Chapter 4

Integrated inventory model for n production batch

4.1 Introduction

This chapter is mainly concerned with integrated inventory policy for n batch production cycle under time-varying demand process. Both vendor and buyer will collaborate and get their benefit through this policy. We will show how the formulation of the model is derived in details. The basic of the model is similar as in Chapter 3 but now we extend it to n batch production which consists of the final batch at the end of the production cycle. We will consider two different cases, that is $h_1 < h_2$ and $h_1 > h_2$. Both cases will discuss the equal and unequal cycle time, T_i and shipment sizes, $q_{i,j}$. We show how the solution of the model may be derived when the shipment sizes and periods are equal or unequal. We illustrate this policy with numerical examples.

4.2 Mathematical formulation

The cost factors considered here are

- the fixed production set up cost, A_1
- the fixed order/shipment cost, A_2
- the fixed inventory holding cost for the vendor, h_1
- the fixed inventory holding cost for the buyer, h_2

4.2.1 Notation

Let $i = 1, 2, \ldots, n$ be the number of batches, $j = 1, 2, \ldots, m$ be the number of shipments and H is the finite planning horizon.

- The demand rate for the finished product at time t is $f(t)$ for $t \in (0, H)$.
- P units per unit time is the finite production rate. The value of the production rate is greater than the demand rate, $P > D$.
- x_i is the initial stock held at the buyer for each batch.

- T_i is the cycle time (the length of time for the ith batch). It is the time calculated between successive production start-ups that is from $t_{i,0}$ to $t_{i+1,0}$.
- $q_{i,j}$ is the size of each shipment.
- $t_{i,j}$ is the replenishment time with $t_{1,0} = 0$.
- D_i is the demand for ith batch.

4.2.2 Assumption

- In these models, demand at any time t is given by the function $f(t)$. It could be linearly decreasing ($f(t) = a - bt$) or linearly increasing ($f(t) = a + bt$) or exponentially decreasing ($f(t) = ae^{-bt}$). We choose $f(t) = a - bt$ as an example in every model in this chapter. Here, a is the initial rate demand with $a > 0$ and b is the slope with $b > 0$.
- We are currently at time zero and wish to determine the stock replenishment which minimises the expected total relevant cost.
- The set-up and ordering costs are fixed throughout the planning horizon.
- The production rate, P is also fixed throughout the planning horizon.

- There are no limitation on the order size.
- The transportation cost per unit time is ignored since we are assuming that it is constant and independent from the ordering quantity.
- x_i is greater than zero and depends on the size of the first shipment for every batch.
- The finished product is transferred from the vendor to the buyer in n batches and m shipments.
- No shortages are allowed.

4.3 Case 1 : $h_1 < h_2$

We first consider the case where the vendor's holding cost is less than the buyer's. This problem has received a great deal of attention by many researches, with much of their work concentrating on fixed demand. In this case, the buyer will keep their inventory level as low as possible by receiving their finished product when their inventory level fall to zero or as late as possible. The location of the stock is preferably at the vendor's warehouse. In Chapter 2, we also discussed this case, $h_1 < h_2$ but that is for the final batch of the production cycle. Here, we develop a complete n batch which have $n - 1$ repeated batch and will end up with the final batch.

Figure 4.1: Plot of the inventory level against time when $n = 3$ and $m = 5$

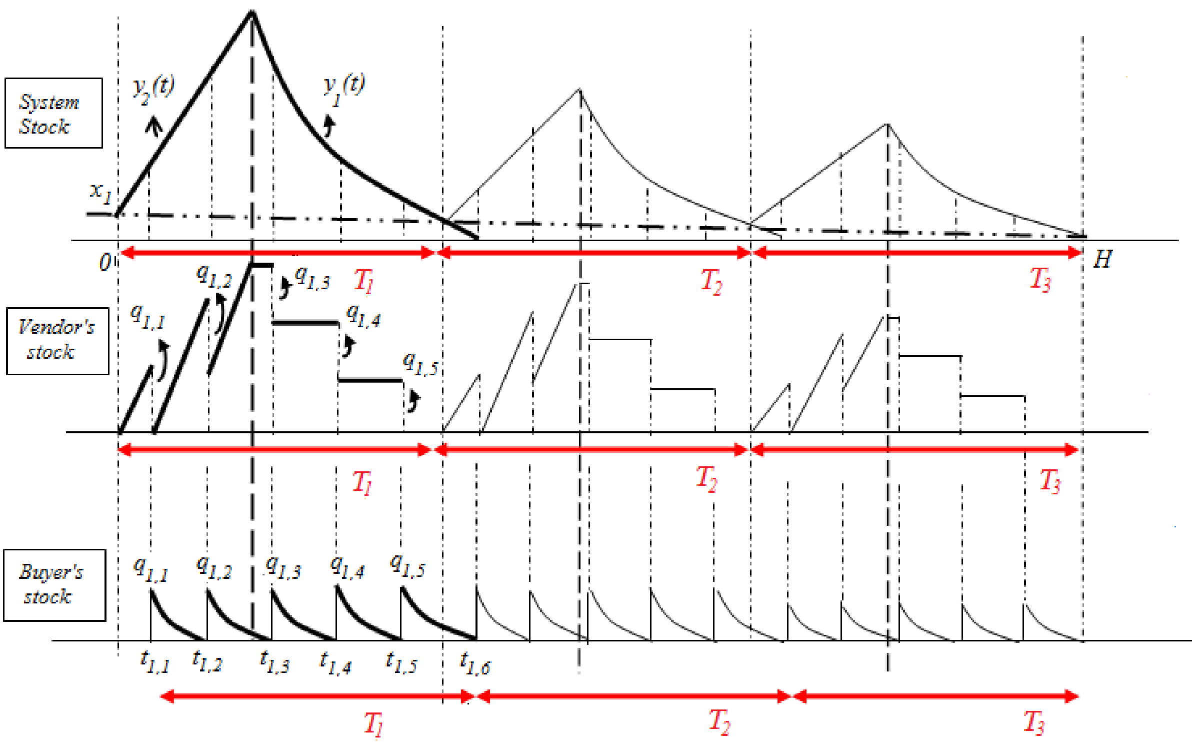


Figure 4.1 gives the graphical representation for the model with 3 batches and 5 shipments. At the beginning of the cycle, the vendor begins production at a fixed production rate P, and inventory begins to accumulate up to production time. Here we assume that the demand rate for the previous batch is constant. Therefore, the initial stock at the buyer when the production is about to start is x_1 and given as follow

$$x_1 = a(\frac{q_{1,1}}{P}) \tag{4.1}$$

where $q_{1,1}/P$ is the time to produce the first shipment quantity, $q_{1,1}$.

However, for x_i, $i = 2, 3, \cdots, n$, the demand rate is considered decreasing with time, so,

$$x_i = \int_{t_{x_i}}^{t_{i-1,m+1}=t_{i,1}} f(t)dt \tag{4.2}$$

where t_{x_i} can be obtained from the following equation :

$$\begin{aligned} P(t_{i,1} - t_{x_i}) &= q_{i,1} \\ t_{x_i} &= t_{i,1} - \frac{q_{i,1}}{P}. \end{aligned} \tag{4.3}$$

The production batch will starts at $t_{i,0}$ for every batch until production uptime $t_{i,p}$. The first shipment is at $t_{i,1}$ ($t_{1,1} = 0$ according to buyer's time) and follows at time $t_{i,2}, t_{i,3}, \ldots, t_{i,m}$, with the shipment sizes $q_{i,1}, q_{i,2}, \ldots, q_{i,m}$ where $i = 1, 2, \ldots, n$.

$y_1(t)$ represents the stock level at time t in the interval $(t_{i,p}, t_{i+1,0})$ which can be express as

$$\int_{t_{i,p}}^{t_{i+1,0}} f(t)\; dt - \int_{t_{i,p}}^{t} f(t)dt. \tag{4.4}$$

Then,

$$y_1(t) = \int_{t}^{t_{i+1,0}} f(t)dt. \tag{4.5}$$

$y_2(t)$ is the stock level during production time, $(t_{i,0}, t_{i,p})$ and given by

$$y_2(t) \quad = \quad P(t - t_{i,0}) - \int_{t_{i,0}}^{t} f(t)dt. \tag{4.6}$$

The total demand for the complete production cycle, D is

$$\int_{t_{1,0}=0}^{H} f(t)dt \tag{4.7}$$

where $\sum_{i=1}^{n} D_i = D$ and $D_i = \int_{t_{i,0}}^{t_{i+1,0}} f(t)dt$.

4.3.1 Total time-weighted system stock

The total system stock is represent by the area under the curve $y_1(t)$ and $y_2(t)$ in $(t_{i,p}, t_{i+1,0})$ and $(t_{i,0}, t_{i,p})$ respectively. Hence, we have the total time-weighted system stock, TSS, as

$$\sum_{i=1}^{n} \int_{t_{i,0}}^{t_{i,p}} y_2(t)dt + \sum_{i=1}^{n} \int_{t_{i,p}}^{t_{i+1,0}} y_1(t)\; dt + \sum_{i=1}^{n} (x_i + x_{i+1}) \frac{T_i}{2} \tag{4.8}$$

where $t_{n,m+1} = H$ and $x_{n+1} = 0$.

The production uptime, $t_{i,p}$ can be obtained from the following :

$$\begin{aligned}
P(t_{i,p} - t_{i,0}) &= \int_{t_{i,0}}^{t_{i+1,0}} f(t)dt \\
t_{i,p} &= \frac{1}{P}\int_{t_{i,0}}^{t_{i+1,0}} f(t)dt + t_{i,0} \\
&= \frac{D_i}{P} + t_{i,0} \qquad (4.9)
\end{aligned}$$

4.3.2 Total time-weighted buyer stock

In figure 4.1, $q_{1,j}$ $j = 1, 2, 3, 4, 5$ is the shipment sizes for each shipment for the first batch. Similarly, $q_{i,j}$, $i = 2$ and $j = 1, 2, 3, 4, 5$ is the shipment sizes for each shipment for the second batch. Therefore, the shipment sizes for each shipment for every batch is given by

$$q_{i,j} = \int_{t_{i,j}}^{t_{i,j+1}} f(t)dt \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m. \qquad (4.10)$$

Let $I(t)$ be the inventory level at any time t and it given by

$$\int_{t}^{t_{i,j+1}} f(t)\ dt \qquad (4.11)$$

Hence, the buyer stock can be calculated by the area under the curve $I(t)$ in the period $(t_{i,j}, t_{i,j+1})$. It follows that the total time-weighted buyer stock from jth shipment, TBS is

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{\int_{t_{i,j}}^{t_{i,j+1}}\left[\int_{t}^{t_{i,j+1}} f(t)\ dt\right] dt\right\} \qquad (4.12)$$

It follows, the total cost for this policy, TC is in the term of n (discrete variable), m (discrete variable) and $\vec{t} = t_{i,2}, t_{i,3}, \ldots, t_{i,m}$ (real variables) which is given by

$$TC = n(A_1) + nm(A_2) + h_1 TSS + (h_2 - h_1)TBS \tag{4.13}$$

For example, let the demand rate is linearly decreasing over the period $(0, H)$ that is

$$f(t) = a - bt \qquad a > 0; b > 0; t > 0; H > 0. \tag{4.14}$$

Substituting (4.14) into (4.5), (4.6), (4.7), (4.8), (4.9) and (4.12) we have

$$y_1 = a(t_{i+1,0} - t) - \frac{b}{2}(t_{i+1,0}^2 - t^2), \tag{4.15}$$

$$y_2 = (P - a)(t - t_{i,0}) + \frac{b}{2}(t^2 - t_{i,0}^2), \tag{4.16}$$

$$D = H\left(a - \frac{b}{2}H\right), \tag{4.17}$$

$$\begin{aligned} TSS &= \sum_{i=1}^{n} \left\{ \frac{b}{6}(t_{i,p}^3 - t_{i,0}^3) + \frac{(P-a)}{2}(t_{i,p}^2 - t_{i,0}^2) \right. \\ &\qquad \left. - [(P-a)t_{i,0} + \frac{b}{2}t_{i,0}^2](t_{i,p} - t_{i,0}) \right\} \\ &+ \sum_{i=1}^{n} \left\{ \frac{b}{6}(t_{i+1,0}^3 - t_{i,p}^3) - \frac{a}{2}(t_{i+1,0}^2 - t_{i,p}^2) \right. \\ &\qquad \left. + (at_{i+1,0} - \frac{b}{2}t_{i+1,0}^2)(t_{i+1,0} - t_{i,p}) \right\} \\ &+ \sum_{i=1}^{n}(x_i + x_{i+1})\frac{T_i}{2}, \end{aligned} \tag{4.18}$$

$$
\begin{aligned}
TBS &= \sum_{i=1}^{n}\sum_{j=1}^{m} a t_{i,j+1}(t_{i,j+1}-t_{i,j}) - \frac{a}{2}\sum_{i=1}^{n}\left[t_{i-1,m+1}^2 - t_{i,1}^2\right] \\
&\quad -\sum_{i=1}^{n}\frac{b}{2}\sum_{j=1}^{m} t_{i,j+1}^2(t_{i,j+1}-t_{i,j}) + \frac{b}{6}\sum_{i=1}^{n}\left[t_{i-1,m+1}^3 - t_{i,1}^3\right].
\end{aligned} \tag{4.19}
$$

Finally, substituting (4.18) and (4.19) into (4.13) we have,

$$
\begin{aligned}
TC &= n(A_1) + nm(A_2) + h_1\left\{\sum_{i=1}^{n}\left\{\frac{b}{6}(t_{i,p}^3 - t_{i,0}^3) + \frac{(P-a)}{2}(t_{i,p}^2 - t_{i,0}^2)\right.\right. \\
&- \left.[(P-a)t_{i,0} + \frac{b}{2}t_{i,0}^2](t_{i,p}-t_{i,0})\right\} + \sum_{i=1}^{n}\left\{\frac{b}{6}(t_{i+1,0}^3 - t_{i,p}^3)\right. \\
&- \left.\left.\frac{a}{2}(t_{i+1,0}^2 - t_{i,p}^2) + (a t_{i+1,0} - \frac{b}{2}t_{i+1,0}^2)(t_{i+1,0}-t_{i,p})\right\} + \sum_{i=1}^{n}(x_i + x_{i+1})\frac{T_i}{2}\right\} \\
&+ (h_2 - h_1)\left\{a\sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j+1}(t_{i,j+1}-t_{i,j}) - \frac{a}{2}\sum_{i=1}^{n}\left[t_{i,m+1}^2 - t_{i,1}^2\right]\right. \\
&- \left.\frac{b}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j+1}^2(t_{i,j+1}-t_{i,j}) + \frac{b}{6}\sum_{i=1}^{n}\left[t_{i,m+1}^3 - t_{i,1}^3\right]\right\}
\end{aligned} \tag{4.20}
$$

Denote TC as $TC(n, m, \vec{t})$ where n and m are discrete variables and $\vec{t} = t_{i,2}, t_{i,3}, \ldots, t_{i,m}$ is a real variables.

We will explore the best solution of the above total cost based on two policies with 2 cases of the shipment sizes:

1. Policy 1 : Equal cycle time
   (a) Equal shipment sizes
   (b) Unequal shipment sizes and unequal shipment periods
2. Policy 2 : Unequal cycle time
   (a) Equal shipment sizes

(b) Unequal shipment sizes and unequal shipment periods

The objective in every policies is to find an optimal $t_{i,j}$ and $q_{i,j}$ for a given n and m which gives the minimum total cost, TC.

4.3.3 Policy 1 : Equal cycle times

In this policy, the value of T_i is assumed to be equal. So, we have fixed value of T_i which can be calculated as follow

$$T_i = \frac{H}{n} \tag{4.21}$$

where the sum of all T_i must be equal to H, that is

$$\sum_{i=1}^{n} T_i = H. \tag{4.22}$$

We will explore this policy with equal and unequal shipment sizes.

4.3.3.1 Policy 1 (a) : Equal shipment sizes

In this policy the shipment sizes are equal for every batch. Hence, we have fixed value of $q_{i,j}$. Therefore, the demand of i batch, D_i will be divided by the number of shipments, m, that is given by

$$q_{i,j} = \frac{D_i}{m} \qquad i = 1, 2, \ldots, n \tag{4.23}$$

The buyer receives an equal quantity for each shipment and takes $(t_{i,j+1} - t_{i,j})$ amount of times to use up $q_{i,j}$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Hence, we have

$$\int_{t_{i,j}}^{t_{i,j+1}} f(t)dt = \frac{D_i}{m}. \tag{4.24}$$

For example, as shown in Figure 4.1, the vendor delivers five equal shipments. The first shipment for the first batch is at time $t_{1,1}$ ($t_{1,1} = 0$ according to buyer's time), it will be finished at time $t_{1,2}$ where the second shipment is arrive. This can be written as

$$q_{1,1} = \int_{t_{1,1}}^{t_{1,2}} f(t)\ dt = \frac{D_1}{n}. \tag{4.25}$$

From equation (4.25), we get

$$t_{1,2} = \frac{a}{b}\left\{1 - \sqrt{a - \frac{2b}{a^2}\left[-\frac{b}{2}{t_{1,1}}^2 + at_{1,1} + \left(\frac{D_1}{n}\right)\right]}\right\}. \tag{4.26}$$

Similarly, in the period time, $(t_{1,2}, t_{1,3})$, the buyer will use up $q_{1,2}$ while the vendor will continue producing and deliver the third shipment which will arrive at the buyer exactly just before the second shipment is finished at $t_{1,3}$. This can be written as

$$q_{1,2} = \int_{t_{1,2}}^{t_{1,3}} f(t)\ dt = \frac{D_1}{n}. \tag{4.27}$$

From equation (4.27), we get

$$t_{1,3} = \frac{a}{b}\left\{1 - \sqrt{a - \frac{2b}{a^2}\left[-\frac{b}{2}{t_{1,2}}^2 + at_{1,2} + \left(\frac{D_1}{n}\right)\right]}\right\}. \tag{4.28}$$

This process is repeated until the end of the planning horizon, H. Generally, the shipment times is

$$t_{i,j+1} = \frac{a}{b}\left\{1-\sqrt{a-\frac{2b}{a^2}\left[-\frac{b}{2}t_{i,j}^2+at_{i,j}+\left(\frac{D_i}{n}\right)\right]}\right\}, \qquad (4.29)$$
$$\text{where} \quad i=1,2,\ldots,n, \quad j=1,2,\cdots,m-1.$$

Substituting (4.29) into (4.20) will give the total cost, $TC_{1(a)}$ for this policy.

4.3.3.2 Solution procedure

The computer algorithm of the solution procedure is outline below :

1. Let $n=1$
2. Let $m=1$
3. Set $T_i = H/n \quad i=1,2,\ldots,n$
4. Set $t_{i,1}=0$, $t_{n,m+1}=H$
5. Compute $q_{i,j} = D_i/n$, $(i=1,2,\cdots,n$ and $j=1,2,\ldots m)$
6. Compute $t_{i,j+1}$, $\quad (i=1,2,\cdots,n, \quad j=1,2,\cdots,m-1)$ using (4.29) and $TC_{1(a)}(n,m,\ \vec{t}\,)$ using (4.20)
7. Set $TC_{1(a)}(n,m,\ \vec{t}\,)$ as $TC_{1(a)}(n,m^*,\ \vec{t}\,)$. Increase m by 1 and repeat step 5 to 6. Stop when $TC_{1(a)}(n,m,\ \vec{t}\,) \geq TC_{1(a)}(n,m^*,\ \vec{t}\,)$
8. Increase n by 1 and repeat step 5 to 7. Set $TC_{1(a)}(n,m^*,\ \vec{t}\,)$ as $TC_{1(a)}(n^*,m^*,\ \vec{t}\,)$. Stop when $TC_{1(a)}(n,m^*,\ \vec{t}\,) \geq TC_{1(a)}(n^*,m^*,\ \vec{t}\,)$

The basic idea of the above algorithm is to start with $n = 1$ and $m = 1$. Next, we increase m to improve the total system cost until the first $m = m^*$ that satisfies the conditions $TC_{1(a)}(n, m^*, \vec{t}\,) < TC_{1(a)}(n, m^* - 1, \vec{t}\,)$ and $TC_{1(a)}(n, m^*, \vec{t}\,) < TC_{1(a)}(n, m^* + 1, \vec{t}\,)$. Then we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_{1(a)}(n^*, m^*, \vec{t}\,) < TC_{1(a)}(n^* - 1, m^* - 1, \vec{t}\,)$ and $TC_{1(a)}(n^*, m^*, \vec{t}\,) < TC_{1(a)}(n^* + 1, m^* + 1, \vec{t}\,)$.

4.3.3.3 Policy 1 (b) : Unequal shipment sizes and unequal shipment periods

In this policy the shipment sizes are unequal, so, $q_{i,j}$ is now a variable. Figure 4.2 shows the illustration of this policy.

Since stockout is not allowed, the time for the vendor to produce $q_{i,j+1}$ must be less than the time for the buyer to finish up $q_{i,j}$. For example, Figure 4.3 shows the illustration of the inventory level at the buyer for the first and second shipments for the first batch.

The time to produce the second shipment, $q_{1,2}$ must be less than the time for the buyer to finish up the first shipment, $q_{1,1}$, so we can write it as follows

$$\begin{aligned} P(t_{1,2} - t_{1,1}) &\geq q_{1,2} \\ t_{1,2} - t_{1,1} &\geq \frac{q_{1,2}}{P} \end{aligned}$$

Figure 4.2: Illustration of the inventory level at the vendor and buyer for the first batch

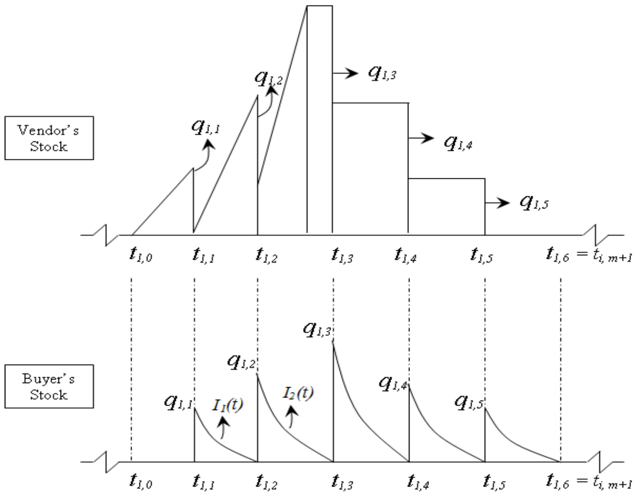
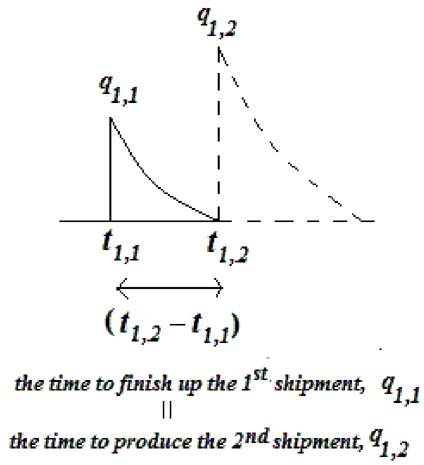


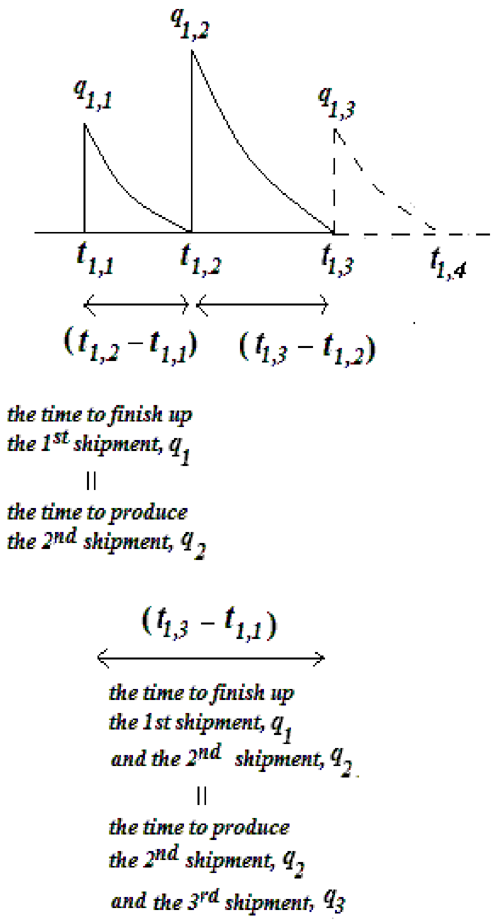
Figure 4.3: The first and second shipments for the first batch



$$
\begin{aligned}
\text{where} \qquad q_{1,1} &= \int_{t_{1,1}}^{t_{1,2}} f(t)dt. \\
t_{1,2} &= \frac{a}{b}\left\{1 - \sqrt{a - \frac{2b}{a^2}\left[at_{1,1} - \frac{b}{2}{t_{1,1}}^2 + q_{1,1}\right]}\right\}.
\end{aligned}
$$

Figure 4.4 shows the illustration of the inventory level at the buyer for the first, second and third shipments for the first batch.

Figure 4.4: The first, second and third shipments



Similarly, the time to produce the third shipment, $q_{1,3}$, must be less than the time for the buyer to finish up the second shipment, $q_{1,2}$ and we have

$$\begin{aligned}
P(t_{1,3} - t_{1,2}) &\geq q_{1,3} \\
P(t_{1,3} - t_{1,1} + t_{1,1} - t_{1,2}) &\geq q_{1,3} \\
P(t_{1,3} - t_{1,1}) &\geq P(t_{1,2} - t_{1,1}) + q_{1,3} \\
P(t_{1,3} - t_{1,1}) &\geq q_{1,2} + q_{1,3} \\
(t_{1,3} - t_{1,1}) &\geq \frac{q_{1,2} + q_{1,3}}{P} \qquad (4.30)
\end{aligned}$$

where
$$\begin{aligned}
q_{1,2} &= \int_{t_{1,2}}^{t_{1,3}} f(t)dt \\
t_{1,3} &= \frac{a}{b}\left\{1 - \sqrt{1 - \frac{2b}{a^2}\left[at_{1,2} - \frac{b}{2}t_{1,2}^2 + q_{1,2}\right]}\right\}
\end{aligned}$$

Generally, the time to produce the $j+1$ shipment, $q_{i,j+1}$ must be less than the time for the buyer to finish up the j shipment, $q_{i,j}$, that is

$$\begin{aligned}
P(t_{i,j+1} - t_{i,1}) &\geq \sum_1^j q_{i,j+1} \\
t_{i,j+1} - t_{i,1} &\geq \sum_1^j \frac{q_{i,j+1}}{P}.
\end{aligned}$$

where
$$\begin{aligned}
q_{i,j} &= \int_{t_{i,j}}^{t_{i,j+1}} f(t)dt \\
t_{i,j+1} &= \frac{a}{b}\left\{1 - \sqrt{1 - \frac{2b}{a^2}\left[at_{i,j} - \frac{b}{2}t_{i,j}^2 + q_{i,j}\right]}\right\}.
\end{aligned}$$

The total quantity delivered to the buyer must be equal to the total demand for each batch, that is

$$\sum_{j=1}^{m} q_{i,j} = D_i \ , \quad i = 1, 2, \cdots, n. \tag{4.31}$$

From these arguments, we can establish the following constraint optimization problem,

$$\text{Minimum} \qquad TC_{1(b)}(n, m, \vec{t}\,) \tag{4.32}$$

Subject to

$$t_{i,j+1} - t_{i,1} \ \geq \ \sum_{1}^{j} \frac{q_{i,j+1}}{P} \tag{4.33}$$

$$\sum_{j=1}^{m} q_{i,j} \ = \ D_i \tag{4.34}$$

where

$$t_{i,j+1} = \frac{a}{b} \left\{ 1 - \sqrt{1 - \left(\frac{2b}{a^2}\right) \left[a t_{i,j} - \frac{b}{2} t_{i,j}^2 + q_{i,j} \right]} \ \right\}. \tag{4.35}$$

Our objective is to minimize the total system cost, that is equation (4.32) subject to the constraints (4.33), and (4.34).

4.3.3.4 Solution procedure

We derived the following algorithm and use the Microsoft Excel Solver as a solution tool:

1. Let $n = 1$
2. Let $m = 1$

3. Set $T_i = H/n \quad i = 1, 2, \ldots, n$

4. Set $t_{i,1} = 0$, $t_{n,m+1} = H$

5. Determine $q_{i,j}$, $(i = 1, 2, \cdots, n$ and $j = 1, 2, \ldots m)$ which satisfied constraints (4.33) and (4.34), if it exists

6. Compute $t_{i,j+1}$, $(i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1)$ using (4.35) and $TC(n, m, \vec{t}\,)$ using (4.13)

7. Set $TC_{1(b)}(n, m, \vec{t}\,)$ as $TC_{1(b)}(n, m^*, \vec{t}\,)$. Increase m by 1 and repeat step 5 to 6. Stop when $TC_{1(b)}(n, m, \vec{t}\,) \geq TC_{1(b)}(n, m^*, \vec{t}\,)$

8. Increase n by 1 and repeat step 5 to 7. Set $TC_{1(b)}(n, m^*, \ \vec{t}\,)$ as $TC_{1(b)}(n^*, m^*, \ \vec{t}\,)$. Stop when $TC_{1(b)}(n, m, \vec{t}\,) \geq TC_{1(b)}(n^*, m^*, \vec{t}\,)$

Similarly, the basic idea of the above algorithm is to start with $n = 1$ and $m = 1$. Next, we increase m to improve the total system cost until the first $m = m^*$ that satisfies the conditions $TC_{1(b)}(n, m^*, \vec{t}\,) < TC_{1(b)}(n, m^* - 1, \vec{t}\,)$ and $TC_{1(b)}(n, m^*, \vec{t}\,) < TC_{1(b)}(n, m^* + 1, \vec{t}\,)$. Then we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_{1(b)}(n^*, m^*, \ \vec{t}\,) < TC_{1(b)}(n^* - 1, m^* - 1, \ \vec{t}\,)$ and $TC_{1(b)}(n^*, m^*, \ \vec{t}\,) < TC_{1(b)}(n^* + 1, m^* + 1, \ \vec{t}\,)$.

4.3.4 Policy 2 : Unequal cycle times

In this policy the value of T_i is unknown. The sum of all $T_i's$ must be equal to the finite time horizon, H that is

$$\sum_{i=1}^{n} T_i = H. \tag{4.36}$$

We need to find the optimal value of T_i that gives the minimum total cost. We will explore this policy with equal and unequal shipment sizes.

4.3.4.1 Policy 2 (a) : Equal shipment sizes

In this policy the shipment sizes is assumed to be equal. This policy is similar to the Policy 1(a) in Section 4.3.3.3 therefore we can adopt Equation (4.23) and (4.24) for solving this policy. The value of T_i will be set as a changing variable, Solver will give the optimal value of T_i which consider fixed equal shipments, and the minimum total cost of the system for a given n and m. The constraint optimization problem is as follows :

$$\text{Minimum} \qquad TC_{2(a)}(n, m, \vec{t}\,) \tag{4.37}$$

Subject to

$$\sum_{i=1}^{n} T_i = H \tag{4.38}$$

$$\sum_{j=1}^{m} q_{i,j} = D_i \tag{4.39}$$

where

$$t_{i,j+1} = \frac{a}{b}\left\{1 - \sqrt{1 - \left(\frac{2b}{a^2}\right)\left[at_{i,j} - \frac{b}{2}t_{i,j}^2 + q_{i,j}\right]}\ \right\}. \tag{4.40}$$

Our objective is to minimize the total system cost, that is equation (4.37) subject to the constraints (4.38) and (4.39).

4.3.4.2 Solution procedure

We derived the following algorithm and use the Microsoft Excel Solver as a solution tool:

1. Let $n = 1$
2. Let $m = 1$
3. Set $t_{i,1} = 0$, $t_{n,m+1} = H$
4. Determine T_i, $\quad i = 1, 2, \cdots, n$ which satisfied constraints (4.38) and (4.39), if it exists
5. Compute $t_{i,j+1}$, $\quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1$ using (4.40) and $TC_{2(a)}(n, m, \vec{t}\,)$ using (4.37)
6. Set $TC_{2(a)}(n, m, \vec{t}\,)$ as $TC_{2(a)}(n, m^*, \vec{t}\,)$. Increase m by 1 and repeat step 4 to 5. Stop when $TC_{2(a)}(n, m, \vec{t}\,) \geq TC_{2(a)}(n, m^*, \vec{t}\,)$
7. Increase n by 1 and repeat step 4 to 6. Set $TC_{2(a)}(n, m^*,\ \vec{t}\,)$ as $TC_{2(a)}(n^*, m^*,\ \vec{t}\,)$. Stop when $TC_{2(a)}(n, m^*, \vec{t}\,) \geq TC_{2(a)}(n^*, m^*, \vec{t}\,)$

Again, the basic idea of the above algorithm is to start with $n = 1$ and $m = 1$. Next, we increase m to improve the total system cost until the first $m = m^*$ that satisfies the conditions $TC_{2(a)}(n, m^*, \vec{t}\ ) < TC_{2(a)}(n, m^* - 1)$ and $TC_{2(a)}(n, m^*, \vec{t}\ ) < TC_{2(a)}(n, m^* + 1)$. Then we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_{2(a)}(n^*, m^*,\ \vec{t}\ ) < TC_{2(a)}(n^* - 1, m^* - 1,\ \vec{t}\ )$ and $TC_{2(a)}(n^*, m^*,\ \vec{t}\ ) < TC_{2(a)}(n^* + 1, m^* + 1,\ \vec{t}\ )$.

4.3.4.3 Policy 2 (b) : Unequal shipment sizes and unequal shipment periods

Finally, we consider the case where both the cycle time and the shipment sizes are unequal. We set T_i and $q_{i,j}$ as the changing variables. The constraint optimization problem is as follow :

$$\text{Minimum} \qquad TC_{2(b)}(n, m, , \vec{t}\ ) \tag{4.41}$$

Subject to

$$t_{i,j} - t_{i,j-1} \geq \sum_{1}^{j} \frac{q_{i,j+1}}{P} \tag{4.42}$$

$$\sum_{i=1}^{n} T_i = H \tag{4.43}$$

$$\sum_{j=1}^{m} q_{i,j} = D_i \tag{4.44}$$

where

$$t_{i,j+1} = \frac{a}{b}\left\{1 - \sqrt{1 - \left(\frac{2b}{a^2}\right)\left[at_{i,j} - \frac{b}{2}t_{i,j}^2 + q_{i,j}\right]}\ \right\}. \tag{4.45}$$

Our objective is to minimize the total system cost, that is equation (4.41) subject to the constraints (4.42), (4.43) and (4.44).

4.3.4.4 Solution procedure

We derived the following algorithm and use the Microsoft Excel Solver as a solution tool:

1. Let $n = 1$
2. Let $m = 1$
3. Set $t_{i,1} = 0$, $t_{n,m+1} = H$
4. Determine T_i, $i = 1, 2, \cdots, n$ and $q_{i,j}$ which satisfied constraints (4.42), (4.43) and (4.44), if they exist
5. Compute $t_{i,j+1}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m-1$ using (4.45) and $TC_{2(b)}(n, m, \vec{t}\,)$ using (4.41)
6. Set $TC_{2(b)}(n, m, \vec{t}\,)$ as $TC_{2(b)}(n, m^*, \vec{t}\,)$. Increase m by 1 and repeat steps 4 to 5. Stop when $TC_{2(b)}(n, m, \vec{t}\,) \geq TC_{2(b)}(n, m^*, \vec{t}\,)$
7. Increase n by 1 and repeat steps 4 to 6. Set $TC_{2(b)}(n, m^*, \vec{t}\,)$ as $TC_{2(b)}(n^*, m^*, \vec{t}\,)$. Stop when $TC_{2(b)}(n, m, \vec{t}\,) \geq TC_{2(b)}(n^*, m^*, \vec{t}\,)$

4.3.5 Numerical examples and sensitivity analysis

To demonstrate the effectiveness of the proposed models, we present some numerical examples for every cases which uses similar parameter values as follows:

$$A_1 = 400, \quad A_2 = 25, \quad H = 5, \quad P = 1000$$

$$h_1 = 4, \qquad h_2 = 5, \quad a = 200, \quad b = 20.$$

Note that these values are similar as in numerical example in final batch inventory model in Section 3.3.6.

Table 4.1 gives the total cost for Policy 1(a) and Policy 1(b) for some combinations of n and m, where $n = 1, 2, \cdots, 6$ and $m = 1, 2, \cdots, 13$. The value of the total cost for Policy 1(b) is given in the parenthesis. The underlined values represent the minimum total cost for a given m while increasing the value of n and the double underlined values represent the minimum total cost for a given n while increasing the value of n. Overall, it shows that when n and m increase, the total cost decreases until it reached the minimum value.

For example, when $n = 1$, the minimum total cost is at $m = 12$ for Policy 1(a) and $m = 10$ for Policy 1(b) respectively. If we consider $m = 1$ while increasing the value of n, the minimum total cost is at $n = 5$ for both Policies 1(a) and 1(b).

We observed that the value of the total cost when $m = 1$ for every n gives the same value for both Policies 1(a) and 1(b). This is because there is only one shipment for every batch and T_1 must be equal to H. Therefore, the minimum shipment size, $q_{i,1}$ is always equal to the total demand, D_i. Hence, it produces the same result.

As expected, the total cost for Policy 1(b) is always lower than Policy 1(a). This result suggests that unequal shipment sizes policy performs very well when compared to the equal shipment sizes policy. The optimal total costs for both policies are at $n = 4$ and $m = 3$. These values are given in bold in Table 4.1. However, Policy 1(b) gives a better solution which is 86.10 less than Policy 1(a) where $TC^*_{1(a)}$ is 3757.77 while $TC^*_{1(b)}$ is 3671.67.

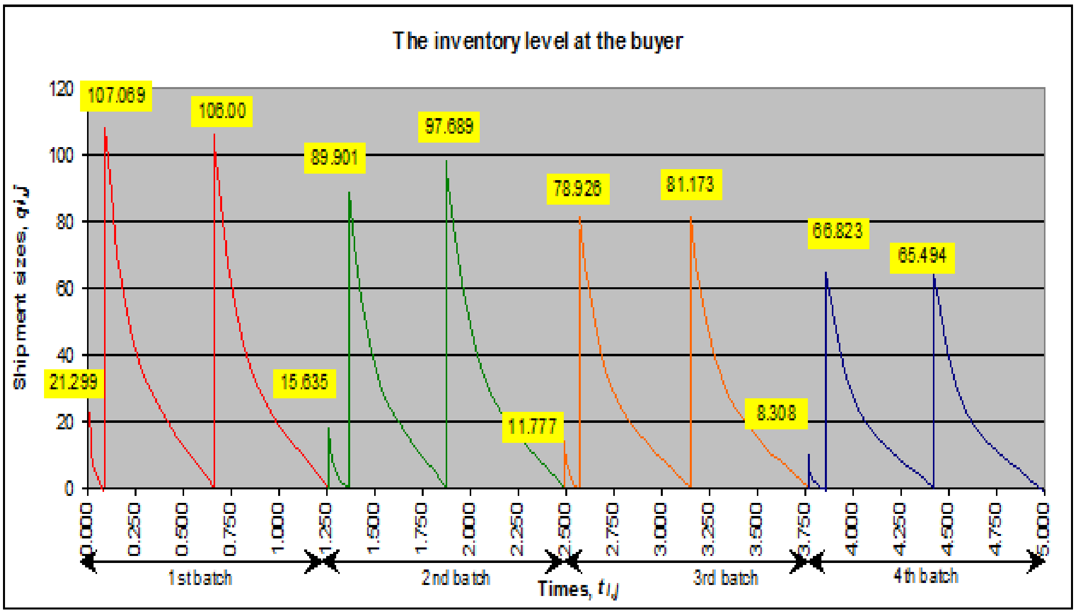
Table 4.1: The minimum total cost for Policies 1(a) and 1(b)

Total relevant cost						
$m\downarrow \quad \rightarrow n$	**1**	**2**	**3**	**4**	**5**	**6**
1	9133.33 (9133.33)	5614.97 (5614.97)	4557.92 (4557.92)	4203.88 (4203.88)	4148.55 (4148.55)	4247.80 (4247.80)
2	7622.97 (7477.66)	4774.55 (4646.12)	4005.55 (3914.10)	3841.50 (3745.78)	3879.880 (3829.67)	4066.04 (4012.592)
3	7116.16 (6911.15)	4523.27 (4380.31)	3869.64 (3761.76)	**3757.77** (**3671.67**)	3873.09 (3803.00)	4105.09 (4077.69)
4	6870.95 (6675.63)	4421.74 (4290.71)	3842.795 (3752.77)	3776.48 (3698.85)	3932.09 (3867.82)	4199.54 (4144.73)
5	6732.37 (6556.44)	4380.54 (4264.701)	3850.26 (3764.27)	3827.65 (3759.53)	4017.46 (3961.81)	4316.83 (4268.19)
6	6647.69 (6489.75)	4369.614 (4266.85)	3882.82 (3806.77)	3895.07 (3834.90)	4098.78 (4066.34)	- -
7	6594.05 (6451.25)	4376.04 (4283.94)	3927.48 (3859.47)	3971.79 (3918.03)	4222.13 (4177.73)	- -
8	6559.89 (6429.66)	4393.32 (4309.92)	3979.71 (3918.22)	4059.83 (4005.74)	4344.52 (4302.73)	- -

9	6538.78	4417.86	4037.00	4140.74	4446.94	-
	(6419.022)	(4341.61)	(3981.32)	(4096.40)	(4410.52)	-
10	6526.828	4447.48	4097.83	4229.86	4563.12	-
	(6415.87)	(4377.19)	(4046.12)	(4189.05)	(4529.44)	-
11	6521.55	4480.79	4161.22	4320.96	4680.90	-
	(6418.07)	(4415.54)	(4113.28)	(4283.16)	(4649.71)	-
12	6521.29	4516.88	4226.56	4413.55	4799.88	-
	(6424.23)	(4455.94)	(4181.86)	(4378.33)	(4770.84)	-
13	6524.90	4555.11	4293.37	4507.27	-	-
	(6433.44)	(4497.91)	(4251.51)	(4474.30)	-	-

We illustrate the inventory level at the buyer and vendor for Policy 1(b) by plotting the optimal solution as given by Figure 4.5 and 4.6 respectively.

Figure 4.5: Buyer's inventory level



As depicted in Figure 4.5, the optimal value of shipment sizes, $q^*_{i,j}$ where i=1, 2, 3, 4 and j=1, 2, 3 are given by (21.299, 107.069, 106.007), (15.635, 89.801, 97.689), (11.777, 78.926, 81.173) and (8.308, 66.823, 65.494). The value of the optimal initial stock at the buyer, x_i are (4.260, 2.738, 1.768, 1.039). As expected, it shows that $x_i > x_{i+1}$ due to the decreasing demand. The optimal shipment times given by $t^*_{i,j}$, where $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ are (0, 0.107, 0.664), (1.250, 1.340, 1.875), (2.500, 2.579, 3.131) and (3.750, 3.817, 4.383). The $t^*_{i,p}$ for i=1, 2, 3, 4 is (0.234, 1.453, 2.672, 3.891). The value of the optimal T_i is obtain from H/n, therefore the optimal T_i^* where $i = 1, 2, 3, 4$ is 1.250.

Figure 4.6: Vendor's inventory level

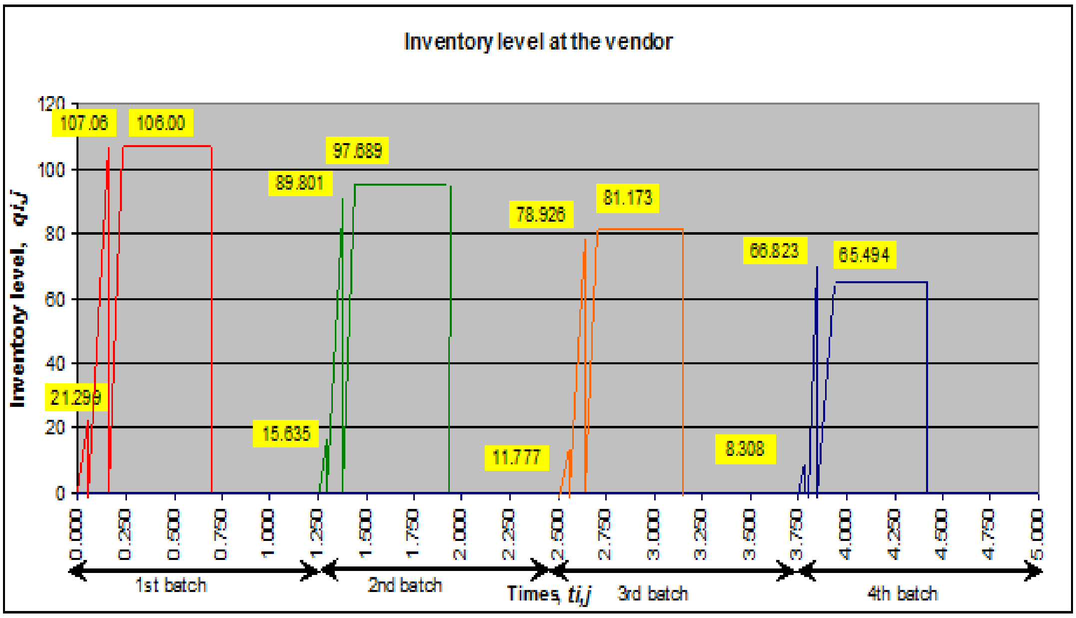


Figure 4.6 shows the amount which is produced and shipped at the respective shipment times. For example, at the first batch, the vendor start their production at $t_{1,0} = 0$ until $t_{1,p} = 0.234$. They have to produce 234.38 in order to satisfy the demand for this batch. The first shipment, $q_{1,1}$=21.299 is produced and shipped to the buyer at $t_{1,1} = 0.021$. They continue producing and deliver the second shipment, $q_{1,1}$= 107.069 at $t_{1,2} = 0.128$. The third shipment, $q_{1,3}$=106.007 is produced at the production uptime, $t_{1,p}$. However, this amount is held by the vendor and delivered when the buyer finished

up the second shipment at $t_{1,3} = 0.6852$. The second batch will start at $t_{2,0} = 1.289$ that is 0.016 unit of time before the third shipment is finished up by the buyer.

Table 4.2 gives the total cost for Policy 2(a) and Policy 2(b) for some combinations of n and m, where $n = 1, 2, \cdots, 6$ and $m = 1, 2, \cdots, 13$. The value of the total cost for Policy 2(b) is given in the parenthesis. The underlined values represent the minimum total cost for a given m while increasing the value of m, and the double underlined values represent the minimum total cost for a given m while increasing the value of n.

Similarly, as in Policies 1(a) and 1(b), when n and m increase, the total cost decreases until it reached the minimum solution. For example, when $n = 1$, the minimum total cost is reached at $m = 12$ for Policy 2(a) and $m = 10$ for Policy 2(b). If we consider $m = 1$ while increasing the value of n, the minimum total cost is reached at $n = 5$ for both Policies 2(a) and 2(b).

As expected, the total cost for Policy 2(b) is always lower than Policy 2(a) where the different shipments size policy performs very well when compared to the equal shipment sizes policy. The optimal total costs for both policies are reached at $n = 4$ and $m = 3$. However, Policy 2(b) gives a better solution which is 84.29 less than Policy 2(a) where $TC^*_{2(a)}$ is 3742.99 and $TC^*_{2(b)}$ is 3658.70. The optimal cycle time is increasing due to the decreasing demand where T^*_i, are 1.1233, 1.1812, 1.2751 and 1.4205.

Table 4.2: The minimum total cost for Policies 2(a) and 2(b)

Total relevant cost						
$m\downarrow \quad \rightarrow n$	**1**	**2**	**3**	**4**	**5**	**6**
1	9133.33 (9133.33)	5261.61 (5587.71)	4529.39 (4529.39)	4179.29 (4179.29)	4127.55 (4127.55)	4229.64 (4229.64)
2	7622.97 (7477.66)	4751.52 (4624.45)	3984.93 (3895.91)	3819.57 (3731.16)	3865.57 (3829.67)	4053.794 (4002.35)
3	7116.16 (6911.15)	4501.80 (4358.99)	3851.44 (3745.18)	**3742.99** (**3658.70**)	3860.82 (3792.17)	4094.65 (4072.53)
4	6870.95 (6675.63)	4401.08 (4270.27)	3824.575 (3729.854)	3762.80 (3686.56)	3920.81 (3857.85)	4189.97 (4136.37)
5	6732.37 (6556.44)	4360.36 (4244.76)	3833.90 (3745.32)	3814.61 (3747.63)	4006.74 (3952.21)	4307.65 (4260.13)
6	6647.69 (6489.75)	4349.762 (4247.23)	3866.91 (3791.76)	3882.46 (3823.26)	4088.01 (4056.82)	- -
7	6594.05 (6451.25)	4356.42 (4283.94)	3911.89 (3844.66)	3959.48 (3906.56)	4212.06 (4168.46)	- -
8	6559.89 (6429.66)	4373.88 (4290.69)	3964.43 (3903.57)	4052.95 (3994.41)	4334.54 (4293.46)	- -

9	6538.78	4398.56	4021.89	4133.03	4437.21	-
	(6419.02)	(4322.51)	(3966.65)	(4085.16)	(4401.41)	-
10	6526.83	4428.28	4084.80	4222.22	4553.51	-
	(6415.87)	(4358.19)	(4031.68)	(4177.89)	(4520.44)	-
11	6521.55	4461.69	4149.47	4309.28	4675.64	-
	(6418.07)	(4396.63)	(4098.91)	(4272.06)	(4640.76)	-
12	6521.29	4497.85	4214.60	4405.99	4790.47	-
	(6424.23)	(4437.09)	(4167.55)	(4367.28)	(4761.93)	-
13	6524.90	4536.14	4281.72	4495.75	-	-
	(6433.44)	(4479.11)	(4237.25)	(4463.29)	-	-

The results shown in Table 4.1 and Table 4.2 conclude that Policy 2(b) always gives the minimum total cost compared to the Policies 1(a), 1(b) and 2(a).

4.3.5.1 Sensitivity analysis

To study the effect of $TC_1(a)$, $TC_1(b)$, $TC_2(a)$ and $TC_2(b)$, we analyze these four policies by varying some parameter values. We perform a numerical sensitivity analysis by varying the value of b, P, a, h_1/h_2 and A_2/A_2 for these four policies. We use the following parameter values as the standard values of the parameter:

$$A_1 = 400, \quad A_2 = 25, \quad H = 5, \quad P = 1000$$

$$h_1 = 4, \quad h_2 = 5, \quad a = 200, \quad b = 20.$$

In this example, the value of b is 20. Note that $\frac{2(aH-P)}{H^2} < b < \frac{A}{h}$ because of $D < P$ and $a - bH > 0$. Therefore, we vary the value of b from 5 to 35 for all policies to see the changes of the TC. The other standard parameters remain the same. These results are illustrated in Figure 4.7.

We found that the larger the value of b, the lower the total cost of all policies. When b is lower, the total cost of Policy 1(a) almost gives the same value to the total cost of Policy 2(a). For example, when $b = 5$, $TC_{1(a)}$ and $TC_{2(a)}$ are 4230.00 and 4229.11 respectively. Policies 1(b) and 2(b) also show the same pattern. However, as b becomes larger, $TC_{1(b)}$ is close to $TC_{2(a)}$ and they intersect at a certain value of b. After that, the value of $TC_{1(b)} > TC_{2(a)}$.

Figure 4.7: Plot of the total cost with different b

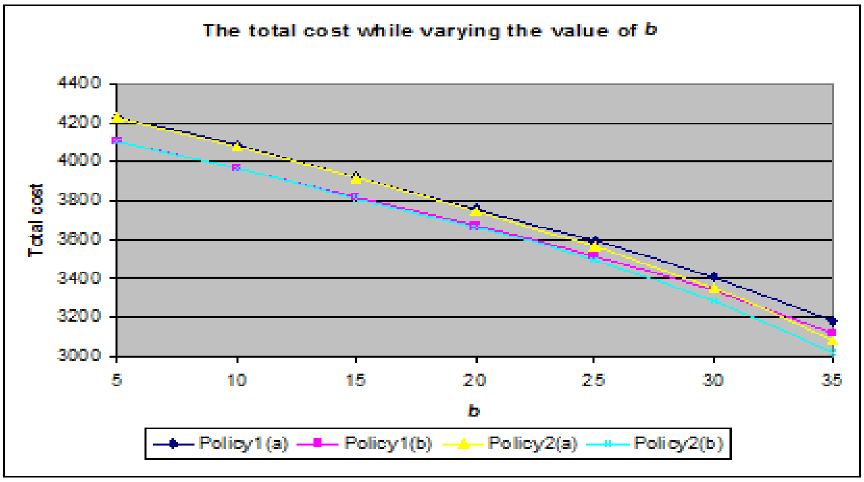
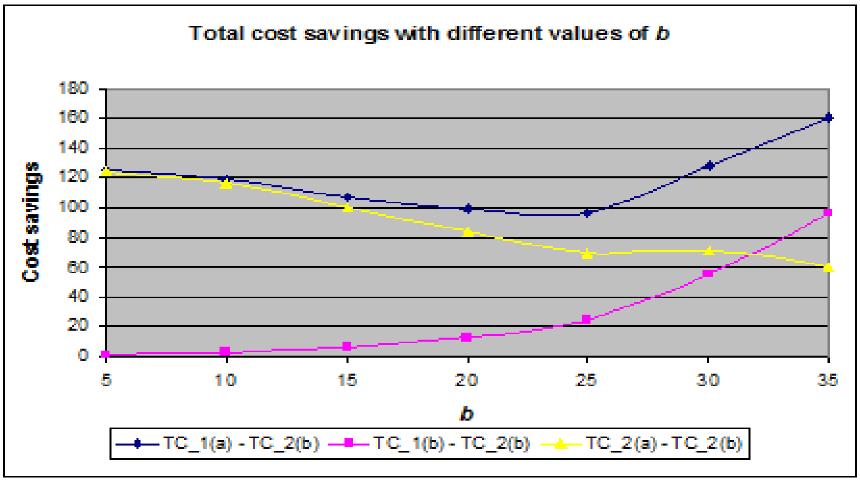


Figure 4.8 illustrates the cost savings obtained by using Policy 2(b) rather than the other three policies (Policies 1(a), 1(b) and 2(a)) for different value of b. The blue, red and yellow lines represent the total savings by evaluating $(TC_{1(a)} - TC_{2(b)})$, $(TC_{1(b)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$ respectively. For the smaller value of b, the red line gives the lowest total cost savings. It shows that the difference between equal and unequal shipments policy is more significant than the difference between equal and unequal cycle times. For example, when $b = 5$, $TC_{1(a)} - TC_{2(b)}$ is 125.61 and $TC_{2(a)} - TC_{2(b)})$ is 124.72 while $TC_{1(b)} - TC_{2(b)}$ is 0.54.

Figure 4.8: Plot of the total cost savings with different b



However, when $b = 35$, $(TC_{2(a)} - TC_{2(b)})$ decreases to 60.810 while $(TC_{1(a)} - TC_{2(b)})$ and $(TC_{1(b)} - TC_{2(b)})$ increase to 160.76 and 97.16 respectively. This result suggests that the larger the value of b, the larger the total cost savings can be obtained by implementing the unequal cycle times policy rather than the equal cycle times.

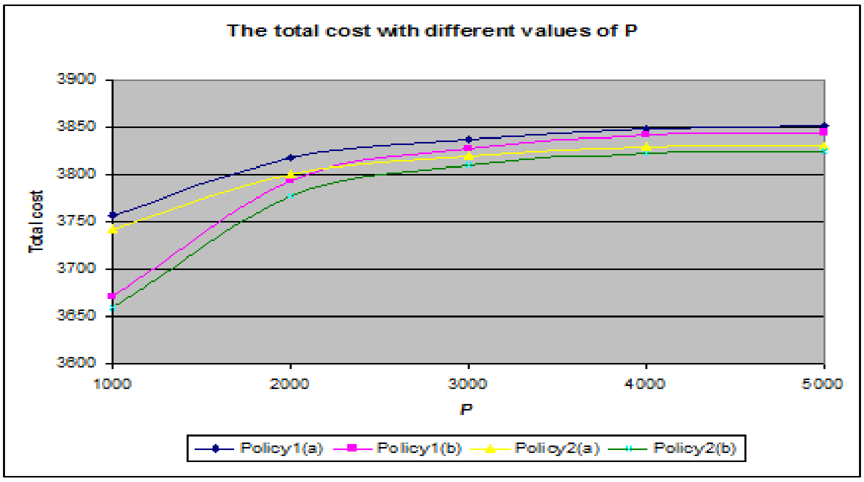
We also found that for Policy 2(b) the larger b, the larger the difference between T_i^*. For example, when $b = 5$ the optimal T_i are 1.2303, 1.2352, 1.2531 and 1.2814 while when $b = 35$, the optimal T_i are 1.254, 1.457 and 2.289.

The curves in Figure 4.8 are not smooth at $b = 25$ because of the changes in the number of batches and the number of shipment. For example, the minimum total cost when $b = 25$ is at $n = 4$ and $m = 3$ for all policies.

However, when $b = 30$, the minimum total cost for Policy 1(a), 1(b) and 2(a) is at $n = 3$ and $m = 4$ while Policy 2(b) is at $n = 3$ and $m = 3$.

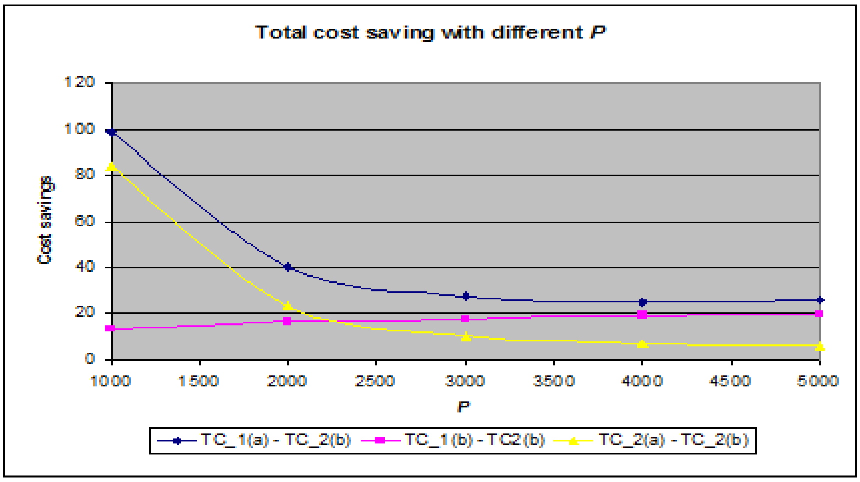
Now we increase the value of $P = 1000$ up to 5000 while the other standard parameter values remain the same. The result is shown in Figure 4.9. As expected, when P increases, the total cost increases. It also shows that the difference between the total cost of Policies 1(a) and 1(b) becomes smaller when P becomes larger. Similarly, Policy 2(a) and 2(b) also shows the same pattern. For example, when $P = 1000$, the difference between the total cost for Policies 2(a) and 2(b) is 84.30 whereas, when $P = 5000$, it is only 5.71.

Figure 4.9: The total cost with different P



We plotted the cost savings which is obtained while varying the value of P in Figure 4.10. We found that as P increases, the total cost savings decrease for $(TC_{1(a)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$. However, when P becomes larger, $(TC_{1(b)} - TC_{2(b)})$ increases and close to the value of $(TC_{1(a)} - TC_{2(b)})$.

Figure 4.10: The total cost saving with different P



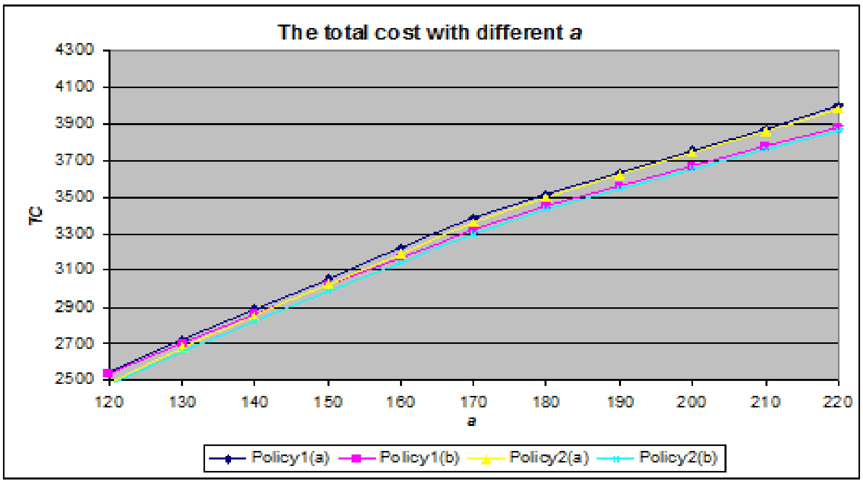
The optimal total cost for Policy 2(a) when $P = 1000$ is at $n = 4$ and $m = 3$, whereas when $P = 5000$ it is at $n = 4$ and $m = 2$. The shipment sizes for each batch is given by Table 4.3. This table suggest that when P is larger, the first shipment for each batch becomes larger compared to smaller P. This is because the production process move on faster, therefore the number of shipments will be less.

Now, let us consider a different value of a where all the other standard parameter values remain the same. Note that the value of a must be greater than bH and less than $\frac{P}{H} + \frac{bH}{2}$ because the demand, $a - bH > 0$ and $aH - \frac{bH^2}{2} \leq P$. The result is given in Figure 4.11.

Table 4.3: Shipment sizes for optimal total cost for Policy 2(b) when $P =$ 1000 and 5000

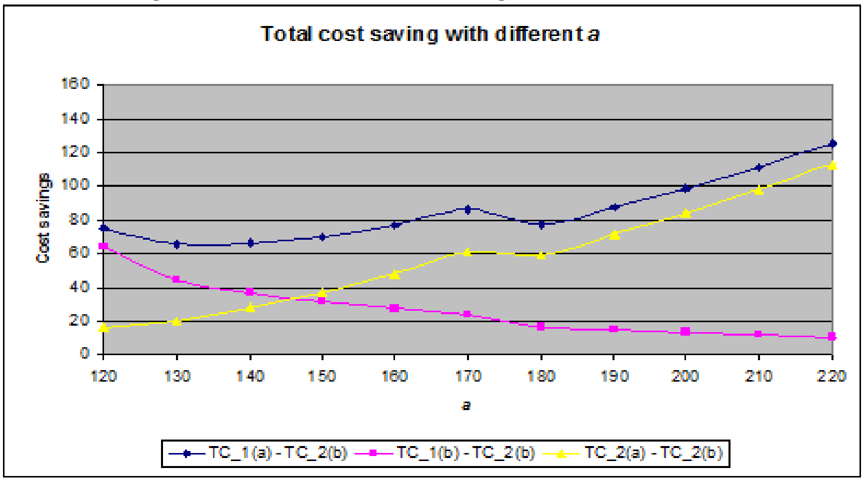
P	n^*	m^*	TC^*	$q^*_{i,j}$	1st batch	2nd batch	3rd batch	4th batch
1000	4	3	3658.70	$q^*_{i,1}$	19.26	15.28	12.63	9.86
				$q^*_{i,2}$	96.75	6.51	82.50	7.22
				$q^*_{i,3}$	96.048	93.932	84.88	75.15
5000	4	2	3825.55	$q^*_{i,1}$	97.241	85.044	81.118	76.329
				$q^*_{i,2}$	111.606	109.890	99.933	88.838

Figure 4.11: The total cost with different a



As depicted in Figure 4.11, the total cost for all policies increases as the value of a increases. The lower total cost is given by Policies 1(b) and 2(b). This result suggests that, with the larger initial demand rate, the different shipment sizes policy is more effective compared to the equal shipment sizes policy. We also found that the larger a, the smaller the difference between the total cost for Policies 1(a) and 2(a). This also happens to the Policies 1(b) and 2(b). For example, when $a = 120$ the difference between the total cost for Policies 1(a) and 2(a) is 59.63 while for Policies 1(b) and 2(b) is 64.02. When $a = 240$, the differences decrease to 9.63 and 8.77 respectively.

Figure 4.12: The cost savings with different a



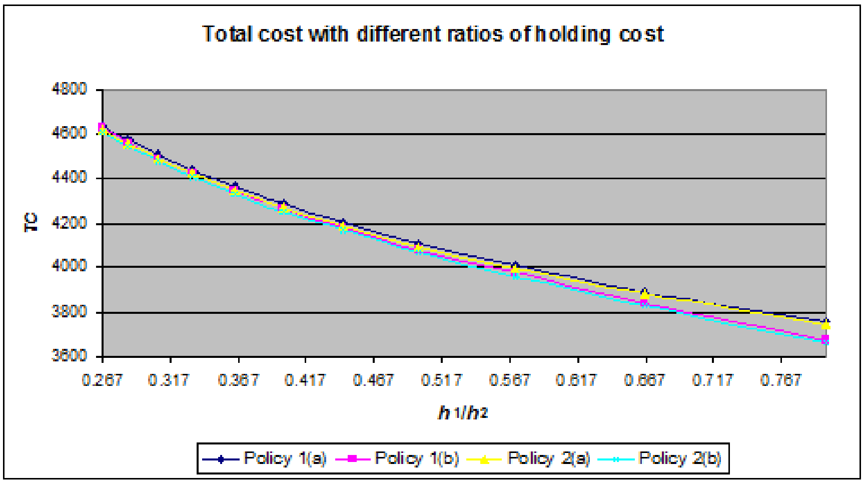
We plotted the total cost savings which is obtained while varying the value of a in Figure 4.12. It suggests that the larger the value of a, the larger cost savings are obtained by implementing Policy 2(b) rather than Policies

1(a) and 2(a). However, the total cost savings decrease for $(TC_{1(b)} - TC_{2(b)})$. It shows that, as a increases the value of the total cost for equal cycle time with unequal shipment sizes policy and unequal cycle times with unequal shipment sizes policy become closer to each other.

The curves in Figure 4.12 are not smooth at a=170 because of the increment in the number of batches. For example, the minimum total cost for all policies when a=170 is at n=3 and m=3. However, when a=180, the minimum total cost for all policies is at $n = 4$ and $m = 3$.

Next, we decrease the ratio of the holding cost, h_1/h_2 by increasing h_2 from 5 up to 15 while $h_1 = 4$. All the other standard parameter values remain the same. The corresponding results are displayed in Figure 4.13.

Figure 4.13: The total cost with different ratios of h_1/h_2



We observe that all policies show the similar pattern where the larger h_1/h_2, the lower the total cost for all policies. However, Policy 2(b) always gives the minimum total cost. For example, when $h_1/h_2 = 0.267$, the total cost for Policies 1(a), 1(b), 2(a) and 2(b) are 4638.76, 4627.54, 4620.53 and 4610.66 respectively. However, when $h_1/h_2 = 0.8$, these total costs decrease to 3757.77, 3671.67, 3742.99 and 3658.70 respectively.

Figure 4.14: Total cost savings with different ratios of h_1/h_2

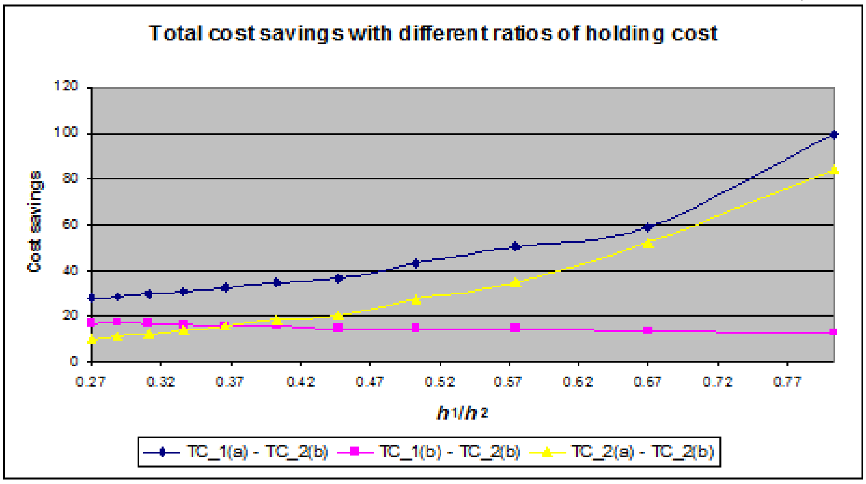
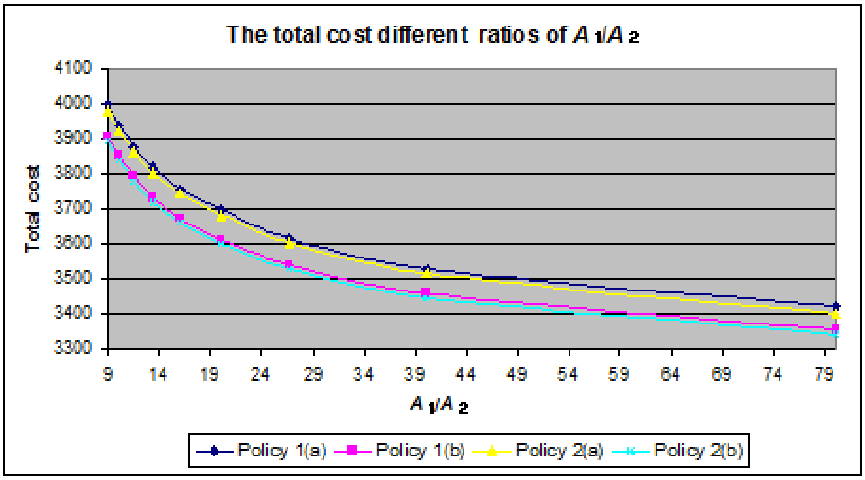


Figure 4.14 gives a diagrammatic plot of the total cost savings obtained by implementing Policy 2(b) rather than the other three policies. We found that the larger the value of h_1/h_2 the larger $(TC_{1(a)} - TC_{2(b)})$ and ($TC_{2(a)}$ - $TC_{2(b)}$) but the lower $(TC_{1(b)} - TC_{2(b)})$. For example, when $h_1/h_2 = 0.267$, $(TC_{1(a)} - TC_{2(b)})$, $(TC_{1(b)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$ are 28.10, 16.88 and 9.87. However, when $h_1/h_2 = 0.8$, ($TC_{1(a)}$ - $TC_{2(b)}$) and $(TC_{2(a)} - TC_{2(b)})$

increase to 99.07 and 84.29 while $(TC_{1(b)} - TC_{2(b)})$ decreases to 12.97.

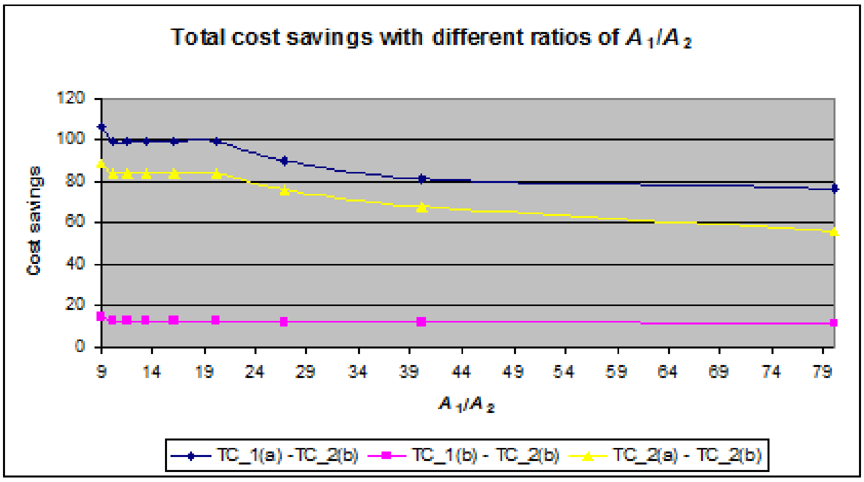
Figure 4.15: The total cost with different ratios of A_1/A_2



Now, we vary the ratios of A_1/A_2 by changing the value of A_2 from 5 to 45 while $A_1 = 400$. Similarly, the other standard parameter values remain the same. The results are given by Figure 4.15. We conclude that the larger the ratios of A_1/A_2, the lower total cost for all policies. The total cost for Policy 1(a) is close to Policy 2(a). Similarly, the total cost for Policy 1(b) is close to the Policy 2(b). As we have concluded previously, Policy 2(b) always gives the best minimum total cost.

Lastly, the cost savings obtained by implementing Policy 2(b) rather than the other three policies is plotted in Figure 4.16. We observed that as A_1/A_2 increases there is a slight decreasing for all $(TC_{1(a)} - TC_{2(b)})$, $(TC_{1(b)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$. However, $(TC_{2(a)} - TC_{2(b)})$ always gives the lowest

Figure 4.16: Total cost savings with different ratios of A_1/A_2



value.

4.4 Case 2 : $h_1 > h_2$

In this section, we will consider the case where the vendor's holding cost is greater than the buyer's. As discussed in Chapter 3 section 3.4, it is optimal for the vendor to keep their inventory level as low as possible by moving all the stock to the buyer's premises. This policy is based on mutual collaboration and integration between vendor and buyer which is defined as a consignment policy.

Figure 4.17: Plot of the inventory level for the consignment policy with $n = 2$, $m = 5$

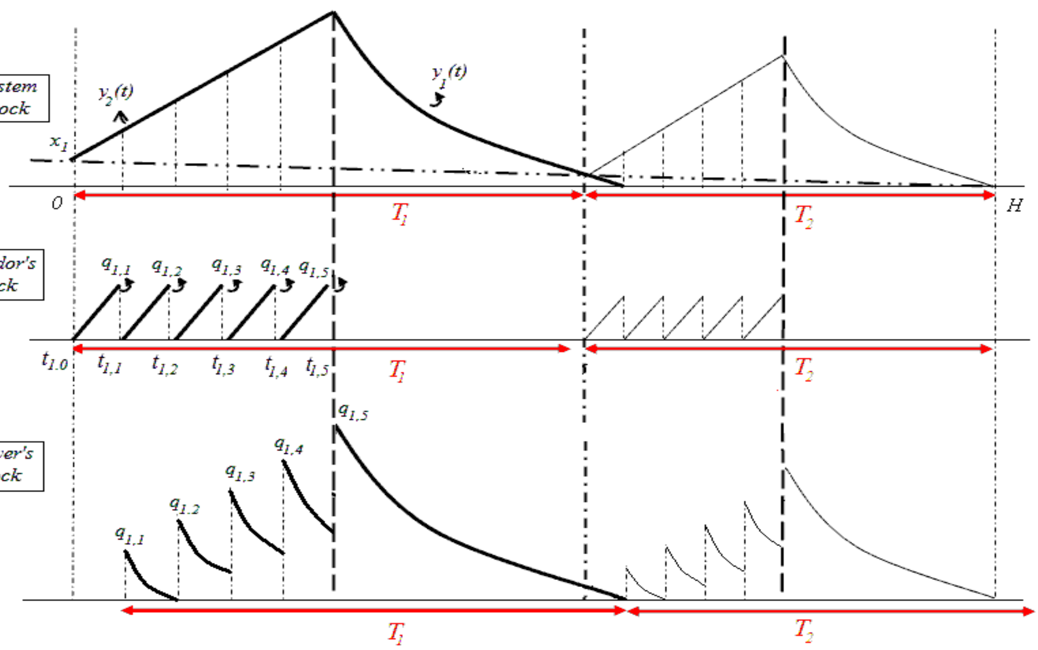


Figure 4.17 shows the illustration of the model with two batches and five shipments which represent the stock level of the system, vendor and buyer. The production batch will starts at $t_{i,0}$ until the production uptime, $t_{i,p}$. The first shipments is at time $t_{1,1}$ and follows at time $t_{i,2}, t_{i,3}, \ldots, t_{i,m}$ with the shipment sizes $q_1, q_2, \ldots, q_n$. $y_1(t)$, $y_2(t)$ and D is similar as equation (4.5), (4.6) and (4.7) in Section 4.3.

4.5 Total time-weighted system stock

The total system stock represented by the area under the curves $y_1(t)$ and $y_2(t)$ in $(t_{i,p}, H)$ and $(0, t_{i,p})$ respectively. The total time weighted system stock for this case is similar to the Case 1 which is given by equation (4.8) where $t_{i,p}$ is similar with equation (4.9).

4.6 Total time-weighted vendor stock

The total time-weighted vendor stock, TVS is the total area under the triangles. In Figure 4.17, the triangles are represent by $t_{i,0}q_{1,1}t_{1,1}$, $t_{1,1}q_2t_{1,2}$, $t_{1,2}q_{1,3}t_{1,3}$, $t_{1,3}q_{1,4}t_{1,4}$ and $t_{1,4}q_{1,5}t_{1,5}$. Hence, we have

$$TVS = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} q_{i,j}(t_{i,j} - t_{i,j-1}). \tag{4.46}$$

It follows that the total cost for this model, $TC(n)$, is given by

$$TC = n(A_1) + nm(A_2) + h_2 TSS + (h_1 - h_2)TVS \tag{4.47}$$

Structurally the cost function is identical to the equation (4.13) in the previous model (Case $h_1 < h_2$). The constants h_1 and h_2 are interchanged and the last term on the right hand side is now multiplied by the total time-weighted vendor stock, TVS instead of TBS.

For example, let the demand rate is linearly decreasing over the period $(0, H)$ that is

$$f(t) = a - bt \qquad a > 0; b > 0; t > 0; H > 0. \tag{4.48}$$

Substituting (4.48) into (4.5), (4.6), (4.7), (4.8), (4.9) and (4.12) we have

$$y_1 = a(t_{i+1,0} - t) - \frac{b}{2}(t_{i+1,0}^2 - t^2), \tag{4.49}$$

$$y_2 = (P - a)(t - t_{i,0}) + \frac{b}{2}(t^2 - t_{i,0}^2), \tag{4.50}$$

$$D = H\left(a - \frac{b}{2}H\right), \tag{4.51}$$

$$
\begin{aligned}
TSS &= \sum_{i=1}^{n}\left\{\frac{b}{6}(t_{i,p}^3 - t_{i,0}^3) + \frac{(P-a)}{2}(t_{i,p}^2 - t_{i,0}^2)\right. \\
&\qquad \left. - [(P-a)t_{i,0} + \frac{b}{2}t_{i,0}^2](t_{i,p} - t_{i,0})\right\} \\
&+ \sum_{i=1}^{n}\left\{\frac{b}{6}(t_{i+1,0}^3 - t_{i,p}^3) - \frac{a}{2}(t_{i+1,0}^2 - t_{i,p}^2)\right. \\
&\qquad \left. + (at_{i+1,0} - \frac{b}{2}t_{i+1,0}^2)(t_{i+1,0} - t_{i,p})\right\} \\
&+ \sum_{i=1}^{n}(x_i + x_{i+1})\frac{T_i}{2}. \qquad (4.52)
\end{aligned}
$$

Finally, substituting (4.52) and (4.46) into (4.47) we have,

$$
\begin{aligned}
TC &= n(A_1) + nm(A_2) + h_2\sum_{i=1}^{n}\left\{\frac{b}{6}(t_{i,p}^3 - t_{i,0}^3) + \frac{(P-a)}{2}(t_{i,p}^2 - t_{i,0}^2)\right. \\
&- \left.[(P-a)t_{i,0} + \frac{b}{2}t_{i,0}^2](t_{i,p} - t_{i,0})\right\} + \sum_{i=1}^{n}\left\{\frac{b}{6}(t_{i+1,0}^3 - t_{i,p}^3)\right. \\
&- \left.\frac{a}{2}(t_{i+1,0}^2 - t_{i,p}^2) + (at_{i+1,0} - \frac{b}{2}t_{i+1,0}^2)(t_{i+1,0} - t_{i,p})\right\} \\
&+ \sum_{i=1}^{n}(x_i + x_{i+1})\frac{T_i}{2} + (h_1 - h_2)\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} q_{i,j}(t_{i,j} - t_{i,j-1})\right\}. (4.53)
\end{aligned}
$$

Denote TC as $TC(n, m, \vec{t})$ where n and m are discrete variables and $\vec{t} = t_{i,2}, t_{i,3}, \ldots, t_{i,m}$ is a real variables.

As in the previous section, we will explore the best solution of the above total cost based on two policies with 2 cases of the shipment sizes:

1. Policy 1 : Equal cycle time
   (a) Equal shipment sizes
   (b) Unequal shipment sizes and unequal shipment periods
2. Policy 2 : Unequal cycle time

(a) Equal shipment sizes

(b) Unequal shipment sizes and unequal shipment periods

The objective in every policies is to find an optimal $t_{i,j}$ and $q_{i,j}$ for a given n and m which gives the minimum total cost, TC.

4.6.1 Policy 1 : Equal cycle times

In this policy, the value of T_i is assumed to be equal. Equation (4.21) and Equation (4.22) in Section 4.3.3 are remain. We will explore this policy with equal shipment sizes and unequal shipment sizes and unequal shipment periods.

4.6.1.1 Policy 1 (a) : Equal shipment sizes

In this policy the shipment sizes and shipment periods are equal for every batch. So, we have fixed value of $q_{i,j}$ which is similar to the equation (4.23) in Section 4.3.3.1.

The illustration of this policy is given by Figure 4.17. The shipment times can be expressed as

$$t_{i,j} = \frac{q_{i,j}}{P} + t_{i,j-1}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m \tag{4.54}$$

Substituting (4.54) into (4.53) will give the total cost for this policy.

4.6.1.2 Solution procedure

The computer algorithm of the solution procedure is outline below :

1. Let $n = 1$
2. Let $m = 1$
3. Set $T_i = H/n, \quad i = 1, 2, \ldots, n$
4. Set $t_{i,1} = 0$, $t_{n,m+1} = H$
5. Set $q_{i,j} = D_i/n, \quad (i = 1, 2, \cdots, n$ and $j = 1, 2, \ldots m)$
6. Compute $t_{i,j}, \quad (i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m)$ using (4.54) and $TC_{1(a)}(n, m, \vec{t}\,)$ using (4.53)
7. Set $TC_{1(a)}(n, m, \vec{t}\,)$ as $TC_{1(a)}(n, m^*, \vec{t}\,)$. Increase m by 1 and repeat step 3 to 6. Stop when $TC_{1(a)}(n, m, \vec{t}\,) \geq TC_{1(a)}(n, m^*, \vec{t}\,)$
8. Increase n by 1 and repeat step 2 to 7. Set $TC_{1(a)}(n, m^*, \ \vec{t}\,)$ as $TC_{1(a)}(n^*, m^*, \ \vec{t}\,)$.Stop when $TC_{1(a)}(n, m^*, \vec{t}\,) \geq TC_{1(a)}(n^*, m^*, \vec{t}\,)$

The basic idea of the above algorithm is to start with $n = 1$ and $m = 1$. Next, we increase m to improve the total system cost until the first $m = m^*$ that satisfies the conditions $TC_{1(a)}(n, m^*, \vec{t}\,) < TC_{1(a)}(n, m^* - 1, \vec{t}\,)$ and $TC_{1(a)}(n, m^*, \vec{t}\,) < TC_{1(a)}(n, m^* + 1, \vec{t}\,)$. Then we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_{1(a)}(n^*, m^*, \ \vec{t}\,) < TC_{1(a)}(n^* - 1, m^* - 1, \ \vec{t}\,)$ and $TC_{1(a)}(n^*, m^*, \ \vec{t}\,) < TC_{1(a)}(n^* + 1, m^* + 1, \ \vec{t}\,)$.

4.6.1.3 Policy 1(b) : Unequal shipment sizes and unequal shipment periods

In this policy the shipment sizes are assumed to be unequal. We set $q_{i,j}$ as the changing variable. The constraint optimization problem is as follow :

$$
\begin{aligned}
\text{Minimum} \qquad & TC_{1(b)}(n,m,\vec{t}\,) && (4.55) \\
\text{Subject to} \qquad & && \\
\frac{a}{b}\left\{1-\sqrt{1-\left(\frac{2b}{a^2}\right)\left[at_{i,j-1}-\frac{b}{2}t_{i,j-1}^2+q_{i,j}\right]}\right\}-t_{i,0} \;\geq\; & \sum_{1}^{j}\frac{q_{i,j+1}}{P} && (4.56) \\
\sum_{j=1}^{m} q_{i,j} \;=\; & D_i && (4.57)
\end{aligned}
$$

Our objective is to minimize the total system cost, that is equation (4.55) subject to the constraints (4.56) and (4.60).

4.6.1.4 Solution procedure

The algorithm is given as follow :

1. Let $n = 1$
2. Let $m = 1$
3. Set $T_i = H/n, \quad i = 1, 2, \ldots, n$
4. Set $t_{i,0} = 0$, $t_{i,m+1} = H$

5. Determine $q_{i,j}$ $i = 2, 3, \cdots, n$ which satisfied the constraints (4.56) and (4.60)
6. Compute $t_{i,j}$, $(i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m)$ using (4.54) and $TC_{1(b)}(n, m, \vec{t}\ )$ using (4.53)
7. Set $TC_{1(b)}(n, m, \vec{t}\ )$ as $TC_{1(b)}(n, m^*, \vec{t}\ )$. Increase m by 1 and repeat step 5 to 6. Stop when $TC_{1(b)}(n, m, \vec{t}\ ) \geq TC_{1(b)}(n, m^*, \vec{t}\ )$
8. Increase n by 1 and repeat step 5 to 7. Set $TC_{1(b)}(n, m^*, \ \vec{t}\ )$ as $TC_{1(b)}(n^*, m^*, \ \vec{t}\ )$. Stop when $TC_{1(b)}(n, m^*, \vec{t}\ ) \geq TC_{1(b)}(n^*, m^*, \vec{t}\ )$

Similarly, the basic idea of the above algorithm is to start with $n = 1$ and $m = 1$. Next, we increase m to improve the total system cost until the first $m = m^*$ that satisfies the conditions $TC_{1(b)}(n, m^*, \vec{t}\ ) < TC_{1(b)}(n, m^* - 1, \vec{t}\ )$ and $TC_{1(b)}(n, m^*, \vec{t}\ ) < TC_{1(b)}(n, m^* + 1, \vec{t}\ )$. Then we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_{1(b)}(n^*, m^*, \ \vec{t}\ ) < TC_{1(b)}(n^* - 1, m^* - 1, \ \vec{t}\ )$ and $TC_{1(b)}(n^*, m^*, \ \vec{t}\ ) < TC_{1(b)}(n^* + 1, m^* + 1, \ \vec{t}\ )$.

4.6.2 Policy 2 : Unequal cycle time

In this policy, the value of T_i is unknown. We used equation (4.36) in Section 3.3.4 and find the optimal value of T_i which gives the minimum total cost. We will explore this policy with equal shipment sizes and unequal shipment sizes and unequal shipment periods.

4.6.2.1 Policy 2 (a) : Equal shipment sizes

In this policy, the shipment sizes are assumed to be equal for every batch. Similarly, as in Section 4.6.1.1, the fixed value of $q_{i,j}$ is similar to the Equation (4.23) in Section 4.3.3.1. The shipment times, $t_{i,j}$ is similar to the Equation (4.54) in Section 4.6.1.1. Substituting (4.54) into (4.53) will give the total cost for this policy.

The constraint optimization problem is as follow :

$$\text{Minimum} \qquad TC_{2(a)}(n, m, \vec{t}\,) \tag{4.58}$$

Subject to

$$T_i = \sum_{i=1}^{n} H \tag{4.59}$$

$$\sum_{j=1}^{m} q_{i,j} = D_i \tag{4.60}$$

Our objective is to minimize the total system cost, that is equation (4.58) subject to the constraints (4.59) and (4.60).

4.6.2.2 Solution procedure

The computer algorithm of the solution procedure is outline below :

1. Let $n = 1$
2. Let $m = 1$
3. Set $t_{i,0} = 0$, $t_{n,m+1} = H$
4. Set $q_{i,j} = D_i/n$, $(i = 1, 2, \cdots, n$ and $j = 1, 2, \ldots m)$
5. Determine T_i, $i = 1, 2, \ldots, n$ which satisfied constraint (4.59) and (4.60), if it exists.
6. Compute $t_{i,j}$, $(i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m)$ using (4.54) and $TC_{2(a)}(n, m, \vec{t}\,)$ using (4.53)
7. Set $TC_{2(a)}(n, m, \vec{t}\,)$ as $TC_{2(a)}(n, m^*, \vec{t}\,)$. Increase m by 1 and repeat step 5 to 6. Stop when $TC_{2(a)}(n, m, \vec{t}\,) \geq TC_{2(a)}(n, m^*, \vec{t}\,)$
8. Increase n by 1 and repeat step 5 to 7. Set $TC_{2(a)}(n, m^*, \ \vec{t}\,)$ as $TC_{2(a)}(n^*, m^*, \ \vec{t}\,)$. Stop when $TC_{2(a)}(n, m^*, \vec{t}\,) \geq TC_{2(a)}(n^*, m^*, \vec{t}\,)$

The basic idea of the above algorithm is to start with $n = 1$ and $m = 1$. Next, we increase m to improve the total system cost until the first $m = m^*$ that satisfies the conditions $TC_{2(a)}(n, m^*, \vec{t}\,) < TC_{2(a)}(n, m^* - 1, \vec{t}\,)$ and $TC_{2(a)}(n, m^*, \vec{t}\,) < TC_{2(a)}(n, m^* + 1, \vec{t}\,)$. Then we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_{2(a)}(n^*, m^*, \ \vec{t}\,) < TC_{2(a)}(n^* - 1, m^* - 1, \ \vec{t}\,)$ and $TC_{2(a)}(n^*, m^*, \ \vec{t}\,) < TC_{2(a)}(n^* + 1, m^* + 1, \ \vec{t}\,)$.

4.6.2.3 Policy 2(b) : Unequal shipment sizes and unequal shipment periods

Finally, we consider the case where the cycle times, T_i, the shipment sizes and shipment periods are unequal. The constraint optimization problem is as follow :

$$\text{Minimum :} \qquad TC_{2(b)}(n, m, \vec{t}\,) \tag{4.61}$$

Subject to

$$
\begin{aligned}
T_i &= \sum_{i=1}^{n} H & (4.62)\\
\frac{a}{b}\left\{1-\sqrt{1-\left(\frac{2b}{a^2}\right)\left[at_{i,j-1}-\frac{b}{2}t_{i,j-1}^2+q_{i,j}\right]}\right\}-t_{i,0} &\geq \sum_{1}^{j}\frac{q_{i,j+1}}{P} & (4.63)\\
\sum_{j=1}^{m} q_{i,j} &= D_i & (4.64)
\end{aligned}
$$

Our objective is to minimize the total system cost, that is equation (4.61) subject to the constraints (4.62), (4.63) and (4.64).

4.6.2.4 Solution procedure

The algorithm is given as follow :

1. Let $n = 1$
2. Let $m = 1$
3. Set $t_{i,0} = 0$, $t_{i,m+1} = H$

4. Determine T_i and $q_{i,j}$ $i = 2, 3, \cdots, n$ and $j = 1, 2, \cdots, m$ which satisfied the constraints (4.62), (4.63) and (4.64) if they exist

5. Compute $t_{i,j}$, $(i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m)$ using (4.54) and $TC_{2(b)}(n, m, \vec{t}\,)$ using (4.53)

6. Set $TC_{2(b)}(n, m, \vec{t}\,)$ as $TC_{2(b)}(n, m^*, \vec{t}\,)$. Increase m by 1 and repeat step 5 to 6. Stop when $TC_{2(b)}(n, m, \vec{t}\,) \geq TC_{2(b)}(n, m^*, \vec{t}\,)$

7. Increase n by 1 and repeat step 5 to 7. Set $TC_{2(b)}(n, m^*, \ \vec{t}\,)$ as $TC_{2(b)}(n^*, m^*, \ \vec{t}\,)$. Stop when $TC_{2(b)}(n, m^*, \vec{t}\,) \geq TC_{2(b)}(n^*, m^*, \vec{t}\,)$

Similarly, the basic idea of the above algorithm is to start with $n = 1$ and $m = 1$. Next, we increase m to improve the total system cost until the first $m = m^*$ that satisfies the conditions $TC_{2(b)}(n, m^*, \vec{t}\,) < TC_{2(b)}(n, m^* - 1, \vec{t}\,)$ and $TC_{2(b)}(n, m^*, \vec{t}\,) < TC_{2(b)}(n, m^* + 1, \vec{t}\,)$. Then we increase n to improve the total system cost until the first $n = n^*$ that satisfies the conditions $TC_{2(b)}(n^*, m^*, \ \vec{t}\,) < TC_{2(b)}(n^* - 1, m^* - 1, \ \vec{t}\,)$ and $TC_{2(b)}(n^*, m^*, \ \vec{t}\,) < TC_{2(b)}(n^* + 1, m^* + 1, \ \vec{t}\,)$.

4.6.3 Numerical examples and sensitivity analysis

To show the effectiveness of the proposed policies, we adopt the same numerical examples as in case $h_1 < h_2$ except for the value of h_1. For easy reference, the parameter values are restated here :

$$a{=}200,\quad b{=}20,\quad H{=}5,\quad h_2 = 5,\quad P{=}1000,\quad A_1{=}400,\quad A_2{=}25,\quad D{=}750.$$

Note that in this example, $h_1 = 6$ is selected, as in the consignment policy, h_1 must be greater than h_2.

Table 4.4 gives the total cost for Policy 1(a) and Policy 1(b) for some combinations of n and m where $n = 1, 2, \cdots, 6$ and $m = 1, 2, \cdots, 11$. Similarly, the value of the total cost for Policy 1(b) is given in the parenthesis. The underlined values represent the minimum total cost for a given m while increasing the value of n, and the double underlined values represent the minimum total cost for n while increasing the value of m . Overall, it shows that when n and m increase, the total cost decreases until it reached the minimum value.

For example, when $n = 1$, the minimum total cost reached at $m = 10$ for both Policies 1(a) and 1(b). If we consider $m = 1$ while increasing the value of n, the minimum total cost is at $n = 5$ for both Policies 1(a) and 1(b).

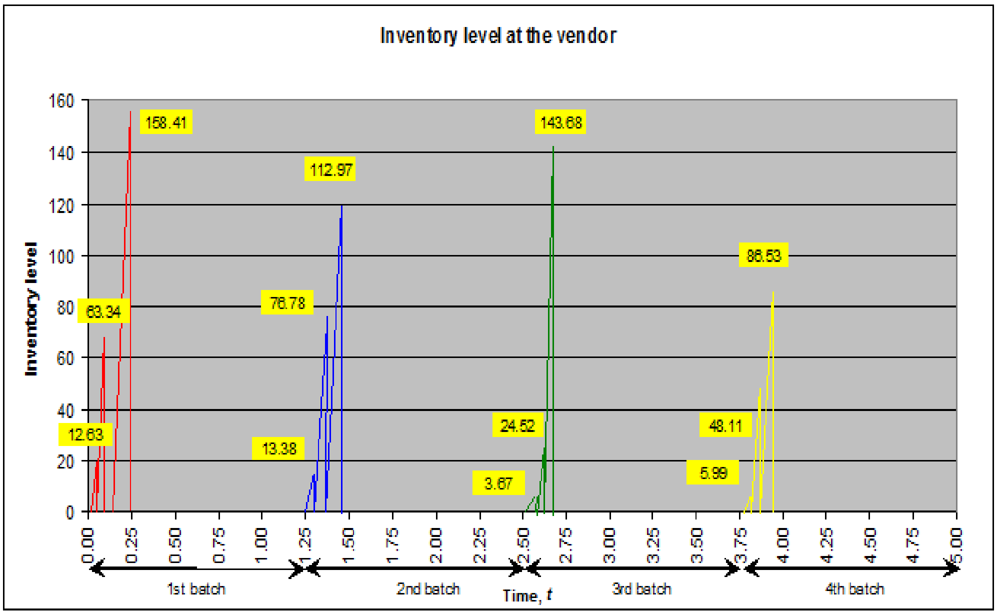
As expected, the total cost for Policy 1(b) is always lower than Policy 1(a). This result suggests that unequal shipment sizes policy performs very well when compared to the equal shipment sizes policy. The optimal total cost for Policy 1(a) reached at $n = 4$ and $m = 3$ and Policy 1(b) reach its minimum at $n = 4$ and $m = 3$. It shows that Policy 1(b) gives a better solution which is 144.95 less than Policy 1(a) where the optimal total cost,

Table 4.4: The minimum total costs for Policies 1(a) and (b)

Total relevant cost						
$m \downarrow \quad \rightarrow n$	**1**	**2**	**3**	**4**	**5**	**6**
1	9508.33	5819.56	4706.46	4321.62	4253.51	4358.84
	(9508.33)	(5819.56)	(4706.46)	(4321.62)	(4253.51)	(4331.45)
2	8455.21	5240.05	4523.68	4069.66	4086.32	4236.85
	(7886.14)	(4876.54)	(4256.93)	(**3859.45**)	(3910.17)	(4089.48)
3	8120.83	5078.85	4254.06	**4004.40**	4113.47	4304.98
	(7650.55)	(4782.39)	(4037.11)	(3873.75)	(3970.50)	(4185.37)
4	7966.15	5023.00	4253.24	4092.62	4189.46	-
	(7576.23)	(4780.67)	(4076.59)	(3953.83)	(4073.46)	-
5	7883.33	5009.41	4282.69	4157.13	4285.02	-
	(7557.53)	(4808.09)	(4136.23)	(4042.15)	(4189.03)	-
6	7836.46	4968.32	4327.30	4233.45	4390.39	-
	(7558.01)	(4844.14)	(4202.72)	(4135.94)	(4308.82)	-
7	7810.12	5036.66	4380.58	4310.94	4501.36	-
	(7567.29)	(4887.58)	(4272.32)	(4231.42)	(4430.53)	-
8	7796.61	5063.90	4439.29	4398.95	4615.83	-
	(7581.35)	(4932.05)	(4343.59)	(4328.68)	(4551.81)	-
9	7791.67	5096.20	4501.61	4489.61	4732.65	-
	(7598.30)	(4978.00)	(4415.86)	(4426.67)	(4676.60)	-
10	7789.90	5132.04	4566.47	4582.15	-	-
	(7617.19)	(5024.90)	(4488.80)	(4525.14)	-	-
11	7793.46	5170.45	4614.59	4676.04	-	-
	(7637.50)	(5072.49)	(4563.80)	(4623.93)	-	-

$TC^*_{Policy\ 1(b)}$ is 3859.45.

Figure 4.18: Plot of the inventory level for the consignment policy at the vendor

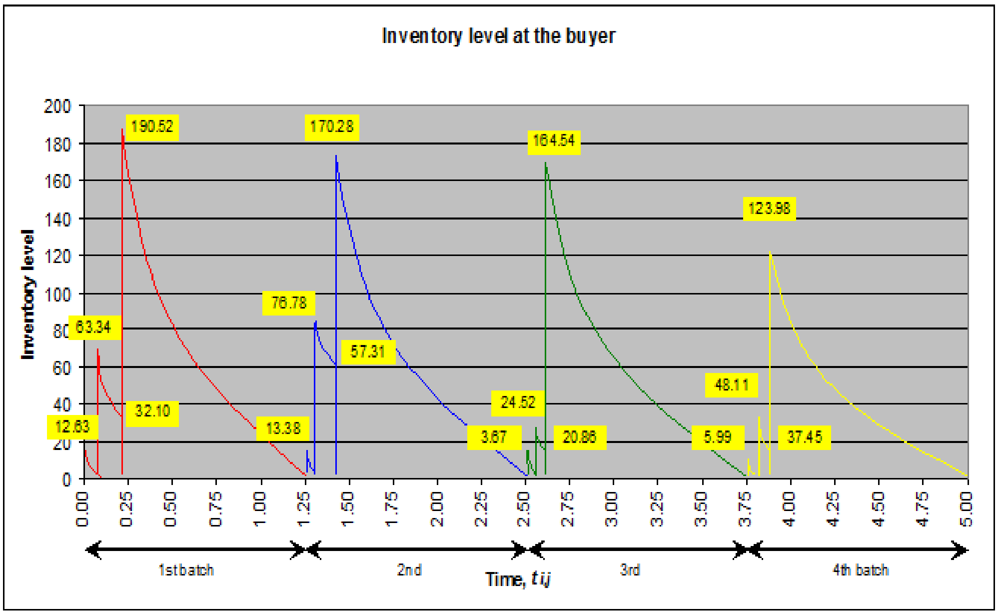


We plotted the inventory level at the vendor for $n = 4$ and $m = 3$ in Figure 4.18. For example, at the first batch, the vendor produces and delivers the first shipment, $q_{1,1}$=12.63 at $t_{1,1}$= 0.013. They continue producing and delivering 63.34 units as the second shipment at $t_{1,2}$=0.076. The last shipment is at $t_{1,3} = t_{1,p} = 0.234$. There will be no production after this time until they start their production for the next batch at $t_{2,0} = T_1 = 1.250$.

We also plotted the inventory level at the buyer as in Figure 4.19. For example, in the first batch, the buyer finished up the first shipment $q_{1,1}$=12.63 until the vendor replenish the buyer's inventory in the second shipment,

$q_{1,2}$=63.34. The buyer used up 31.23 between $t_{1,1}$ and $t_{1,2}$, therefore they still have 32.10 at $t_{1,2}$. At $t_{1,2}$ the vendor deliver 158.41, so that the total inventory at the buyer at $t_{1,2}$ is 190.52. The buyer finished up this stock until the second batch at $t_{2,0} = 1.250$. This process is repeated until the end of the production cycle, H.

Figure 4.19: Plot of the inventory level for the consignment policy at the buyer



Next, Table 4.5 gives the result for Policies 2(a) and 2(b). It shows the same pattern as in Policies 1(a) and 1(b). The minimum total cost for Policy 2(a) is at $n = 4$ and $m = 3$, while Policy 2(b) reached its minimum at $n = 4$ and $m = 2$. Similarly, Policy 2(b) gives the best minimum solution compared to Policy 2(a). $TC^*_{2(b)}$ is 3843.00 which is 145.820 less than $TC^*_{2(a)}$.

Table 4.5: The minimum total costs for Policies 2(a) and (b)

Total relevant cost						
$m \downarrow \quad \rightarrow n$	**1**	**2**	**3**	**4**	**5**	**6**
1	9508.33	5786.61	4673.79	4293.73	4233.54	4312.84
	(9508.33)	(5787.20)	(4673.79)	(4293.73)	(4233.55)	(4310.90)
2	8455.21	5212.90	4518.22	4049.88	4071.39	4222.67
	(7886.14)	(4850.36)	(4254.38)	(**3843.00**)	(3897.02)	(4078.09)
3	8120.83	5053.32	4232.58	**3988.82**	4100.06	4292.70
	(7650.55)	(4757.64)	(4018.20)	(3859.14)	(3958.68)	(4175.41)
4	7966.15	4998.22	4232.92	4076.34	4176.77	-
	(7576.23)	(4756.54)	(4058.25)	(3939.65)	(4062.04)	-
5	7883.33	4985.06	4263.04	4141.49	4272.79	-
	(7557.53)	(4784.33)	(4118.21)	(4028.23)	(4177.81)	-
6	7836.46	4931.84	4308.08	4218.23	4378.40	-
	(7558.01)	(4820.13)	(4184.88)	(4122.22)	(4297.73)	-
7	7810.12	5012.79	4361.74	4297.35	4489.60	-
	(7567.29)	(4864.18)	(4254.60)	(4217.76)	(4419.52)	-
8	7796.61	5040.18	4420.60	4385.40	4604.19	-
	(7581.35)	(4908.76)	(4325.95)	(4315.09)	(4540.64)	-
9	7791.67	5072.60	4483.14	4476.10	4721.11	
	(7598.30)	(4954.78)	(4398.28)	(4413.13)	(4665.68)	-
10	7789.90	5108.52	4548.12	4568.65	-	-
	(7617.19)	(5001.74)	(4471.27)	(4511.63)	-	-
11	7793.46	5147.00	4591.85	4662.56	-	-
	(7637.50)	(5049.38)	(4546.27)	(4610.47)	-	-

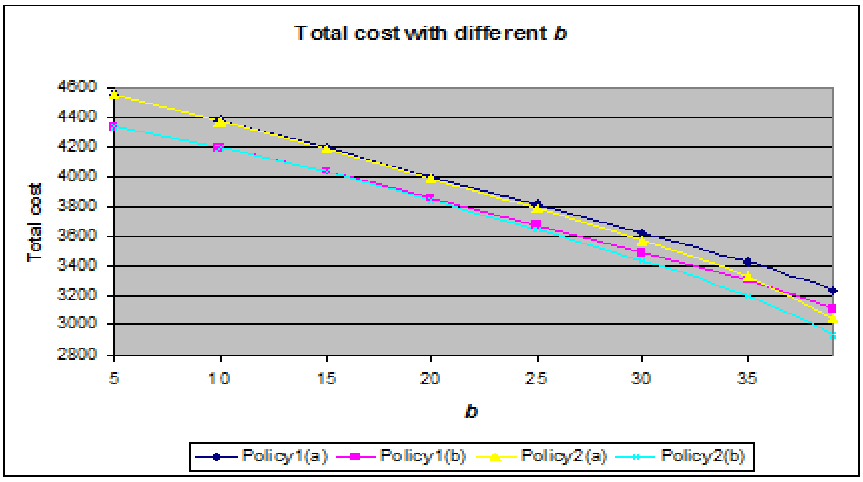
4.6.3.1 Sensitivity analysis

As in Case 1 ($h_1 < h_2$), to study the effect of $TC_{1(a)}$, $TC_{1(b)}$, $TC_{2(a)}$ and $TC_{2(b)}$, we analyze these four policies by varying some parameter values. We perform a numerical sensitivity analysis by varying the value of b, P, a, h_1/h_2 and A_1/A_2 for these policies. We use the following parameter values as the standard values of the parameter.

$$a{=}200, \quad b{=}20, \quad H{=}5, \quad h_2 = 5, \quad P{=}1000, \quad A_1{=}400, \quad A_2{=}25, \quad D{=}750.$$

In this example, the value of b is 20. As in the case of $h_1 < h_2$, $\frac{2(aH-P)}{H^2} < b < \frac{A}{h}$ because of $D < P$ and $a - bH > 0$. Therefore we vary the value of b from 5 to 39 for all policies to see the changes of the TC. The other standard parameter values remain the same. These results are illustrated in Figure 4.20.

Figure 4.20: The total cost with different b



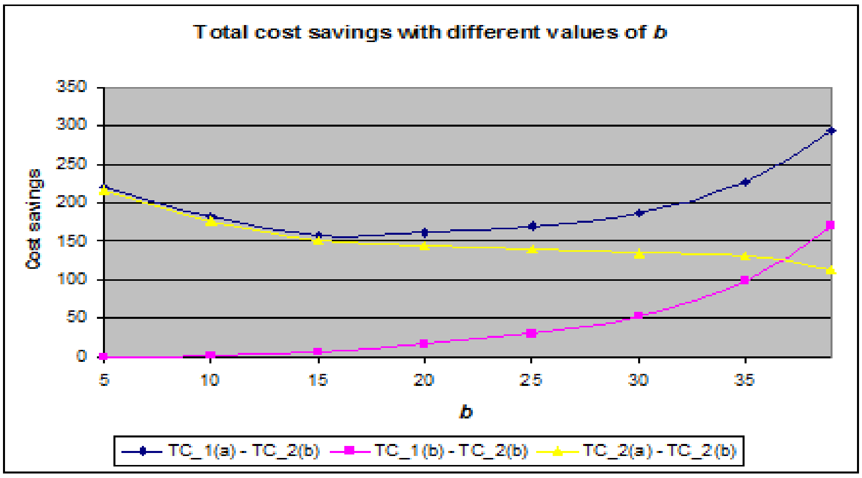
As expected, $TC_{2(b)}$ is always minimum compared to the other policies. In this example, when b increases, the value of D decreases. Hence, with the same value of P, the number of batches will be decreased with optimal number of shipments. For example, for Policy 2(b), when $b = 5$, the demand, $D = 937.50$, the minimum total cost is at $n = 5$ and $m = 2$, whereas when $b = 39$, the demand decrease to 512.50, and the minimum total cost is at $n = 3$ and $m = 3$.

We observed that when b is lower, the total cost of Policy 1(a) gives almost the same value as the total cost of Policy 2(a). For example, when $b = 5$, $TC_{1(a)}$ and $TC_{2(a)}$ are 4557.69 and 4554.48, respectively. Policies 1(b) and 2(b) also show the same pattern. For example, when $b = 5$ is $TC_{1(b)}$ and $TC_{2(b)}$ are 4337.51 and 4337.50, respectively.

However, as b becomes larger, the value of $TC_{1(b)}$ is close to the value of $TC_{2(a)}$ and intersect at the same value of b and TC. After that $TC_{1(a)} > TC_{2(a)}$. Moreover, when b becomes larger, the increasing order of the total cost, $TC_{2(b)} < TC_{2(a)} < TC_{1(b)} < TC_{1(a)}$. It suggests that when b becomes larger, the unequal shipment periods policy is always better than the equal shipment periods policy.

Figure 4.21 illustrates the cost savings obtained by using Policy 2(b) rather than the other three policies (Policy 1(a), 1(b) and 2(a)) for different values of b. The blue, red and yellow lines represent the total savings by evaluating $(TC_{1(a)} - TC_{2(b)})$, $(TC_{1(b)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$, respectively. When b is smaller, the red line gives the lowest total cost saving. For example, when $b = 5$, $(TC_{1(a)} - TC_{2(b)})$ is 125.61, $(TC_{2(a)} - TC_{2(b)})$ is 124.72 and $(TC_{1(b)} - TC_{2(b)})$ is 0.54. However, when $b = 39$, $(TC_{2(a)} - TC_{2(b)})$ decreases to 113.194 while $(TC_{1(a)} - TC_{2(b)})$ and $(TC_{1(b)} - TC_{2(b)})$ increase to 293.019 and 170.553, respectively. This result suggests that the larger the value of b, the larger the total cost savings can be obtained by implementing the unequal cycle times policy rather than the equal cycle times.

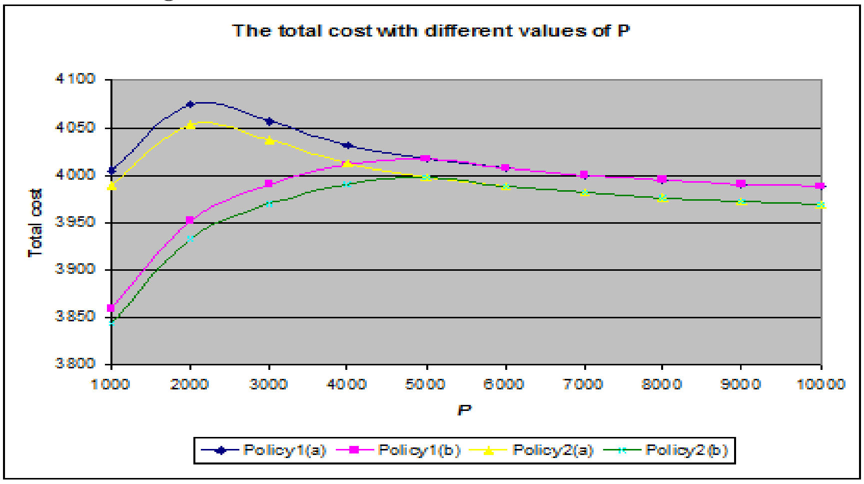
Figure 4.21: Plot of the total cost savings while varying b



Now, we increase the value of $P = 1000$ up to 10000 while the other standard parameter values remain the same. The result is shown in Figure 4.22. We observed that the total costs of Policies 1(a) and 2(a) increase when P increases form 1000 up to 2000. However, it decreases when P increases from

3000 up to 10000. While the total costs for Policies 1(b) and 2(b) increase when P increases from 1000 until 5000 and after that they decrease until P=10000.

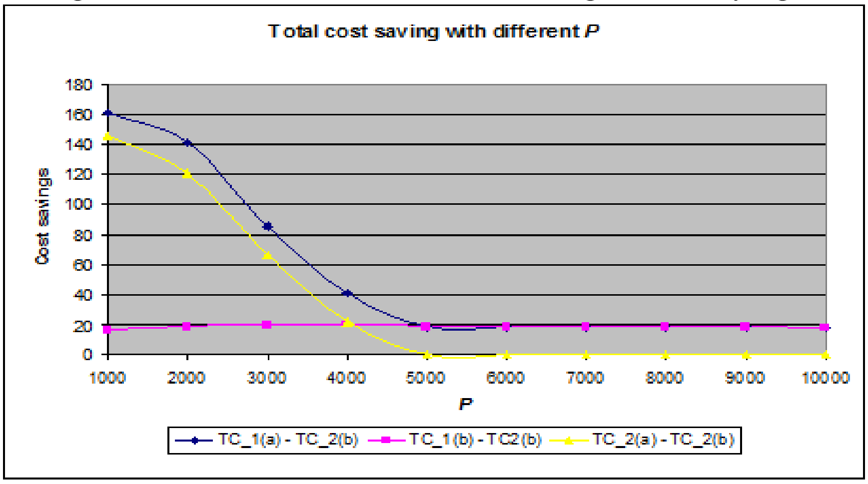
Figure 4.22: The total cost with different P



At P=5000, the total cost of both Policies 1(a) and 2(a) give the same value because the optimal total cost for both policies is at $n = 5$ and $m = 1$. Therefore the blue line overlapped with the red line. This also happen to Policies 1(b) and 2(b). It suggests that when the production rate, P is too high, the production move on faster, and hence the vendor only needs a single shipment to deliver the product.

Figure 4.23 plotted the cost savings which were obtained while varying the value of P. We found that as P increases, the total cost saving decreases for all policies. Furthermore, $(TC_{1(b)} - TC_{2(b)})$ and $(TC_{1(b)} - TC_{2(b)})$ overlapped

Figure 4.23: Plot of the total cost saving while varying P

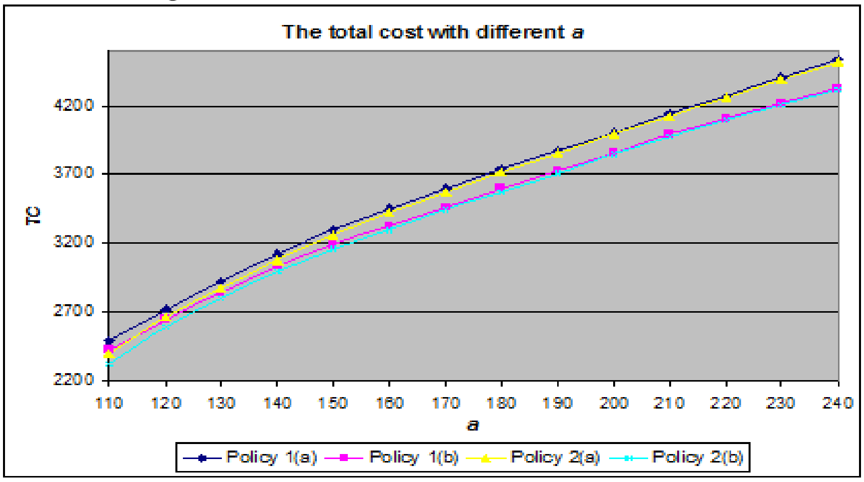


at $P = 5000$ until 10000 while $(TC_{2(a)} - TC_{2(b)})$ produces zero cost saving. This is because $TC_{1(a)} = TC_{1(b)}$ and $TC_{Policy2(a)} = TC_{2(b)}$.

Let us consider a different value of a where all the other standard parameter values remain the same. As noted in the case of $h_1 < h_2$, the value of a must be greater than bH and less than $\frac{P}{H} + \frac{bH}{2}$ because the demand $a - bH > 0$ and $aH - \frac{bH^2}{2} \leq P$. Therefore, in this example, we choose $a = 110, 120, \cdots, 240$. The result is shown in Figure 4.24.

We found that the total cost for all policies increases as the value of a increases. The lowest total cost is given by Policies 1(b) and 2(b). This result suggests that, with the larger initial demand rate, a, the different shipment sizes policy is more effective compared to the equal shipment sizes. We also found that the larger the value of a the smaller the difference between the

Figure 4.24: The total cost with different a



total costs for Policies 1(a) and 2(a). This also happens to Policies 1(b) and 2(b). For example, when $a = 110$ the difference between the total costs for Policies 1(a) and 2(a) is 91.91 while for Policies 1(b) and 2(b) is 102.12. When $a = 240$, the difference decreases to 11.84 and 9.81 respectively.

Figure 4.25: The cost saving with different a

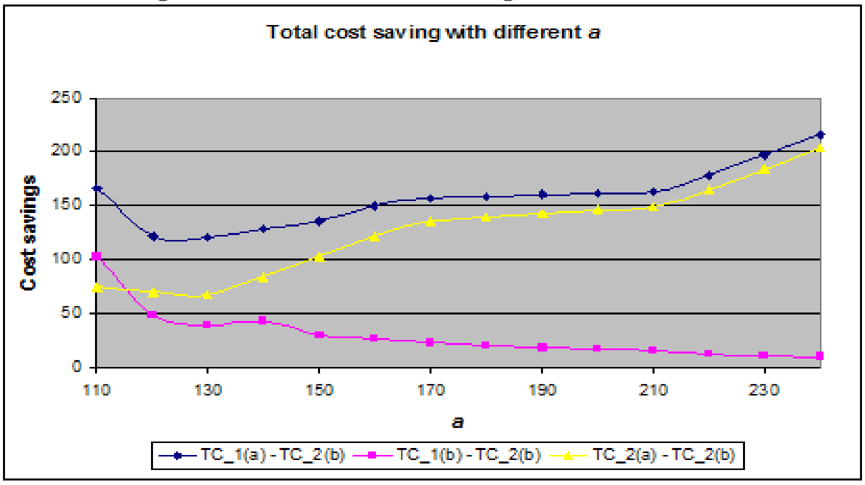
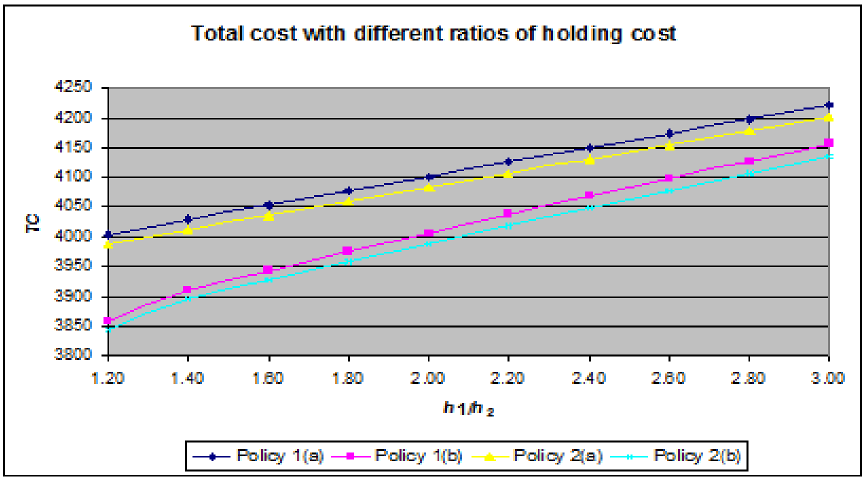


Figure 4.25 shows the diagrammatic plot of the total cost savings which is obtained while varying the value of a. The total cost savings for $(TC_{1(a)} - TC_{2(b)})$, $(TC_{1(b)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$ decrease when $110 \leq a \leq 130$. When $140 \leq a \leq 240$, the difference increases for $(TC_{1(a)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$ while decreases for $(TC_{1(b)} - TC_{2(b)})$.

Next, we increase the ratio of the holding cost, h_1/h_2 by increasing h_1 from 6 up to 15 while $h_2 = 5$. All the other standard parameter values remain the same as in the previous example. The corresponding results are displayed in Figure 4.26.

Figure 4.26: The total cost with different h_1/h_2



It shows that the larger the ratios of h_1/h_2, the larger the total cost for all policies. Again, Policy 2(b) always gives the best minimum total cost followed by Policies 1(b), 2(a) and 1(a). This result suggests that unequal shipment sizes and unequal shipment periods case always better than the equal shipment sizes policy.

Figure 4.27: The cost savings with different h_1/h_2

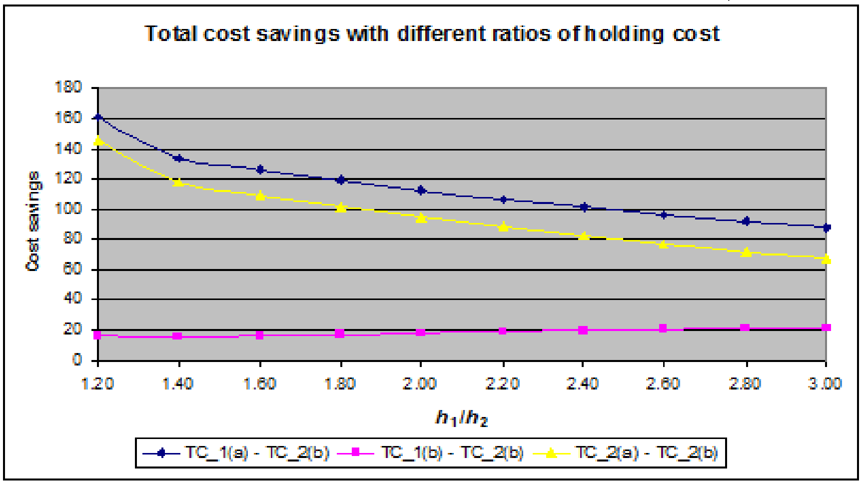
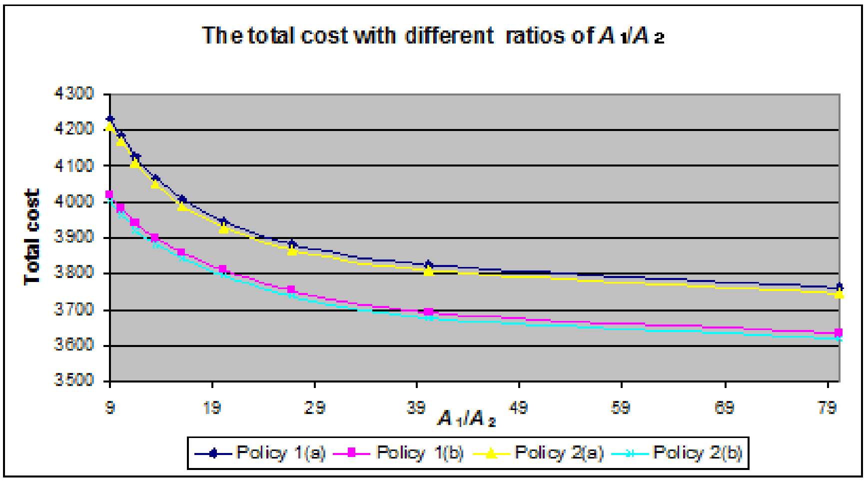


Figure 4.27 illustrated the total cost savings obtained by implementing Policy 2(a) rather than the other three policies. We found that the larger the value of h_1/h_2 the lower $(TC_{1(a)} - TC_{2(b)})$ and $(TC_{Policy\ 2(a)} - TC_{2(b)})$ but a slight improvement for $(TC_{1(b)} - TC_{2(b)})$.

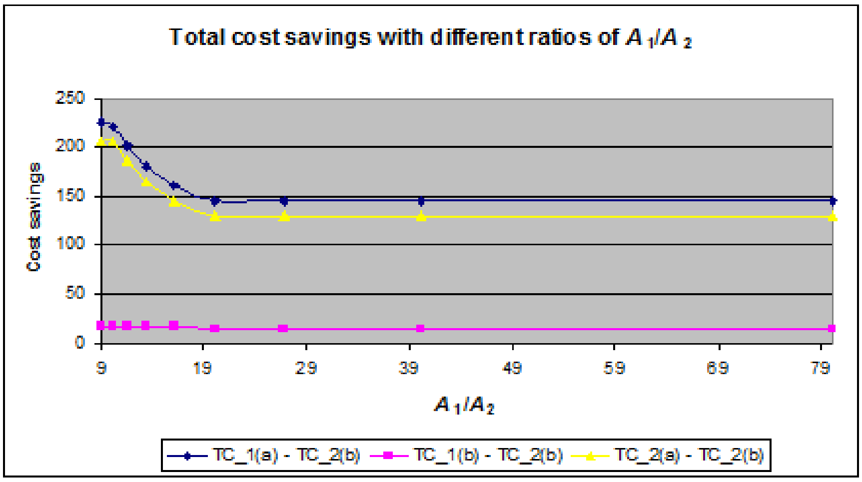
Finally, we vary the ratio of the production set up and shipment cost, A_1/A_2 by varying the value of A_2 while A_1 is 400. All the other standard parameter values remain the same. Figure 4.28 shows that the larger the ratio of A_1/A_2, the lower the total cost of all policies where Policy 2(b) gives the lowest total cost.

Figure 4.28: The total cost with different A_1/A_2



The total cost saving by varying the value of $A_1/A_2$is given by Figure 4.29. When $8.89 \leq A_1/A_2 \leq 20$, the total cost savings is decrease for $(TC_{1(a)} - TC_{2(b)})$, $(TC_{1(b)} - TC_{2(b)})$ and $(TC_{2(a)} - TC_{2(b)})$. However, the total cost savings are constant when $20 \leq A_1/A_2 \leq 80$.

Figure 4.29: The cost savings with different A_1/A_2



4.7 Conclusion

In this chapter, we have considered the integrated inventory model for n production batch where $h_1 < h_2$ with equal and unequal cycle time, T_i policies. Both of these policies discussed the case of equal shipment sizes and unequal shipment sizes and unequal shipment periods. We have concluded that the unequal cycle time with unequal shipment sizes and unequal shipment periods (Policy 2(b)) is always superior compared to the other three policies.

We have extended this model to the case of $h_1 > h_2$. We have also discussed the equal shipment sizes policy and unequal shipment sizes and unequal shipment periods policy and found that the best minimum solution is also given by the unequal cycle time with unequal shipment sizes and unequal shipment periods policy (Policy 2(b)).