CHAPTER 5

THE QUEUE LENGTH DISTRIBUTION IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/*c*/*c* QUEUES

5.1 INTRODUCTION

Consider a system of two Hypo(2)/Hypo(2)/c/c queues in which the customer in queue m may stay back in queue m with probability q_{mm} or may cross over to the other queue with probability $q_{mm'}$, m, m' = 1, 2, $m' \neq m$.

When all the servers in queue m are busy, the next arriving customer seeking service in queue m is blocked and will get lost in the system, $m \in \{1, 2\}$. The incident of a customer seeking service in queue m is blocked and gets lost could arise in one of the following ways:

(1) A customer who arrives at queue m decides to stay back but only to find that all servers in queue m are busy.

(2) A customer who arrives at queue m' ($m' \neq m$) decides to cross over to queue m but only to find that all servers in queue m are busy.

A method is proposed in Sections 5.2 and 5.3 to derive the stationary joint queue length distribution in a system of two dependent Hypo(2)/Hypo(2)/2/2 queues.

In Section 5.4, we discuss a simulation procedure that may also be used to find the joint queue length distribution. In Section 5.5, we show some numerical results for the joint distribution of the queue length and states of arrival and service processes obtained by using the proposed method and the simulation procedure.

In Section 5.6, we describe how the method in Sections 5.2 and 5.3 may be adapted to find the joint queue length distribution in a system of two dependent Hypo(2)/Hypo(2)/*c*/*c* queues.

5.2 DERIVATION OF THE FORWARD EQUATIONS IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/2/2 QUEUES

Consider a system of two Hypo(2)/Hypo(2)/2/2 queues in which the customer who arrives at queue m has a probability of $q_{mm'} \geq 0$ of joining queue m', where $m \in \{1, 2\}$, $m' \in \{1, 2\}$ and $\sum_{m'=1}^{2} q_{mm'} = 1$.

Let $P^{(k)}_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ be the probability that at the end of the interval $\tau_k = ((k-1)\Delta t, k\Delta t]$, the number of customers in queue m is n_m (including the customer that is being served), the service process in queue m is in the state i_{ml} for server l, and the arrival process in queue m is in the state j_m, $l \in \{1, 2\}$, $m \in \{1, 2\}$, $i_{ml} \in \{0, 1, 2\}$ and $j_m \in \{1, 2\}$. If a server in queue m is idle, then we may define the state of server to be zero.

Assume that

$$P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2] = \lim_{k\to\infty} P^{(k)}_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$$

exists.

Let $\mathbf{h}^{(k)}$ be the vector

$$\mathbf{h}^{(k)} = \left(i^{(k)}_{11},\ i^{(k)}_{12},\ j^{(k)}_{1},\ i^{(k)}_{21},\ i^{(k)}_{22},\ j^{(k)}_{2},\ n^{(k)}_{1},\ n^{(k)}_{2} \right)$$

of which the components are respectively the values of i_{11}, i_{12}, j_1, i_{21}, i_{22}, j_2, n_1, n_2 at the end of τ_k. We refer to $\mathbf{h}^{(k)}$ as the vector of characteristics of the queueing system at the end of τ_k.

The value $\mathbf{h}^{(k)}$ may be developed from $\mathbf{h}^{(k-1)}$ after some appropriate activities in the interval τ_k. The set of possible activities may be denoted by the set A = {A_1, A_2, …, A_{36}}. The elements in A are shown below.

$$A_1 = (1, 0, 0, 0, 0, 0, -1, -1)$$

$$A_2 = (0, 1, 0, 0, 0, 0, -1, -1)$$

$$A_3 = (0, 0, 1, 0, 0, 0, -1, -1)$$

$$A_4 = (0, 0, 1, 0, 0, 0, 11, -1)$$

$$A_5 = (0, 0, 1, 0, 0, 0, 12, -1)$$

$$A_6 = (0, 0, 0, 1, 0, 0, -1, -1)$$

$$A_7 = (0, 0, 0, 0, 1, 0, -1, -1)$$

$$A_8 = (0, 0, 0, 0, 0, 1, -1, -1)$$

$$A_9 = (0, 0, 0, 0, 0, 1, -1, 21)$$

$$A_{10} = (0, 0, 0, 0, 0, 1, -1, 22)$$

$$A_{11} = (0, 0, 0, 0, 0, 0, -1, -1)$$

$$A_{12} = (-1, -1, 0, 0, 0, 0, -1, -1)$$

$$A_{13} = (-1, -1, 1, 0, 0, 0, -1, -1)$$

$$A_{14} = (-1, -1, 1, 0, 0, 0, 11, -1)$$

$$A_{15} = (-1, -1, 1, 0, 0, 0, 12, -1)$$

$$A_{16} = (-1, -1, 0, 1, 0, 0, -1, -1)$$

$$A_{17} = (-1, -1, 0, 0, 1, 0, -1, -1)$$

$$A_{18} = (-1, -1, 0, 0, 0, 1, -1, -1)$$

$$A_{19} = (-1, -1, 0, 0, 0, 1, -1, 21)$$

$$A_{20} = (-1, -1, 0, 0, 0, 1, -1, 22)$$

$$A_{21} = (1, 0, 0, -1, -1, 0, -1, -1)$$

$$A_{22} = (0, 1, 0, -1, -1, 0, -1, -1)$$

$$A_{23} = (0, 0, 1, -1, -1, 0, -1, -1)$$

$$A_{24} = (0, 0, 1, -1, -1, 0, 11, -1)$$

$$A_{25} = (0, 0, 1, -1, -1, 0, 12, -1)$$

$$A_{26} = (0, 0, 0, -1, -1, 0, -1, -1)$$

$$A_{27} = (0, 0, 0, -1, -1, 1, -1, -1)$$

$$A_{28} = (0, 0, 0, -1, -1, 1, -1, 21)$$

$$A_{29} = (0, 0, 0, -1, -1, 1, -1, 22)$$

$$A_{30} = (-1, -1, 0, -1, -1, 0, -1, -1)$$

$$A_{31} = (-1, -1, 1, -1, -1, 0, -1, -1)$$

$$A_{32} = (-1, -1, 1, -1, -1, 0, 11, -1)$$

$$A_{33} = (-1, -1, 1, -1, -1, 0, 12, -1)$$

$$A_{34} = (-1, -1, 0, -1, -1, 1, -1, -1)$$

$$A_{35} = (-1, -1, 0, -1, -1, 1, -1, 21)$$

$$A_{36} = (-1, -1, 0, -1, -1, 1, -1, 22)$$

The meanings of the first six components in A_w are explained in Table 5.2.1.

Table 5.2.1 : The meanings of the components in A_w.

Position of component	Value of component	Meaning
1	1	A transition in the state of server 1 in queue 1 occurs in τ_k.
1	0	A transition in the state of server 1 in queue 1 does not occur in τ_k.
1	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of server 1 in queue 1 has occurred in τ_k is not relevant.
2	1	A transition in the state of server 2 in queue 1 occurs in τ_k.
2	0	A transition in the state of server 2 in queue 1 does not occur in τ_k.
2	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of server 2 in queue 1 has occurred in τ_k is not relevant.
3	1	A transition in the state of the arrival process in queue 1 occurs in τ_k.
3	0	A transition in the state of the arrival process in queue 1 does not occur in τ_k.
4	1	A transition in the state of server 1 in queue 2 occurs in τ_k.
4	0	A transition in the state of server 1 in queue 2 does not occur in τ_k.
4	–1	Queue 2 is empty at the end of τ_{k-1} and whether a transition in the state of server 1 in queue 2 has occurred in τ_k is not relevant.
5	1	A transition in the state of server 2 in queue 2 occurs in τ_k.
5	0	A transition in the state of server 2 in queue 2 does not occur in τ_k.
5	–1	Queue 2 is empty at the end of τ_{k-1} and whether a transition in the state of server 2 in queue 2 has occurred in τ_k is not relevant.
6	1	A transition in the state of the arrival process in queue 2 occurs in τ_k.
6	0	A transition in the state of the arrival process in queue 2 does not occur in τ_k.

The meanings of the seventh and eighth components (A_{w7} and A_{w8}) of A_w are explained below:

$$A_{w7} = \begin{cases} 11, & \text{if the arriving customer in queue 1 stays back in queue 1.} \\ 12, & \text{if the arriving customer in queue 1 goes to queue 2.} \\ -1, & \text{no customer arrives in queue 1 and it is not relevant to find out whether} \\ & \text{the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w8} = \begin{cases} 21, & \text{if the arriving customer in queue 2 goes to queue 1.} \\ 22, & \text{if the arriving customer in queue 2 stays back in queue 2.} \\ -1, & \text{no customer arrives in queue 2 and it is not relevant to find out whether} \\ & \text{the arriving customer is staying back or going elsewhere.} \end{cases}$$

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form. An example of the codes is given in Table 5.2.1.

In Table 5.2.1,

Columns 1 – 8 give the components of $\mathbf{h}^{(k)}$.

Columns 9 – 16 give the components of $\mathbf{h}^{(k-1)}$.

Columns 17 – 44 give respectively the powers of $(1-\mu_{11}\Delta t)$, $(1-\mu_{12}\Delta t)$, $(1-\mu_{11}\Delta t)$, $(1-\mu_{12}\Delta t)$, $(1-\lambda_{11}\Delta t)$, $(1-\lambda_{12}\Delta t)$, $(1-\mu_{21}\Delta t)$, $(1-\mu_{22}\Delta t)$, $(1-\mu_{21}\Delta t)$, $(1-\mu_{22}\Delta t)$, $(1-\lambda_{21}\Delta t)$, $(1-\lambda_{22}\Delta t)$, $(\mu_{11}\Delta t)$, $(\mu_{12}\Delta t)$, $(\mu_{11}\Delta t)$, $(\mu_{12}\Delta t)$, $(\lambda_{11}\Delta t)$, $(\lambda_{12}\Delta t)$, $(\mu_{21}\Delta t)$, $(\mu_{22}\Delta t)$, $(\mu_{21}\Delta t)$, $(\mu_{22}\Delta t)$, $(\lambda_{21}\Delta t)$, $(\lambda_{22}\Delta t)$, (q_{11}), $(1-q_{11})$, (q_{22}) and $(1-q_{22})$.

Table 5.2.2 : An example of the codes of $\mathbf{h}^{(k)}, \mathbf{h}^{(k-1)}$ and probability of the corresponding event in the interval τ_k .

$\mathbf{h}^{(k)}$	$\mathbf{h}^{(k-1)}$	**Power**		
2 2 1 2 1 1 2 2	2 2 1 2 1 1 2 2	0 1 0 1 1 0 0 1 1 0 1 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0
2 2 1 2 1 1 2 2	1 2 1 2 1 1 2 2	0 0 0 1 1 0 0 1 1 0 1 0	1 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0
2 2 1 2 1 1 2 2	2 1 1 2 1 1 2 2	0 1 0 0 1 0 0 1 1 0 1 0	0 0 1 0 0 0 0 0 0 0 0 0	0 0 0 0
2 2 1 2 1 1 2 2	2 2 2 2 0 1 2 1	0 1 0 1 0 0 0 1 0 0 1 0	0 0 0 0 0 1 0 0 0 0 0 0	0 1 0 0
2 2 1 2 1 1 2 2	2 2 1 1 1 1 2 2	0 1 0 1 1 0 0 0 1 0 1 0	0 0 0 0 0 0 1 0 0 0 0 0	0 0 0 0
2 2 1 2 1 1 2 2	2 2 1 2 0 2 2 1	0 1 0 1 1 0 1 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 1	0 0 1 0

The multiplication of (1-$\mu_{11}\Delta t$), (1-$\mu_{12}\Delta t$), (1-$\mu_{11}\Delta t$), (1-$\mu_{12}\Delta t$), (1-$\lambda_{11}\Delta t$), (1-$\lambda_{12}\Delta t$), (1-$\mu_{21}\Delta t$), (1-$\mu_{22}\Delta t$), (1-$\mu_{21}\Delta t$), (1-$\mu_{22}\Delta t$), (1-$\lambda_{21}\Delta t$), (1-$\lambda_{22}\Delta t$), ($\mu_{11}\Delta t$), ($\mu_{12}\Delta t$), ($\mu_{11}\Delta t$), ($\mu_{12}\Delta t$), ($\lambda_{11}\Delta t$), ($\lambda_{12}\Delta t$), ($\mu_{21}\Delta t$), ($\mu_{22}\Delta t$), ($\mu_{21}\Delta t$), ($\mu_{22}\Delta t$), ($\lambda_{21}\Delta t$), ($\lambda_{22}\Delta t$), (q_{11}), (1-q_{11}), (q_{22}), (1-q_{22}) raised respectively to the corresponding powers will represent the probability of occurrence of the corresponding event which may be represented by an element in A.

The information represented by the above codes may be used to form the following equation:

$$\begin{aligned} P^{(k)}_{221211}[2][2] \cong\ & P^{(k-1)}_{221211}[2][2](1-2\mu_{12}\Delta t-\lambda_{11}\Delta t-\mu_{22}\Delta t-\mu_{21}\Delta t-\lambda_{21}\Delta t) \\ & + P^{(k-1)}_{121211}[2][2](\mu_{11}\Delta t) + P^{(k-1)}_{211211}[2][2](\mu_{11}\Delta t) \\ & + P^{(k-1)}_{222201}[2][1](\lambda_{12}\Delta t)(1-q_{11}) + P^{(k-1)}_{221111}[2][2](\mu_{21}\Delta t) \\ & + P^{(k-1)}_{221202}[2][1](\lambda_{22}\Delta t)q_{22} \end{aligned}$$

(5.2.1)

The derivation of Equation (5.2.1) may also be illustrated by Figure 5.2.2

Subtracting the term $P^{(k-1)}_{221211}[2][2]$ from both sides of (5.2.1), dividing both sides of the resulting equation by Δt, and letting $\Delta t \to 0$, and later letting $k \to \infty$, we get the balance equation

$$\begin{aligned} 0 \cong\ & P_{221211}[2][2](-2\mu_{12}-\lambda_{11}-\mu_{22}-\mu_{21}-\lambda_{21}) \\ & + P_{121211}[2][2]\mu_{11} + P_{211211}[2][2]\mu_{11} \\ & + P_{222201}[2][1]\lambda_{12}(1-q_{11}) + P_{221111}[2][2]\mu_{21} \\ & + P_{221202}[2][1]\lambda_{22}q_{22} \end{aligned}$$

(5.2.2)

Equation (5.2.2) may be represented in a coded form as shown in Table 5.2.2.

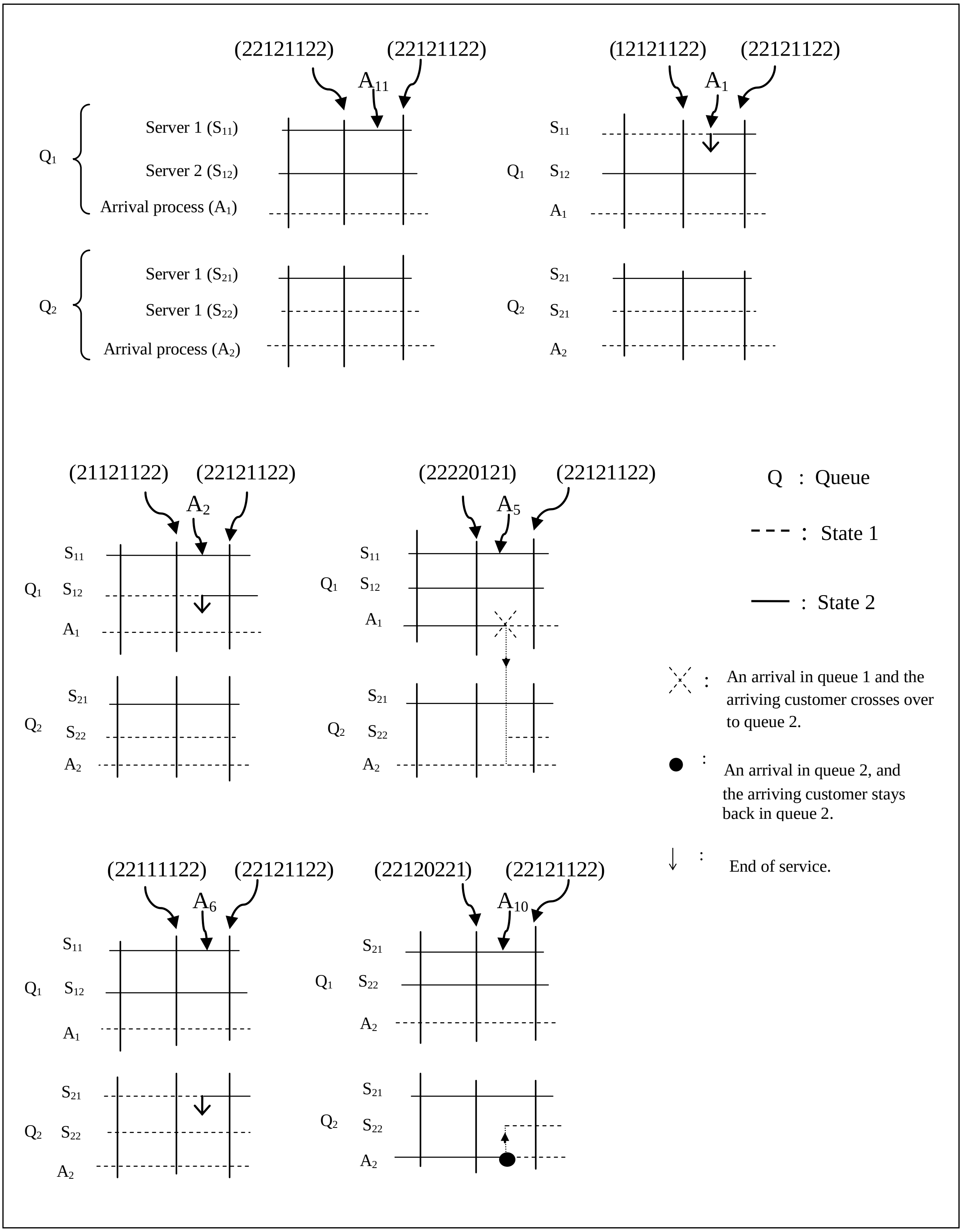

Figure 5.2.1 : The values of $\mathbf{h}^{(k-1)}$ and A_w which lead to the given value of $\mathbf{h}^{(k)}$.

Table 5.2.3 : Representation of balance equation in (5.2.2) by codes.

Constant	**h**	**Power**
-1	2 2 1 2 1 1 2 2	0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
-1	2 2 1 2 1 1 2 2	0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
-1	2 2 1 2 1 1 2 2	0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0
-1	2 2 1 2 1 1 2 2	0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
-1	2 2 1 2 1 1 2 2	0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
-1	2 2 1 2 1 1 2 2	0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
1	1 2 1 2 1 1 2 2	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1	2 1 1 2 1 1 2 2	0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
1	2 2 2 2 0 1 2 1	0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0
1	2 2 1 1 1 1 2 2	0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0
1	2 2 1 2 0 2 2 1	0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0
1*		

In Table 5.2.2,

Column 1 gives a coefficient value.

Columns 2 – 9 give the components of $\mathbf{h}$.

Columns 10 – 25 give respectively the powers of (μ_{11}), (μ_{12}), (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (q_{11}), (1-q_{11}), (q_{22}), and (1-q_{22}).

The symbol " 1* " in the last line denotes the end of the equation.

For each row in Table 5.2.2, we form a product of

(i) the coefficient in column 1,

(ii) the term $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ of which the values i_{11}, i_{12}, j_1, i_{21}, i_{22}, j_2, n_1, n_2, are given by $\mathbf{h}$, and

(iii) the product of (μ_{11}), (μ_{12}), (μ_{11}), (μ_{12}), (λ_{11}), (λ_{12}), (μ_{21}), (μ_{22}), (μ_{21}), (μ_{22}), (λ_{21}), (λ_{22}), (q_{11}), (1-q_{11}), (q_{22}) and (1-q_{22}) raised respectively to the corresponding powers.

We then equate the sum of the products for all the rows in Table 5.2.2 to zero to form Equation (5.2.2).

For other given value of $\mathbf{h}^{(k)}$, we may likewise use a computer to search all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may again be summarized and recorded in a coded form.

Next, the codes for the corresponding balance equations similar to (5.2.2) may be obtained. The resulting table of codes for $0 \leq n_1 + n_2 \leq 4$ can be found in the file *TwoQueueLossSystem_codes.txt* in the CD attached.

5.3 COMPUTATION OF THE VALUE OF $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$

Before solving the balance equations to obtain the stationary queue length distribution, we first introduce the following notations. Let

(a) $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n]$ be a value of $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ of which $n_1 + n_2 = n$.

(b) $\{P\,[n_1][n_2]\}$ the set consisting of all the possible $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$.

(c) $\{P\,[n]\}$ a set formed by the $\{P\,[n_1][n_2]\}$ of which $n_1 + n_2 = n$.

(d) $\{P\,[n],\ \ P\,[n+1],\ \ P\,[n+2]\}$ the set of equations of the form

$$\begin{aligned}&\sum_{i_{11}=0}^{2}\sum_{i_{12}=0}^{2}\sum_{j_1=1}^{2}\sum_{i_{21}=0}^{2}\sum_{i_{22}=0}^{2}\sum_{j_2=1}^{2} a_{i_{11}i_{12}j_1i_{21}i_{22}j_2} P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n]\\ &+\sum_{i_{11}=0}^{2}\sum_{i_{12}=0}^{2}\sum_{j_1=1}^{2}\sum_{i_{21}=0}^{2}\sum_{i_{22}=0}^{2}\sum_{j_2=1}^{2} b_{i_{11}i_{12}j_1i_{21}i_{22}j_2} P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n+1]\\ &+\sum_{i_{11}=0}^{2}\sum_{i_{12}=0}^{2}\sum_{j_1=1}^{2}\sum_{i_{21}=0}^{2}\sum_{i_{22}=0}^{2}\sum_{j_2=1}^{2} c_{i_{11}i_{12}j_1i_{21}i_{22}j_2} P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n+2] = 0\end{aligned}$$

where $a_{i_{11}i_{12}j_1i_{21}i_{22}j_2}, b_{i_{11}i_{12}j_1i_{21}i_{22}j_2}, c_{i_{11}i_{12}j_1i_{21}i_{22}j_2}$ are constants.

(e) $\left(P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2] \;\middle|\; \{P\,[0]\},\ \{P\,[n+1]\}\right)$ an equation of the form

$$P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2] = \sum_{i'_{12}=0}^{2}\sum_{i'_{12}=0}^{2}\sum_{j'_1=1}^{2}\sum_{i'_{21}=0}^{2}\sum_{i'_{22}=0}^{2}\sum_{j'_2=1}^{2} d_{i'_{11}i'_{12}j'_1i'_{21}i'_{22}j'_2}\, P_{{}_{11}i'_{12}j'_1i'_{21}i'_{22}j'_2}[0]$$
$$+ \sum_{i'_{12}=0}^{2}\sum_{i'_{12}=0}^{2}\sum_{j'_1=1}^{2}\sum_{i'_{21}=0}^{2}\sum_{i'_{22}=0}^{2}\sum_{j'_2=1}^{2} e_{i'_{11}i'_{12}j'_1i'_{21}i'_{22}j'_2}\, P_{{}_{11}i'_{12}j'_1i'_{21}i'_{22}j'_2}[n+1]$$

where $d_{i'_{11}i'_{12}j'_1i'_{21}i'_{22}j'_2}, e_{i'_{11}i'_{12}j'_1i'_{21}i'_{22}j'_2}$ are constants.

With the above notations, the balance equations represented by the codes in the file *TwoQueueLossSystem_codes.txt* in the CD can be represented as

$$\{P\,[0],\ \ P\,[1]\}, \tag{5.3.1}$$

$$\{P\,[n-1],\ \ P\,[n],\ \ P\,[n+1]\} \quad ,\ \ n = 1,\ 2,\ 3\ , \tag{5.3.2}$$

and
$$\{P\,[4],\ \ P\,[3]\}. \tag{5.3.3}$$

From the Equations (5.3.1) to (5.3.3), we get

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0] \;\big|\; \{P\,[1]\}\}, \tag{5.3.4}$$

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n] \;\big|\; \{P\,[n-1]\},\ \{P\,[n+1]\}\}\ ,\ \ n = 1,\ 2,\ 3, \tag{5.3.5}$$

and
$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[4] \;\big|\; \{P\,[3]\}\}. \tag{5.3.6}$$

Substituting the left side of (5.3.6) into the equations in (5.3.5) for n = 3, we get

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[3] \;\big|\; \{P\,[2]\}\}. \tag{5.3.7}$$

Substituting the left side of (5.3.7) into the equations in (5.3.5) for n = 2, we get

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[2] \;\big|\; \{P\,[1]\}\}\ . \tag{5.3.8}$$

Substituting the left side of (5.3.8) into the equations in (5.3.5) for $n = 1$, we get

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[1] \mid \{P[0]\}\}. \tag{5.3.9}$$

Substituting the left side of (5.3.9) into the equations in (5.3.4), we obtain

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0] \mid \{P[0]\}\}. \tag{5.3.10}$$

An inspection of (5.3.10) reveals that there are 4 equations in (5.3.10), and among the 4 equations, only 3 of them are linearly independent. Hence, we need to include another linearly independent equation so that the resulting system of equations has a unique solution.

By substituting the left side of (5.3.9) into the equations in (5.3.8), we get

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[2] \mid \{P[0]\}\}. \tag{5.3.11}$$

By substituting the left side of (5.3.11) into the equations in (5.3.7), we get

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[3] \mid \{P[0]\}\}. \tag{5.3.12}$$

By substituting the left side of (5.3.12) into the equations in (5.3.6), we get

$$\{P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[4] \mid \{P[0]\}\}. \tag{5.3.13}$$

Equating the sum of the left sides of (5.3.9), (5.3.11), (5.3.12) and (5.3.13) to the sum of the corresponding right sides, we get an equation of the form

$$\begin{aligned}
&\sum_{i_{11}=0}^{2}\sum_{i_{12}=0}^{2}\sum_{j_1=1}^{2}\sum_{i_{21}=0}^{2}\sum_{i_{22}=0}^{2}\sum_{j_2=1}^{2}\sum_{\substack{n_1=0\\ n_1+n_2\ge 1}}^{2}\sum_{n_2=0}^{2} P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2] \\
&= \sum_{i_{11}=0}^{2}\sum_{i_{12}=0}^{2}\sum_{j_1=1}^{2}\sum_{i_{21}=0}^{2}\sum_{i_{22}=0}^{2}\sum_{j_2=1}^{2} C_{i_{11}i_{12}j_1i_{21}i_{22}j_2} P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0][0]
\end{aligned}$$

where the $C_{i_{11}i_{12}j_1i_{21}i_{22}j_2}$ are constants, or

$$\begin{aligned}
&1-\sum_{i_{11}=0}^{2}\sum_{i_{12}=0}^{2}\sum_{j_1=1}^{2}\sum_{i_{21}=0}^{2}\sum_{i_{22}=0}^{2}\sum_{j_2=1}^{2} P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0][0] \\
&= \sum_{i_{11}=0}^{2}\sum_{i_{12}=0}^{2}\sum_{j_1=1}^{2}\sum_{i_{21}=0}^{2}\sum_{i_{22}=0}^{2}\sum_{j_2=1}^{2} C_{i_{11}i_{12}j_1i_{21}i_{22}j_2} P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0][0]
\end{aligned}$$

which is also of the form

$$(P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0] \,\big|\, \{P[0]\}). \tag{5.3.14}$$

Equation (5.3.14) together with 3 equations chosen from the equations of the form given by (5.3.10) will form a set of equations in 4 unknowns. Solving the set of 4 equations, we get the numerical answers for the $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0][0]$.

Then, from (5.3.9), (5.3.11), (5.3.12), (5.3.13) and the values of the $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[0][0]$, we can get the numerical answers for $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$, for the case when $n_1 + n_2 \le 4$.

5.4 SIMULATED VALUE OF $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$

The probability $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ may also be estimated by using a simulation procedure described below.

In the simulation procedure, we need to know the approximate probability that an event from set A will occur in τ_k given the conditions of the system at the end of τ_{k-1}. The above conditional probabilities are given in Table 5.4.1.

As in Section 2.4, let $N_T > 0$ be an integer such that $N_T\Delta t$ corresponds to a time which is "very long" after $t = 0$. Suppose that at the end of τ_1,

$$\mathbf{h}^{(1)} = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1). \tag{5.4.1}$$

The events which may occur in τ_2 are A_1, A_2, A_3, A_6, A_7, A_8 and A_{11}. The probability of each of these events and the resulting value $\mathbf{h}^{(2)}$ are as shown in Table 5.4.2.

To generate $\mathbf{h}^{(2)}$, we may first generate a random number $U^{(1)}$ from the U(0,1) distribution. Let $P_0^{(1)} = 0$. If $\sum_{i=0}^{j-1} P_i^{(1)} < U^{(1)} < \sum_{i=0}^{j} P_i^{(1)}$, then event $A^{(2)} = E_j^{(1)}$ is said to have occurred, j=1, 2, ..., 7 and the resulting $\mathbf{h}^{(2)}$ is as given in Table 5.4.2.

Similarly given a value of $\mathbf{h}^{(k-1)}$, we may first find out the set of possible events $E_1^{(k-1)}$, $E_2^{(k-1)}$, ..., $E_L^{(k-1)}$ which can occur in τ_k. Suppose $E_i^{(k-1)}$ occurs with probability $P_i^{(k-1)}$. To generate $\mathbf{h}^{(k)}$, we may first generate a random number $U^{(k-1)}$ from the U(0,1) distribution. Let $P_0^{(k-1)} = 0$. If $\sum_{i=0}^{j-1} P_i^{(k-1)} < U^{(k-1)} < \sum_{i=0}^{j} P_i^{(k-1)}$, then event $A^{(k)} = E_j^{(k-1)}$ is said to have occurred and the resulting $\mathbf{h}^{(k)}$ can be determined.

In short, we generate $\left(A^{(2)}, A^{(3)}, \Lambda\ , A^{(N_T)}\right)$ starting from $\underset{\sim}{h}^{(1)}$ given by (5.4.1). We repeat the generation of $\left(A^{(2)}, A^{(3)}, \Lambda\ , A^{(N_T)}\right)$ for N_s number of times. The probability $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ is then given approximately by the proportion of times the vector of characteristics given by $\mathbf{h} = \left(i_{11},\ i_{12},\ j_1,\ i_{21},\ i_{22},\ j_2,\ n_1,\ n_2 \right)$ is obtained at $t = N_T\Delta t$.

Table 5.4.1 : The approximate probability that event A will occur in τ_k given the conditions of system at the end of τ_{k-1} [v_i =1 or 2, $1 \le i \le 6$].

Conditions of system at the end of τ_{k-1}	Event occurring in τ_k	Approximate probability that event in Column 2 will occur given the conditions in Column 1.
$i_{11}^{(k-1)} = 1$	A_1	$\mu_{11}\Delta t$
$i_{11}^{(k-1)} = 2$	A_1	$\mu_{12}\Delta t$
$i_{12}^{(k-1)} = 1$	A_2	$\mu_{11}\Delta t$
$i_{12}^{(k-1)} = 2$	A_2	$\mu_{12}\Delta t$
$j_1^{(k-1)} = 1$	A_3	$\lambda_{11}\Delta t$
$j_1^{(k-1)} = 2$	A_4	$\lambda_{12}\Delta t q_{11}$
$j_1^{(k-1)} = 2$	A_5	$\lambda_{12}\Delta t(1-q_{11})$
$i_{21}^{(k-1)} = 1$	A_6	$\mu_{21}\Delta t$
$i_{21}^{(k-1)} = 2$	A_6	$\mu_{22}\Delta t$
$i_{22}^{(k-1)} = 1$	A_7	$\mu_{21}\Delta t$
$i_{22}^{(k-1)} = 2$	A_7	$\mu_{22}\Delta t$
$j_2^{(k-1)} = 1$	A_8	$\lambda_{21}\Delta t$
$j_2^{(k-1)} = 2$	A_9	$\lambda_{22}\Delta t(1-q_{22})$
$j_2^{(k-1)} = 2$	A_{10}	$\lambda_{22}\Delta t q_{22}$
$i_{11}^{(k-1)} = v_1, i_{12}^{(k-1)} = v_2$, $j_1^{(k-1)} = v_3, i_{21}^{(k-1)} = v_4$, $i_{22}^{(k-1)} = v_5, j_2^{(k-1)} = v_6$.	A_{11}	$1-(\mu_{1v_1}+\mu_{1v_2}+\lambda_{1v_3}+\mu_{2v_4}+\mu_{2v_5}+\lambda_{2v_6})\Delta t$
$i_{11}^{(k-1)} = 0, i_{12}^{(k-1)} = 0$, $j_1^{(k-1)} = v_3, i_{21}^{(k-1)} = v_4$, $i_{22}^{(k-1)} = v_5, j_2^{(k-1)} = v_6$.	A_{12}	$1-(\lambda_{1v_3}+\mu_{2v_4}+\mu_{2v_5}+\lambda_{2v_6})\Delta t$
$i_{11}^{(k-1)} = 0, i_{12}^{(k-1)} = 0$, $j_1^{(k-1)} = 1$.	A_{13}	$\lambda_{11}\Delta t$
$i_{11}^{(k-1)} = 0, i_{12}^{(k-1)} = 0$, $j_1^{(k-1)} = 2$.	A_{14}	$\lambda_{12}\Delta t q_{11}$
$i_{11}^{(k-1)} = 0, i_{12}^{(k-1)} = 0$, $j_1^{(k-1)} = 2$.	A_{15}	$\lambda_{12}\Delta t(1-q_{11})$

Table 5.4.1, continued

Conditions of system at the end of τ_{k-1}	Event occurring in τ_k	Approximate probability that event in Column 2 occur will given the conditions in Column 1.
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=1.$	A_{16}	$\mu_{21}\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=2.$	A_{16}	$\mu_{22}\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{22}^{(k-1)}=1.$	A_{17}	$\mu_{21}\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{22}^{(k-1)}=2.$	A_{17}	$\mu_{22}\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $j_{2}^{(k-1)}=1.$	A_{18}	$\lambda_{21}\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $j_{2}^{(k-1)}=2.$	A_{19}	$\lambda_{22}\Delta t(1-q_{22})$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $j_{2}^{(k-1)}=2.$	A_{20}	$\lambda_{22}\Delta t q_{22}$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $i_{11}^{(k-1)}=1.$	A_{21}	$\mu_{11}\Delta t$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $i_{11}^{(k-1)}=2.$	A_{21}	$\mu_{12}\Delta t$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $i_{12}^{(k-1)}=1.$	A_{22}	$\mu_{11}\Delta t$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $i_{12}^{(k-1)}=2.$	A_{22}	$\mu_{12}\Delta t$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0, j_{1}^{(k-1)}=1.$	A_{23}	$\lambda_{11}\Delta t$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0, j_{1}^{(k-1)}=2.$	A_{24}	$\lambda_{12}\Delta t q_{11}$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0, j_{1}^{(k-1)}=2.$	A_{25}	$\lambda_{12}\Delta t(1-q_{11})$
$i_{11}^{(k-1)}=v_1, i_{12}^{(k-1)}=v_2,$ $j_{1}^{(k-1)}=v_3, i_{21}^{(k-1)}=0,$ $i_{22}^{(k-1)}=0, j_{2}^{(k-1)}=v_6.$	A_{26}	$1-(\mu_{2v_1}+\mu_{2v_2}\lambda_{1v_3}+\lambda_{2v_6})\Delta t$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_{2}^{(k-1)}=1.$	A_{27}	$\lambda_{21}\Delta t$

Table 5.4.1, continued

Conditions of system at the end of τ_{k-1}	Event occurring in τ_k	Approximate probability that event in Column 2 occur will given the conditions in Column 1.
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_1^{(k-1)}=2.$	A_{28}	$\lambda_{22}\Delta t(1-q_{22})$
$i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_2^{(k-1)}=2.$	A_{29}	$\lambda_{22}\Delta t q_{22}$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $j_1^{(k-1)}=v_3, i_{21}^{(k-1)}=0,$ $i_{22}^{(k-1)}=0, j_2^{(k-1)}=v_6.$	A_{30}	$1-(\lambda_{1v_3}+\lambda_{2v_6})\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_1^{(k-1)}=1.$	A_{31}	$\lambda_{11}\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_1^{(k-1)}=2.$	A_{32}	$\lambda_{12}\Delta t q_{11}$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_1^{(k-1)}=2.$	A_{33}	$\lambda_{12}\Delta t(1-q_{11})$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_2^{(k-1)}=1.$	A_{34}	$\lambda_{21}\Delta t$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_2^{(k-1)}=2.$	A_{35}	$\lambda_{22}\Delta t(1-q_{22})$
$i_{11}^{(k-1)}=0, i_{12}^{(k-1)}=0,$ $i_{21}^{(k-1)}=0, i_{22}^{(k-1)}=0,$ $j_2^{(k-1)}=2.$	A_{36}	$\lambda_{22}\Delta t q_{22}$

Table 5.4.2 : The approximate probability that event A will occur in τ_2 given the conditions of system at the end of τ_1.

i	Event, $E_i^{(1)}$	Probability, $P_i^{(1)}$	$\mathbf{h}^{(2)}$
1	$E_1^{(1)} = \mathrm{A}_1$	$P_1^{(1)} = \mu_{11}\Delta t$	(2, 1, 1, 1, 1, 1, 1, 1)
2	$E_2^{(1)} = \mathrm{A}_2$	$P_2^{(1)} = \mu_{11}\Delta t$	(1, 2, 1, 1, 1, 1, 1, 1)
3	$E_3^{(1)} = \mathrm{A}_3$	$P_3^{(1)} = \lambda_{11}\Delta t$	(1, 1, 2, 1, 1, 1, 1, 1)
4	$E_4^{(1)} = \mathrm{A}_6$	$P_4^{(1)} = \mu_{21}\Delta t$	(1, 1, 1, 2, 1, 1, 1, 1)
5	$E_5^{(1)} = \mathrm{A}_7$	$P_5^{(1)} = \lambda_{21}\Delta t$	(1, 1, 1, 1, 2, 1, 1, 1)
6	$E_6^{(1)} = \mathrm{A}_8$	$P_6^{(1)} = \lambda_{21}\Delta t$	(1, 1, 1, 1, 1, 2, 1, 1)
7	$E_7^{(1)} = \mathrm{A}_{11}$	$P_7^{(1)} = 1-(\mu_{1v_1} + \mu_{1v_2} + \lambda_{1v_3} + \mu_{2v_4} + \mu_{2v_5} + \lambda_{2v_6})\Delta t$	(1, 1, 1, 1, 1, 1, 1, 1)

5.5 NUMERICAL RESULTS FOR DISTRIBUTION OF QUEUE LENGTH AND STATES OF ARRIVAL AND SERVICE PROCESSES IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/2/2 QUEUES

Suppose $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (2, 6, 2, 6), and q_{11} = q_{22} = 0.6. The traffic intensities ρ_1, ρ_2 in the two queues are then given respectively by

$$\rho_1 = (\mu_{11}{}^{-1} + \mu_{12}{}^{-1}) \big/ (\lambda_{11}{}^{-1} + \lambda_{12}{}^{-1}) = 0.225 \ ,$$

$$\rho_2 = (\mu_{21}{}^{-1} + \mu_{22}{}^{-1}) \big/ (\lambda_{21}{}^{-1} + \lambda_{22}{}^{-1}) = 0.225 .$$

The probabilities $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ computed by using the method in Section 5.3 and the simulation procedure in Section 5.4 are represented in Table (5.5.1) and Figure (5.5.1).

Next, Table (5.5.2) and Figure (5.5.2) show the results for the case when $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (4, 5, 3, 8), q_{11} = q_{22} = 0.9, $\rho_1 = 0.33333$ and $\rho_2 = 0.32727$.

The tables and figures show that the results for $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ found by the proposed method agree well with those found by the simulation procedure.

Table 5.5.1 : Comparison of results for $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ based on the proposed method and simulation procedure $[(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (2, 6, 2, 6), $q_{11} = q_{22} = 0.6$, $\rho_1 = 0.225$, $\rho_2 = 0.225$ and $N_s = 50000$].

n_1n_2	$i_{11}i_{12}j_1i_{21}i_{22}j_2$	Proposed method	Simulation procedure
00	001001	0.31757	0.31726
	001002	0.12968	0.12698
	002001	0.12539	0.12666
	002002	0.04956	0.05202
01	001011	0.00444	0.00412
	001012	0.00106	0.00102
	001021	0.00283	0.00292
	001022	0.00075	0.00078
	001101	0.06340	0.06362
	001102	0.01480	0.01492
	001201	0.02900	0.02982
	001202	0.00780	0.00784
	002011	0.00117	0.00102
	002012	0.00028	0.00022
	002021	0.00081	0.00098
	002022	0.00021	0.00014
	002101	0.01824	0.01916
	002102	0.00316	0.00314
	002201	0.00906	0.00944
	002202	0.00211	0.00206
10	011001	0.00443	0.00372
	011002	0.00117	0.00114
	012001	0.00105	0.00094
	012002	0.00027	0.00016
	021001	0.00282	0.00246
	021002	0.00081	0.00072
	022001	0.00075	0.00068
	022002	0.00021	0.00022
	101001	0.06304	0.06450
	101002	0.01820	0.01734
	102001	0.01476	0.01464
	102002	0.00315	0.00280
	201001	0.02884	0.03024
	201002	0.00903	0.00888
	202001	0.00778	0.00680
	202002	0.00211	0.00184
02	001111	0.00449	0.00444
	001112	0.00063	0.00066
	001121	0.00150	0.00198
	001122	0.00027	0.00016
	001211	0.00287	0.00350
	001212	0.00046	0.00046
	001221	0.00101	0.00098
	001222	0.00019	0.00016
	002111	0.00078	0.00078
	002112	0.00009	0.00006
	002121	0.00031	0.00036
	002122	0.00005	0.00006
	002211	0.00057	0.00064
	002212	0.00007	0.00004
	002221	0.00023	0.00022
	002222	0.00004	0.00002
11	011011	0.00002	0
	011012	4.37E-06	0
	011021	1.54E-05	2.00E-05
	011022	3.19E-06	0
	011101	5.06E-04	5.00E-04
	011102	8.34E-05	1.00E-04
	011201	2.28E-04	4.00E-05
	011202	4.45E-05	2.00E-05

Table 5.5.1, continued

n_1n_2	$i_{11}i_{12}j_1i_{21}i_{22}j_2$	Proposed method	Simulation procedure
	012011	4.37E-06	0
11	012012	8.23E-07	0
	012021	3.08E-06	0
	012022	6.34E-07	0
	012101	9.42E-05	8.00E-05
	012102	1.32E-05	2.00E-05
	012201	4.60E-05	0
	012202	8.51E-06	0
	021011	1.54E-05	2.00E-05
	021012	3.08E-06	0
	021021	1.00E-05	0
	021022	2.20E-06	0
	021101	3.48E-04	3.20E-04
	021102	6.13E-05	6.00E-05
	021201	1.53E-04	1.00E-04
	021202	3.19E-05	2.00E-05
	022011	3.19E-06	0
	022012	6.34E-07	0
	022021	2.20E-06	0
	022022	4.85E-07	0
	022101	7.13E-05	6.00E-05
	022102	1.04E-05	0
	022201	3.42E-05	4.00E-05
	022202	6.65E-06	2.00E-05
	101011	5.07E-04	4.80E-04
	101012	9.44E-05	2.00E-05
	101021	3.49E-04	3.20E-04
	101022	7.14E-05	2.00E-05
	101101	0.00865	0.00870
	101102	0.00132	0.00138
	101201	0.00415	0.00438
	101202	7.80E-04	7.20E-04
	102011	8.36E-05	8.00E-05
	102012	1.32E-05	0
	102021	6.15E-05	4.00E-05
	102022	1.04E-05	0
	102101	0.00132	0.00122
	102102	1.68E-04	8.00E-05
	102201	7.12E-04	6.80E-04
	102202	1.13E-04	6.00E-05
	201011	2.28E-04	1.80E-04
	201012	4.61E-05	8.00E-05
	201021	1.54E-04	1.60E-04
	201022	3.42E-05	2.00E-05
	201101	0.00414	0.00370
	201102	7.12E-04	5.80E-04
	201201	0.001927	0.00164
	201202	3.97E-04	4.40E-04
	202011	4.45E-05	8.00E-05
	202012	8.52E-06	0
	202021	3.19E-05	0
	202022	6.66E-06	0
	202101	7.79E-04	7.60E-04
	202102	1.12E-04	1.40E-04
	202201	3.97E-04	4.20E-04
	202202	7.47E-05	4.00E-05
	111001	0.00448	0.00480
	111002	7.82E-04	6.00E-04
	112001	6.32E-04	6.00E-04
	112002	9.02E-05	6.00E-05
	121001	0.001497	0.00138
	121002	3.11E-04	2.60E-04
	122001	2.64E-04	1.60E-04
	122002	4.96E-05	4.00E-05
	211001	0.00286	0.00268
	211002	5.68E-04	6.20E-04
20	212001	4.60E-04	3.20E-04
	212002	7.15E-05	4.00E-05
	221001	0.00101	0.00106
	221002	2.29E-04	2.40E-04
	222001	1.96E-04	4.00E-05
	222001	4.01E-05	4.00E-05
	Total	**0.99639**	**0.99650**

Table 5.5.2 : Comparison of results for $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ based on the proposed method and simulation procedure $[(\mu_{11},\mu_{12},\mu_{21},\mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22})$ = (4, 5, 3, 8), $q_{11} = q_{22} = 0.9$, $\rho_1 = 0.33333$, $\rho_2 = 0.32727$ and $N_s = 50000$].

n_1n_2	$i_{11}i_{12}j_1i_{21}i_{22}j_2$	Proposed method	Simulation procedure
00	001001	0.154967	0.154700
	001002	0.079565	0.074560
	002001	0.176347	0.179680
	002002	0.082711	0.082800
01	001011	0.004130	0.003560
	001012	1.12E-03	0.001140
	001021	0.002555	0.002660
	001022	7.92E-04	7.20E-04
	001101	0.050224	0.049580
	001102	0.010346	0.009440
	001201	0.022045	0.021920
	001202	0.006014	0.006020
	002011	0.003871	0.003420
	002012	1.03E-03	7.60E-04
	002021	0.002487	0.001960
	002022	7.66E-04	8.40E-04
	002101	0.049634	0.050940
	002102	0.009004	0.008720
	002201	0.022668	0.024400
	002202	0.005867	0.005520
10	011001	0.004725	0.003340
	011002	1.98E-03	0.001880
	012001	2.24E-03	0.001780
	012002	9.26E-04	7.60E-04
	021001	0.002873	0.002780
	021002	0.001235	0.001180
	022001	1.63E-03	0.001520
	022002	6.96E-04	5.40E-04
	101001	0.062573	0.064180
	101002	0.028070	0.029400
	102001	0.021006	0.019900
	102002	0.008323	0.007860
	201001	0.026763	0.028360
	201002	0.012170	0.012640
	202001	0.012762	0.012460
	202002	0.005494	0.004940
02	001111	0.004475	0.00460
	001112	6.20E-04	6.40E-04
	001121	1.51E-03	0.00162
	001122	2.81E-04	1.60E-04
	001211	0.002900	0.003420
	001212	4.50E-04	5.80E-04
	001221	9.96E-04	9.00E-04
	001222	2.08E-04	2.80E-04
	002111	3.38E-03	0.003720
	002112	3.98E-04	3.40E-04
	002121	1.24E-03	0.001320
	002122	2.11E-04	2.00E-04
	002211	0.002392	0.002360
	002212	3.11E-04	3.20E-04
	002221	8.63E-04	9.00E-04
	002222	1.67E-04	1.40E-04
11	011011	8.34E-05	6.00E-05
	011012	2.12E-05	0
	011021	5.33E-05	2.00E-05
	011022	1.56E-05	4.00E-05
	011101	1.18E-03	8.60E-04
	011102	2.11E-04	2.80E-04
	011201	5.25E-04	4.00E-04
	011202	1.30E-04	1.20E-04

Table 5.5.2, continued

n_1n_2	$i_{11}i_{12}j_1i_{21}i_{22}j_2$	Proposed method	Simulation procedure
	012011	3.66E-05	0
	012012	9.21E-06	0
11	012021	2.37E-05	2.00E-05
	012022	6.86E-06	0
	012101	5.28E-04	5.20E-04
	012102	9.02E-05	1.20E-04
	012201	2.38E-04	2.00E-04
	012202	5.81E-05	4.00E-05
	021011	5.18E-05	4.00E-05
	021012	1.33E-05	2.00E-05
	021021	3.28E-05	4.00E-05
	021022	9.69E-06	0
	021101	7.37E-04	5.60E-04
	021102	1.35E-04	8.00E-05
	021201	3.26E-04	3.40E-04
	021202	8.21E-05	1.00E-04
	022011	2.72E-05	0
	022012	6.93E-06	0
	022021	1.75E-05	0
	022022	5.15E-06	0
	022101	3.97E-04	2.40E-04
	022102	6.83E-05	6.00E-05
	022201	1.78E-04	1.60E-04
	022202	4.39E-05	2.00E-05
	101011	1.31E-03	0.001360
	101012	3.38E-04	4.00E-04
	101021	8.40E-04	7.40E-04
	101022	2.51E-04	2.80E-04
	101101	0.017054	0.018160
	101102	3.08E-03	0.002720
	101201	0.007750	0.009480
	101202	1.96E-03	0.002060
	102011	3.81E-04	4.20E-04
	102012	9.35E-05	4.00E-05
	102021	2.50E-04	2.00E-04
	102022	7.01E-05	6.00E-05
	102101	0.004991	0.004980
	102102	8.49E-04	8.80E-04
	102201	2.32E-03	0.002260
	102202	5.54E-04	5.00E-04
	201011	5.59E-04	3.40E-04
	201012	1.46E-04	1.20E-04
	201021	3.56E-04	5.20E-04
	201022	1.08E-04	1.80E-04
	201101	0.007395	0.008460
	201102	1.36E-03	0.001340
	201201	0.003328	0.003800
	201202	8.53E-04	9.00E-04
	202011	2.39E-04	1.60E-04
	202012	6.11E-05	6.00E-05
	202021	1.55E-04	2.20E-04
	202022	4.57E-05	1.00E-04
	202101	0.003215	0.003100
	202102	5.61E-04	5.80E-04
	202201	0.001468	0.001620
	202202	3.65E-04	4.00E-04
	111001	0.005198	0.005080
	111002	1.87E-03	0.002020
	112001	1.08E-03	6.00E-04
	112002	3.40E-04	3.00E-04
	121001	1.74E-03	0.001780
	121002	6.67E-04	5.00E-04
	122001	5.11E-04	4.60E-04
	122002	1.83E-04	1.20E-04
	211001	0.003437	0.003400
	211002	1.33E-03	0.001440
20	212001	8.07E-04	7.00E-04
	212002	2.67E-04	1.80E-04
	221001	1.16E-03	0.001260
	221002	4.63E-04	4.80E-04
	222001	3.89E-04	3.20E-04
	222001	1.46E-04	1.00E-04
	Total	**0.987255**	**0.987880**

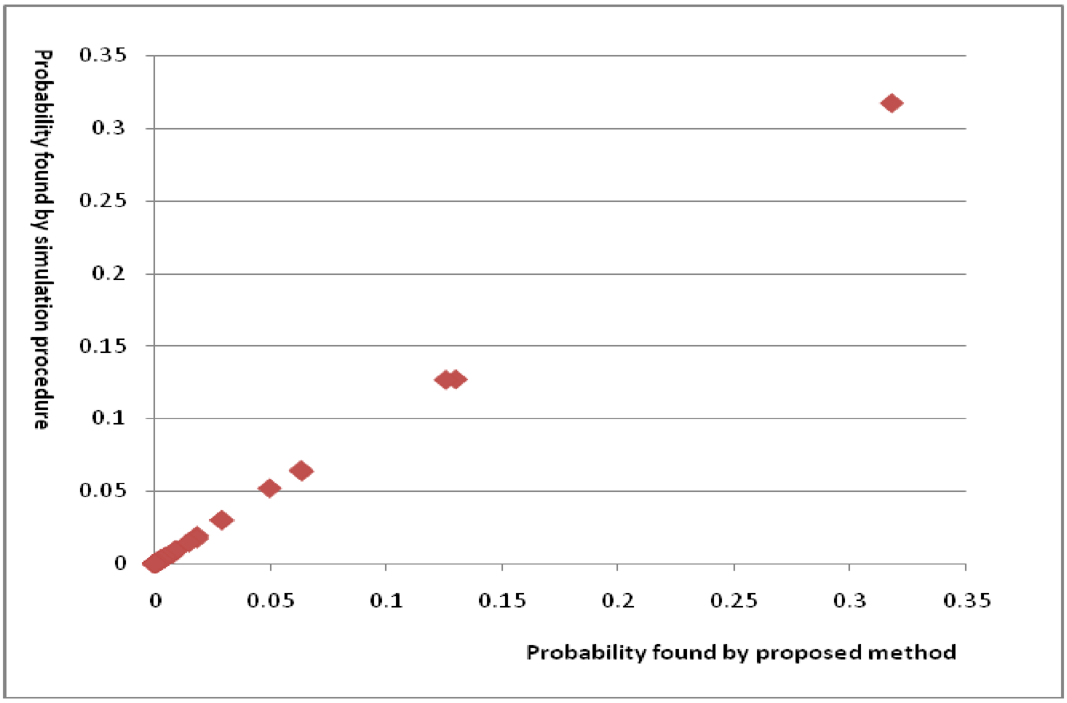

Figure 5.5.1 : Comparison of results for $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ based on the proposed method and simulation procedure [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (2, 6, 2, 6), $q_{11} = q_{22}$ = 0.60, $\rho_1 = 0.225$, $\rho_2 = 0.225$ and N_s = 50000].

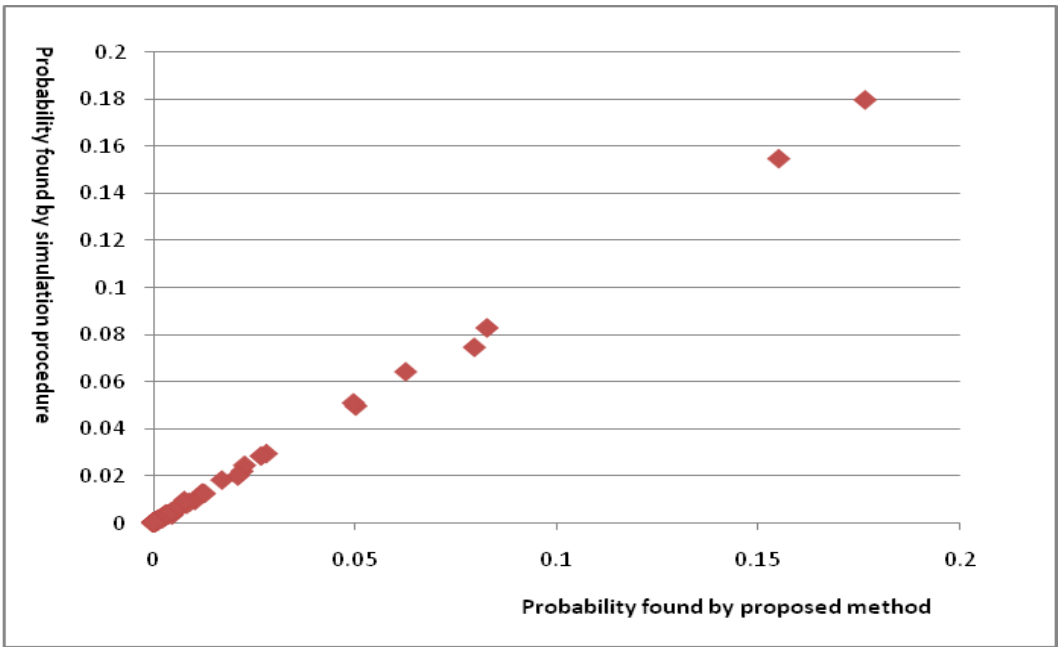

Figure 5.5.2 : Comparison of results for $P_{i_{11}i_{12}j_1i_{21}i_{22}j_2}[n_1][n_2]$ based on the proposed method and simulation procedure [$(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})$ = (10, 20, 10, 20), $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})$ = (4, 5, 3, 8), $q_{11} = q_{22}$ = 0.90, $\rho_1 = 0.33333$, $\rho_2 = 0.32727$ and N_s = 50000].

5.6 DERIVATION OF THE FORWARD EQUATIONS IN A SYSTEM OF TWO DEPENDENT HYPO(2)/HYPO(2)/*C*/*C* QUEUES

Consider a system of two Hypo(2)/Hypo(2)/*c*/*c* queues in which the customer who arrives at queue m has a probability of $q_{mm'} \geq 0$ of joining queue m', where $m \in \{1, 2\}$, $m' \in \{1, 2\}$ and $\sum_{m'=1}^{2} q_{mm'} = 1$.

As in Section 2.10, let $P^{(k)}_{i_{11}i_{12}\ldots i_{1c}j_1 i_{21}i_{22}\ldots i_{2c}j_2}[n_1][n_2]$ be the probability that at the end of the interval τ_k, the number of customers in queue m is n_m, the service process in queue m is in the state i_{ml} for server l, and the arrival process in queue m is in the state j_m, $m \in \{1, 2\}$, $1 \leq l \leq c$, $i_{ml} \in \{0, 1, 2\}$ and $j_m \in \{1, 2\}$. Assume that

$$P_{i_{11}i_{12}\ldots i_{1c}j_1 i_{21}i_{22}\ldots i_{2c}j_2}[n_1][n_2] = \lim_{k\to\infty} P^{(k)}_{i_{11}i_{12}\ldots i_{1c}j_1 i_{21}i_{22}\ldots i_{2c}j_2}[n_1][n_2]$$

exists.

Let $\mathbf{h}^{(k)}$ be the vector

$$\mathbf{h}^{(k)} = \left(i^{(k)}_{11},\ i^{(k)}_{12},\ \ldots,\ i^{(k)}_{1c}, j^{(k)}_{1},\ i^{(k)}_{21},\ i^{(k)}_{22},\ \ldots,\ i^{(k)}_{2c}, j^{(k)}_{2},\ n^{(k)}_{1}, n^{(k)}_{2} \right)$$

of which the components are respectively the value of i_{11}, i_{12}, …, i_{1c}, j_1, i_{21}, i_{22}, …, i_{2c}, j_2, n_1, n_2 at the end of τ_k. Again we refer to $\mathbf{h}^{(k)}$ as the vector of characteristics of the queueing system at the end of τ_k.

The value $\mathbf{h}^{(k)}$ may be developed from $\mathbf{h}^{(k-1)}$ after some appropriate activities in the interval τ_k. The set of possible activities may be denoted by a set A = {A_1, A_2, …, A_W} where A_w = { A_{w1}, $A_{w2, \ldots,}$ $A_{w(2c+4)}$ }, $1 \leq w \leq W$.

The meanings of the components in A_w are explained in Table 5.6.1.

Table 5.6.1 : The meanings of the components A_{wj} in A_w, $1 \le j \le 2(c+1)$.

j	A_{wj}	**Meaning**
1	1	A transition in the state of server 1 in queue 1 occurs in τ_k.
1	0	A transition in the state of server 1 in queue 1 does not occur in τ_k.
1	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of server 1 in queue 1 has occurred in τ_k is not relevant.
2	1	A transition in the state of server 2 in queue 1 occurs in τ_k.
2	0	A transition in the state of server 2 in queue 1 does not occur in τ_k.
2	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of server 2 in queue 1 has occurred in τ_k is not relevant.
Λ	Λ	Λ
c	1	A transition in the state of server r in queue 1 occurs in τ_k.
c	0	A transition in the state of server r in queue 1 does not occur in τ_k.
c	–1	Queue 1 is empty at the end of τ_{k-1} and whether a transition in the state of server r in queue 1 has occurred in τ_k is not relevant.
$(c+1)$	1	A transition in the state of the arrival process in queue 1 occurs in τ_k.
$(c+1)$	0	A transition in the state of the arrival process in queue 1 does not occur in τ_k.
$(c+2)$	1	A transition in the state of server 1 in queue 2 occurs in τ_k.
$(c+2)$	0	A transition in the state of server 1 in queue 2 does not occur in τ_k.
$(c+2)$	–1	Queue 2 is empty at the end of τ_{k-1} and whether a transition in the state of server 1 in queue 2 has occurred in τ_k is not relevant.
$(c+3)$	1	A transition in the state of server 2 in queue 2 occurs in τ_k.

Table 5.6.1, continued

j	A_{wj}	**Meaning**
$(c+3)$	0	A transition in the state of server 2 in queue 2 does not occur in τ_k .
$(c+3)$	–1	Queue 2 is empty at the end of τ_{k-1} and whether a transition in the state of server 2 in queue 1 has occurred in τ_k is not relevant.
Λ	Λ	Λ
$(2c+1)$	1	A transition in the state of server c in queue 2 occurs in τ_k .
$(2c+1)$	0	A transition in the state of server c in queue 2 does not occur in τ_k .
$(2c+1)$	–1	Queue 2 is empty at the end of τ_{k-1} and whether a transition in the state of server c in queue 2 has occurred in τ_k is not relevant.
$(2c+2)$	1	A transition in the state of the arrival process in queue 2 occurs in τ_k .
$(2c+2)$	0	A transition in the state of the arrival process in queue 2 does not occur in τ_k .

The meanings of A_{wj} in for $(2c+3) \leq j \leq (2c+4)$ are given below:

$$A_{w(2c+3)} = \begin{cases} 11, & \text{if the arriving customer in queue 1 stays back in queue 1.} \\ 12, & \text{if the arriving customer in queue 1 goes to queue 2.} \\ -1, & \text{no customer arrives in queue 1 and it is not relevant to find out} \\ & \text{whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

$$A_{w(2c+4)} = \begin{cases} 21, & \text{if the arriving customer in queue 2 goes to queue 1.} \\ 22, & \text{if the arriving customer in queue 2 stays back in queue 2.} \\ -1, & \text{no customer arrives in queue 2 and it is not relevant to find out} \\ & \text{whether the arriving customer is staying back or going elsewhere.} \end{cases}$$

For a given value of $\mathbf{h}^{(k)}$, we may use a computer to search all the possible combinations of $\mathbf{h}^{(k-1)}$ and A_w which lead to $\mathbf{h}^{(k)}$. The results of the search may be summarized and recorded in a coded form as has been done in Section 5.2.

Next, the codes for the corresponding balance equations similar to (5.2.2) may be obtained. The method in Section 5.3 may now be used to solve the balance equations so that the joint queue length distribution may be computed.

A simulation procedure similar to that given in Section 5.4 may also be used to find the joint queue length distribution.