CHAPTER 4

WAITING TIME DISTRIBUTION IN A SYSTEM OF *M* HYPO(2)/HYPO(2)/1 DEPENDENT QUEUES

4.1 INTRODUCTION

Consider again the system of M Hypo(2)/ Hypo(2)/1 queues which follow the First Interaction Scheme in which the customer who arrives at queue m has a probability of $q_{mm'} \geq 0$ of joining queue m', where $m \in \{1, 2, \ldots, M\}$, $m' \in \{1, 2, \ldots, M\}$ and $\sum_{m'=1}^{M} q_{mm'} = 1$.

Suppose a customer seeks service in queue m when the system is in a stationary state. The incident of a customer seeking service in queue m could arise in one of the following ways:

(1) A customer arrives at queue m and decides to stay in the queue.

(2) A customer arrives at queue m' ($m' \neq m$) and decides to cross over to queue m.

Let the time of occurrence of the incident of a customer seeking service in queue m be denoted as $t = 0$. Furthermore, let W_m be the time the customer needs to wait before being served and

$$W^{(m)}(t) = P(W_m \leq t)$$

the cumulative distribution function (cdf) of the waiting time W_m .

In Section 4.2, we find the cdf $W^{(m)}(t)$ in queue m in a system of 2 dependent Hypo(2)/Hypo(2)/1 queues. In Sections 4.5 and 4.7, we describe how the method in

Section 4.2 can be adapted to find the cdf $W^{(m)}(t)$ in queue m in a system of 3 dependent and M dependent Hypo(2)/ Hypo(2)/1 queues respectively.

In Sections 4.8 and 4.9, we change the interaction scheme from Scheme 1 to Scheme 2 and perform the corresponding derivation of the cdf $W^{(1)}(t)$.

4.2 DERIVATION OF THE WAITING TIME DISTRIBUTION IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE FIRST INTERACTION SCHEME

Without loss of generality, assume that we are interested in finding $W^{(1)}(t)$ in queue 1 in the system.

As in Section 2.2, let Δt be a small increment in time. The possible ways to get an incident of a customer seeking service in queue 1 at time $t = 0$ which is inside an interval τ of length Δt are as follows:

(i) A customer arrives at queue 1 and the arriving customer stays back in queue 1.

(ii) A customer arrives at queue 2 and the arriving customer crosses over to queue 1.

When the arrival process in queue 1 is in state 2, an arrival of a customer inside an interval τ in queue 1 may occur with an approximate probability of $\lambda_{12}\Delta t$. Meanwhile at the beginning of τ,

(a) the service process in queue 1 will be in one of the states in $\{0, 1, 2\}$

(b) the service process in queue 2 will be in one of the states in $\{0, 1, 2\}$ and

(c) the arrival process in queue 2 will be in one of the states in $\{1, 2\}$.

Thus the probability that

(1) the queue size in queue 1 is n_1 at the beginning of τ,

(2) the service process in queue 1 is in state i_1 at the beginning of τ,

(3) the arrival process in queue 1 is in state 2 at the beginning of τ,

(4) the queue size in queue 2 is n_2 at the beginning of τ ,

(5) the service process in queue 2 is in state i_2 at the beginning of τ,

(6) the arrival process in queue 2 is in state j_2 at the beginning of τ,

(7) a customer arrives at queue 1 in τ, and

(8) the customer in (7) stays back in queue 1

is given approximately by

$$P_{i_1 2 i_2 j_2}[n_1][n_2](\lambda_{12}\Delta t)q_{11}.$$

When the arrival process in queue 2 is in state 2, an arrival of a customer inside an interval τ in queue 2 may occur with an approximate probability of $\lambda_{22}\Delta t$.

Thus the probability that

(9) the queue size in queue 1 is n_1 at the beginning of τ,

(10) the service process in queue 1 is in state i_1 at the beginning of τ,

(11) the arrival process in queue 1 is in state j_1 at the beginning of τ,

(12) the queue size in queue 2 is n_2 at the beginning of τ,

(13) the service process in queue 2 is in state i_2 at the beginning of τ,

(14) the arrival process in queue 2 is in state 2 at the beginning of τ,

(15) a customer arrives at queue 2 in τ, and

(16) the customer in (15) crosses over to queue 1

is given approximately by

$$P_{i_1 j_1 i_2 2}[n_1][n_2](\lambda_{22}\Delta t)q_{21}.$$

Let $g_{1i_1}(t)$ be the pdf of the service time in queue 1 given that the service process in queue 1 is in state i_1 at the beginning of τ and $g_1^{(n_1-1)}(t)$ the (n_1-1) fold convolution of $g_1(t)$ (The definition of $g_1(t)$ is given in Section 2.2). The customer who seeks service in queue 1 in τ will have a waiting time of zero if $n_1 = 0$, and a waiting time of which the probability density function (pdf) is given by the convolution $g_{1i_1}(t) * g_1^{(n_1-1)}(t)$ if $n_1 \geq 1$. The cdf $W^{(1)}(t)$ is then given by

$$W^{(1)}(t) = \lim_{\Delta t \to 0} \left[\frac{P_a^{(2)}}{P_b^{(2)}} \right] \tag{4.2.1}$$

where $P_a^{(2)}$ and $P_b^{(2)}$ are given below.

$$\begin{aligned}
P_a^{(2)} &= \sum_{j_2=1}^{2} P_{020j_2}[0][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2} P_{0j_1 02}[0][0](\lambda_{22}\Delta t)q_{21} \\
&+ \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_2=1}^{\infty} P_{02i_2 j_2}[0][n_2](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{n_2=1}^{\infty} P_{0j_1 i_2 2}[0][n_2](\lambda_{22}\Delta t)q_{21} \\
&+ \sum_{i_1=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 20 j_2}[n_1][0](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
&+ \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 j_1 02}[n_1][0](\lambda_{22}\Delta t)q_{21}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
&+ \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 2 i_2 j_2}[n_1][n_2](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}du \\
&+ \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 j_1 i_2 2}[n_1][n_2](\lambda_{22}\Delta t)q_{21}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}du
\end{aligned}$$

and

$$P_b^{(2)} = \sum_{j_2=1}^{2} P_{020j_2}[0][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2} P_{0j_102}[0][0](\lambda_{22}\Delta t)q_{21}$$
$$+ \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_2=1}^{\infty} P_{02i_2j_2}[0][n_2](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{n_2=1}^{\infty} P_{0j_1i_22}[0][n_2](\lambda_{22}\Delta t)q_{21}$$
$$+ \sum_{i_1=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty} P_{i_120j_2}[n_1][0](\lambda_{12}\Delta t)q_{11} + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1j_102}[n_1][0](\lambda_{22}\Delta t)q_{21}$$
$$+ \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_12i_2j_2}[n_1][n_2](\lambda_{12}\Delta t)q_{11}$$
$$+ \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1j_1i_22}[n_1][n_2](\lambda_{22}\Delta t)q_{21} \quad .$$

4.3 SIMULATED WAITING TIME DISTRIBUTION IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE FIRST INTERACTION SCHEME

The waiting time distribution in a system of two dependent Hypo(2)/Hypo(2)/1 queues which follow the First Interaction Scheme can be estimated by using simulation. The simulation process is described below.

As in Section 2.4, let $N_T > 0$ be an integer such that $N_T\Delta t$ corresponds to a time which is "very long" after $t = 0$. Suppose that at the end of τ_1,

$$\mathbf{h}^{(1)} = (1,\ 1,\ 1,\ 1,\ 1,\ 1) \tag{4.3.1}$$

then we may generate an event in τ_2 according to the probabilities specified in Table 2.4.1.

Similarly, given the value of $\mathbf{h}^{(k-1)} = \left(\ i_1^{(k-1)},\ j_1^{(k-1)},\ i_2^{(k-1)},\ j_2^{(k-1)},\ n_1^{(k-1)},\ n_2^{(k-1)}\right)$ we generate an event $A^{(k)}$ in τ_k according to the probability in Table 2.4.1,

k = 3, 4, …, N_T, and update the value of $\mathbf{h}^{(k)} = \left(i_1^{(k)},\ j_1^{(k)},\ i_2^{(k)},\ j_2^{(k)},\ n_1^{(k)},\ n_2^{(k)} \right)$. In short, we generate $\left(A^{(2)}, A^{(3)}, \Lambda\ , A^{(N_T)}\right)$ starting from $\mathbf{h}^{(1)}$ given by (4.3.1).

In the process of generating $A^{(k)}$ in τ_k and updating the value of $\mathbf{h}^{(k)}$ at the end of τ_k, we record the times of arrival of the customers who seek service in queue 1 and the service times taken up by the customers.

Let $T_i^{(1)}$ be the interarrival time between the arrivals of the (i–1)-th and i-th customers after t = 0 seeking service in queue 1 and $S_i^{(1)}$ the service time of the i-th customer after t = 0 seeking service in queue 1, i =1 , 2, … .

Suppose a total of N_s pairs of arrival and service times are recorded:

$$(0, S_0^{(1)}),\ (T_1^{(1)}, S_1^{(1)}),\ \Lambda\ ,\ (T_{N_s}^{(1)}, S_{N_s}^{(1)}).$$

Then the waiting time for the i-th customer who seeks service in queue 1 is given by

$$W_0^{(1)} = 0$$

$$W_i^{(1)} = \begin{cases} 0 & ,\ \text{if}\ \ S_{i-1}^{(1)} < (T_i^{(1)} - W_{i-1}^{(1)}) \\ S_{i-1}^{(1)} - (T_i^{(1)} - W_{i-1}^{(1)}) & ,\ \text{if}\ \ S_{i-1}^{(m)} > (T_i^{(1)} - W_{i-1}^{(1)}),\ \ 1 \le i \le N_s \end{cases} \tag{4.3.1}$$

(see Figure 4.3.1).

Let $N^{(1)}(t)$ be the number of $W_i^{(1)}$ which are less than t. An estimate of $W_i^{(1)}(t)$ is then given by $\dfrac{N^{(1)}(t)}{N_s}$.

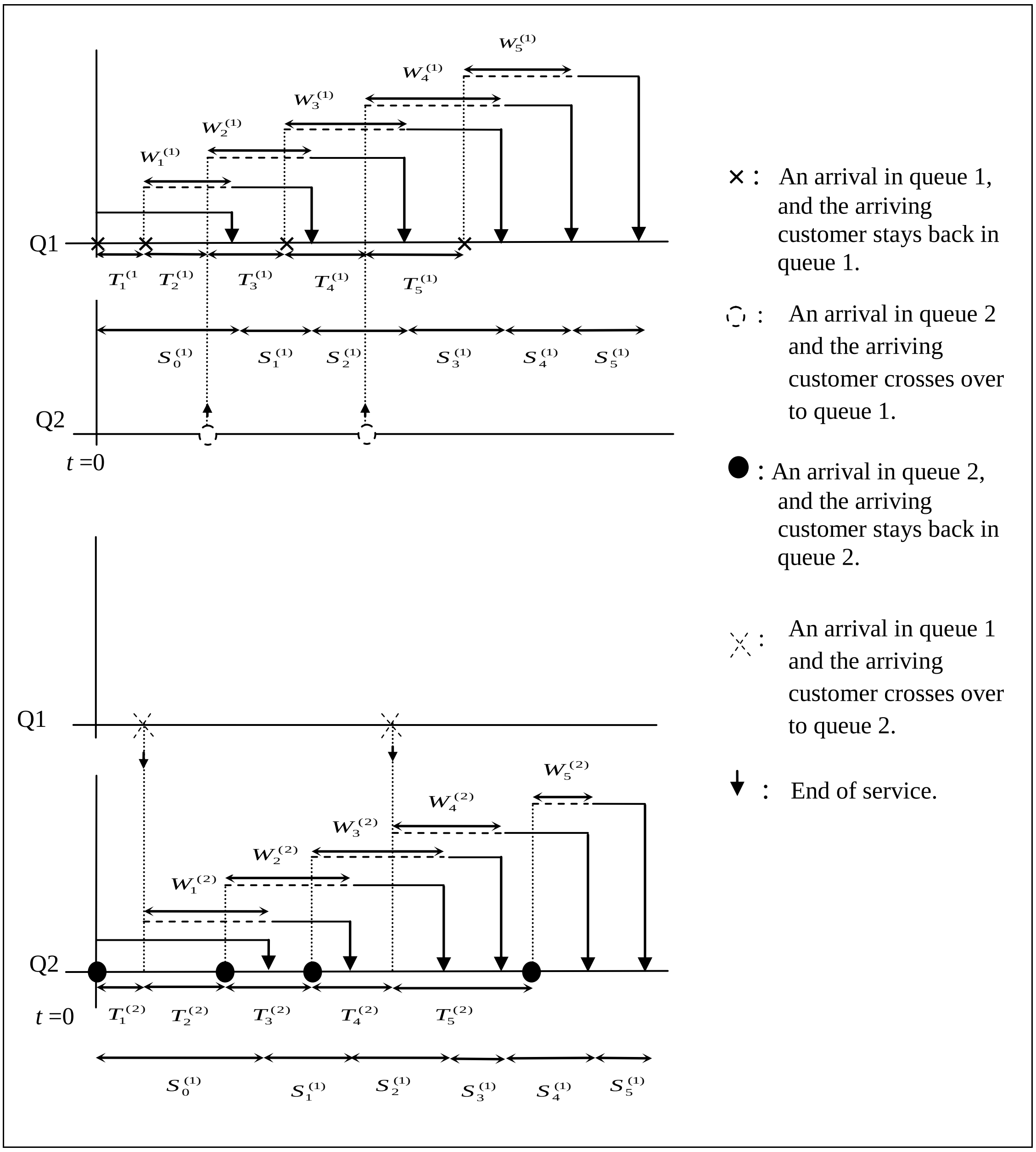

Figure 4.3.1 : The waiting time in queue *m* for a system of 2 dependent queues which follow the First Interaction Scheme.

4.4 NUMERICAL RESULTS FOR WAITING TIME DISTRIBUTION IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE FIRST INTERACTION SCHEME

The results for the cdf $W^{(m)}(t)$ obtained using the numerical method in Section 4.2 and the simulation procedure in Section 4.3 are presented in Table 4.4.1. The table shows that the values of $W^{(m)}(t)$ obtained by the numerical method agree fairly well with those obtained by the simulation procedure.

Table 4.4.1 : Comparison of $W^{(m)}(t)$ obtained by the numerical method and simulation procedure for m = 1, 2 [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 10, 10, 10), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (1, 1, 1, 1), $q_{11} = q_{22} = 0.9$, $\Delta t = 0.01$, $\rho_1 = 0.1$, $\rho_2 = 0.1$, N = 13 and Ns = 10000000].

t	$W^{(1)}(t)$ Numerical method	$W^{(1)}(t)$ Simulation procedure	$W^{(2)}(t)$ Numerical method	$W^{(2)}(t)$ Simulation procedure
0.01	0.9644487	0.9675080	0.9642906	0.9671771
0.02	0.9665024	0.9689539	0.9663445	0.9689585
0.03	0.9684871	0.9714440	0.9683309	0.9712723
0.04	0.9703974	0.9723477	0.9702440	0.9721733
0.05	0.9722295	0.9740345	0.9720798	0.9739547
0.06	0.9739810	0.9764042	0.9738356	0.9759408
0.07	0.9756507	0.9780107	0.9755102	0.9773946
0.08	0.9772382	0.9788139	0.9771029	0.9783774
0.09	0.9787440	0.9803000	0.9786143	0.9798312
0.10	0.9801694	0.9816454	0.9800454	0.9816331
0.11	0.9815160	0.9825089	0.9813978	0.9823498
0.12	0.9827859	0.9841556	0.9826735	0.9840288
0.13	0.9839815	0.9851798	0.9838748	0.9848273
0.14	0.9851054	0.9865252	0.9850044	0.9861787
0.15	0.9861605	0.9870674	0.9860651	0.9867111
0.16	0.9871496	0.9879310	0.9870597	0.9878373
0.17	0.9880758	0.9890154	0.9879912	0.9888201
0.18	0.9889421	0.9895977	0.9888625	0.9894139
0.19	0.9897515	0.9904411	0.9896768	0.9902739
0.20	0.9905070	0.9912243	0.9904370	0.9911134
0.21	0.9912116	0.9918870	0.9911460	0.9916663
0.22	0.9918680	0.9922284	0.9918066	0.9920348
0.23	0.9924790	0.9928509	0.9924217	0.9927515
0.24	0.9930474	0.9933730	0.9929939	0.9933043
0.25	0.9935757	0.9935739	0.9935258	0.9934272
0.26	0.9940665	0.9941562	0.9940200	0.9941439
0.27	0.9945220	0.9944976	0.9944787	0.9948605
0.28	0.9949445	0.9951804	0.9949043	0.9952086
0.29	0.9953363	0.9954013	0.9952989	0.9953519
0.30	0.9956992	0.9957427	0.9956645	0.9957000

4.5 DERIVATION OF THE WAITING TIME DISTRIBUTION IN A SYSTEM OF THREE DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE FIRST INTERACTION SCHEME

Consider in the system of three Hypo(2)/Hypo(2)/1 queues which follow the First Interaction Scheme. Similarly without loss of generality, assume that we are interested in finding the cdf $W^{(1)}(t)$ in queue 1.

Again let Δt be a small increment in time. The following are three possible ways to get an incident of a customer seeking service in queue 1 at time $t = 0$ which is inside an interval τ of length Δt:

(i) A customer arrives at queue 1 and the arriving customer stays back in queue 1.

(ii) A customer arrives at queue 2 and the arriving customer crosses over to queue 1.

(iii) A customer arrives at queue 3 and the arriving customer crosses over to queue 1.

When the arrival process in queue 1 is in state 2, an arrival of a customer inside an interval τ in queue 1 may occur with an approximate probability of $\lambda_{12}\Delta t$. Meanwhile at the beginning of τ,

(a) the service process in queue 1 will be in one of the states in {0, 1, 2}

(b) the service process in queue 2 will be in one of the states in {0, 1, 2}

(c) the arrival process in queue 2 will be in one of the states in {1, 2}

(d) the service process in queue 3 will be in one of the states in {0, 1, 2} and

(e) the arrival process in queue 3 will be in one of the state in {1, 2}.

Thus the probability that

(1) the queue size in queue 1 is n_1 at the beginning of τ,

(2) the service process in queue 1 is in state i_1 at the beginning of τ,

(3) the arrival process in queue 1 is in state 2 at the beginning of τ,

(4) the queue size in queue 2 is n_2 at the beginning of τ,

(5) the service process in queue 2 is in state i_2 at the beginning of τ,

(6) the arrival process in queue 2 is in state j_2 at the beginning of τ,

(7) the queue size in queue 3 is n_3 at the beginning of τ,

(8) the service process in queue 3 is in state i_3 at the beginning of τ,

(9) the arrival process in queue 3 is in state j_3 at the beginning of τ,

(10) a customer arrives at queue 1 in τ, and

(11) the customer in (10) stays back in queue 1

is given approximately by

$$P_{i_1 2 i_2 j_2 i_3 j_3}[n_1][n_2][n_3](\lambda_{12}\Delta t)q_{11}.$$

When the arrival process in queue 2 is in state 2, an arrival of a customer inside an interval τ in queue 2 may occur with an approximate probability of $\lambda_{22}\Delta t$.

Thus the probability that

(12) the queue size in queue 1 is n_1 at the beginning of τ,

(13) the service process in queue 1 is in state i_1 at the beginning of τ,

(14) the arrival process in queue 1 is in state j_1 at the beginning of τ,

(15) the queue size in queue 2 is n_2 at the beginning of τ,

(16) the service process in queue 2 is in state i_2 at the beginning of τ,

(17) the arrival process in queue 2 is in state 2 at the beginning of τ,

(18) the queue size in queue 3 is n_3 at the beginning of τ,

(19) the service process in queue 3 is in state i_3 at the beginning of τ,

(20) the arrival process in queue 3 is in state j_3 at the beginning of τ,

(21) a customer arrives at queue 2 in τ, and

(22) the customer in (21) crosses over to queue 1

is given approximately by

$$P_{i_1 j_1 i_2 2 i_3 j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)q_{21}.$$

Next, when the arrival process in queue 3 is in state 2, an arrival of a customer inside an interval τ in queue 3 may occur with an approximate probability of $\lambda_{32}\Delta t$.

Thus the probability that

(23) the queue size in queue 1 is n_1 at the beginning of τ,

(24) the service process in queue 1 is in state i_1 at the beginning of τ,

(25) the arrival process in queue 1 is in state j_1 at the beginning of τ,

(26) the queue size in queue 2 is n_2 at the beginning of τ,

(27) the service process in queue 2 is in state i_2 at the beginning of τ,

(28) the arrival process in queue 2 is in state j_2 at the beginning of τ,

(29) the queue size in queue 3 is n_3 at the beginning of τ,

(30) the service process in queue 3 is in state i_3 at the beginning of τ,

(31) the arrival process in queue 3 is in state 2 at the beginning of τ,

(32) a customer arrives at queue 3 in τ, and

(33) the customer in (32) crosses over to queue 1

is given approximately by

$$P_{i_1 j_1 i_2 j_2 i_3 2}[n_1][n_2][n_3](\lambda_{32}\Delta t)q_{31}.$$

The customer who seeks service in queue 1 in τ will have a waiting time of zero if $n_1 = 0$, and a waiting time of which the probability density function (pdf) is given by the convolution $g_{1i_1}(t) * g_1^{(n_1-1)}(t)$ if $n_1 \geq 1$. The cdf $W^{(1)}(t)$ is then given by

$$W^{(1)}(t) = \lim_{\Delta t \to 0} \left[\frac{P_a^{(3)}}{P_b^{(3)}} \right] \tag{4.5.1}$$

where $P_a^{(3)}$ and $P_b^{(3)}$ are given by

$$
\begin{aligned}
P_a^{(3)} = \ & \sum_{j_2=1}^{2}\sum_{j_3=1}^{2} P_{020j_2 0j_3}[0][0][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{j_3=1}^{2} P_{0j_1 020j_3}[0][0][0](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2} P_{0j_1 0j_2 02}[0][0][0](\lambda_{32}\Delta t)q_{31} + \sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{020j_2 i_3 j_3}[0][0][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0j_1 02i_3 j_3}[0][0][n_3](\lambda_{22}\Delta t)q_{21} + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0j_1 0j_2 i_3 2}[0][0][n_3](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{02i_2 j_2 0j_3}[0][n_2][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{0j_1 i_2 20j_3}[0][n_2][0](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_2=1}^{\infty} P_{0j_1 i_2 j_2 02}[0][n_2][0](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 20j_2 0j_3}[n_1][0][0](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 j_1 020j_3}[n_1][0][0](\lambda_{22}\Delta t)q_{21}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 j_1 0j_2 02}[n_1][0][0](\lambda_{32}\Delta t)q_{31}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 2i_2 j_2 0j_3}[n_1][n_2][0](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 j_1 i_2 20j_3}[n_1][n_2][0](\lambda_{22}\Delta t)q_{21}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 j_1 i_2 j_2 02}[n_1][n_2][0](\lambda_{32}\Delta t)q_{31}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 20j_2 i_3 j_3}[n_1][0][n_3](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 02i_3 j_3}[n_1][0][n_3](\lambda_{22}\Delta t)q_{21}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 0j_2 i_3 2}[n_1][0][n_3](\lambda_{32}\Delta t)q_{31}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{02i_2 j_2 i_3 j_3}[0][n_2][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0j_1 i_2 2i_3 j_3}[0][n_2][n_3](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0j_1 i_2 j_2 i_3 2}[0][n_2][n_3](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 2i_2 j_2 i_3 j_3}[n_1][n_2][n_3](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 i_2 2i_3 j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)q_{21}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 i_2 j_2 i_3 2}[n_1][n_2][n_3](\lambda_{32}\Delta t)q_{31}\int_{u=0}^{t} g_{1i_1}(u) * g_1^{(n_1-1)}(u)du
\end{aligned}
$$

and

$$
\begin{aligned}
P_b^{(3)} = {} & \sum_{j_2=1}^{2}\sum_{j_3=1}^{2} P_{020j_2 0 j_3}[0][0][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{j_3=1}^{2} P_{0 j_1 020 j_3}[0][0][0](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2} P_{0 j_1 0 j_2 02}[0][0][0](\lambda_{32}\Delta t)q_{31} + \sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{020 j_2 i_3 j_3}[0][0][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0 j_1 02 i_3 j_3}[0][0][n_3](\lambda_{22}\Delta t)q_{21} + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0 j_1 0 j_2 i_3 2}[0][0][n_3](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{02 i_2 j_2 0 j_3}[0][n_2][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{0 j_1 i_2 20 j_3}[0][n_2][0](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_2=1}^{\infty} P_{0 j_1 i_2 j_2 02}[0][n_2][0](\lambda_{32}\Delta t)q_{31} + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 20 j_2 0 j_3}[n_1][0][0](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 j_1 020 j_3}[n_1][0][0](\lambda_{22}\Delta t)q_{21} + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 j_1 0 j_2 02}[n_1][0][0](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 2 i_2 j_2 0 j_3}[n_1][n_2][0](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 j_1 i_2 20 j_3}[n_1][n_2][0](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 j_1 i_2 j_2 02}[n_1][n_2][0](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 20 j_2 i_3 j_3}[n_1][0][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 02 i_3 j_3}[n_1][0][n_3](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 0 j_2 i_3 2}[n_1][0][n_3](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{02 i_2 j_2 i_3 j_3}[0][n_2][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0 j_1 i_2 2 i_3 j_3}[0][n_2][n_3](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0 j_1 i_2 j_2 i_3 2}[0][n_2][n_3](\lambda_{32}\Delta t)q_{31} \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 2 i_2 j_2 i_3 j_3}[n_1][n_2][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 i_2 2 i_3 j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)q_{21} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 i_2 j_2 i_3 2}[n_1][n_2][n_3](\lambda_{32}\Delta t)q_{31}
\end{aligned}
$$

The waiting time $W^{(1)}(t)$ may also be estimated by using a simulation procedure similar to that described in Section 4.3.

4.6 NUMERICAL RESULTS FOR WAITING TIME DISTRIBUTION IN A SYSTEM OF THREE DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE FIRST INTERACTION SCHEME

The results for $W^{(m)}(t)$ obtained using the numerical method in Section 4.5 and a simulation procedure similar to that given in Section 4.3 are presented in Table 4.6.1.

Again we see that the values of $W^{(m)}(t)$ obtained by the numerical method agree fairly well with those obtained by the simulation procedure.

Table 4.6.1 : Comparison of $W^{(m)}(t)$ obtained by the numerical method and simulation procedure for m = 1, 2, 3 [$(\mu_{11},\mu_{12},\mu_{21},\mu_{22},\mu_{31},\mu_{32})$ = (10, 10, 10, 10, 10, 10), $(\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22},\lambda_{31},\lambda_{32})$ = (1, 1, 1, 1, 1, 1), q_{11} = q_{22} = q_{33} = 0.9, Δt = 0.01, $\rho_1 = 0.1$, $\rho_2 = 0.1$, $\rho_3 = 0.1$, N = 3 and Ns = 10000000].

t	$W^{(1)}(t)$		$W^{(2)}(t)$		$W^{(3)}(t)$	
	Numerical method	**Simulation procedure**	**Numerical method**	**Simulation procedure**	**Numerical method**	**Simulation procedure**
0.02	0.9711083	0.9693379	0.9712623	0.9699058	0.9712418	0.9676170
0.04	0.9744671	0.9724704	0.9745960	0.9731178	0.9745779	0.9712197
0.06	0.9775575	0.9765064	0.9776654	0.9771276	0.9776495	0.9750488
0.08	0.9803665	0.9790164	0.9804568	0.9793576	0.9804428	0.9775193
0.10	0.9828946	0.9817473	0.9829700	0.9824468	0.9829579	0.9805867
0.12	0.9851513	0.9844983	0.9852143	0.9847381	0.9852038	0.9831806
0.14	0.9871519	0.9870082	0.9872045	0.9870908	0.9871954	0.9854246
0.16	0.9889151	0.9883737	0.9889590	0.9888093	0.9889512	0.9872979
0.18	0.9904612	0.9896186	0.9904979	0.9903641	0.9904911	0.9889449
0.20	0.9918111	0.9912049	0.9918417	0.9917553	0.9918359	0.9907359
0.22	0.9929850	0.9922491	0.9930105	0.9925941	0.9930055	0.9919094
0.24	0.9940024	0.9937752	0.9940236	0.9941284	0.9940194	0.9931446
0.26	0.9948814	0.9946989	0.9948991	0.9949877	0.9948955	0.9941945
0.28	0.9956388	0.9957229	0.9956536	0.9959697	0.9956505	0.9952032
0.30	0.9962899	0.9962249	0.9963022	0.9963788	0.9962995	0.9956562
0.32	0.9968482	0.9967470	0.9968584	0.9968903	0.9968562	0.9963561
0.34	0.9973260	0.9973494	0.9973345	0.9973608	0.9973326	0.9969119
0.36	0.9977341	0.9977510	0.9977412	0.9977291	0.9977396	0.9974884
0.38	0.9980822	0.9981727	0.9980881	0.9980769	0.9980867	0.9981677
0.40	0.9983785	0.9984337	0.9983834	0.9981996	0.9983822	0.9984354
0.42	0.9986304	0.9987550	0.9986345	0.9985883	0.9986335	0.9987236
0.44	0.9988442	0.9988755	0.9988476	0.9987315	0.9988468	0.9989706
0.46	0.9990256	0.9991365	0.9990284	0.9988747	0.9990277	0.9990941
0.48	0.9991791	0.9992972	0.9991815	0.9989975	0.9991809	0.9991765
0.50	0.9993090	0.9993775	0.9993110	0.9990589	0.9993105	0.9992794

4.7 DERIVATION OF THE WAITING TIME DISTRIBUTION IN A SYSTEM OF *M* DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE FIRST INTERACTION SCHEME

Assume that we are interested in finding the cdf $W^{(1)}(t)$ in queue 1 in a system of M Hypo(2)/Hypo(2)/1 queues which follow the First Interaction Scheme.

There are now M possible ways to get an incident of a customer seeking service in queue 1 at time $t = 0$ which is inside an interval τ of length Δt:

(i) A customer arrives at queue 1 and the arriving customer stays back in queue 1.

(ii) A customer arrives at queue m' ($m' \neq 1$) and the arriving customer crosses over to queue 1.

When the arrival process in queue 1 is in state 2, an arrival of a customer inside an interval τ in queue 1 will occur with an approximate probability of $\lambda_{12}\Delta t$. Meanwhile at the beginning of τ,

(a) the service process in queue 1 will be in one of the states in {0, 1, 2}.

(b) the service process in queue m' will be in one of the states in {0, 1, 2} and

(c) the arrival process in queue m' will be in one of the states in {1, 2}.

Thus the probability that

(1) the queue size in queue 1 is n_1 at the beginning of τ,

(2) the service process in queue 1 is in state i_1 at the beginning of τ,

(3) the arrival process in queue 1 is in state 2 at the beginning of τ,

(4) the queue size in queue m' is $n_{m'}$ at the beginning of τ,

(5) the service process in queue m' is in state $i_{m'}$ at the beginning of τ,

(6) the arrival process in queue m' is in state $j_{m'}$ at the beginning of τ,

(7) a customer arrives at queue 1 in τ, and

(8) the customer in (7) stays back in queue 1

is given approximately by

$$P_{i_1 2 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M](\lambda_{12}\Delta t)q_{11}.$$

When the arrival process in queue m' is in state 2, an arrival of a customer inside an interval τ in queue m' may occur with an approximate probability of $\lambda_{\mathrm{m}'2}\Delta t$.

Thus the probability that

(9) the queue size in queue 1 is n_1 at the beginning of τ,

(10) the service process in queue 1 is in state i_1 at the beginning of τ,

(11) the arrival process in queue 1 is in state j_1 at the beginning of τ,

(12) the queue size in queue m' is $n_{m'}$ at the beginning of τ,

(13) the service process in queue m' is in state $i_{m'}$ at the beginning of τ,

(14) the arrival process in queue m' is in state 2 at the beginning of τ,

(15) a customer arrives at queue m' in τ, and

(16) the customer in (15) crosses over to queue 1

is given approximately by

$$P_{i_1 j_1 i_2 j_2 \Lambda\, i_{m'} 2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_{m'}]\Lambda\,[n_M](\lambda_{m'2}\Delta t)q_{m'1}.$$

The customer who seeks service in queue 1 in τ will have a waiting time of zero if $n_1 = 0$, and a waiting time of which the probability density function (pdf) is given by the convolution $g_{1i_1}(t) * g_1^{(n_1-1)}(t)$ if $n_1 \geq 1$. The cdf $W^{(1)}(t)$ is then given by

$$W^{(1)}(t) = \lim_{\Delta t \to 0} \left[\frac{P_a^{(M)}}{P_b^{(M)}} \right] \tag{4.7.1}$$

where $P_a^{(M)}$ and $P_b^{(M)}$ may be derived in a way similar to that used for deriving $P_a^{(M)}$, $P_b^{(M)}$ when $M = 2$, 3.

4.8 DERIVATION OF THE WAITING TIME DISTRIBUTION IN A SYSTEM OF THREE DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE SECOND INTERACTION SCHEME

Assume that we are interested in finding the cdf $W^{(1)}(t)$ in queue 1 in a system of three Hypo(2)/Hypo(2)/1 queues which follow the Second Interaction Scheme in which the customer who arrives at queue m will stay back in queue m with probability q_{mm} or cross over to one of the I_s shortest queues (among the remaining $M - 1$ queues) with probability $(1 - q_{mm})/I_s$.

There are now three possible ways to get an incident of a customer seeking service in queue 1 at time $t = 0$ which is inside an interval τ of length Δt:

(i) A customer arrives at queue 1 and the arriving customer stays back in queue 1.

(ii) A customer arrives at queue 2 and the arriving customer crosses over to queue 1 after knowing that the length n_1 of queue 1 at the beginning of τ is less than or equal the length n_3 of queue 3 at the beginning of τ.

(iii) A customer arrives at queue 3 and the arriving customer crosses over to queue 1 after knowing that the length n_1 of queue 1 at the beginning of τ is less than or equal the length n_2 of queue 2 at the beginning of τ.

When the arrival process in queue 1 is in state 2, an arrival of a customer inside an interval τ in queue 1 may occur with an approximate probability of $\lambda_{12}\Delta t$. Meanwhile at the beginning of τ,

(a) the service process in queue 1 will be in one of the states in {0, 1, 2}

(b) the service process in queue 2 will be in one of the states in {0, 1, 2}

(c) the arrival process in queue 2 will be in one of the states in {1, 2}

(d) the service process in queue 3 will be in one of the states in {0, 1, 2}

and (e) the arrival process in queue 3 will be in one of the state in {1, 2}.

Thus the probability that

(1) the queue size in queue 1 is n_1 at the beginning of τ,

(2) the service process in queue 1 is in state i_1 at the beginning of τ,

(3) the arrival process in queue 1 is in state 2 at the beginning of τ,

(4) the queue size in queue 2 is n_2 at the beginning of τ ,

(5) the service process in queue 2 is in state i_2 at the beginning of τ,

(6) the arrival process in queue 2 is in state j_2 at the beginning of τ,

(7) the queue size in queue 3 is n_3 at the beginning of τ ,

(8) the service process in queue 3 is in state i_3 at the beginning of τ,

(9) the arrival process in queue 3 is in state j_3 at the beginning of τ,

(10) a customer arrives at queue 1 in τ,

and (11) the customer in (10) stays back in queue 1

is given approximately by

$$P_{i_1 2 i_2 j_2 i_3 j_3}[n_1][n_2][n_3](\lambda_{12}\Delta t)q_{11}.$$

When the arrival process in queue 2 is in state 2, an arrival of a customer inside an interval τ in queue 2 may occur with an approximate probability of $\lambda_{22}\Delta t$.

Thus the probability that

(12) the queue size in queue 1 is n_1 at the beginning of τ,

(13) the service process in queue 1 is in state i_1 at the beginning of τ,

(14) the arrival process in queue 1 is in state j_1 at the beginning of τ,

(15) the queue size in queue 2 is n_2 at the beginning of τ,

(16) the service process in queue 2 is in state i_2 at the beginning of τ,

(17) the arrival process in queue 2 is in state 2 at the beginning of τ,

(18) the queue size in queue 3 is n_3 at the beginning of τ,

(19) the service process in queue 3 is in state i_3 at the beginning of τ,

(20) the arrival process in queue 3 is in state j_3 at the beginning of τ,

(21) a customer arrives at queue 2 in τ,

and (22) the customer in (21) crosses over to queue 1

is given approximately by

$$P_{i_1 j_1 i_2 2 i_3 j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)(1-q_{22})/2\,, \quad \text{if } n_1 = n_3\,,$$

or

$$P_{i_1 j_1 i_2 2 i_3 j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)(1-q_{22})\,, \qquad \text{if } n_1 < n_3\,.$$

Next, when the arrival process in queue 3 is in state 2, an arrival of a customer inside an interval τ in queue 3 may occur with an approximate probability of $\lambda_{32}\Delta t$.

Thus the probability that

(23) the queue size in queue 1 is n_1 at the beginning of τ,

(24) the service process in queue 1 is in state i_1 at the beginning of τ,

(25) the arrival process in queue 1 is in state j_1 at the beginning of τ,

(26) the queue size in queue 2 is n_2 at the beginning of τ,

(27) the service process in queue 2 is in state i_2 at the beginning of τ,

(28) the arrival process in queue 2 is in state j_2 at the beginning of τ,

(29) the queue size in queue 3 is n_3 at the beginning of τ,

(30) the service process in queue 3 is in state i_3 at the beginning of τ,

(31) the arrival process in queue 3 is in state 2 at the beginning of τ,

(32) a customer arrives at queue 3 in τ,

and (33) the customer in (32) crosses over to queue 1

is given approximately by either

$$P_{i_1 j_1 i_2 j_2 i_3 2}[n_1][n_2][n_3](\lambda_{32}\Delta t)(1-q_{33})/2, \quad \text{if } n_1 = n_2,$$

or $$P_{i_1 j_1 i_2 j_2 i_3 2}[n_1][n_2][n_3](\lambda_{32}\Delta t)(1-q_{33}), \qquad \text{if } n_1 < n_2.$$

The customer who seeks service in queue 1 in τ will have a waiting time of zero if $n_1 = 0$, and a waiting time of which the probability density function (pdf) is given by the convolution $g_{1i_1}(t) * g_1^{(n_1-1)}(t)$ if $n_1 \geq 1$. The cdf $W^{(1)}(t)$ is then given by

$$W^{(1)}(t) = \lim_{\Delta t \to 0} \left[\frac{P_{a'}^{(3)}}{P_{b'}^{(3)}} \right] \tag{4.8.1}$$

where $P_{a'}^{(3)}$ and $P_{b'}^{(3)}$ are given by

$$
\begin{aligned}
P_{a'}^{(3)} = {} & \sum_{j_2=1}^{2}\sum_{j_3=1}^{2} P_{020j_20j_3}[0][0][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{j_3=1}^{2} P_{0j_1020j_3}[0][0][0](\lambda_{22}\Delta t)(1-q_{22})/2 \\
& + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2} P_{0j_10j_202}[0][0][0](\lambda_{32}\Delta t)(1-q_{33})/2 + \sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{020j_2i_3j_3}[0][0][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0j_102i_3j_3}[0][0][n_3](\lambda_{22}\Delta t)(1-q_{22}) + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0j_10j_2i_32}[0][0][n_3](\lambda_{32}\Delta t)(1-q_{33})/2 \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{02i_2j_20j_3}[0][n_2][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{0j_1i_220j_3}[0][n_2][0](\lambda_{22}\Delta t)(1-q_{22})/2 \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_2=1}^{\infty} P_{0j_1i_2j_202}[0][n_2][0](\lambda_{32}\Delta t)(1-q_{33}) \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty} P_{i_120j_20j_3}[n_1][0][0](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_12i_2j_20j_3}[n_1][n_2][0](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{n_1=1\\ n_1<n_2}}^{\infty}\sum_{n_2=2}^{\infty} P_{i_1j_1i_2j_202}[n_1][n_2][0](\lambda_{32}\Delta t)(1-q_{33})\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{n_1=1\\ n_1=n_2}}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1j_1i_2j_202}[n_1][n_2][0](\lambda_{32}\Delta t)((1-q_{33})/2)\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_120j_2i_3j_3}[n_1][0][n_3](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{\substack{n_1=1\\ n_1<n_3}}^{\infty}\sum_{n_3=2}^{\infty} P_{i_1j_102i_3j_3}[n_1][0][n_3](\lambda_{22}\Delta t)(1-q_{22})\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{\substack{n_1=1\\ n_1=n_3}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1j_102i_3j_3}[n_1][0][n_3](\lambda_{22}\Delta t)((1-q_{22})/2)\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{02i_2j_2i_3j_3}[0][n_2][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0j_1i_22i_3j_3}[0][n_2][n_3](\lambda_{22}\Delta t)(1-q_{22}) \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0j_1i_2j_2i_32}[0][n_2][n_3](\lambda_{32}\Delta t)(1-q_{33}) \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\;\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_12i_2j_2i_3j_3}[n_1][n_2][n_3](\lambda_{12}\Delta t)q_{11}\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\;\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=1\\ n_1<n_3}}^{\infty}\sum_{n_3=2}^{\infty} P_{i_1j_1i_22i_3j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)(1-q_{22})\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\;\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=1\\ n_1=n_3}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1j_1i_22i_3j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)((1-q_{22})/2)\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\;\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=2\\ n_1<n_2}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1j_1i_2j_2i_32}[n_1][n_2][n_3](\lambda_{32}\Delta t)(1-q_{33})\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\;\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=1\\ n_1=n_2}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1j_1i_2j_2i_32}[n_1][n_2][n_3](\lambda_{32}\Delta t)((1-q_{33})/2)\int_{u=0}^{t} g_{1i_1}(u)*g_1^{(n_1-1)}(u)du
\end{aligned}
$$

$$
\begin{aligned}
P_{b'}^{(3)} = & \sum_{j_2=1}^{2}\sum_{j_3=1}^{2} P_{020j_2 0 j_3}[0][0][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{j_3=1}^{2} P_{0 j_1 020 j_3}[0][0][0](\lambda_{22}\Delta t)(1-q_{22})/2 \\
& + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2} P_{0 j_1 0 j_2 02}[0][0][0](\lambda_{32}\Delta t)(1-q_{33})/2 + \sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{020 j_2 i_3 j_3}[0][0][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0 j_1 02 i_3 j_3}[0][0][n_3](\lambda_{22}\Delta t)(1-q_{22}) + \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_3=1}^{\infty} P_{0 j_1 0 j_2 i_3 2}[0][0][n_3](\lambda_{32}\Delta t)(1-q_{33})/2 \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{02 i_2 j_2 0 j_3}[0][n_2][0](\lambda_{12}\Delta t)q_{11} + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty} P_{0 j_1 i_2 20 j_3}[0][n_2][0](\lambda_{22}\Delta t)(1-q_{22})/2 \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{n_2=1}^{\infty} P_{0 j_1 i_2 j_2 02}[0][n_2][0](\lambda_{32}\Delta t)(1-q_{33}) \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty} P_{i_1 20 j_2 0 j_3}[n_1][0][0](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 2 i_2 j_2 0 j_3}[n_1][n_2][0](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{n_1=1\\ n_1<n_2}}^{\infty}\sum_{n_2=2}^{\infty} P_{i_1 j_1 i_2 j_2 02}[n_1][n_2][0](\lambda_{32}\Delta t)(1-q_{33}) \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{\substack{n_1=1\\ n_1=n_2}}^{\infty}\sum_{n_2=1}^{\infty} P_{i_1 j_1 i_2 j_2 02}[n_1][n_2][0](\lambda_{32}\Delta t)((1-q_{33})/2) \\
& + \sum_{i_1=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 20 j_2 i_3 j_3}[n_1][0][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{\substack{n_1=1\\ n_1<n_3}}^{\infty}\sum_{n_3=2}^{\infty} P_{i_1 j_1 02 i_3 j_3}[n_1][0][n_3](\lambda_{22}\Delta t)(1-q_{22}) \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{\substack{n_1=1\\ n_1=n_3}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 02 i_3 j_3}[n_1][0][n_3](\lambda_{22}\Delta t)((1-q_{22})/2) \\
& + \sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{02 i_2 j_2 i_3 j_3}[0][n_2][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0 j_1 i_2 2 i_3 j_3}[0][n_2][n_3](\lambda_{22}\Delta t)(1-q_{22}) \\
& + \sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{0 j_1 i_2 j_2 i_3 2}[0][n_2][n_3](\lambda_{32}\Delta t)(1-q_{33}) \\
& + \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 2 i_2 j_2 i_3 j_3}[n_1][n_2][n_3](\lambda_{12}\Delta t)q_{11} \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=1\\ n_1<n_3}}^{\infty}\sum_{n_3=2}^{\infty} P_{i_1 j_1 i_2 2 i_3 j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)(1-q_{22}) \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{j_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=1\\ n_1=n_3}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 i_2 2 i_3 j_3}[n_1][n_2][n_3](\lambda_{22}\Delta t)((1-q_{22})/2) \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=2\\ n_1<n_2}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 i_2 j_2 i_3 2}[n_1][n_2][n_3](\lambda_{32}\Delta t)(1-q_{33}) \\
& + \sum_{i_1=1}^{2}\sum_{j_1=1}^{2}\sum_{i_2=1}^{2}\sum_{j_2=1}^{2}\sum_{i_3=1}^{2}\sum_{n_1=1}^{\infty}\sum_{\substack{n_2=1\\ n_1=n_2}}^{\infty}\sum_{n_3=1}^{\infty} P_{i_1 j_1 i_2 j_2 i_3 2}[n_1][n_2][n_3](\lambda_{32}\Delta t)((1-q_{33})/2)
\end{aligned}
$$

The waiting time $W^{(1)}(t)$ may also be estimated by using a simulation procedure similar to that described in Section 4.3.

4.9 DERIVATION OF THE WAITING TIME DISTRIBUTION IN A SYSTEM OF *M* DEPENDENT HYPO(2)/HYPO(2)/1 QUEUES WHICH FOLLOW THE SECOND INTERACTION SCHEME

Similarly without loss of generality, assume that we are interested in finding the cdf $W^{(1)}(t)$ in queue 1 in a system of M Hypo(2)/Hypo(2)/1 queues which follow the Second Interaction Scheme.

There are now M possible ways to get an incident of a customer seeking service in queue 1 at time $t = 0$ which is inside an interval τ of length Δt:

(i) A customer arrives at queue 1 and the arriving customer stays back in queue 1.

(ii) A customer arrives at queue m' ($m' \neq 1$) and the arriving customer crosses over to queue 1.

When the arrival process in queue 1 is in state 2, an arrival of a customer inside an interval τ in queue 1 may occur with an approximate probability of $\lambda_{12}\Delta t$. Meanwhile at the beginning of τ,

(a) the service process in queue 1 will be in one of the states in {0, 1, 2}

(b) the service process in queue m' will be in one of the states in {0, 1, 2} and

(c) the arrival process in queue m' will be in one of the states in {1, 2}.

Thus the probability that

(1) the queue size in queue 1 is n_1 at the beginning of τ,

(2) the service process in queue 1 is in state i_1 at the beginning of τ,

(3) the arrival process in queue 1 is in state 2 at the beginning of τ,

(4) the queue size in queue m' is $n_{m'}$ at the beginning of τ,

(5) the service process in queue m' is in state $i_{m'}$ at the beginning of τ,

(6) the arrival process in queue m' is in state $j_{m'}$ at the beginning of τ,

(7) a customer arrives at queue 1 in τ,

and (8) the customer in (7) stays back in queue 1

is given approximately by

$$P_{i_1 2 i_2 j_2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_M](\lambda_{12}\Delta t)q_{11}.$$

When the arrival process in queue m' is in state 2, an arrival of a customer inside an interval τ in queue m' may occur with an approximate probability of $\lambda_{m'2}\Delta t$.

Thus the probability that

(9) the queue size in queue 1 is n_1 at the beginning of τ,

(10) the service process in queue 1 is in state i_1 at the beginning of τ,

(11) the arrival process in queue 1 is in state j_1 at the beginning of τ,

(12) the queue size in queue m' is $n_{m'}$ at the beginning of τ,

(13) the service process in queue m' is in state $i_{m'}$ at the beginning of τ,

(14) the arrival process in queue m' is in state 2 at the beginning of τ,

(15) a customer arrives at queue m' in τ,

and (16) the customer in (15) crosses over to queue 1 is given approximately by

$$P_{i_1 j_1 i_2 j_2 \Lambda\, i_{m'} 2 \Lambda\, i_M j_M}[n_1][n_2]\Lambda\,[n_{m'}]\Lambda\,[n_M](\lambda_{m'2}\Delta t)(1-q_{m'm'})/I_s\,,$$

if $n_1 \le n_{m'}$, $m' \ne 1$ and I_s is the number of elements in $\{n_2, n_3, \Lambda\, , n_M\}$ which are equal to n_1.

The customer who seeks service in queue 1 in τ will have a waiting time of zero if $n_1 = 0$, and a waiting time of which the probability density function (pdf) is given by the convolution $g_{1i_1}(t) * g_1^{(n_1-1)}(t)$ if $n_1 \ge 1$. The cdf $W^{(1)}(t)$ is then given by

$$W^{(1)}(t) = \lim_{\Delta t \to 0} \left[\frac{P_{a'}^{(M)}}{P_{b'}^{(M)}} \right] \tag{4.9.1}$$

where $P_{a'}^{(M)}$ and $P_{b'}^{(M)}$ may be derived in a way similar to that used for deriving $P_{a'}^{(3)}$ and $P_{b'}^{(3)}$.