Chapter 3

Integrated inventory model for final production batch

3.1 Introduction

The integrated inventory model has been widely discussed by many researchers since past four decades. The literature review for this model has been presented in Chapter 2. Most of the model considered the constant demand rate. Recently, researcher realized that the constant demand rate is no more realistic because the demand is always changing, for example it is either increasing or decreasing with time.

In this chapter, we will discuss various of inventory policies regarding the inventory model under time varying demand rate. The discussion starts with the integrated policy for shipping a vendor's final production batch to a single buyer under linearly decreasing demand rate. The reason why the final batch is important to be discussed is that just before the equipment for manufacturing the product is dismantled, there is always one final opportunity to make enough stock to meet all the remaining demand. As usual, costs are attached to the manufacturing batch set up, the delivery of a shipment and stockholding at the vendor and buyer. For a final batch, the objective is to determine the size of batch together with the number and shipments size which minimize the total cost, assuming that the vendor and buyer collaborate and find a way of sharing the consequent benefits.

Most previous work has been based on the assumption that unit stockholding costs increase as stock moves down the supply chain, but recent research has suggested that the opposite may sometimes hold [29]. Motivated from this ideas, both situations in which the buyer's holding cost is higher than the vendor's and the reverse situation will be discussed . We show how the solution policy may be derived when the shipment sizes and periods are equal or unequal. We illustrate this policy with numerical examples.

3.2 Mathematical formulation

The cost factors considered here are

1. the fixed shipment set up cost, A_2
2. the inventory holding cost for the vendor, h_1
3. the inventory holding cost for the buyer, h_2

Note that the production set up cost can be ignored since we are only making one batch of production. Here, we state the general notations and assumptions which will be used throughout this chapter.

3.2.1 Notation

Let $i = 1, 2, \ldots, n$ be the number of shipments and H is the finite planning horizon.

- The demand rate for the finished product at time t is $f(t)$ for $t \in (0, H)$.
- P units per unit time is the finite production rate. The value of the production rate is greater than the demand rate, $P > D$.
- x is the initial stock held at the buyer when the final production is about to start.
- q_i is the size of each shipment.

- D is the total demand in the interval $(0, H)$.

3.2.2 Assumption

- The general notation for demand function is $f(t)$. It could be linearly decreasing ($f(t) = a - bt$) or linearly increasing ($f(t) = a + bt$) or exponentially decreasing ($f(t) = ae^{-bt}$). We choose $f(t) = a - bt$ as an example in our models in this chapter. Here, a is the initial rate demand with $a > 0$ and b is the slope with $b > 0$.

- We are currently at time zero and wish to determine the stock replenishment which minimises the total relevant cost.

- The set-up and ordering costs are fixed throughout the planning horizon.

- The production rate, P is also fixed throughout the planning horizon.

- There are no limitation on the order size.

- The transportation cost per unit time is ignored since we are assuming that it is constant and independently from the ordering quantity.

- x is greater than zero and depends on the size of the first shipment.

- The finished product is transferred from the vendor to the buyer in n shipments during the production up-time and down-time.
- No shortages are allowed.

3.3 Case 1 : $h_1 < h_2$

This case has been widely studied in the literature since 1977 where the assumption is that the buyer's holding cost is higher than the vendor's. In Hill and Omar [29], there are two reasons generally used to justify this assumption; that is (i) the stock increases in value as it moves down the supply chain, and (ii) the vendor may be more likely to use cheap bulk store facilities (particularly in a retail distribution chain). Due to $h_1 < h_2$, the buyer wishes as little stock as possible at their store and the vendor delivers a shipment only when the buyer's inventory is just about to run out.

The initial stock, x is the amount which the buyer needs at the beginning of a production cycle to meet demand during the time it takes for the vendor to manufacture the quantity of stock which will make up the first shipment. Based on Omar's model [37], the initial stock at the buyer, x is given. However, in this model, we assume that the value of x depends on the first shipments size, q_1. We also assume that the demand rate before the final batch is constant at rate a. Therefore, the initial stock at the buyer, x

is given by

$$x = a(\frac{q_1}{P}) \tag{3.1}$$

where q_1/P is the time to produce the first shipment quantity, q_1.

Figure 3.1: Plot of the inventory level against time when $n = 4$

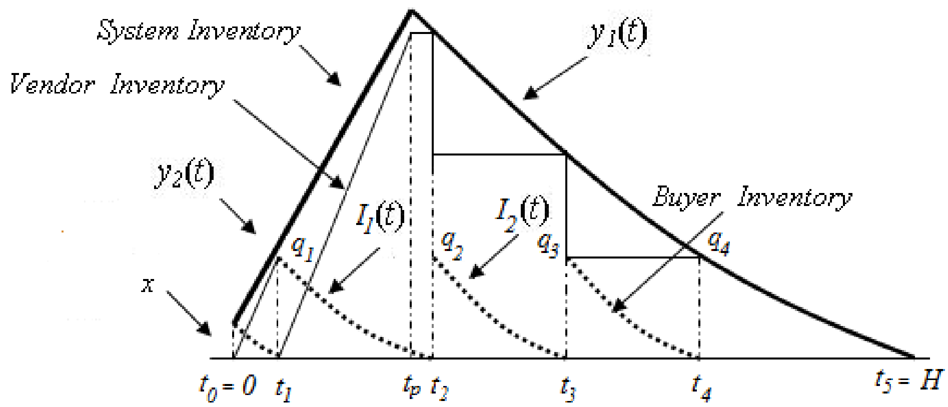


Figure 3.1 shows the illustration of the model with four equal shipments which represents the stock level of the system, vendor and buyer. The production batch will starts at $t_0 = 0$ until the production uptime, t_p. The first shipment is at time t_1 ($t_1 = 0$ according to buyer's time) and follows at time $t_2, t_3, \ldots, t_n$ with the shipment sizes $q_1, q_2, \ldots, q_n$.

In the Figure 3.1, $y_1(t)$ represents the remaining stock level at time t in the interval (t_p, H) which can be expressed as

$$\int_{t_p}^{H} f(t)\ dt - \int_{t_p}^{t} f(t)dt. \tag{3.2}$$

Then,

$$y_1(t) = \int_{t}^{H} f(t)\ dt. \tag{3.3}$$

$y_2(t)$ is the stock level during production time, $(0, t_p)$ and given by

$$y_2(t) = Pt - \int_{0}^{t} f(t)\ dt. \tag{3.4}$$

The total demand during the planning horizon,

$$D = \int_{0}^{H} f(t)\ dt. \tag{3.5}$$

3.3.1 Total time-weighted system stock

The total system stock is represent by the area under the curve $y_1(t)$ and $y_2(t)$ in (t_p, H) and $(0, t_p)$ respectively. Hence, we have the total time-weighted system stock, TSS, as

$$\int_{0}^{t_p} y_2\ dt + \int_{t_p}^{H} y_1\ dt + \frac{1}{2}xH. \tag{3.6}$$

The production uptime, t_p can be obtained from the following :

$$\begin{aligned} Pt_p &= \int_{0}^{H} f(t)\ dt \\ t_p &= \frac{D}{P}. \end{aligned} \tag{3.7}$$

3.3.2 Total time-weighted buyer stock

In figure 3.1, q_i, $i = 1, 2, 3, 4$ represent the shipment sizes for each shipment. We also have

$$q_i = \int_{t_i}^{t_{i+1}} f(t)dt. \tag{3.8}$$

Let $I_i(t)$ be the inventory level for i-shipment at any time t and it given by

$$I_i(t) = \int_{t}^{t_{i+1}} f(t)dt. \tag{3.9}$$

Hence, the buyer stock can be calculated by the area under the curve $I_i(t)$ in the period (t_i, t_{i+1}). It follows that the total time-weighted buyer stock from ith shipment, TBS is

$$\sum_{i=1}^{n} \left\{ \int_{t_i}^{t_{i+1}} \left[\int_{t}^{t_{i+1}} f(t)\ dt \right]\ dt \right\}. \tag{3.10}$$

It follows, the total cost for this model, TC, is given by

$$\begin{aligned} TC &= nA_2 + h_1(TSS - TBS) + h_2 TBS \\ &= nA_2 + h_1 TSS + (h_2 - h_1)TBS. \end{aligned} \tag{3.11}$$

For example, let the demand rate is linearly decreasing over the period $(0, H)$ that is

$$f(t) = a - bt \qquad a > 0; b > 0; t > 0; H > 0. \tag{3.12}$$

Substituting (3.12) into (3.3), (3.4), (3.5), (3.6), (3.7) and (3.10) we have

$$y_1(t) = a(H - t) - \frac{b}{2}(H^2 - t^2). \tag{3.13}$$

$$y_2(t) = (P - a)t + \frac{b}{2}t^2. \tag{3.14}$$

$$D = H\left(a - \frac{b}{2}H\right). \tag{3.15}$$

$$TSS = \frac{Pt_p^2}{2} + H(\frac{bH}{2} - a)t_p + H^2(\frac{a}{2} - \frac{bH}{3}) + \frac{xH}{2}. \tag{3.16}$$

$$t_p = \frac{H}{P}(a - \frac{b}{2}H). \tag{3.17}$$

$$\begin{aligned} TBS &= a\sum_{i=1}^{n}(t_{i+1}^2 - t_i t_{i+1}) - \frac{a}{2}(H^2 - t_1^2) - \frac{b}{2}\sum_{i=1}^{n}(t_{i+1}^3 - t_i t_{i+1}^2) \\ &\quad + \frac{b}{6}(H^3 - t_1^3). \end{aligned} \tag{3.18}$$

Finally, substituting (3.16) and (3.18) into (3.11) we have,

$$\begin{aligned} TC &= nA_2 + h_1\left\{\frac{Pt_p^2}{2} + H(\frac{bH}{2} - a)t_p + H^2(\frac{a}{2} - \frac{bH}{3}) + \frac{xH}{2}\right\} \\ &+ (h_2 - h_1)\left\{a\sum_{i=1}^{n}(t_{i+1}^2 - t_i t_{i+1}) - \frac{a}{2}(H^2 - t_1^2)\right. \\ &- \left.\frac{b}{2}\sum_{i=1}^{n}(t_{i+1}^3 - t_i t_{i+1}^2) + \frac{b}{6}(H^3 - t_1^3)\right\}. \end{aligned} \tag{3.19}$$

TC is in the function of n (discrete variable) and a vector $\vec{t} = t_1, t_2, \ldots, t_n$ where $t_1 = 0$ and $t_{n+1} = H$.

We will explore the best solution of the above total cost based on three policies :

1. Policy 1 : Equal shipment sizes
2. Policy 2 : Equal shipment periods
3. Policy 3 : Unequal shipment sizes and periods

The objective in every policy is to find an optimal t_i (or q_i) for a given n which gives the minimum total cost, $TC(n, \vec{t}\,)$.

3.3.3 Policy 1 : Equal shipment sizes

In this policy the shipment sizes, q_i are assumed to be equal. So, we have fixed value of q_i which can be calculated as follow

$$q_i \;=\; \frac{D}{n} \qquad i = 1, 2, \cdots, n. \tag{3.20}$$

The buyer receives an equal quantity for each shipment and takes $(t_{i+1} - t_i)$ amount of time to use up q_i, where $i = 1, 2, \ldots, n$. Hence, we have

$$\int_{t_i}^{t_{i+1}} f(t)\; dt = \frac{D}{n}. \tag{3.21}$$

For example, as shown in Figure 3.1, the vendor delivers four equal shipments. The first shipment is at time t_1 ($t_1 = 0$ according to buyer's time), and in the period (t_1, t_2), the buyers will use up q_1 until the second shipment is arrive just before the first shipment is finished at t_2. This can be written as

$$q_1 = \int_{t_1}^{t_2} f(t)\ dt = \frac{D}{n}. \tag{3.22}$$

From equation (3.22), we get

$$t_2 = \frac{a}{b}\left\{1 - \sqrt{a - \frac{2b}{a^2}\left[-\frac{b}{2}{t_1}^2 + at_1 + \left(\frac{D}{n}\right)\right]}\right\}. \tag{3.23}$$

Similarly, in the period time, (t_2, t_3), the buyer will use up q_2 while the vendor will continue producing and deliver the third shipment which will arrive at the buyer exactly just before the second shipment is finished at t_3. This can be written as

$$q_2 = \int_{t_2}^{t_3} f(t)\ dt = \frac{D}{n}. \tag{3.24}$$

From equation (3.24), we get

$$t_3 = \frac{a}{b}\left\{1 - \sqrt{a - \frac{2b}{a^2}\left[-\frac{b}{2}{t_2}^2 + at_2 + \left(\frac{D}{n}\right)\right]}\right\}. \tag{3.25}$$

This process is repeated until the end of the planning horizon, H. Generally, the shipment times is

$$t_{i+1} = \frac{a}{b}\left\{1 - \sqrt{a - \frac{2b}{a^2}\left[-\frac{b}{2}{t_i}^2 + at_i + \left(\frac{D}{n}\right)\right]}\right\} \quad i = 1, 2, \cdots, n-1. \tag{3.26}$$

Substituting (3.26) into (3.19) gives the total cost, $TC_1(n,\ \vec{t}\ )$ for this policy.

3.3.3.1 Solution procedure

The computer algorithm of the solution procedure is outline below :

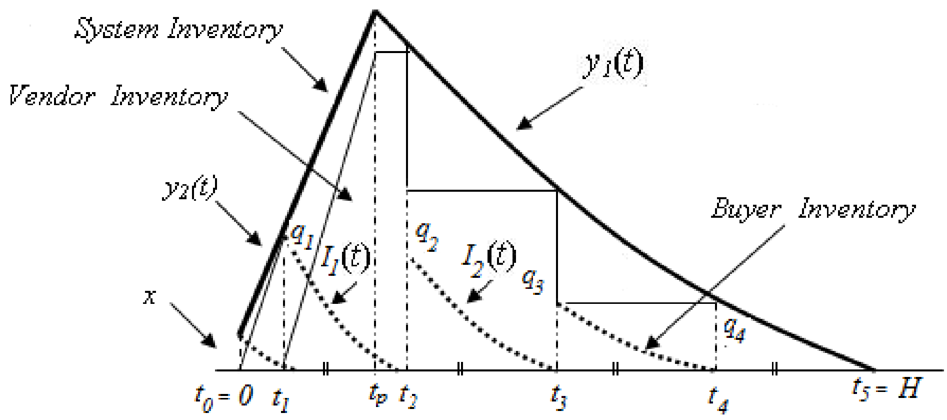
1. Let $n = 1$
2. Set $t_1 = 0$, $t_{n+1} = H$
3. Set $q_i = D/n, \quad i = 1, 2, \ldots, n$
4. Compute $t_{i+1}, \quad i = 1, 2 \cdots, n-1$ using (3.26) and $TC_1(n, \vec{t}\,)$ using (3.19)
5. Set $TC_1(n, \vec{t}\,)$ as $TC_1(n^*, \vec{t}\,)$. Increase n by 1 and repeat step 4. Stop when $TC_1(n, \vec{t}\,) \geq TC_1(n^*, \vec{t}\,)$.

The basic idea of the above algorithm is to start with $n = 1$. Next, we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_1(n^*, \vec{t}\,) < TC_1(n^* - 1, \vec{t}\,)$ and $TC_1(n^*, \vec{t}\,) < TC_1(n^* + 1, \vec{t}\,)$.

3.3.4 Policy 2 : Equal shipment periods

In this policy, the periods between shipments are assumed to be equal. Figure 3.2 gives the graphical representation for this policy.

Figure 3.2: Plot of the inventory level against time when $n = 4$ with equal shipment periods policy



Note that the value of H is fixed and $t_1 = 0$ according to buyer's time. Then, we have

$$\begin{aligned}
t_2 &= t_1 + \frac{H}{n} &= \frac{H}{n} \\
t_3 &= t_2 + \frac{H}{n} &= 2\frac{H}{n} \\
t_4 &= t_3 + \frac{H}{n} &= 3\frac{H}{n} \\
&\vdots \\
t_{i+1} &= t_i + \frac{H}{n} &= i\frac{H}{n}, \qquad i = 1, 2, \cdots, n-1. \qquad (3.27)
\end{aligned}$$

It follows

$$q_i = \int_{t_i}^{t_{i+1}} f(t)dt. \qquad (3.28)$$

The total cost for this policy can be calculated by substituting (3.27) into

(3.19). Then we have,

$$\begin{aligned} TC_2(n,\vec{t}\,) &= nA_2 + h_1\left\{\frac{Pt_p^2}{2} + H(\frac{bH}{2} - a)t_p + H^2(\frac{a}{2} - \frac{bH}{3}) + \frac{xH}{2}\right\} \\ &+ \frac{(h_2 - h_1)H^2}{2n}\left\{a - \frac{bH}{6n}(3n+1)\right\}. \end{aligned} \tag{3.29}$$

The total cost, $TC_2(n, \vec{t}\,)$ for this policy is in the term of n (discrete variable) and $\vec{t} = t_2, t_3, \ldots, n$ (real variables).

3.3.4.1 Solution procedure

Similarly, the computer algorithm of the solution procedure is outline below:

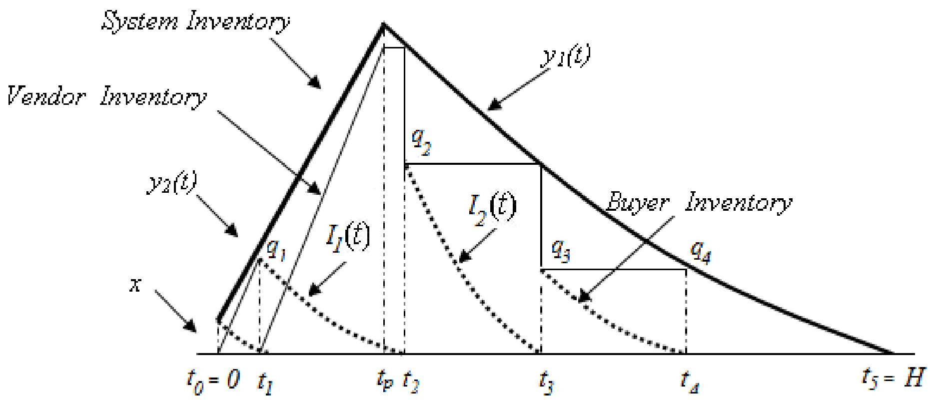
1. Let $n = 1$
2. Set $t_1 = 0$, $t_{n+1} = H$
3. Set $t_{i+1}, \quad i = 1, 2, \cdots, n-1$ as (3.27)
4. Compute $TC_2(n, \vec{t}\,)$ using (3.29)
5. Set $TC_2(n, \vec{t}\,)$ as $TC_2(n^*, \vec{t}\,)$. Increase n by 1 and repeat step 4. Stop when $TC_2(n, \vec{t}\,) \geq TC_2(n^*, \vec{t}\,)$

Similarly, the basic idea of the above algorithm is to start with $n = 1$. Next, we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_2(n^*, \vec{t}\,) < TC_2(n^* - 1, \vec{t}\,)$ and $TC_2(n^*, \vec{t}\,) < TC_2(n^* + 1, \vec{t}\,)$.

3.3.5 Policy 3 : Unequal shipment sizes and unequal shipment periods

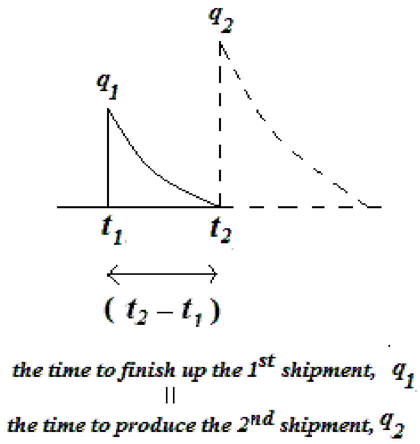
In this policy both shipment sizes and periods are assumed to be varied. Figure 3.3 gives the graphical representation for this policy with four unequal shipments.

Figure 3.3: Plot of the inventory level against time when $n = 4$ with unequal shipment sizes and periods policy



Since stockout is not allowed, the time for the vendor to produce q_{i+1} must be less than the time for the buyer to finish up q_i. For example, Figure 3.4 shows the illustration of the inventory level at the buyer for the first and second shipments.

Figure 3.4: The first and second shipments



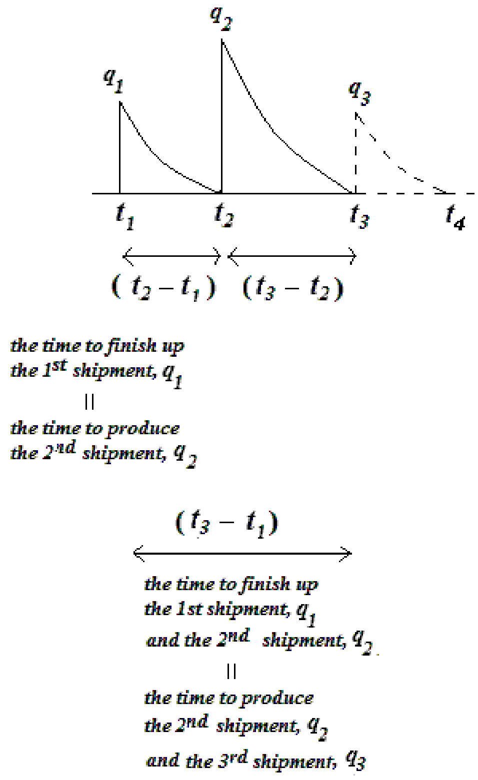
The time to produce the second shipment, q_2 must be less than the time for the buyer to finish up the first shipment, q_1. It follows that

$$
\begin{aligned}
P(t_2 - t_1) &\geq q_2 \\
t_2 - t_1 &\geq \frac{q_2}{P}
\end{aligned}
$$

where
$$
\begin{aligned}
q_1 &= \int_{t_1}^{t_2} f(t)dt \\
t_2 &= \frac{a}{b}\left\{1 - \sqrt{1 - \frac{2b}{a^2}\left[at_1 - \frac{b}{2}t_1^2 + q_1\right]}\right\}
\end{aligned}
$$

Figure 3.5 shows the illustration of the inventory level at the buyer for the first, second and third shipments.

Figure 3.5: The first, second and third shipments



Similarly, the time to produce the third shipment, q_3, must be less than the time for the buyer to finish up the second shipment, q_2, and we have

$$\begin{aligned}
P(t_3 - t_2) &\geq q_3 \\
P(t_3 - t_1 + t_1 - t_2) &\geq q_3 \\
P(t_3 - t_1) &\geq P(t_2 - t_1) + q_3 \\
P(t_3 - t_1) &\geq q_2 + q_3 \\
(t_3 - t_1) &\geq \frac{q_2 + q_3}{P} \qquad (3.30)
\end{aligned}$$

$$\text{where} \qquad \begin{aligned}
q_2 &= \int_{t_2}^{t_3} f(t)dt \\
t_3 &= \frac{a}{b}\left\{1 - \sqrt{1 - \frac{2b}{a^2}\left[at_2 - \frac{b}{2}t_2^2 + q_2\right]}\right\}
\end{aligned}$$

Generally, the time to produce the $i+1$ shipment, q_{i+1} must be less than the time for the buyer to finish up the i shipment, q_i, that is

$$P(t_{i+1} - t_1) \geq \sum_1^i q_{i+1}$$

$$\text{where} \qquad \begin{aligned}
q_i &= \int_{t_i}^{t_{i+1}} f(t)dt \qquad (3.31) \\
t_{i+1} &= \frac{a}{b}\left\{1 - \sqrt{1 - \frac{2b}{a^2}\left[at_i - \frac{b}{2}t_i^2 + q_i\right]}\right\} \qquad (3.32) \\
& i = 1, 2, \cdots, n-1.
\end{aligned}$$

The total quantity delivered to the buyer must be equal to the total demand, that is

$$\sum_{i=1}^{n} q_i = D \ , \quad i = 1, 2, \cdots, n \tag{3.33}$$

From these arguments, we can establish the following constraint optimization problem,

Minimum : $\quad TC_2(n, \vec{t}\,)$

Subject to

$$\begin{aligned} \frac{a}{b}\left\{1 - \sqrt{1 - \frac{2b}{a^2}\left[at_i - \frac{b}{2}t_i^2 + q_i\right]}\right\} \ &\geq \ \frac{1}{P}\sum_{1}^{i} q_{i+1}, \qquad (3.34) \\ & \quad i = 1, 2, \cdots, n \\ \sum_{i=1}^{n} q_i \ &= \ D. \qquad (3.35) \end{aligned}$$

3.3.5.1 Solution procedure

We derived the following algorithm and use the Microsoft Excel Solver as a solution tool:

1. Let $n = 1$
2. Set $t_1 = 0$, $t_{n+1} = H$
3. Set $q_i, \quad i = 1, 2, \cdots, n$ as changing variables. The Microsoft Excel Solver will find an optimal solution of q_i which satisfy constraints (3.34) and (3.35), if exist.

4. Compute $t_{i+1}, \quad i = 1, 2, \cdots, n-1$ using (3.33) and $TC_2(n, \vec{t}\,)$ using (3.19)

5. Set $TC_2(n, \vec{t}\,)$ as $TC_2(n^*, \vec{t}\,)$. Increase n by 1 and repeat step 4. Stop when $TC_2(n, \vec{t}\,) \geq TC_2(n^*, \vec{t}\,)$.

Again, the basic idea of the above algorithm is to start with $n = 1$. Next, we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC(n^*, \vec{t}\,) < TC(n^* - 1, \vec{t}\,)$ and $TC(n^*, \vec{t}\,) < TC(n^* + 1, \vec{t}\,)$.

3.3.6 Numerical examples and sensitivity analysis

To show the effectiveness of the proposed policies we adopt the same numerical examples as Omar [37] except the value of a, b and D. For easy reference, the parameter values are restated here:

$$A_2 = 25, \;\; a = 200, \quad b = 20 \quad H = 5, \quad h_1 = 4, \quad h_2 = 5$$
$$D = 750, \quad P = 1000$$

Tables 3.1, 3.2 and 3.3 give the minimum total cost and its minimum shipment sizes for Policy 1, 2 and 3. For example, in Table 3.1, when $n = 2$, the total minimum cost for this policy is 7222.97 with the initial inventory at the buyer, x is 75 and two equal shipments, where $q_1 = q_2 = 375.00$. Policy

1 reached the optimal total cost, $TC_1^* = 6121.29$ with 12 equal shipments where $q_i^* = 62.50$, for $i = 1, 2, \cdots, 12$.

Table 3.2 gives the minimum total cost for Policy 2. For example, when $n = 3$, the total cost is 6829.63 with its initial inventory level at the buyer, x is 61.11. The shipment sizes, (q_1, q_2, q_3) are (305.56, 250.00, 194.44) with the shipment times, (t_1, t_2, t_3) of (0.000, 1.667, 3.333) respectively. The minimum total cost for Policy 2, $TC_2^* = 6159.66$ is also at $n^* = 12$.

Table 3.3 gives the minimum total cost for Policy 3. For example, when $n = 3$, the total cost is 6511.15 where its initial inventory level at the buyer, x is 13.95, and the shipment sizes, (q_1, q_2, q_3) are (69.75, 355.05, 325.20) with the shipment times, (t_1, t_2, t_3) of (0.000, 0.355, 2.416) respectively. We observed that the total cost for Policy 3 is always better than Policy 1 and 2 for all $n = 1, 2, \cdots, 14$. Policy 3 reached the optimal total cost, $TC_3 = 6015.87$ with 10 shipments. The total cost savings which can be obtained from implementing Policy 3 rather than Policy 1 and 2 are 105.42 and 143.79 respectively.

Generally, the results given by Table 3.1 3.2 and 3.3 show the convexity of the total cost function respect to n.

Table 3.1: The total cost for case 1 : $h_1 < h_2$, Policy 1-Equal shipment sizes

n	Total Cost	x	q
1	8733.33	150.00	750.00(1)
2	7222.97	75.00	375.00(2)
3	6716.16	50.00	250.00(3)
4	6470.95	37.50	187.50(4)
5	6332.37	150.00	150.00(5)
6	6247.69	25.00	125.00(6)
7	6194.05	21.43	107.14(7)
8	6159.89	18.75	93.75(8)
9	6138.78	16.67	83.33(9)
10	6126.83	15.00	75.00(10)
11	6121.55	13.64	68.18(11)
12	**6121.29**	**12.50**	**62.50(12)**
13	6124.90	11.54	57.69(13)
14	6131.54	10.71	53.57(14)

Table 3.2: The total cost for case 1 : $h_1 < h_2$, Policy 2-Equal shipment periods

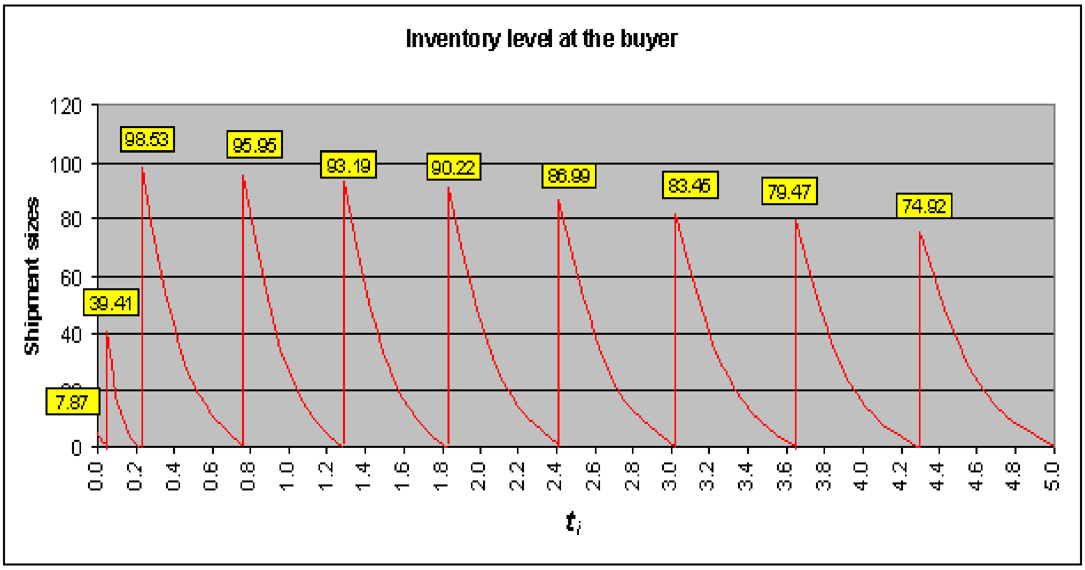
n	Total cost	x	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}
1	8733.33	150.000	750.000	-	-	-	-	-	-	-	-	-	-	-	-	-
2	7352.08	87.50	437.50	312.50	-	-	-	-	-	-	-	-	-	-	-	-
3	6829.63	61.11	305.56	250.00	194.44	-	-	-	-	-	-	-	-	-	-	-
4	6566.15	46.88	234.38	203.13	171.88	140.63	-	-	-	-	-	-	-	-	-	-
5	6413.33	38.00	190.00	170.00	150.00	130.00	110.00	-	-	-	-	-	-	-	-	-
6	6317.82	31.94	159.72	145.83	131.94	118.06	104.17	90.28	-	-	-	-	-	-	-	-
7	6255.78	27.55	137.76	127.55	117.35	107.14	96.94	86.73	76.53	-	-	-	-	-	-	-
8	6214.97	24.22	121.09	113.28	105.47	97.66	89.84	82.03	74.22	66.41	-	-	-	-	-	-
9	6188.477	21.60	108.02	101.85	95.68	89.51	83.33	77.16	70.99	64.81	58.64	-	-	-	-	-
10	6172.08	19.50	97.50	92.50	87.50	82.50	77.50	72.50	67.50	62.50	57.50	52.50	-	-	-	-
11	6163.09	17.77	88.84	84.71	80.58	76.45	72.31	68.18	64.05	59.92	55.79	51.65	47.52	-	-	-
12	**6159.66**	**16.32**	**81.60**	**78.13**	**74.65**	**71.18**	**67.71**	**64.24**	**60.76**	**57.29**	**53.82**	**50.35**	**46.88**	**43.40**	**-**	**-**
13	6160.55	15.09	75.44	72.49	69.53	66.57	63.61	60.65	57.69	54.73	51.78	48.82	45.86	42.90	9.94	-
14	6164.84	14.03	70.15	67.60	65.05	62.50	59.95	57.40	54.85	52.30	49.74	47.19	44.64	42.09	39.54	36.99

Table 3.3: The total cost for case 1 : $h_1 < h_2$, Policy 3-Unequal shipment sizes and periods

n	Total cost	x	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}
1	8733.33	150.00	750.00			-	-	-	-	-	-	-	-	-	-	-
2	7077.66	40.41	202.04	547.96	-	-	-	-	-	-	-	-	-	-	-	-
3	6511.15	13.95	69.75	355.05	325.20	-	-	-	-	-	-	-	-	-	-	-
4	6275.63	9.298	46.49	235.21	247.13	221.18	-	-	-	-	-	-	-	-	-	-
5	6156.44	6.64	33.20	167.39	198.16	184.03	167.27	-	-	-	-	-	-	-	-	-
6	6089.75	4.92	24.58	123.66	165.05	156.23	146.10	134.39	-	-	-	-	-	-	-	-
7	6051.25	3.71	18.54	93.11	141.35	135.31	128.53	120.89	112.26	-	-	-	-	-	-	-
8	6029.66	2.81	14.06	70.53	123.40	119.09	114.37	109.13	103.19	96.23	-	-	-	-	-	-
9	6019.02	2.12	10.61	53.18	109.57	106.30	102.75	98.89	94.63	89.82	84.25	-	-	-	-	-
10	**6015.87**	**1.57**	**7.87**	**39.41**	**98.53**	**95.95**	**93.19**	**90.22**	**86.99**	**83.45**	**79.47**	**74.92**	**-**	**-**	**-**	**-**
11	6018.07	1.13	5.63	28.21	89.50	87.42	85.21	82.85	80.32	77.58	4.58	71.24	67.45	-	-	-
12	6024.23	0.76	3.79	18.98	82.17	80.40	78.52	76.53	74.42	72.17	69.79	67.25	64.51	61.45	-	-
13	6033.44	0.59	2.97	14.85	74.63	73.81	72.29	70.70	69.03	67.25	65.35	63.31	61.11	58.69	56.01	-
14	6045.14	0.54	2.69	13.45	67.58	68.23	66.93	65.57	64.13	62.62	61.03	59.35	57.58	55.70	53.69	51.47

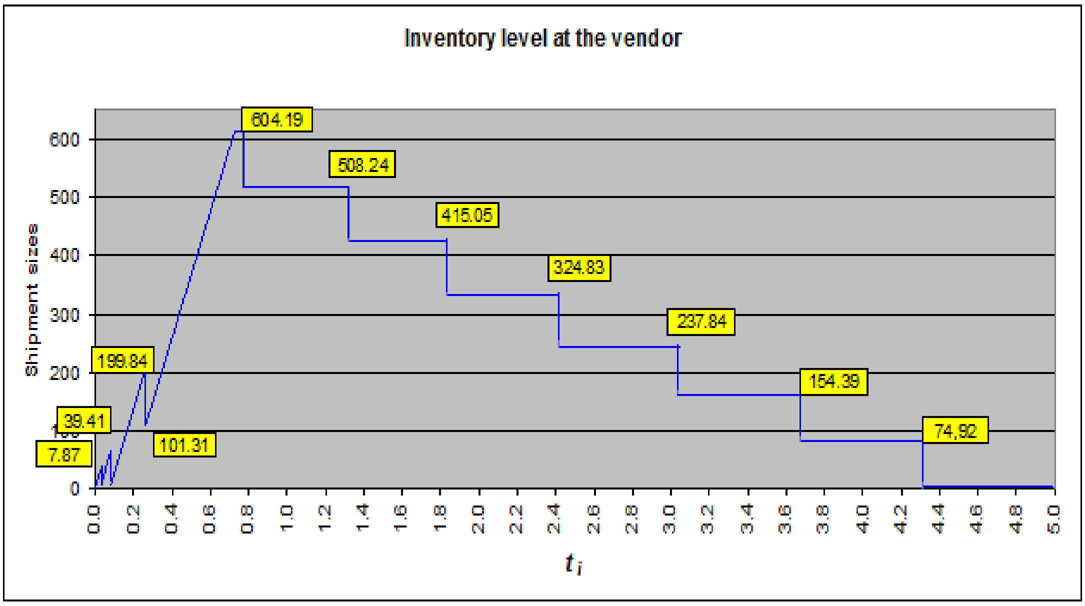
Figure 3.6 gives the diagrammatic plot of the inventory level at the buyer for Policy 3 with 10 shipments. All shipment sizes and periods are different. The optimal shipment sizes, $(q_1^*, q_2^*, q_3^*, q_4^*, q_5^*, q_6^*, q_7^*, q_8^*, q_9^*, q_{10}^*)$ are (7.87, 39.41, 98.53, 95.95, 93.19, 90.22, 86.99, 83.45, 79.47, 74.92) with its optimal shipment times, $(t_1^*, t_2^*, t_3^*, t_4^*, t_5^*, t_6^*, t_7^*, t_8^*, t_9^*, t_{10}^*)$ are at (0, 0.039, 0.239, 0.758, 1.292, 1.845, 2.418, 3.015, 3.641, 4.300) respectively.

Figure 3.6: Inventory level at the buyer for Policy 3 with $n^* = 10$



For Policy 3, the illustration of the inventory level at the vendor is shown in Figure 3.7. The first, second and third shipments are delivered during the production time until it reached the production up-time, $t_p = 0.75$ with inventory level at the vendor is 604.19. The vendor deliver the fourth, fifth, $\cdots$, tenth shipments during the production downtime until the end of the production cycle, $t_n = H = 5$.

Figure 3.7: Inventory level at the vendor for Policy 3 with $n^* = 10$



From our numerical results, Policy 3 always gives the best minimum total cost compared to the Policy 1 and Policy 2.

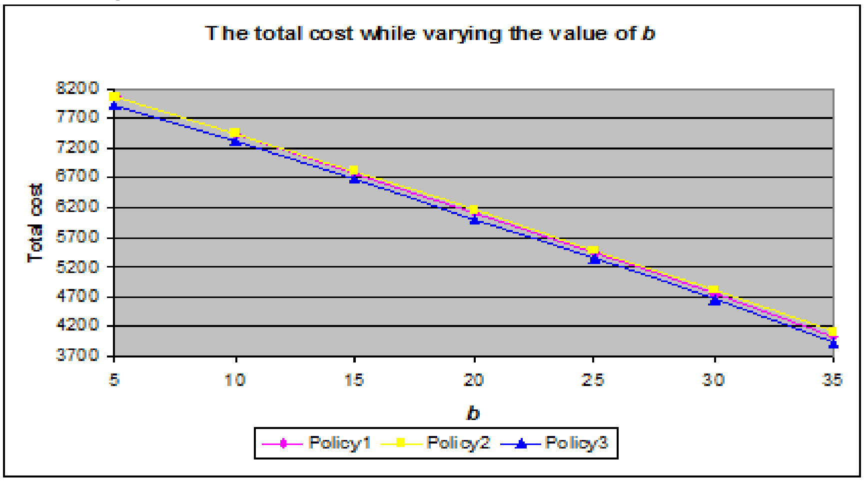
3.3.6.1 Sensitivity analysis

To study the effect of the total costs for Policy, 1, 2 and 3 that is TC_1, TC_2 and TC_3, we analyze these three policies by varying some parameter values.

We perform a numerical sensitivity analysis by varying the value of b, P, a, h_1/h_2 and A_2. We use the following values as the standard values of the parameter:

$$A_2 = 25, \quad h_1 = 4, \quad h_2 = 5, \;\; D(t) = 200 - 20t, \quad P = 1000, \quad H = 5$$

Figure 3.8: The total cost with different values of b



We vary the value of b from 5 to 35 for all policies to see the changes of the TC. The standard values of the other parameters remain the same. These results are illustrated in Figure 3.8. Note that $\frac{2(aH-P)}{H^2} < b < \frac{A}{h}$ because of $D < P$ and $a - bH > 0$.

We found that the larger the value of b, the lower the total cost of all policies. For example, when $b = 5$, TC_1, TC_2 and TC_3 are 8058.06, 8066.94 and 938.57 respectively and when $n = 35$, it decreases to 4029.19, 4108.27 and 3936.30 respectively.

Figure 3.9: The total cost saving while varying the value of b

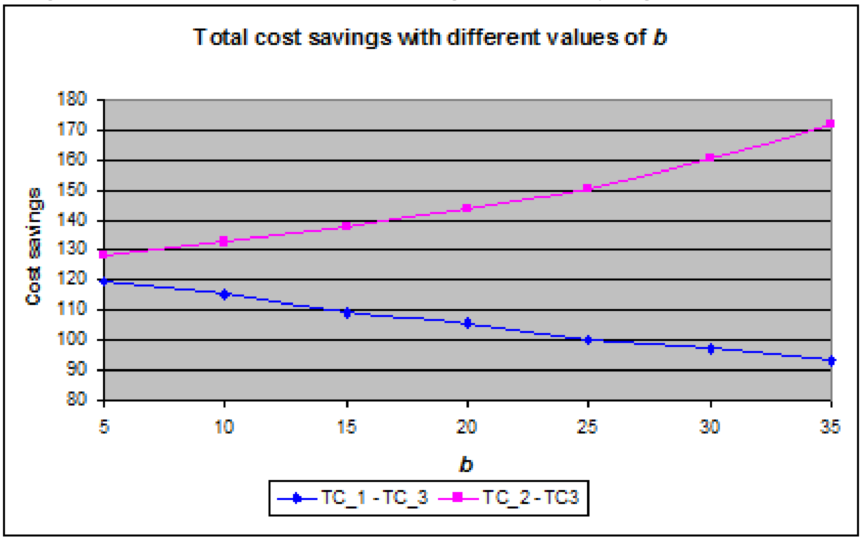
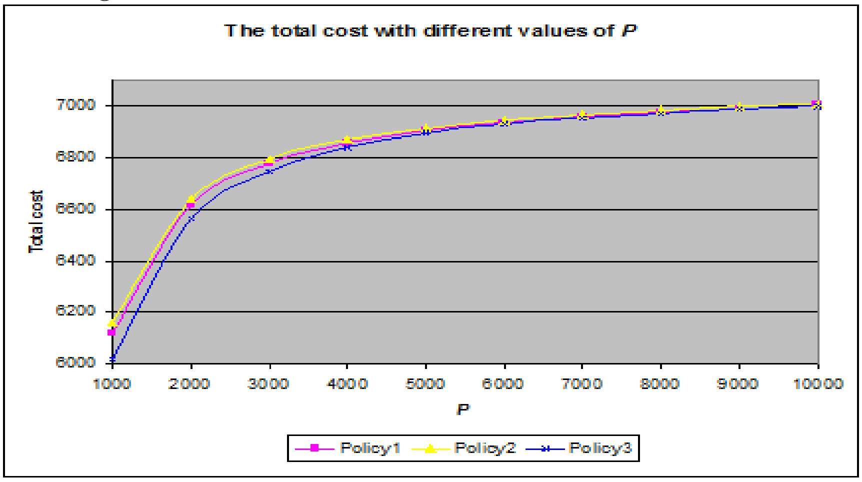


Figure 3.9 illustrates the total cost savings obtained by using Policy 3 rather than the other two policies (Policies 2 and 3) for different values of b. The blue and red lines represent the total savings by evaluating $(TC_1 - TC_3)$ and $(TC_2 - TC_3)$ respectively. The blue line gives the lower total cost savings compared to the red line and decreases as b increases while red line shows the reverse pattern. For example, when $b = 5$, $(TC_1 - TC_3)$ is 119.49 and $(TC_2 - TC_3)$ is 128.37 and when $b = 35$, $(TC_1 - TC_3)$ decrease to 92.89 and $(TC_2 - TC_3)$ increases to 171.97.

Now, we present the implication of varying the value of P to the total cost while the other standard parameter values remain the same. Note that the value of P must be greater than D otherwise shortages will occur. In this example, $D = 750$ therefore we increase $P = 1000$ up to 10000. The result is shown in Figure 3.10.

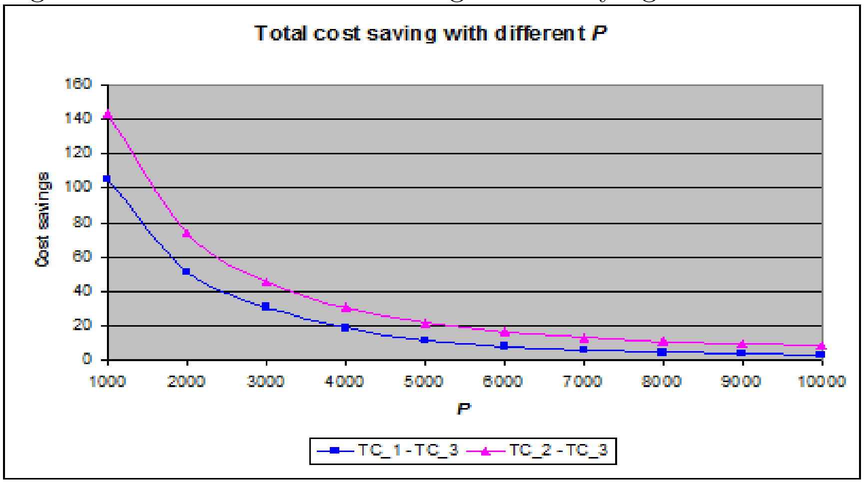
Figure 3.10: The total cost with different values of P



As expected, the larger the value of P, the larger the total cost for all policies and the lower the difference between the total cost of all policies. For example, when $P = 1000$, the total cost for Policies 1, 2 and 3 are 6121.29, 6159.66 and 6015.87 with 12, 12 and 10 shipments respectively and when $P = 10000$ it becomes 7001.28, 7006.53 and 6998.25 respectively with nine shipments for all policies. This result suggests that the faster production rate, P, the smaller the number of shipments and the larger the corresponding cost. It also suggests that when P is very large, the total cost for all policies will converge.

The total cost savings which are obtained while varying the value of P is shown in Figure 3.11. We found that as P increases, the total cost

Figure 3.11: The total cost saving while varying the value of P

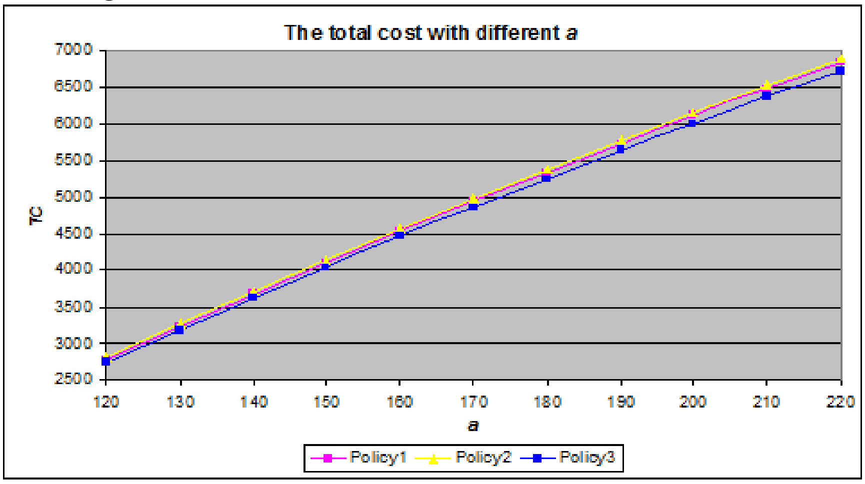


savings decrease for both $(TC_1 - TC_3)$ and $(TC_2 - TC_3)$. For example, when $P = 1000$, $(TC_1 - TC_3)$ and $(TC_2 - TC_3)$ are 105.42 and 143.79 respectively while when $P = 10000$, it decreases to 3.03 and 8.28 respectively. This result suggest that when P is very large, the total cost saving converges and gives the same value.

Let us test the impact of parameter a to the total cost for all policies. The other standard parameter values remain the same. Note that the value of a must be greater than bH and less than $\frac{P}{H} + \frac{bH}{2}$ because the demand, $a - bH > 0$ and $aH - \frac{bH^2}{2} \leq P$. In this example, $b = 20$, $P = 1000$ and $H = 5$, therefore $100 < a \leq 250$. The result is given in Figure 3.12.

It can be seen that the larger the value of a, the larger the total cost for all policies. For example, when $a = 120$, the total cost for Policy 1, 2 and

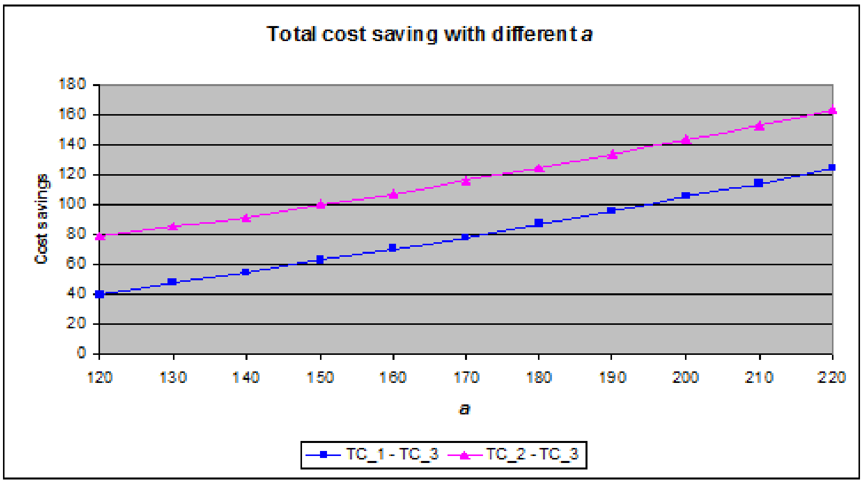
Figure 3.12: The total cost with different values of a



3 are 2774.11, 2813.10 and 2734.13, whereas when $a = 220$, it increases to 6852.63, 6891.79 and 6727.52 respectively. The number of shipments for all policies also increase as the value of a increases. In this example, it increases from 7 to 13 for Policy 1, 8 to 13 for Policy 2 and 7 to 11 for Policy 3. It makes sense that, with the same b, P and H, the greater the value of a, the larger the total demand, D. Therefore, the vendor needs more shipments to satisfy the demand and at the same time, to minimize their total cost.

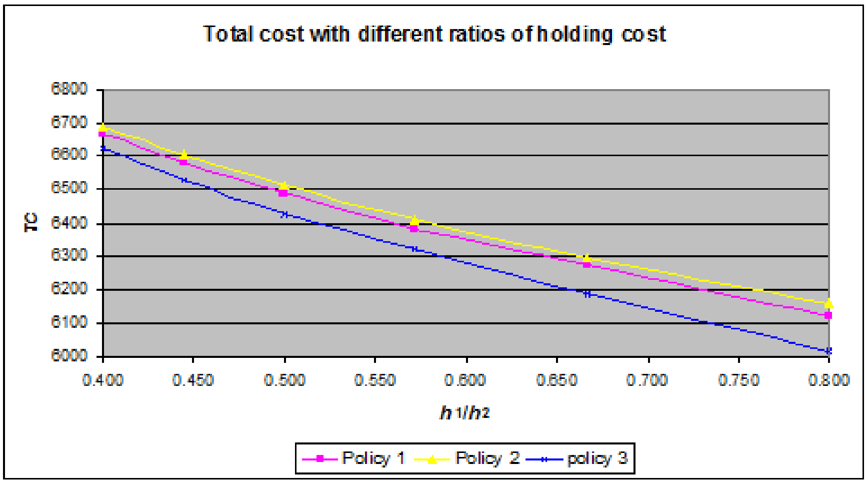
Let us now turn to the impact of a to the total cost savings. As depicted in Figure 3.13, the larger the value of a, the larger the total cost savings can be obtained from implementing Policy 3 rather than Policies 1 and 2. Clearly shown in the figure, $TC_1 - TC_3$ gives the lowest total cost savings compared to $TC_2 - TC_3$. The difference between its respective lines is almost the same for every a which is around 37 to 39.

Figure 3.13: The total cost saving while varying the value of a



Next, we decrease the ratio of the holding cost, h_1/h_2 by increasing h_2 from 5 up to 10 while $h_1 = 4$. All the other standard parameter values remain the same. The corresponding results are displayed in Figure 3.14.

Figure 3.14: The total cost with different values of h_1/h_2



We observe that all policies show similar pattern where the larger h_1/h_2, the lower the total cost for all policies. As we expected, Policy 3 always gives the best minimum total cost. For example, when $h_1/h_2 = 0.400$ the total cost for Policies 1, 2 and 3 are 6668.31, 6689.45 and 6626.06 with 22, 23 and 22 shipments respectively and when $h_1/h_2 = 0.800$, it decreases to 6121.29, 6159.66 and 6015.87 with 12, 12 and 10 shipments respectively. These results support the belief that for a large holding cost of the buyer, h_2, it is better to replenish their inventory in large number of shipments in order to reduce the total cost.

Figure 3.15: The total cost saving while varying the value of h_1/h_2

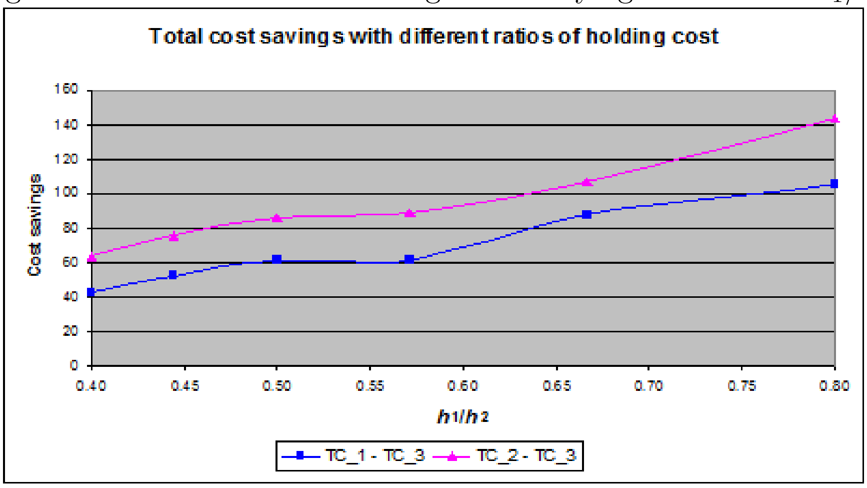
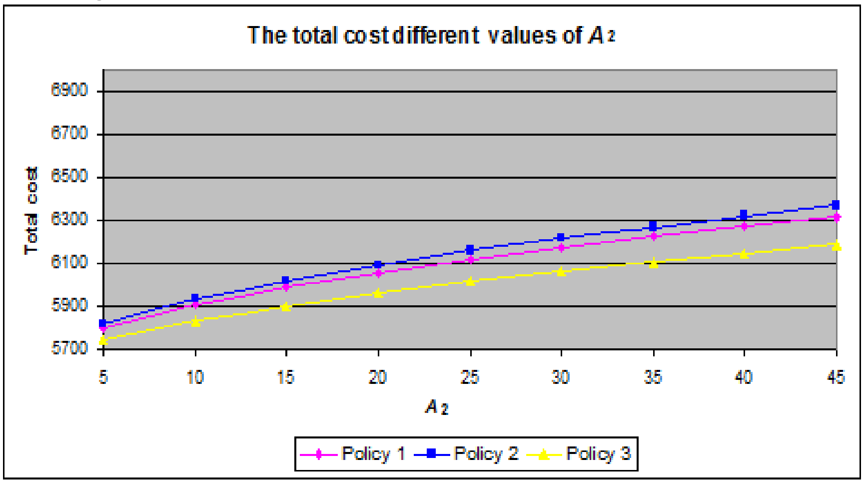


Figure 3.15 gives a diagrammatic plot while varying the value of h_1/h_2. We found that the larger h_1/h_2 the larger $(TC_1 - TC_3)$ and $(TC_2 - TC_3)$. For example, when $h_1/h_2 = 0.4$, $(TC_1 - TC_3)$ and $(TC_2 - TC_3)$ is 42.25 and

63.39. However, when $h_1/h_2 = 0.8$, it increases to 105.42 and 143.79.

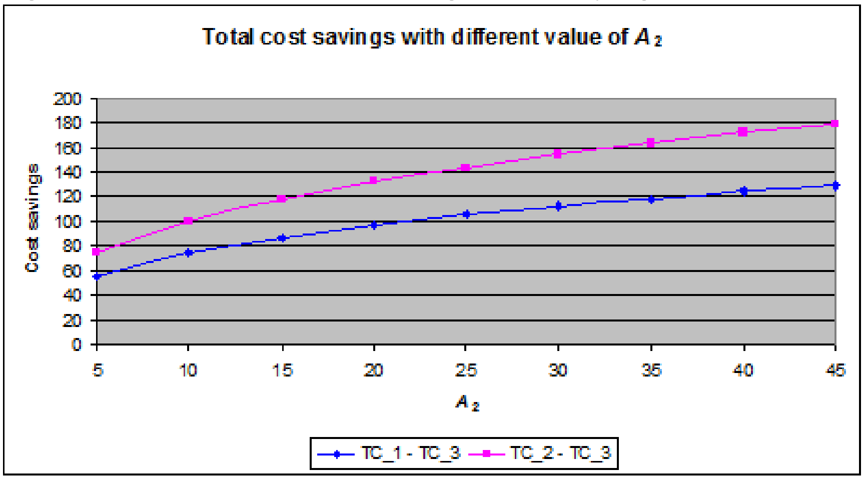
Next, we vary the value of A_2 from 5 to 45. Similarly, the other standard parameter values are remain the same. The results are shown in Figure 3.16. We conclude that the larger the value of A_2, the larger the total cost for all policies. The total cost for Policy 1 is close to Policy 2. In line with our conclusion, Policy 3 always gives the best minimum total cost.

Figure 3.16: The total cost with different values of A_2



Finally, the cost savings obtained by implementing Policy 3 rather than the other two policies is plotted in Figure 3.17. We observed that the larger the value of A_2 the larger $(TC_1 - TC_3)$ and $(TC_2 - TC_3)$.

Figure 3.17: The total cost saving while varying the value of A_2



In the next section we will consider an integrated inventory model for the case where the vendor's holding cost is greater than the buyer.

3.4 Case 2 : $h_1 > h_2$

Consignment Stock, CS policy has been greatly discussed in the literature. Generally, this model is suggested to be applied when the holding cost of the vendor's is greater than the buyer's. In CS approach, it is assumed that the vendor continues to own the stock held by the buyer up to the point when the buyer pays for it. Therefore, the vendor incurs that part of the stockholding cost. In other words, the supplier locates their inventory in the buyer's store or warehouse and allows them to sell or consume directly from his stock. The buyer pays for the inventory only after he has resold or consumed it. The buyer still incurs costs related to storing and managing the inventory.

According to Piasecki [43], the consignment inventory works well for ;

- New and unproven products.
- The introduction of existing product lines into new sales channels.
- Very expensive products where sales are questionable.
- Service parts for critical equipment which the buyer would not stock due to budget constraint or demand uncertainty.

The consignment material is procured via purchase requisition, purchase orders and outline agreements. Both parties need to clearly understand the terms in the agreements, hence there should have been no bearing on how they determine the production and shipment policies.

Valentini and Zavanella [50] list out the obligation that may included in CS agreement such as,

- The agreed lead time in case of sudden demand peaks for the company.
- The level of the safety stock the supplier should maintain in his own depots, taking into account the provisioning time of the item considered. This parameter may also influence the minimum stock level, s and maximum stock level, S values.

- The type and capacity of the pallets for delivery, as s and S values are an integer multiple of it. This parameter is also to be fixed to interface CS standards with the kanban system.

- The company may agree to pay for the goods stored in its warehouse, even if it has not consumed them yet, after a given amount of time.

Most of the previous research regarding CS as discussed in Chapter 2 assumed that the demand rate is constant. As we mentioned before, this assumption is not realistic because the demand rate should be increasing or decreasing with time. Motivated by this situation, we developed a consignment stock model which considers the linearly decreasing demand for the final production batch.

Figure 3.18: Inventory level for the case of $h_1 > h_2$ with 4 shipments

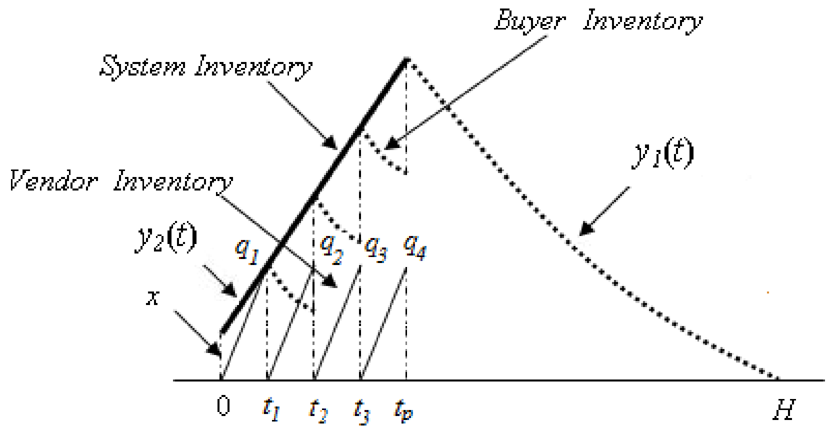


Figure 3.18 shows the illustration of the model with four equal shipments which represents the stock level of the system, vendor and buyer. The production batch will starts at $t_0 = 0$ until the production uptime, t_p. The first shipments is at time t_1 and follows at time $t_2, t_3, \ldots, t_n$ with the shipment sizes $q_1, q_2, \ldots, q_n$. $y_1(t)$, $y_2(t)$ and D is similar as equation (3.3), (3.4) and (3.5) in Section 3.3.

3.5 Total time-weighted system stock

The total system stock represented by the area under the curves $y_1(t)$ and $y_2(t)$ in (t_p, H) and $(0, t_p)$ respectively. The total time weighted system stock for this case is similar to the Case 1 which is given by equation (3.6) where t_p is similar with equation (3.7).

3.6 Total time-weighted vendor stock

The total time-weighted vendor stock, TVS is the total area under the triangles. In Figure 3.18, the triangles are represent by $0q_1t_1$, $t_1q_2t_2$, $t_2q_3t_3$ and $t_3q_4t_p$. Hence, we have

$$TVS = \frac{1}{2}\sum_{i=1}^{n} q_i(t_i - t_{i-1}). \tag{3.36}$$

It follows that the total cost for this model, TC, is in the term of n (discrete variable) and $t_2, t_3, \ldots, t_n$ (real variables) which is given by

$$TC = n(A_2) + h_2 TSS + (h_1 - h_2)TVS. \tag{3.37}$$

Structurally the cost function is identical to the previous model (Case $h_1 < h_2$). The constants h_1 and h_2 are interchanged and the last term on the right hand side is now multiplied by the total time-weighted vendor stock, TVS.

For example, let the demand rate linearly decreasing over the period $(0, H)$ that is

$$f(t) = a - bt \qquad a > 0; b > 0; t > 0; H > 0. \tag{3.38}$$

Substituting (3.38) into (3.3), (3.4), (3.5), (3.6) and (3.7) we have

$$y_1 = a(H - t) - \frac{b}{2}(H^2 - t^2). \tag{3.39}$$

$$y_2 = (P - a)t + \frac{b}{2}t^2. \tag{3.40}$$

$$D = H\left(a - \frac{b}{2}H\right). \tag{3.41}$$

$$TSS = \frac{Pt_p^2}{2} + H(\frac{bH}{2} - a)t_p + H^2(\frac{a}{2} - \frac{bH}{3}) + \frac{xH}{2}. \tag{3.42}$$

$$t_p = \frac{H}{P}(a - \frac{b}{2}H). \tag{3.43}$$

Finally, substituting (3.42) and (3.36) into (3.37) we have,

$$\begin{aligned} TC &= nA_2 + h_2 \left\{ \frac{Pt_p^2}{2} + H(\frac{bH}{2} - a)t_p + H^2(\frac{a}{2} - \frac{bH}{3}) + \frac{xH}{2} \right\} \\ &+ (h_1 - h_2) \left\{ \frac{1}{2} \sum_{i=1}^{n} q_i(t_i - t_{i-1}) \right\}. \end{aligned} \tag{3.44}$$

TC is in the function of n (discrete variable) and a vector $\vec{t} = t_1, t_2, \ldots, t_n$. where $t_1 = 0$ and $t_{n+1} = H$.

We will explore the best solution of the above total cost based on three policies :

1. Policy 1 : Equal shipment sizes
2. Policy 2 : Equal shipment periods
3. Policy 3 : Unequal shipment sizes and periods

The objective in every policies is to find an optimal t_i (or q_i) for a given n which gives the minimum total cost, TC.

3.6.1 Policy 1 : Equal shipment sizes

In this policy the shipment sizes are assumed to be equal. Let q_i be the shipment size of ith shipment. So, we have fixed value of q_i which can be calculated as follow

$$q_i = \frac{D}{n} \qquad i = 1, 2, \ldots, n. \tag{3.45}$$

The illustration of this policy is given by Figure 3.18. The shipment times, t_i is given by

$$t_i = \frac{q}{P} + t_{i-1}, \qquad i = 1, 2, \ldots, n. \tag{3.46}$$

3.6.1.1 Solution procedure

The computer algorithm of the solution procedure is outline below :

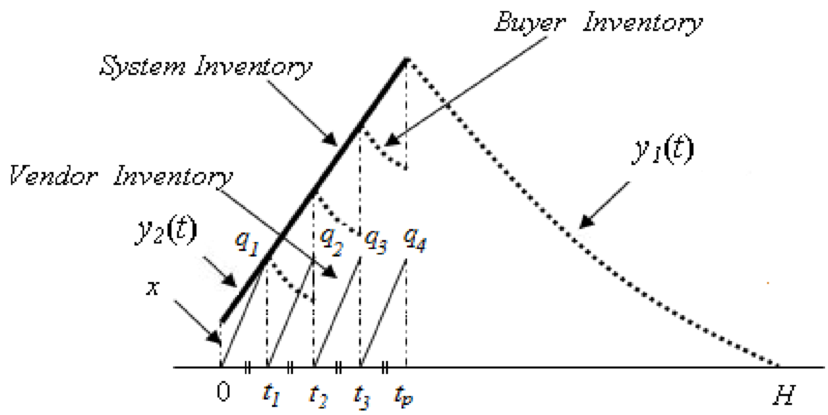
1. Let $n = 1$
2. Set $t_0 = 0$, $t_{n+1} = H$
3. Compute $q_i = D/n, \;\; i = 1, 2, \cdots, n$
4. Compute $t_i, \; i = 1, 2, \cdots, n$ using (3.46) and $TC_1(n, \; \vec{t}\,)$ using (3.44)
5. Set $TC_1(n, \; \vec{t}\,)$ as $TC_1(n^*, \; \vec{t}\,)$. Increase n by 1 and repeat step 3 to 4. Stop when $TC_1(n, \; \vec{t}\,) \geq TC_1(n^*, \; \vec{t}\,)$

The basic idea of the above algorithm is to start with $n = 1$. Next, we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_1(n^*, \; \vec{t}\,) < TC_1(n^* - 1, \; \vec{t}\,)$ and $TC_1(n^*, \; \vec{t}\,) < TC_1(n^* + 1, \; \vec{t}\,)$.

3.6.2 Policy 2 : Equal shipment periods

In this section, we assume that the shipment periods are equal. The illustration of this model is given by Figure 3.19.

Figure 3.19: Inventory level for the case of $h_1 > h_2$ with Policy 2 (Equal shipment periods)



The production uptime t_p will be divided by the number of shipments, n. Then we have,

$$\begin{aligned}
t_1 &= t_0 + \frac{t_p}{n} &=& \frac{t_p}{n} \quad (t_0 = 0) \\
t_2 &= t_1 + \frac{t_p}{n} &=& 2\frac{t_p}{n} \\
t_3 &= t_2 + \frac{t_p}{n} &=& 3\frac{t_p}{n} \\
& \vdots & & \\
t_i &= t_{i-1} + \frac{t_p}{n} &=& i\frac{t_p}{n}, \qquad i = 1, 2, \cdots, n. \qquad (3.47)
\end{aligned}$$

Given that

$$q_i = P(t_i - t_{i-1}). \qquad (3.48)$$

The total cost for this policy can be calculated by substituting (3.47) into equation (3.44). Then we have

$$
\begin{aligned}
TC &= nA_2 + h_2 \left\{ \frac{Pt_p^2}{2} + H(\frac{bH}{2} - a)t_p + H^2(\frac{a}{2} - \frac{bH}{3}) + \frac{xH}{2} \right\} \\
&+ (h_1 - h_2) \left\{ \frac{Dt_p}{2n} \right\}. \qquad (3.49)
\end{aligned}
$$

Note that Policies 1 and 2 will give the same value of the shipment sizes because of the fixed value of production rate, P.

3.6.2.1 Solution procedure

The computer algorithm of the solution procedure is outline below

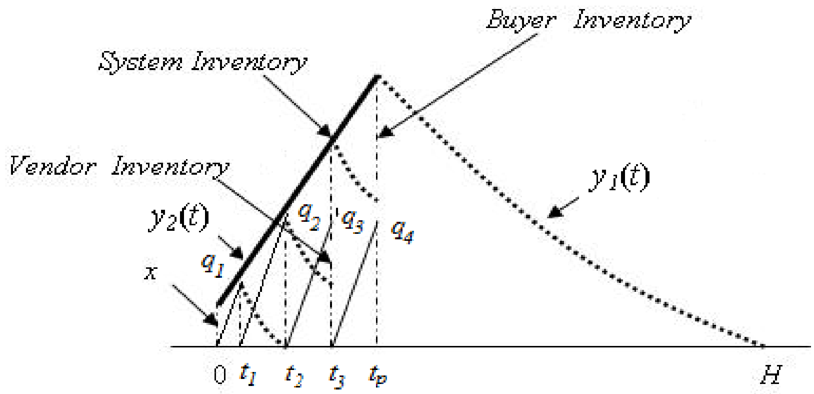
1. Let $n = 1$
2. Set $t_0 = 0$, $t_{n+1} = H$
3. Compute $t_i,\ i = 1, 2, \cdots, n$ using (3.47)
4. Compute $q_i,\ i = 1, 2, \cdots, n$ using (3.48) and $TC_2(n)$ using (3.49)
5. Set $TC_2(n,\ \vec{t}\,)$ as $TC_2(n^*,\ \vec{t}\,)$. Increase n by 1 and repeat step 3 to 4. Stop when $TC_2(n,\ \vec{t}\,) \geq TC_2(n^*,\ \vec{t}\,)$

Similarly, the basic idea of the above algorithm is to start with $n = 1$. Next, we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_2(n^*,\ \vec{t}\,) < TC_2(n^*-1,\ \vec{t}\,)$ and $TC_2(n^*,\ \vec{t}\,) < TC_2(n^*+1,\ \vec{t}\,)$.

3.6.3 Policy 3 : Unequal shipment sizes and unequal shipment periods

In this policy, the shipment sizes and periods are unequal. Figure 3.20 shows the inventory level for this policy.

Figure 3.20: Inventory level for the case of $h_1 > h_2$ with Policy 3 (Unequal shipment sizes)



Substituting (3.6) and (3.36) into (3.44) we get the total cost for the system. We build the following constraint optimization problem for this policy where the objective function, $TC_3(n, \vec{t}\,)$ is refers to equation (3.44). Constraint (3.34) and (3.35) in section 3.3.5 remain the same :

$$\text{Minimize :} \quad TC_3(n,\ \vec{t}\,)$$

Subject to

$$\frac{a}{b}\left\{1-\sqrt{1-\frac{2b}{a^2}\left[a\left(\frac{q_{i-1}}{P}\right)-\frac{b}{2}\left(\frac{q_{i-1}}{P}\right)^2+q_i\right]}\right\} \geq \frac{1}{P}\sum_{1}^{i} q_{i+1}, \quad (3.50)$$

$$i = 2, 3, \cdots, n$$

$$\sum_{i=1}^{n} q_i = D. \quad (3.51)$$

3.6.3.1 Solution procedure

Now, our objective is to minimize the total system cost, that is equation (3.44) subject to the constraints (3.50) and (3.51). The computer algorithm is outline below :

1. Let $n = 1$
2. Set $t_0 = 0$, $t_{n+1} = H$
3. Determine $q_i,\ i = 1, 2, \cdots, n$ which satisfied constraints (3.50) and (3.51), if it exists
4. Compute $t_i,\ i = 1, 2, \cdots, n$ using (3.46) and $TC_3(n,\ \vec{t}\,)$ using (3.44)
5. Set $TC_3(n,\ \vec{t}\,)$ as $TC_3(n^*,\ \vec{t}\,)$. Increase n by 1 and repeat step 3 to 4. Stop when $TC_3(n,\ \vec{t}\,) \geq TC_3(n^*,\ \vec{t}\,)$

Again, the basic idea of the above algorithm is to start with $n = 1$. Next, we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_3(n^*,\ \vec{t}\ ) < TC_3(n^* - 1,\ \vec{t}\ )$ and $TC_3(n^*,\ \vec{t}\ ) < TC_3(n^* + 1,\ \vec{t}\ )$.

3.6.4 Numerical examples and sensitivity analysis

To demonstrate the effectiveness of the proposed policy we present some numerical examples. The parameter values used are :

$$A_2 = 25,\ \ a = 200,\quad b = 20 \quad H = 5,\quad h_1 = 6,\quad h_2 = 5$$

$$D = 750,\quad P = 1000$$

Note that we adopt the same parameter values as in Case 1 $(h_1 < h_2)$ except the value of h_1. In this case, $(h_1 > h_2)$, therefore in this example, we simply choose $h_1 = 6$ which is greater than $h_2 = 5$.

Policies 1 and 2 produce the same result which is given by Table 3.4 while the result for Policy 3 is given by Table 3.5. As expected, Policy 3 where the shipment sizes and periods are unequal is always superior than the other policies. The optimal total cost, $TC_3^* = 7157.53$ is reached at $n^* = 5$, which is 234.14 less than the total costs of each Policy 1 and 2.

Table 3.4: The total cost for case 2 : $h_1 > h_2$, Policy 1-Equal shipment sizes (= Policy 2-Equal shipment periods)

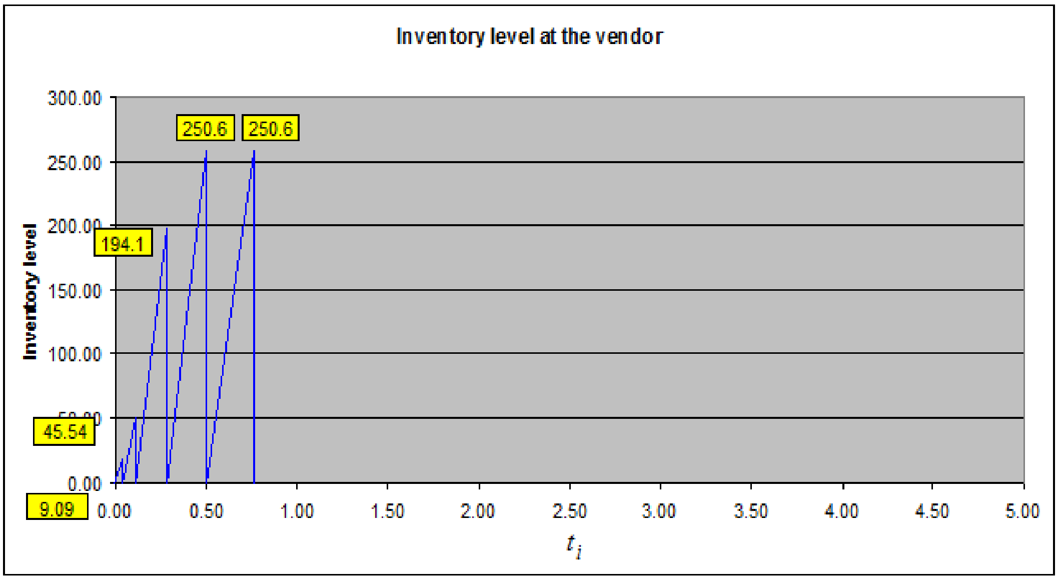
n	Total	x	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}
1	9108.33	150.00	750.00	-	-	-	-	-	-	-	-	-
2	8055.21	75.00	375.00	375.00	-	-	-	-	-	-	-	-
3	7720.83	50.00	250.00	250.00	250.00	-	-	-	-	-	-	-
4	7566.15	37.50	187.50	187.50	187.50	187.50	-	-	-	-	-	-
5	7483.33	30.00	150.00	150.00	150.00	150.00	150.00	-	-	-	-	-
6	7436.46	25.00	125.00	125.00	125.00	125.00	125.00	125.00	-	-	-	-
7	7410.12	21.43	107.14	107.14	107.14	107.14	107.14	107.14	107.14	-	-	-
8	7396.61	18.75	93.75	93.75	93.75	93.75	93.75	93.75	93.75	93.75	-	-
9	**7391.67**	**16.67**	**83.33**	**83.33**	**83.33**	**83.33**	**83.33**	**83.33**	**83.33**	**83.33**	**83.33**	**-**
10	7392.71	15.00	75.00	75.00	75.00	75.00	75.00	75.00	75.00	75.00	75.00	75.00

Table 3.5: The total cost for case 2 : $h_1 > h_2$, Policy 3-Unequal shipment sizes and unequal shipment periods

n	Total	x	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}
1	9108.33	150.00	750.00	-	-	-	-	-	-	-	-	-
2	7486.14	24.34	121.71	628.29	-	-	-	-	-	-	-	-
3	7250.55	6.51	32.54	164.06	553.39	-	-	-	-	-	-	-
4	7176.23	2.91	14.54	72.94	313.77	348.75	-	-	-	-	-	-
5	**7157.53**	**1.82**	**9.09**	**45.54**	**194.11**	**250.63**	**250.63**	**-**	**-**	**-**	**-**	**-**
6	7158.01	1.20	5.98	29.92	126.88	195.74	195.74	195.74	-	-	-	-
7	7167.29	0.79	3.96	19.82	83.76	160.62	160.62	160.62	160.62	-	-	-
8	7181.35	0.51	2.55	12.74	53.73	136.20	136.20	136.20	136.20	136.20	-	-
9	7198.30	0.30	1.50	7.51	31.61	118.23	118.23	118.23	118.23	118.23	118.23	-
10	7217.19	0.23	1.17	5.83	24.53	100.12	103.06	103.06	103.06	103.06	103.06	103.06

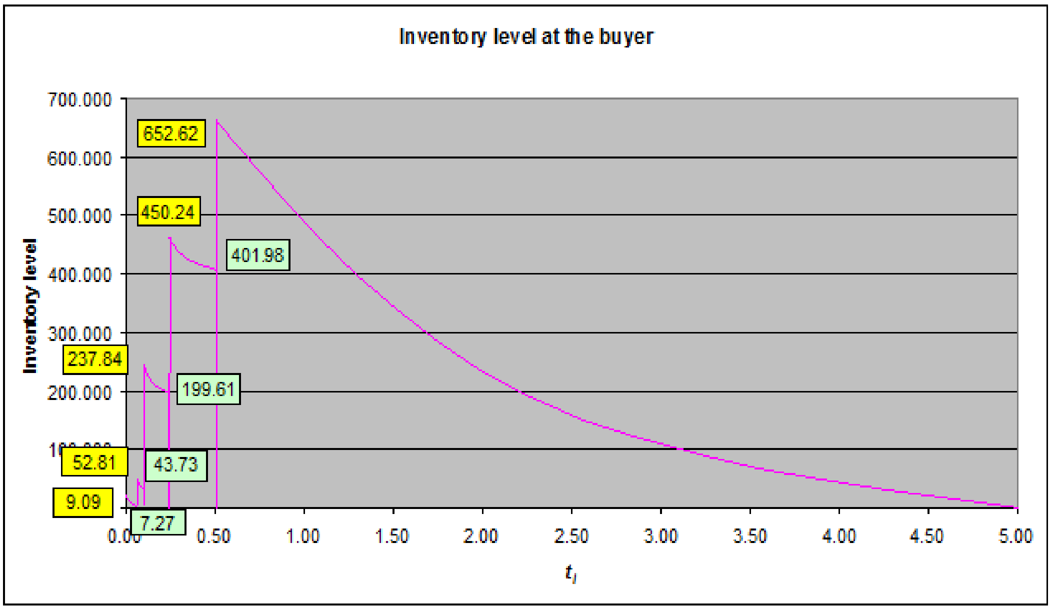
We transform the optimal result from Policy 3 with five shipments into a graph which is given by Figure 3.21. The first, second and third shipments are increasing in size, while q_i, $i \geq 4$ gives equal sizes. The production stop at the production uptime, $t_p = 0.75$. Therefore, there is no inventory after this time until the end of the production cycle, $H = 5$.

Figure 3.21: The inventory level at the vendor



The inventory level at the buyer is plotted in Figure 3.22. The number labeled in yellow, is the total inventory at shipment times whereas the number labeled in green is the remaining inventories which are held at the buyer at $t_2, t_3, \ldots, t_n$. After the production uptime, $t_p = 0.75$, the inventory will continue depleting until it reaches zero level at the end of the planning horizon, $H = 5$.

Figure 3.22: The inventory level at the buyer



As in the previous case ($h_1 < h_2$), we conclude that Policy 3 always gives the best minimum total cost compared to the Policies 1 and 2.

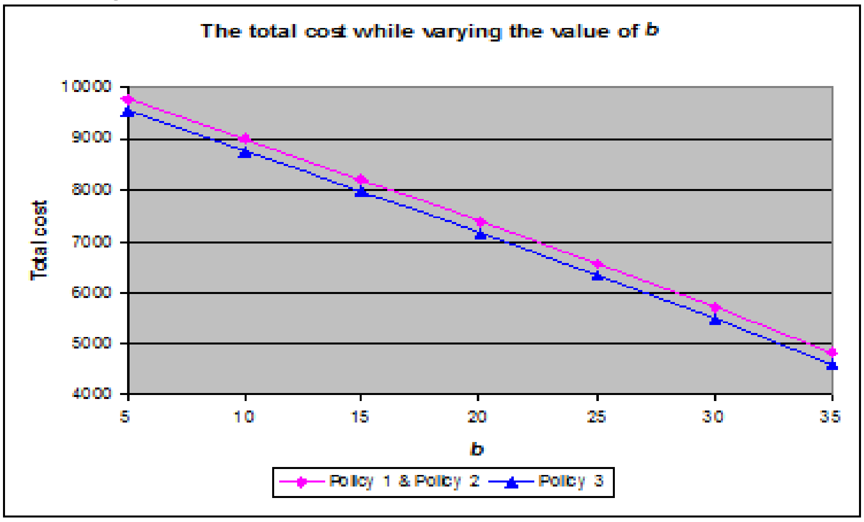
3.6.4.1 Sensitivity analysis

To study the effect of the total costs for Policy, 1, 2 and 3 that is TC_1, TC_2 and TC_3, we analyze these three policies by varying some parameter values. We perform a numerical sensitivity analysis by varying the value of b, P, a, h_1/h_2 and A_2. We use the following values as the standard values of the parameter:

$$A_2 = 25, \quad h_1 = 6, \quad h_2 = 5, \quad D(t) = 200 - 20t, \quad P = 1000, \quad H = 5$$

First, we test the changes in the total cost by varying the value of b from 5 to 35. The other at standard parameter values remain the same as in the previous example. Figure 3.23 illustrate the result.

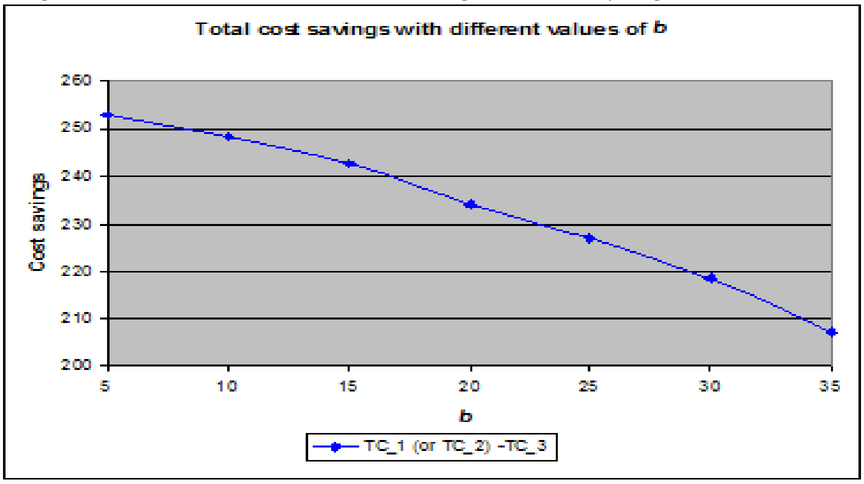
Figure 3.23: The total cost with different value of b



While varying the value of b, we observe that the larger the value of b, the lower the total cost for all policies. However, Policy 3 always generates the lowest total cost. Its minimum shipments size also decreases as b increases. For example, for Policy 3, when $b = 5$ the minimum total cost is at $n = 11$, however, when $b = 35$ it decreases to $n = 8$. This result suggests that the faster production rate, P, the smaller the number of shipments and the lower the corresponding total cost.

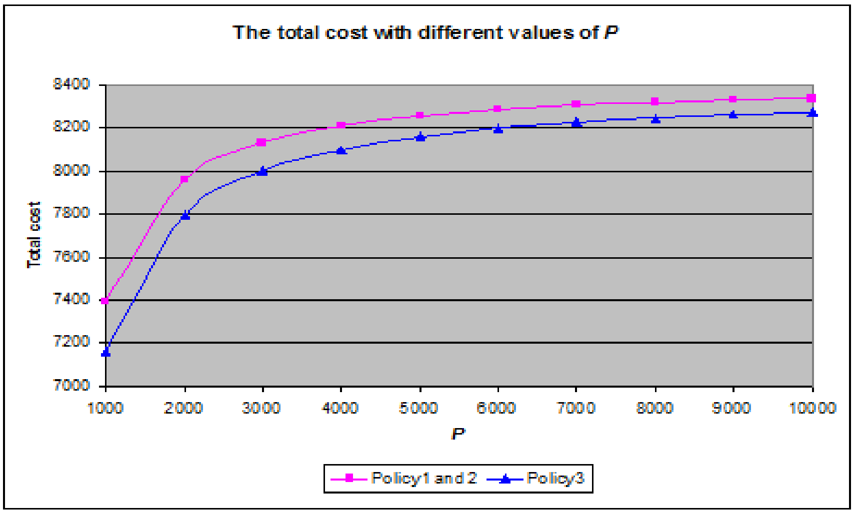
The total cost savings which can be obtained from implementing Policy 3 rather than Policies 1 and 2 while varying the value of b is given by Figure 3.24. This graph presents only one line because $TC_1 = TC_2$. It shows that, when b increases, the total cost saving decreases.

Figure 3.24: The total cost saving while varying the value of b



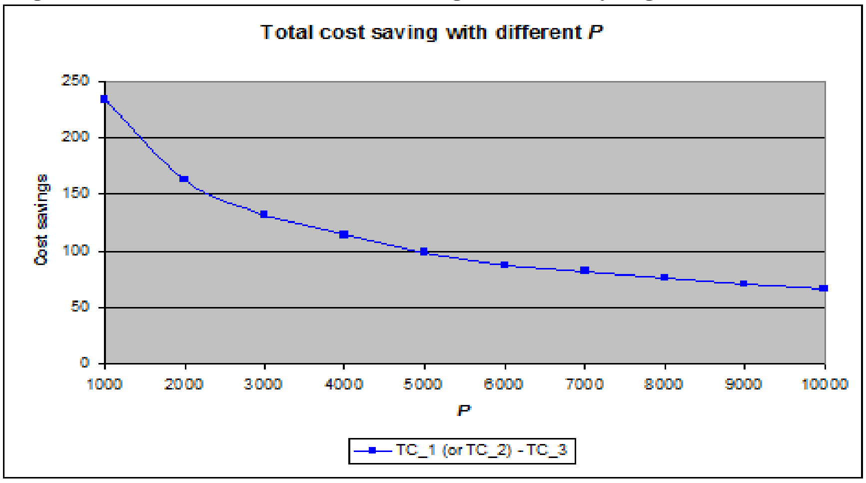
Now, we increase the value of P from 1000 up to 10000 while the other standard parameter values remain the same. The result is given by Figure 3.25. As expected, the total cost for all policies increases as the value of P increases and the number of shipments decreases from 9 to 3 shipments for Policy 1 (or Policy 2) and 5 to 2 for Policy 3. This result suggests that the faster production rate, P, the smaller the number of shipments, n and the larger the corresponding cost, TC.

Figure 3.25: The total cost with different value of P



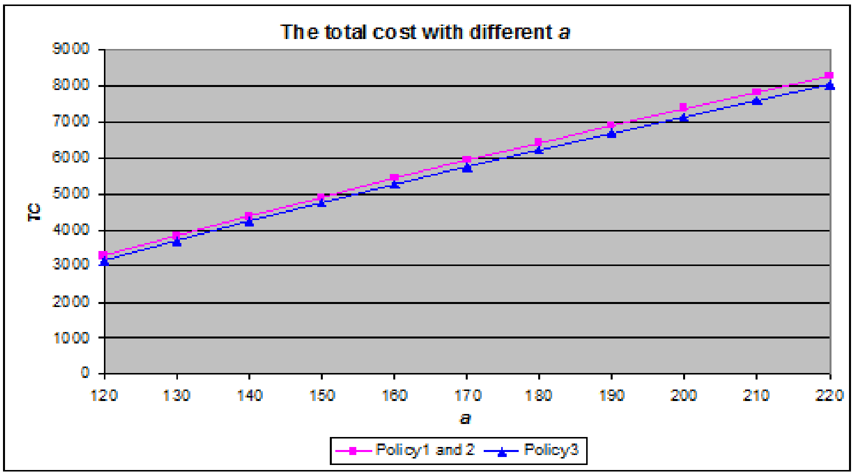
The cost savings while varying P is given by Figure 3.26. We found that the total cost decreases as P increases. For example, when $P = 1000$, $(TC_1(\text{or } TC_2) - TC_3)$ is 234.13 while when $P = 10000$, it decreases to 66.16.

Figure 3.26: The total cost saving while varying the value of P



Let us examine the impact of parameter a to the total cost for all policies. As mentioned in the Section 3.3.6.1, the value of a must be greater than bH and less than $\frac{P}{H} + \frac{bH}{2}$ because the demand, $a - bH > 0$ and $aH - \frac{bH^2}{2} \leq P$. In this example, $b = 20$, $P = 1000$ and $H = 5$, therefore $100 < a \leq 250$. The result is depicted in Figure 3.27. We found that the total cost for all policies increases as the value of a increases. In line with our conclusion, the lowest total cost is given by Policy 3. This result suggests that, with the larger initial demand rate, a, the different shipments size policy is more effective compared to the equal shipments size.

Figure 3.27: The total cost with different value of a

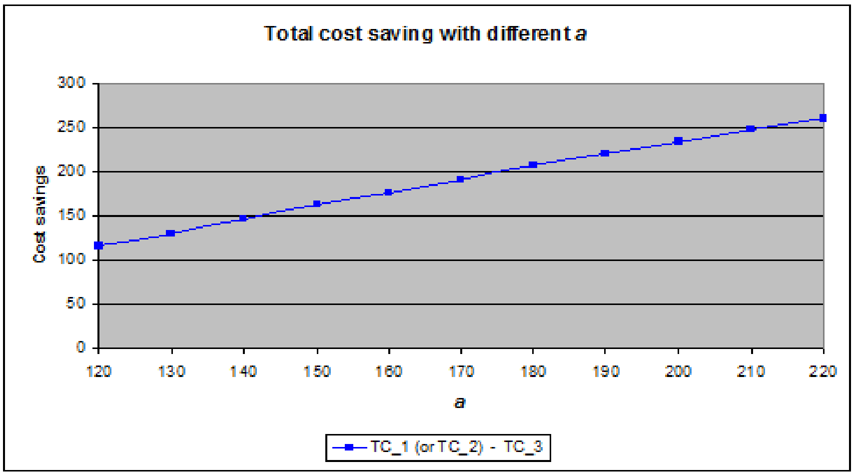


The number of shipments for all policies also increase as the value of a increases. In this example, as a increase from 120 to 220, the number of shipments increases from 5 to 11 for Policy 1 and 2, and 3 to 6 for Policy 3.

It makes sense that, with the same b, P and H, the greater the value of a, the larger total demand, D. Therefore, the vendor needs more shipments to satisfy the demand and at the same time, to minimize their total cost.

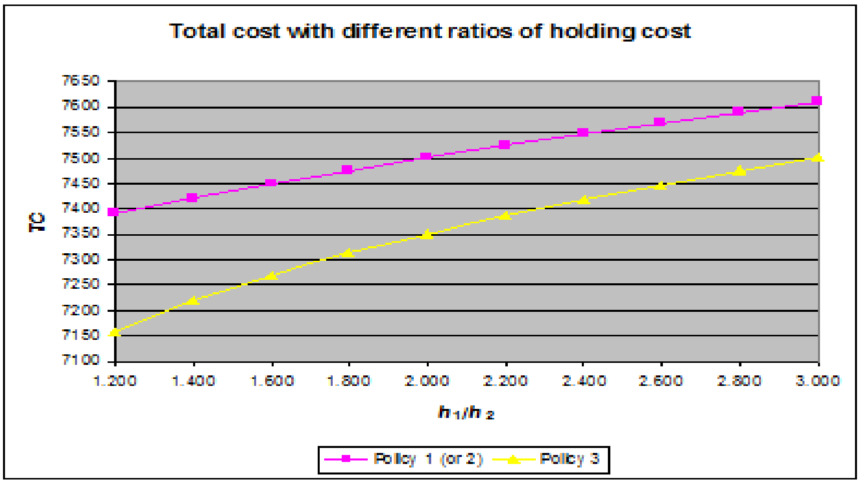
The cost savings while varying a is given by Figure 3.28. We found that the total cost savings increases as the values of a increases. For example, when $a = 120$, $(TC_1(\text{or } TC_2) - TC_3)$ is 116.02 while when $P = 220$, it increases to 260.95.

Figure 3.28: The total cost saving while varying the values of a



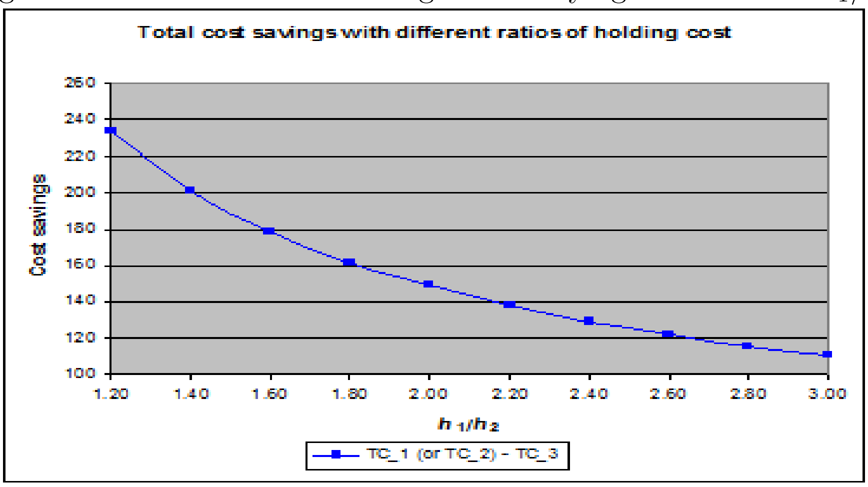
Next, we increase the ratio of the holding cost, h_1/h_2 from 1.2 up to 3.0 by increasing h_1 from 6 up to 15 while $h_2 = 5$. All the other standard parameter values are remain the same as in the previous example. The corresponding results are displayed in Figure 3.29.

Figure 3.29: The total cost with different values of h_1/h_2



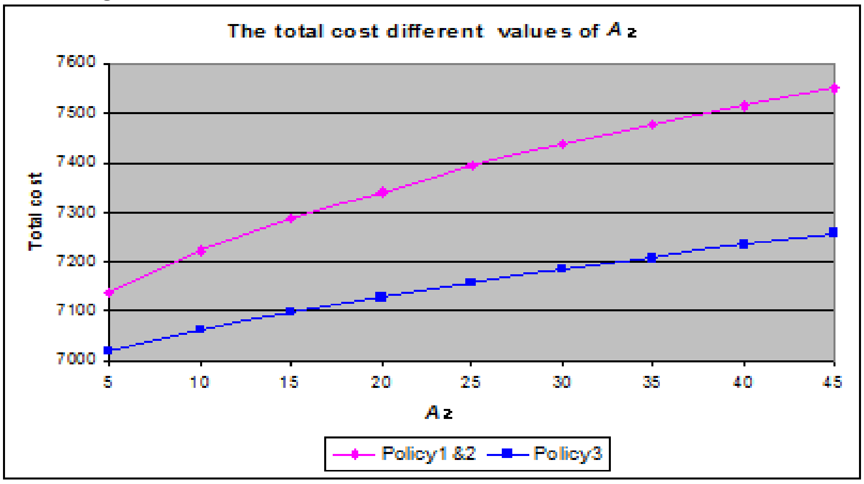
As depicted in the figure, the larger the value of h_1/h_2, the larger the total cost of all policies. For Policy 1, the minimum number of shipments increases from 9 to 14 while for Policy 3 it increases from 5 to 12. The rationale behind this result is that, the vendor will deliver in more numbers of shipments in order to decrease their holding cost.

Figure 3.30: The total cost saving while varying the values of h_1/h_2



The total cost savings obtained by implementing Policy 3 rather than Policies 1 and 2 while varying the value of h_1/h_2 is plotted in Figure 3.30. The larger the ratio of h_1/h_2, the lower the total cost can be obtained. For example, when $h_1/h_2 = 1.20$, (TC_1 (or TC_2)$-TC_3$)$= 234.13$, and it decreases to 110.35 when $h_1/h_2 = 3.00$.

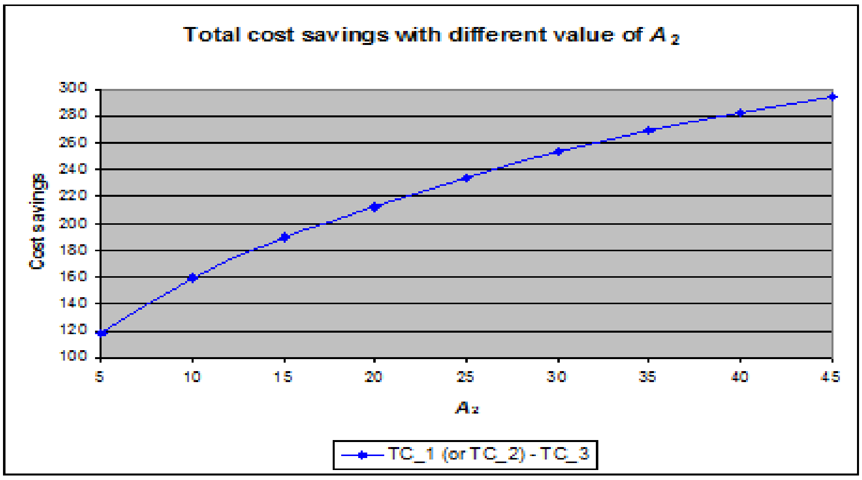
Figure 3.31: The total cost with different value of A_2



Now, we look at the changes of the total cost while varying the value of A_2. All the other standard parameter values remain the same. The result is given in Figure 3.31. The total cost for all policies increases as A_2 increases whereas the number of shipments decreases. For example, when $A_2 = 5$ the minimum number of shipments is 21 but when A_2 increases to 45, n decreases to 7 shipments. We conclude that the larger the value of A_2, the larger the total cost for all policies. In line with our conclusion, Policy 3 always gives the best minimum total cost.

Finally, the total cost savings while varying the value of A_2 is plotted in Figure 3.32. We observed that the larger the value of A_2 the larger the value of (TC_1 (or TC_2) $- TC_3$).

Figure 3.32: The total cost saving while varying the value of A_2



3.7 Conclusion

In this chapter, we have considered the integrated inventory model for final production batch with $h_1 < h_2$ with equal shipment sizes, equal shipment periods and unequal shipment sizes and periods policies. The first two policies are easy to solve. It can be calculated directly because all the parameters are fixed. We have implemented the Microsoft of Excel Solver to solve the third policy. We concluded that the unequal shipments size and period is the best policy compared to the other two policies.

We extended this model with the case of $h_1 > h_2$. We also discussed the three policies as in the case of $h_1 < h_2$ and found that the best minimum solution is also given by the unequal shipments size and period policy.

However a single batch is seldom to be applied in the real problem. Therefore, in the next chapter we will consider the integrated inventory model for n batch production which consists the final production batch at the end of the production cycle.