Chapter 1

Introduction

1.1 Preliminary

When is a torsion-free abelian group free abelian? This problem has been attacked by many mathematicians for many years, but no satisfactory answer has emerged. As a result, they have made the theory of abelian groups rich and diverse. Also, set theory plays an important role in this problem, as pointed out by Thomas [27, 28].

1.2 Abelian Groups

In this section, we will discuss different types of abelian groups.

Free Abelian Group

An abelian group G is said to be *free abelian* if G is a direct sum of infinite cyclic groups. Let K be an index set. If $G = \bigoplus_{k \in K} \langle x_k \rangle$ where $\langle x_k \rangle$ is the infinite cyclic group generated by x_k , then we say G is free on $\{x_k \ : \ k \in K\}$. Moreover, every non-zero element $g \in G$ has the unique expression

$$g = m_{k_1} x_{k_1} + \cdots + m_{k_n} x_{k_n}$$

where the m_{k_i}'s are nonzero integers and the k_i's are distinct. If G is free on $\{x_k : k \in K\}$ and F is free on $\{y_j : j \in J\}$, then the two free abelian groups are isomorphic if and only if J and K have the same number of elements, i.e. there is a one-to-one and onto function (bijection) from J onto K. If $|K| = n$, then we say that G has *rank* n. A non-trivial result will show that a subgroup of a free abelian group is free (see [23, Theorem 9.22 on p. 189]).

Torsion Abelian Group

An abelian group G is said to be *torsion* if every element in G has finite order. This means that, for every $g \in G$, there is an integer $n > 0$ such that $ng = 0$. The structure of finite abelian groups is well-understood. The Basis Theorem of Finite Abelian Groups states that every finite abelian group is a direct sum of cyclic p-groups. Torsion abelian groups have similar structure as finite abelian groups, that is a torsion abelian group is a direct sum of p-groups. Thus, the study of torsion abelian groups is reduced to that of p-groups. Some examples of torsion abelian groups are $\mathbb{Q}/\mathbb{Z}$ and cyclic p-groups.

Torsion-free Abelian Group

An abelian group G is said to be *torsion-free* if it does not have non-identity elements of finite order. This means that, if $ng = 0$ for some $g \in G$, then either

$n = 0$ or $g = 0$.

If G is an abelian group and tG is the torsion subgroup of G, then G/tG is isomorphic to H where H is torsion-free. Note that if H is free abelian, then $G = t(G) \oplus H$, and tG is a direct summand of G. However, this is not true in general, that is when H is not free abelian.

Theorem 1.2.1. [23, Theorem 9.2 on p. 178] *There exists an abelian group G whose torsion subgroup is not a direct summand.*

The counter-example given in the reference for Theorem 1.2.1 is the direct product of cyclic groups of prime order for infinitely many primes.

Torsion-free abelian groups have a useful invariant, which is called *rank*. The rank of a torsion-free abelian group G is defined to be the number of elements in a maximal independent subset of G. Observe that a free abelian group is torsion-free and the two notions of rank coincide. Some examples of torsion-free abelian groups are free abelian groups, $\mathbb{Q}$, and the product of countably infinitely many copies of $\mathbb{Z}$ which is known as the Baer-Specker group. Baer proved that the Baer-Specker group, B, is not free abelian in 1937. Later in 1950, Specker proved that every countable subgroup of B is free abelian.

Theorem 1.2.2. [22, 4.4.4 and 4.4.6 on p. 114 and 115] *The Baer-Specker group, B is not free abelian, but every countable subgroup of B is free abelian.*

Divisible Group

Let n be an integer, we say an element $x \in G$ is *divisible* by n if there is an element $y \in G$ such that $ny = x$. Let p be a prime. The *p-height* of x, denoted by $h_p(x)$, is defined to be the largest positive integer n such that p^n divides x. So, if x is divisible by p^n but not by p^{n+1}, then $h_p(x) = n$.

We say that a group G is *divisible* if every element in G is divisible by every positive integer n. This means that the p-height of every element is infinite for every prime p. If the p-height of an element is infinite, we say that the element is *p-divisible*. Note that a group G is divisible if and only if all its elements are p-divisible for every prime p (see [23, Exercise 9.29 on p. 184]). Some examples of divisible groups are $\mathbb{Q}$, the additive group of real numbers $\mathbb{R}$, and the additive group of complex numbers $\mathbb{C}$.

Pure Subgroup

A subgroup A of G is said to be *pure* in G if an element $a \in A$ is divisible by n in G implies that it is divisible by n in A. Equivalently, A is pure in G if for all integer n,

$$A \cap nG = nA.$$

Note that nA is always a subset of $A \cap nG$, but the reverse inclusion $A \cap nG \subseteq nA$ is not necessarily true. If it is, then A is a pure subgroup of G.

The notion of pure subgroup was introduced by Prüfer who called it "Servanzuntergruppe". This was translated into English as "isolated subgroups" or "serving subgroups". We shall however follow Braconnier's translation, that is a pure subgroup, which is now a common term.

Note that every direct summand of an abelian group G is pure (see [23, Exercise 9.50 on p. 194]). However, a pure subgroup may not be a direct summand. So a pure subgroup can be considered as a generalization of a subgroup which is a direct summand.

A subgroup A is *p-pure* in G if

$$p^k G \cap A = p^k A,$$

for all positive integers k. Note that if A is p-pure in G for every prime p, then A is pure in G (see [7, p. 114]).

Basic Subgroup

A subgroup A is a *p-basic* subgroup of a p-group G, if

(1) A is a direct sum of cyclic p-groups,

(2) A is p-pure in G, and

(3) G/A is p-divisible.

A subgroup A is a *basic* subgroup of a torsion abelian group G, if

(1) A is a direct sum of cyclic p-groups,

(2) A is pure in G, and

(3) G/A is divisible.

Note that every torsion abelian group has a basic subgroup (see [23, Theorem 9.32 on p.197]). This fact was first proved by Kulikov.

Now for a torsion-free abelian group G, a subgroup A of G is said to be a *basic* subgroup if

(1) A is a direct sum of infinite cyclic groups,

(2) A is pure in G, and

(3) G/A is divisible.

Some examples of basic subgroups of a torsion-free abelian group can be found in [5]. In fact, it was shown that all basic subgroups of the same group have the same rank. Furthermore, Dugas and Irwin [5] showed that there exists a torsion-free abelian group that has no basic subgroup. This property distinguishes torsion-free abelian groups from torsion abelian groups, as a torsion abelian group always has a basic subgroup.

1.3 Recent Development

When is a torsion-free abelian group free? In other words, when is a torsion-free abelian group a direct sum of infinite cyclic subgroups? We know that a torsion-free abelian group contains free abelian subgroups. However, a torsion-free abelian group does not necessarily have a largest free abelian subgroup. For example, the additive group $\mathbb{Q}$ has no largest free abelian subgroup. Any free abelian subgroup of $\mathbb{Q}$ has the form $\{nx : n \in \mathbb{Z}\}$ for some x, and by replacing x with $x/2$ gives a strictly larger free abelian subgroup.

Specker [25] produced two criteria under which a torsion-free abelian group is free (see Theorem 1.3.1 and Theorem 1.3.2).

Theorem 1.3.1. *A countable torsion-free abelian group is free if it can be embedded in a product of infinite cyclic groups.*

This means that if there is a monomorphism from a countable torsion-free abelian group into a product of infinite cyclic groups, then the torsion-free abelian group is free. Recall that the product of countably infinitely many copies of $\mathbb{Z}$ is called the Baer-Specker group, B. Let P be the subgroup of bounded sequences of B. If $B = \prod_{i \in \mathbb{I}} \mathbb{Z}$, where $\mathbb{I}$ is an index set, then an element $(n_i)_{i \in \mathbb{I}}$ in B belongs to P if and only if there is a positive integer K such that $|n_i| \leq K$ for all $i \in \mathbb{I}$.

Theorem 1.3.2. *An abelian group of cardinality not exceeding $\aleph_1$ is free if it can be embedded in P.*

The subgroup P is free by Nöbeling[20].

Reid [21] proved the following theorem.

Theorem 1.3.3. *A torsion-free abelian group G can be written as the sum of two free abelian subgroups if and only if G is free or G has infinite rank.*

Note that if G is free, then it can be written as the sum of two free abelian subgroups. However, if G is torsion-free of finite rank and not free, then it cannot be written as a sum of two free abelian subgroups. This gives us more information on the structure of a torsion-free abelian group of infinite rank. In fact, a torsion-free abelian group of infinite rank can be realised as "almost free" by saying that it is the sum of two free abelian subgroups. Note that the term infinite rank can be countably infinite or uncountably infinite but the term infinite rank cannot be dropped.

Another criterion for freeness was given by Nöbeling [20].

Theorem 1.3.4. *If an abelian group can be embedded in the subgroup of bounded sequence P of the Baer-Specker group B, then G is free.*

Note that Theorem 1.3.4 is a generalization of one direction of Theorem 1.3.2.

An *abelian factor set* is a function $f : C \times C \to A$ such that for all $x, y, z \in C$

(i) $f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y) = 0$;

(ii) $f(0,y) = 0 = f(x,0)$;

(iii) $f(x,y) = f(y,x)$

A function $g : C \times C \to A$ is said to be a *coboundary* if there is a function $\alpha : C \to A$ with $\alpha(0) = 0$ for which $g(x,y) = \alpha(y) - \alpha(x+y) + \alpha(x)$. Let

$$\mathrm{Ext}(C,A) = \frac{\mathrm{Z}(C,A)}{\mathrm{B}(C,A)}, \tag{1.1}$$

where $\mathrm{Z}(C,A)$ is the additive group of all abelian factor sets and $\mathrm{B}(C,A)$ is the set of all coboundaries with $\mathrm{B}(C,A) \subseteq \mathrm{Z}(C,A)$.

The following theorem can be found in [23, Theorem 7.15 on p. 147]

Theorem 1.3.5. *The set of equivalence classes of abelian extensions of A by C, is a group isomorphic to* $\mathrm{Ext}(C, A)$*, where the zero element is the class of direct products.*

Griffith [9, Theorem 3.1] gave a criterion for freeness by using Ext.

Theorem 1.3.6. *Let G be a torsion-free abelian group. Then* $\mathrm{Ext}(G, T)$ *is torsion for all torsion groups T if and only if G is free.*

For divisibility and freeness, we have the following criteria (see [23, Corollary 10.12 on p. 219]).

Theorem 1.3.7. *A group A is divisible if and only if* $\mathrm{Ext}(C, A) = 0$ *for every group C. Moreover, a group F is free abelian if and only if* $\mathrm{Ext}(F, A) = 0$ *for every group A.*

Let K be an index set. A set S is an $f\sigma$-union of its subsets S_λ , $\lambda \in K$, if each finite subset of S is contained in some S_λ. The following theorem was proved by Pontryagin as stated by Hill [11].

Theorem 1.3.8. *If a countable, torsion-free abelian group G is the $f\sigma$-union of pure subgroups that are free, then G must be free.*

Hill [11, Theorem 1] then generalized Pontryagin's theorem.

Theorem 1.3.9. *If a torsion-free abelian group G is the $f\sigma$-union of a countable number of pure subgroups that are free abelian, then G must be free.*

Note that in the above generalization, the countable torsion-free abelian group is replaced with just torsion-free abelian group. Another two useful theorems proved by Hill in the same paper are

Theorem 1.3.10. [11, Theorem 2] *If a torsion-free abelian group G is the union of a countable chain of pure free subgroups, then G is free.*

Theorem 1.3.11. [11, Theorem 3] *Let m be a cardinal number that is the limit of an ordinary, countable sequence of smaller cardinals. Suppose that the torsion-free abelian group G has rank m. If each subgroup of G having rank less than m is free, then G must be free.*

An abelian group G is said to be *m-free* if each subgroup of G having cardinality less than m is free. Similarly, we say that G is an m-group, if G has cardinality m and each subgroup of G having cardinality less than m is a direct sum of cyclic groups.

Theorem 1.3.12. [15, Theorem 3] *Let μ be a limit ordinal of cardinality not exceeding $\aleph_1$. Let*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots, \alpha < \mu,$$

be an ascending chain of free subgroups of G, indexed by the ordinal μ. For limit ordinals β, if the following conditions:

(i) $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$;

(ii) $G = \bigcup_{\alpha<\mu} G_\alpha$;

(iii) $|G_\alpha| \leqslant \aleph_1$;

are satisfied, then G is free provided that $G_{\alpha+1}/G_\alpha$ is $\aleph_1$-free for each α.

An ascending chain

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots, \alpha < \mu,$$

of abelian groups, indexed by an ordinal μ, is said to be *smooth* if $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$, whenever β is a limit ordinal less than μ.

Theorem 1.3.13. [13, Theorem 2.1] *Let*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots, \alpha < \mu,$$

be a smooth chain of free subgroups, where μ has cardinality not exceeding $\aleph_1$. If $G_{\alpha+1}/G_\alpha$ is $\aleph_1$-free of cardinality at most $\aleph_1$ for each $\alpha < \mu$, then $G = \bigcup_{\alpha<\mu} G_\alpha$ is necessarily free.

Theorem 1.3.13 is stronger than Theorem 1.3.12 in the sense that the condition (iii) of Theorem 1.3.12 is not needed, but it is replaced with the condition $|G_{\alpha+1}/G_\alpha| \leqslant \aleph_1$.

A group G is said to be *admissible* if

(i) G is not free;

(ii) G is the union of a smooth chain;

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots, \alpha < \omega_2,$$

of free subgroups G_α such that, for each $\alpha < \omega_2$,

(a) $|G_\alpha| \leqslant \aleph_1$ and,

(b) $G_{\alpha+1}/G_\alpha$ is $\aleph_1$-free.

This definition was given by Hill [13, Definition 2.2] and he showed that admissible groups exist and they are $\aleph_2$-free. The proof of the existence of admissible groups can also be found in [15].

Theorem 1.3.14. [13, Theorem 2.2] *Let μ be a limit ordinal of cardinality not exceeding $\aleph_1$. If G is the union of a smooth chain:*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots, \alpha < \mu,$$

of free groups, indexed by μ, such that $G_{\alpha+1}/G_\alpha$ is admissible for each $\alpha < \mu$, then G is free.

For each non-negative integer n, a class F_n of torsion-free abelian groups is defined as follows: The class F_0 consists of all countable torsion-free abelian groups. Inductively, we define F_n to be the class of torsion-free abelian groups G of cardinality not exceeding $\aleph_n$ that can be represented as the union of a smooth chain

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots, \alpha < \mu,$$

of free abelian subgroups G_α where $\mu \leq \omega_n$ such that $G_{\alpha+1}/G_\alpha \in F_{n-1}$. This definition was given by Hill in [14] and he proved the following results.

Theorem 1.3.15. [14, Theorem 1] *Let H be a subgroup of G. If $G \in F_n$, then $H \in F_n$.*

Theorem 1.3.16. [14, Main Theorem] *For every positive integer n, if $G \in F_n$, then G is $\aleph_n$-free.*

Theorem 1.3.17. [14, Lemma] *Let $G \in F_n$ for some positive integer n. Then G can be represented as the union of a smooth chain*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots, \alpha < \omega_n,$$

of free abelian groups G_α such that $|G_\alpha| \leqslant \aleph_{n-1}$ and $G_\beta/G_\alpha \in F_{n-1}$ if $\alpha < \beta < \omega_n$.

Here, we see that if $G \in F_n$, then G can be represented as the union of a smooth chain. It is worth to mention that in Theorem 1.3.12 and Theorem 1.3.14, we just assume that G can be represented as the union of a smooth chain without actually showing that G indeed can be represented as the union of a smooth chain. Hill [14] called groups belonging to class F_n as $\beth_n$-free abelian groups and he proved the following properties for every positive integer n:

(1) A subgroup of a $\beth_n$-free abelian group is $\beth_n$-free.

(2) An extension of a $\beth_n$-free abelian group by a $\beth_n$-free abelian group is $\beth_n$-free.

(3) Any $\beth_n$-free abelian group is $\aleph_n$-free.

The proofs of these properties are complicated. In the same paper, Hill also proved the following results.

Theorem 1.3.18. [14, Theorem 2] *If G is the union of a smooth chain*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots,$$

of length not exceeding $\aleph_n (n < \omega)$ of free abelian groups G_α, then G is free provided that $G_{\alpha+1}/G_\alpha$ is $\beth_n$-free for each α .

Theorem 1.3.19. [14, Theorem 3] *For every positive integer n, there exist $\beth_n$-free abelian groups that are not free.*

Corollary 1.3.20. [14, Corollary 1] *For every positive integer n, there exists an abelian group of cardinality $\aleph_n$ that is not free but $\aleph_n$-free.*

The following problem was posed by Hill [12].

Problem 1.3.21. For which cardinals m is an m-group a direct sum of cyclic groups?

Here, Problem 1.3.21 is stated for p-groups and not for torsion-free abelian groups. Note that an equivalent problem for the torsion-free case is the following.

Problem 1.3.22. For which cardinals m is an m-free abelian group a free abelian group?

It seems that the answer to Problem 1.3.22 is somehow negative by Corollary 1.3.20. However, Hill [12, Theorem 1] showed that if a p-group G is an $\aleph_\omega$-group, then G is necessarily a direct sum of cyclic groups.

A function v from an abelian group G into the set of real numbers $\mathbb{R}$ is said to be a *norm* if

(i) v maps G to the non-negative real numbers $\mathbb{R}$;

(ii) $v(g+h) \leq v(g) + v(h), \forall g, h \in G$;

(iii) $v(mg) = |m|v(g), \forall m \in \mathbb{Z}$.

The norm υ is said to be *discrete* if there is some $\rho > 0$ such that $\upsilon(g) > \rho$ whenever $g \neq 0$. If in addition υ maps to some non-negative integers, then υ is said to be an *integer-valued* norm.

Classification of abelian groups by using norm was started by Lawrence [18]. He proved the following two theorems.

Theorem 1.3.23. [18, Lemma 1] *If G is an abelian group with a non-trivial norm, then G is torsion-free.*

Theorem 1.3.24. [18, Theorem 4] *Let G be a countable abelian group with a discrete norm. Then G is a free abelian group.*

Steprāns [26, Theorem] proved the following theorem, which gives a more complete answer about when an abelian group is free abelian.

Theorem 1.3.25. *An abelian group G is free if and only if it is discretely normed.*

The above result is not practical as the theorem does not tell us how to construct the required norm on an abelian group. It is interesting to note that Steprāns used tools from set theory to prove his result, an indication that set theory has begun to play an important role for this kind of problem.

Note that the structure of a finitely generated torsion-free abelian group is well-understood, as the following theorem shows.

Theorem 1.3.26. [23, Theorem 9.25 on p. 192] *Every finitely generated torsion-free abelian group G is free abelian.*

Recall that Reid [21] proved that if G is a torsion-free abelian group, then G is the sum of two free subgroups if and only if G is free or G has infinite rank (see Theorem 1.3.3). Blass and Irwin [2, Theorem 1.1] later on proved a similar result.

Theorem 1.3.27. *For any torsion-free abelian group G of infinite rank κ, the following two statements are equivalent.*

(1) *G is the sum of two free subgroups, at least one of which is pure in G.*

(2) *G has a pure free subgroup of rank κ.*

Moreover, any subgroup as in (2) can serve as one of the subgroups as in (1).

With a stronger condition on the rank, they obtained the following theorem.

Theorem 1.3.28. [2, Theorem 1.2] *For any torsion-free abelian group G of uncountable rank κ, the following two statements are equivalent.*

(1) *G is the sum of two pure free subgroups.*

(2) *G has a pure free subgroup of rank κ.*

Moreover, any subgroup as in (2) can serve as one of the subgroups in (1).

Let B be a subgroup of an abelian group G. A subgroup H of G is said to be *B-high* if it is maximal among subgroups of G disjoint from B. Disjoint here means that the intersection is $\{0\}$ and not empty, which is impossible for subgroups.

Benabdallah and Irwin [1, Theorem 1] proved the following theorem, which gives a sufficient and necessary condition for when a p-group is a direct sum of cyclic groups.

Theorem 1.3.29. *Let B be a basic subgroup of a p-group G without elements of infinite height. Then all B-high subgroups of G are direct sum of cyclic groups if and only if G is a direct sum of cyclic groups.*

An abelian p-group G is said to be a Σ-*group* if every subgroup of G disjoint from G' is a direct sum of cyclic groups, where G' is the subgroup of elements of infinite height in G.

Theorem 1.3.30. [1, Theorem 2] *Let G be an abelian p-group. Then G contains a basic subgroup B such that all B-high subgroups of G are direct sums of cyclic groups if and only if G is a Σ-group with a finite number of elements of infinite height.*

Later Blass and Irwin [3, Theorem 1.1] proved an analogue of Theorem 1.3.29 for torsion-free abelian groups.

Theorem 1.3.31. *Let G be a torsion-free abelian group such that*

(1) *G has a basic subgroup of infinite rank, and*

(2) *for every basic subgroup B of G, all B-high subgroups of G are free.*

Then G is free.

Note that since subgroups of free abelian groups are free, and since Zorn's lemma allows us to enlarge every subgroup disjoint from B to a B-high subgroup, we can state condition (2) of Theorem 1.3.31 as every subgroup of G disjoint from a basic subgroup B is free.

The following are some variations of Theorem 1.3.31.

Theorem 1.3.32. [3, Corollary 3.1] *Let G be an $\aleph_1$-free group such that*

(1) *G has a basic subgroup, and*

(2) *for every basic subgroup B of G, all B-high subgroups of G are free.*

Then G is free.

Theorem 1.3.33. [3, Corollary 3.3] *Let G be a torsion-free abelian group of uncountable rank κ such that*

(1) *G has a basic subgroup of the same rank κ, and*

(2) *for each basic subgroup B, all the B-high subgroups of G are isomorphic.*

Then G is free.

Theorem 1.3.34. [3, Corollary 3.4] *Let G be a torsion-free group with a pure free subgroup of infinite rank. Assume that for every maximal pure independent subset $I \subseteq G$, all $\langle I \rangle$-high subgroups of G are free. Then G is also free.*

Condition (2) of Theorem 1.3.31 requires that all B-high subgroups of G are free for every basic subgroup B, not just one. This is confirmed by a result of Blass and Shelah [4, Theorem 1.3] recently.

Theorem 1.3.35. *There exists an $\aleph_1$ separable torsion-free abelian group G of size $\aleph_1$, with a basic subgroup B of rank $\aleph_1$ such that all subgroups of G disjoint from B are free but G itself is not free abelian.*

We conclude this chapter by stating the following problem again.

Problem 1.3.36. When is a torsion-free abelian group free abelian?

This has no satisfactory answer. Although Steprāns [26] gave an 'almost' satisfactory answer (see Theorem 1.3.25), it is not practical. It would be interesting to know whether or not an algorithm to decide on whether a torsion-free abelian group is free abelian exists. If it does, then we would have a satisfactory answer to Problem 1.3.36.