +P_{011112}[0][2][1]\lambda_{32}q_{33}
\end{aligned}
\tag{3.2.2}
$$

Equation (3.2.2) may be represented in a coded form as shown in Table 3.2.2.

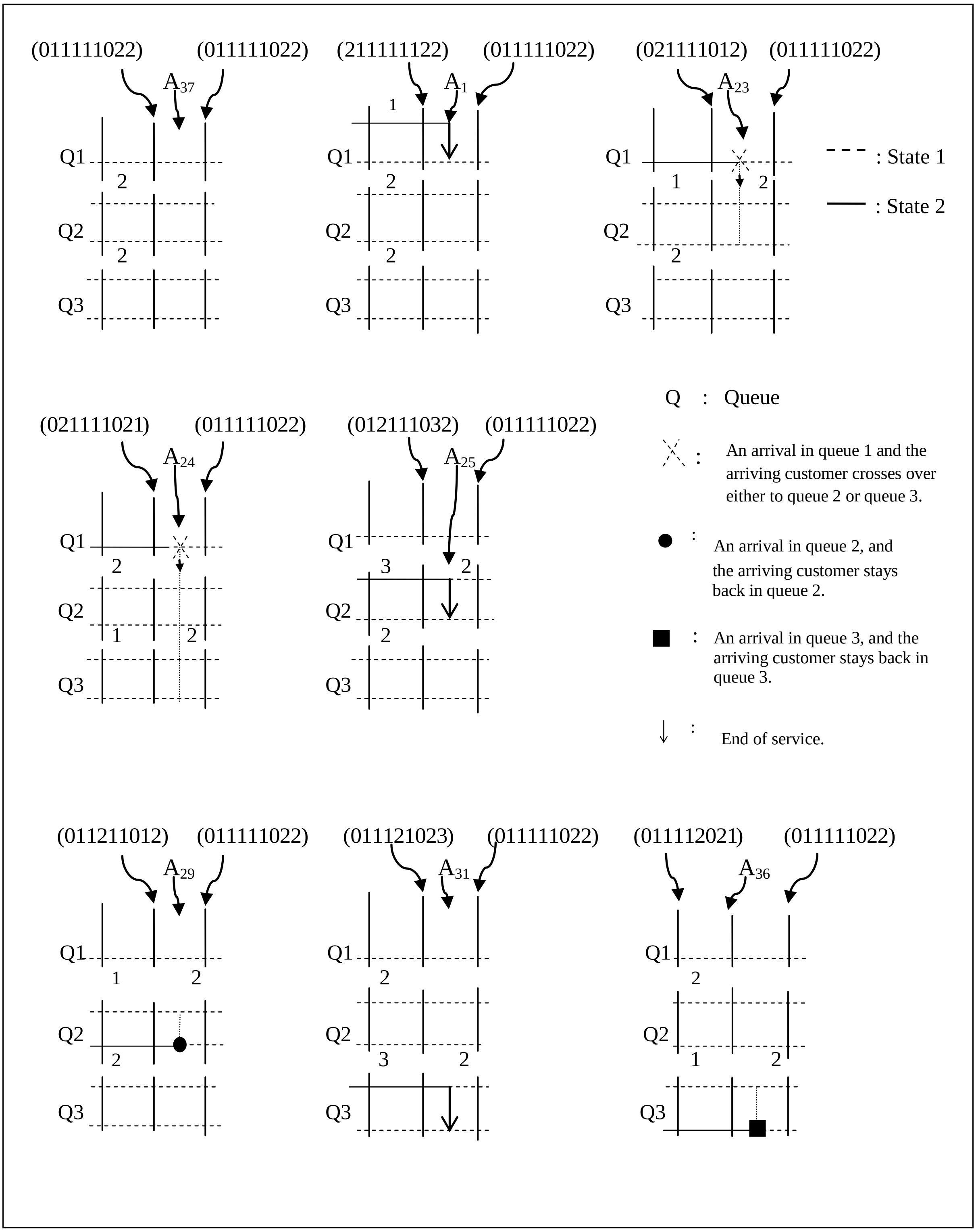

Figure 3.2.2 : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$.

Table 3.2.2 : Representation of balance equation in (3.2.2) by codes.

Constant	h	Power
-1	0 1 1 1 1 1 0 2 2	0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
-1	0 1 1 1 1 1 0 2 2	0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
-1	0 1 1 1 1 1 0 2 2	0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
-1	0 1 1 1 1 1 0 2 2	0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
-1	0 1 1 1 1 1 0 2 2	0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0
1	2 1 1 1 1 1 1 2 2	0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1	0 2 1 1 1 1 0 1 2	0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
1	0 2 1 1 1 1 0 2 1	0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
1	0 1 2 1 1 1 0 3 2	0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1	0 1 1 2 1 1 0 1 2	0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0
1	0 1 1 1 2 1 0 2 3	0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0
1	0 1 1 1 1 2 0 2 1	0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0
1*		

In each row of Table 3.2.2,

Column 1 gives a coefficient value.

Columns 2 – 10 give the components of $\mathbf{h}$.

Columns 11 – 31 give respectively the powers of (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (μ_{31}), (μ_{32}), (λ_{31}), (λ_{32}), (q_{11}), (1-q_{11}), (1-q_{11})/2, (q_{22}), (1-q_{22}), (1-q_{22})/2, (q_{33}), (1-q_{33}) and (1-q_{33})/2.

The symbol " 1* " of the last row denotes the end of the equation.

For each row of Table 3.2.2, we form a product of

(i) the coefficient in column 1,

(ii) the term $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ of which the values i_1, j_1, i_2, j_2, i_3, j_3, n_1, n_2, n_3 are given by $\mathbf{h}$, and

(iii) the product of (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (μ_{31}), (μ_{32}), (λ_{31}), (λ_{32}), (q_{11}), (1-q_{11}), (1-q_{11})/2, (q_{22}), (1-q_{22}), (1-q_{22})/2, (q_{33}), (1-q_{33}) and (1-q_{33})/2 raised respectively to the corresponding powers.

We then equate the sum of the products for all the rows in Table 3.2.2 to zero to form (3.2.2).

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search for all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form as has been discussed in Sections 2.6.

Next, the codes for the corresponding balance equations similar to (3.2.2) may be obtained. The resulting table of codes for $0 \le n_1 + n_2 + n_3 \le 9$ can be found in the file *ThreeQueueSystem_JSQ_codes.txt* in the CD attached.

The method in Section 2.7 may now be used to solve the balance equations so that the joint queue length distribution may be computed.

A simulation procedure is similar to that give in Section 2.8 may also be used to find the joint queue length distribution.

3.3 NUMERICAL RESULTS FOR DISTRIBUTION OF QUEUE LENGTH AND STATES OF ARRIVAL AND SERVICE PROCESSES IN A SYSTEM OF THREE HYPO(2)/HYPO(2)/1 QUEUES WITH INTERACTION SCHEME JOINING THE SHORTER QUEUE

Suppose $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32})$ = (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (1, 2, 1, 2, 1, 2), and $q_{11} = q_{22} = q_{33} = 0.9$. The traffic intensities (ρ_1, ρ_2, ρ_3) in the three queues are then given respectively by

$$\rho_1 = (\mu_{11}{}^{-1} + \mu_{12}{}^{-1}) / (\lambda_{11}{}^{-1} + \lambda_{12}{}^{-1}) = 0.1 \ ,$$

$$\rho_2 = (\mu_{21}{}^{-1} + \mu_{22}{}^{-1}) / (\lambda_{21}{}^{-1} + \lambda_{22}{}^{-1}) = 0.1 \ ,$$

$$\rho_3 = (\mu_{31}{}^{-1} + \mu_{32}{}^{-1}) / (\lambda_{31}{}^{-1} + \lambda_{32}{}^{-1}) = 0.1.$$

By setting $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3] = 0$ when $n_1 + n_2 + n_3 = 4$, the probabilities $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ computed by using the method in Section 2.7 and a simulation procedure similar to that given in Section 2.8 are presented in Table 3.3.1 and Figure 3.3.1.

Next, Table 3.3.2 and Figure 3.3.2 show the results for the case when $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32})$ = (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (3, 4, 3, 4, 3, 4), $q_{11} = q_{22} = q_{33} = 0.9$, $N = 4$, and the traffic intensities are $\rho_1 = 0.2571$, $\rho_2 = 0.2571$, $\rho_3 = 0.2571$.

The tables and figures show that the results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ found by the proposed method agree well with those found by the simulation procedure.

Table 3.3.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32})$ = (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (1, 2, 1, 2, 1, 2), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $\rho_3 = 0.1$, $N = 3$ and $N_s = 50000$].

$n_1n_2n_3$	$i_1j_1i_2j_2i_3j_3$	Proposed method	Simulation procedure
000	010101	0.18960	0.18812
	010102	0.10675	0.10672
	010201	0.10603	0.10948
	010202	0.05993	0.06100
	020101	0.10917	0.10744
	020102	0.06109	0.06154
	020201	0.06080	0.05892
	020202	0.03418	0.03416
001	010111	0.02030	0.01974
	010112	0.00264	0.00224
	010121	0.00958	0.01000
	010122	0.00162	0.00148
	010211	0.01057	0.01100
	010212	0.00120	0.00118
	010221	0.00502	0.00578
	010222	0.00077	0.00094
	020111	0.01104	0.01040
	020112	0.00127	0.00114
	020121	0.00533	0.00580
	020122	0.00084	0.00062
	020211	0.00584	0.00570
	020212	0.00055	0.00052
	020221	0.00280	0.00312
	020222	0.00038	0.00020
010	011101	0.02017	0.02012
	011102	0.01064	0.01058
	011201	0.00263	0.00242
	011202	0.00122	0.00132
	012101	0.00952	0.00936
	012102	0.00507	0.00542
	012201	0.00161	0.00164
	012202	0.00079	0.00076
	021101	0.01098	0.01136
	021102	0.00586	0.00638
	021201	0.00127	0.00110
	021202	0.00055	0.00044
	022101	0.00530	0.00588
	022102	0.00282	0.00300
	022201	0.00083	0.00064
	022202	0.00039	0.00044
100	110101	0.02066	0.01962
	110102	0.01083	0.01080
	110201	0.01072	0.01024
	110202	0.00580	0.00554
	120101	0.00272	0.00246
	120102	0.00124	0.00114
	120201	0.00122	0.00118
	120202	0.00054	0.00048
	210101	0.00974	0.00908
	210102	0.00517	0.00508
	210201	0.00509	0.00522
	210202	0.00277	0.00284
	220101	0.00171	0.00142
	220102	0.00082	0.00064
	220201	0.00079	0.00084
	220202	0.00038	0.00030
	Total	**0.96740**	**0.96498**

Table 3.3.2 : Comparison of results for $P_{i_1j_1i_2j_2i_3j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) =$ (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (1, 2, 2, 4, 3, 6), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.10$, $\rho_2 = 0.20$, $\rho_3 = 0.30$, $N = 4$ and $N_s = 50000]$.

$n_1n_2n_3$	$i_1j_1i_2j_2i_3j_3$	Proposed method	Simulation procedure
000	010101	0.11949	0.12330
	010102	0.07724	0.08030
	010201	0.07153	0.07374
	010202	0.04660	0.04680
	020101	0.07378	0.06978
	020102	0.04755	0.04372
	020201	0.04400	0.04274
	020202	0.02848	0.02606
001	010111	0.03949	0.04080
	010112	0.00853	0.00862
	010121	0.01709	0.01838
	010122	0.00524	0.00474
	010211	0.02213	0.02354
	010212	0.00481	0.0050
	010221	0.00966	0.01118
	010222	0.00296	0.00288
	020111	0.02364	0.02318
	020112	0.00536	0.00468
	020121	0.01048	0.01014
	020122	0.00331	0.00268
	020211	0.01368	0.01270
	020212	0.00292	0.00242
	020221	0.00602	0.00540
	020222	0.00184	0.00120
010	011101	0.02688	0.02642
	011102	0.01557	0.01654
	011201	0.00494	0.00434
	011202	0.00276	0.00254
	012101	0.01203	0.01156
	012102	0.00719	0.00762
	012201	0.00302	0.00274
	012202	0.00178	0.00162
	021101	0.01582	0.01504
	021102	0.00965	0.00956
	021201	0.00306	0.00244
	021202	0.00163	0.00132
	022101	0.00735	0.00728
	022102	0.00450	0.00420
	022201	0.00191	0.00180
	022202	0.00109	0.00082
100	110101	0.01710	0.01432
	110102	0.00903	0.00914
	110201	0.00843	0.00868
	110202	0.00505	0.00500
	120101	0.00333	0.00308
	120102	0.00157	0.00122
	120201	0.00150	0.00106
	120202	0.00064	0.00064
	210101	0.00785	0.00628
	210102	0.00439	0.00436
	210201	0.00397	0.00394
	210202	0.00243	0.00224
	220101	0.00201	0.00148
	220102	0.00102	0.00076
	220201	0.00098	0.00090
	220202	0.00046	0.00028
	Total	**0.87493**	**0.86320**

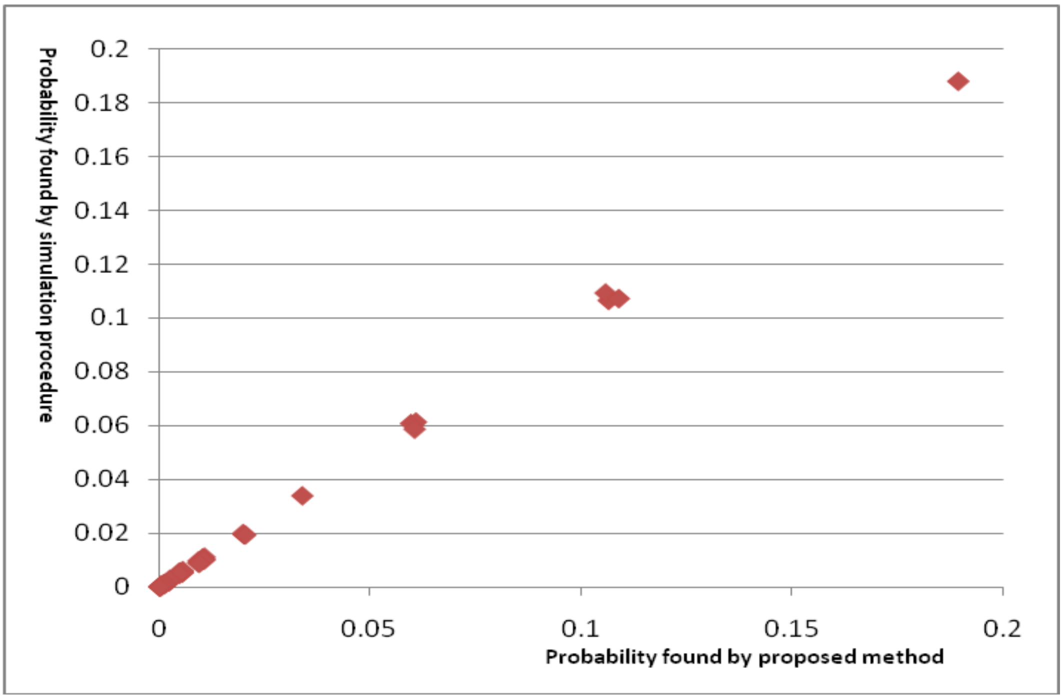

Figure 3.3.1 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) =$ (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (1, 2, 1, 2, 1, 2), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $\rho_3 = 0.1$, $N = 3$ and $N_s = 50000$].

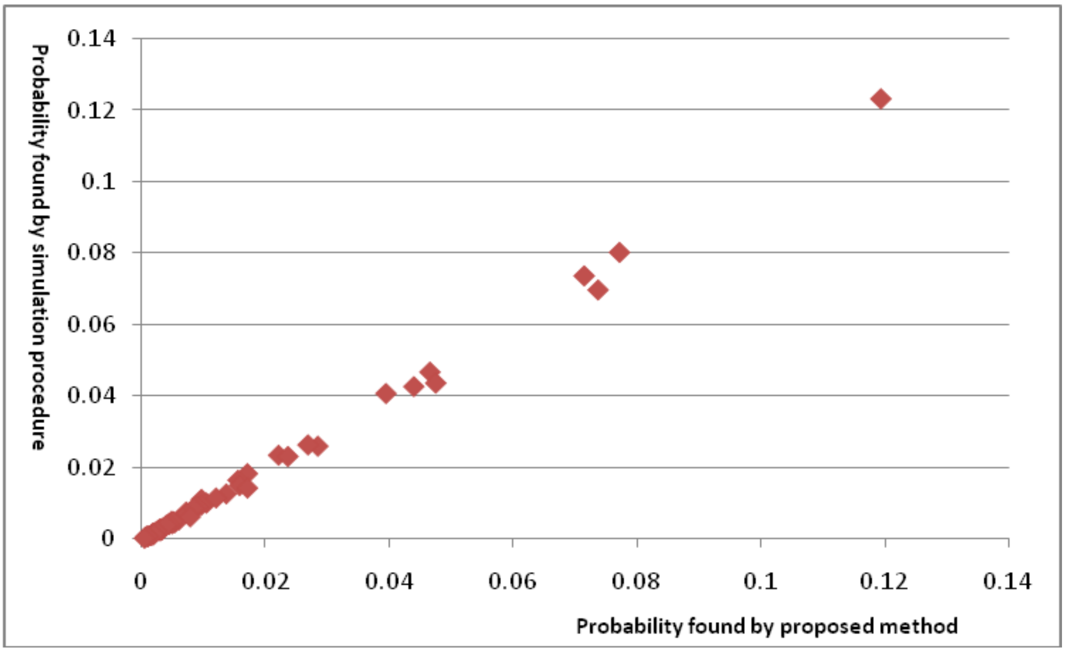

Figure 3.3.2 : Comparison of results for $P_{i_1 j_1 i_2 j_2 i_3 j_3}[n_1][n_2][n_3]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}) =$ (10, 20, 10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32})$ = (1, 2, 2, 4, 3, 6), $q_{11} = q_{22} = q_{33} = 0.9$, $\rho_1 = 0.10$, $\rho_2 = 0.20$, $\rho_3 = 0.30$, $N = 4$ and $N_s = 50000$].

3.4 DERIVATION OF THE FORWARD EQUATIONS IN A SYSTEM OF M HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE SECOND INTERACTION SCHEME

Consider a system of M Hypo(2)/Hypo(2)/1 queues which follow the Second Interaction Scheme.

As in Section 2.10, let $P^{(k)}_{i_1 j_1 i_2 j_2 .. i_M j_M}[n_1][n_2]...[n_M]$ be the probability that at the end of the interval τ_k, the number of customers in the system is n_m in queue m (including the customer that is being served), the service process in queue m is in the state i_m and the arrival process in queue m is in the state j_m, $1 \le m \le M$, $i_m \in \{0, 1, 2\}$ and $j_m \in \{1, 2\}$.

Assume that

$$P_{i_1 j_1 i_2 j_2 .. i_M j_M}[n_1][n_2]...[n_M] = \lim_{k \to \infty} P^{(k)}_{i_1 j_1 i_2 j_2 .. i_M j_M}[n_1][n_2]...[n_M]$$

exists.

Let $\mathbf{h}^{(k)}$ be the vector

$$\mathbf{h}^{(k)} = \left(\; i_1^{(k)}, j_1^{(k)}, \; i_2^{(k)}, j_2^{(k)}, \; ..., \; i_M^{(k)}, j_M^{(k)}, \; n_1^{(k)}, n_2^{(k)}, \; ..., n_M^{(k)} \right)$$

of which the components are respectively the values of $i_1, j_1, i_2, j_2, \ldots, i_M, j_M, n_1, n_2, \ldots, n_M$ at the end of τ_k. Again we refer to $\mathbf{h}^{(k)}$ as the vector of characteristics of the queueing system at the end of τ_k.

The value $\mathbf{h}^{(k)}$ may be developed from $\mathbf{h}^{(k-1)}$ after some appropriate activities in the interval τ_k. The set of possible activities may be denoted by a set A = {A_1, A_2, …, A_W} where A_w = { A_{w1}, $A_{w2, \ldots,}$ A_{w3M} }, $1 \le w \le W$.

The meanings of the components A_{wj} in A_w, $1 \le j \le 2M$ are same as in Table 2.10.1. The meanings of A_{wj} in for $(2M+1) \le j \le 3M$ are given below:

$$A_{w\,(2M+1)} = \begin{cases} 11, & \text{if the arriving customer in queue 1 stays back in queue 1.} \\ 1j, & \text{if the arriving customer in queue 1 goes to queue } j \text{ after noting that} \\ & \text{queue } j \text{ is one of the queues with the shortest queue size.} \\ -1, & \text{no customers arrive in queue 1 and it is not relevant to find out} \\ & \text{whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w\,(2M+2)} = \begin{cases} 22, & \text{if the arriving customer in queue 2 stays back in queue 2.} \\ 2j, & \text{if the arriving customer in queue 2 goes to queue } j \text{ after noting that} \\ & \text{queue } j \text{ is one of the queues with the shortest queue size.} \\ -1, & \text{no customers arrive in queue 2 and it is not relevant to find out} \\ & \text{whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

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$$A_{w\,(3M)} = \begin{cases} MM, & \text{if the arriving customer in queue } M \text{ stays back in queue } M. \\ M\,j, & \text{if the arriving customer in queue } M \text{ goes to queue } j \text{ after noting that} \\ & \text{queue } j \text{ is one of the queues with the shortest queue size.} \\ -1, & \text{no customers arrive in queue } M \text{ and it is not relevant to find out} \\ & \text{whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form as has been done in Section 3.2.

Next, the codes for the corresponding balance equations similar to (3.2.2) may be obtained. The method in Section 2.7 may now be used to solve the balance equations so that the joint queue length distribution may be computed.

A simulation procedure is similar to that give in Section 2.8 may also be used to find the joint queue length distribution.