Chapter 2

Main Results

2.1 Introduction to Set Theory

Set theory is use to provide a solid foundation to all branches of mathematics. It all started with a person, Georg Cantor. Initially, Cantor had developed number theory for his main contribution to mathematical societies. Later on, he moved on his work to trigonometric series. Little did he knew that his work on trigonometric series is going to change the whole course of mathematics. Those paper on trigonometric series contain Cantor's first ideas on set theory. It was in the year 1874 that Cantor published an article in Crelle's Journal which mark the birth of set theory. So Cantor started his investigation on the theory of cardinal and ordinal numbers, as well as the topology of the real line. When Cantor started his investigation in 1874, he proved that the set of all real numbers is uncountable, while the set of all real algebraic numbers is countable. In 1878, he gave the first formulation of the celebrated continuum hypothesis but he was unable to prove it. Today, many mathematicians are happy with the Zermelo-Fraenkel Set Theory with the Axiom of Choice abbreviated ZFC. The results of Gödel (1940) and Cohen (1963) imply that the continuum hypothesis can neither be proved nor be disproved by using the standard ZFC. It is worth mentioning that there are different types of Axiomatic Set Theory, such as New Foundation Set theory, Morse-Kelley Set Theory, and Neumann-Bernays-Gödel Set theory.

We now begin with the axioms of ZFC set theory.

Axioms of ZFC

A Axiom of Extensionality. If X and Y have the same elements, then $X = Y$.

B Axiom of Pairing. For any a and b, there exists a set $\{a, b\}$ that contains exactly a and b.

C Axiom Schema of Separation. If P is a property (with parameter p), then for any X and p, there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.

D Axiom of Union. For any X, there exists a set $Y = \bigcup X$, the union of all the elements of X.

E Axiom of Power Set. For any X, there exists a set $Y = P(X)$, the set of all subsets of X.

F Axiom of Infinity. There exists an infinite set.

G Axiom Schema of Replacement. If a class F is a function, then for any X, there exists a set $Y = F(X) = \{F(x) : x \in X\}$.

H Axiom of Regularity. Every non-empty set has an $\in$-minimal element.

I Axiom of Choice. Every family of non-empty sets has a choice function.

The formulas of set theory are built up from the atomic formulas such as the membership relation $x \in y$ and the equal relation $x = y$ with the use of connectives

$$\varphi \wedge \psi, \quad \varphi \vee \psi, \quad \neg\varphi, \quad \varphi \to \psi, \quad \varphi \leftrightarrow \psi,$$

which are called conjunction, disjunction, negation, implication and equivalence, respectively, and the quantifiers are $\forall x$ and $\exists x$. Note that we only consider the connectives $\neg$ and $\wedge$ as the only primitive connectives because other connectives can be derived from the two connectives above. For example

(1)$\varphi \vee \psi$ for $\neg(\neg\varphi \wedge \neg\psi)$;

(2)$\varphi \to \psi$ for $\neg(\varphi \wedge \neg\psi)$;

(3)$x \neq y$ for $\neg x = y$ and;

(4)$x \notin y$ for $\neg x \in y$.

2.1.1 Ordinal Number

We shall begin with the concept of linear ordering, partial ordering and well-ordering.

Linear and Partial Ordering

Definition 2.1.1. A binary relation $<$ on a set P is a *partial ordering* of P if for all $p, q, r \in P$,

(i) $p \nless p$ for any $p \in P$,

(ii) if $p < q$ and $q < r$, then $p < r$.

$(P, <)$ is called a *partially ordered* set.

Definition 2.1.2. A partial ordering $<$ of P is a *linear ordering* if moreover

(iii) $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$.

Note that if $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$ is a function, then f is called *order preserving*, if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then an order preserving function is also called *increasing*.

A one-to-one and onto function (bijection) of P onto Q is an *isomorphism* of P onto Q, if both f and f^{-1} are order preserving. We say $(P, <)$ is isomorphic to $(Q, <)$. An isomorphism of P onto itself is called an *automorphism* of $(P, <)$.

Definition 2.1.3. A linear ordering $<$ of a set P is a *well-ordering* if every non-empty subset of P has a least element.

Theorem 2.1.4. [ZFC Well-Ordering Theorem] *Every set can be well-ordered.*

Note that the Axiom of Choice is needed to prove Theorem 2.1.4. In fact, it can be shown that the Axiom of Choice, ZFC Well-Ordering Theorem, and Zorn's Lemma are equivalent.

Theorem 2.1.5. [Zorn's Lemma] *If $(P, <)$ is a non-empty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

Zorn's Lemma is very important in algebra, as it is used to prove the existence of certain maximal sets and functions. For instance, the following theorems are proved by using Zorn's Lemma.

Theorem 2.1.6. *Let B be a subgroup of an abelian group A. Let D be an abelian group and $f : B \to D$ be a homomorphism. If D is divisible, then f can be extended to a homomorphism $\tilde{f} : A \to D$.*

Theorem 2.1.7. *Every non-zero commutative ring contains a maximal ideal.*

Theorem 2.1.8. *Every vector space contains a basis.*

Theorem 2.1.9. *Every field has an algebraic closure.*

Definition 2.1.10. A set T is transitive if every element of T is a subset of T, that is, $x \in T \Rightarrow x \subset T$.

As a consequence, $\bigcup T$ and $\bigcap T$ are also transitive.

Definition 2.1.11. A set is an ordinal numbers if it is transitive and well-ordered by $\in$.

These are some properties of ordinal number

(i) $\varnothing$ is an ordinal.

(ii) If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.

(iii) If α, β are ordinals, $\alpha \neq \beta$ and $\alpha \subseteq \beta$, then $\alpha \in \beta$.

(iv) If α, β are ordinals, then either $\alpha \subseteq \beta$, or $\beta \subseteq \alpha$, or $\alpha = \beta$.

Theorem 2.1.12. *Every well-ordered set is isomorphic to a unique ordinal number.*

2.1.2 Cardinal Numbers

The concept of cardinality is central in the study of infinite sets. The idea of cardinal number is due to Cantor 1878. Our discussion of the cardinal numbers begins with the following fact. There are two types of cardinal, one is the cardinal of a set that can be well-ordered, and the other is the cardinal of a set that cannot be well-ordered. Two sets A and B are said to be *equinumerous* if there is a bijection of A onto B.

Definition 2.1.13. The cardinal of x, or synonymously, the cardinality of x, denoted by $|x|$, is

(a) the least ordinal equinumerous to x, if x can be well-ordered;

(b) the set of all sets y of least rank which are equinumerous to x, otherwise.

For definition of rank of set, please refer to [24, on p. 214]. The definitions of cardinality are due to Von Neumann (part (a) of Definition 2.1.13) and Frege-Russell-Scott (part (b) of Definition 2.1.13). Both definitions are defined without the presence of Axiom of Choice. If we assume the presence of Axiom of Choice, then all the cardinals are ordinals, since every set can be well-ordered. Thus, in the presence of Axiom of Choice, the cardinality of x is always the least ordinal equinumerous to x.

The following properties are well-known for finite cardinals.

Theorem 2.1.14. *A set A is finite if and only if it is equinumerous to some natural number.*

Theorem 2.1.15. *Every subset of a finite set is finite, every union of finitely many finite sets is finite and the power set of a finite set is finite.*

Theorem 2.1.16. *Every natural number is a finite cardinal.*

Theorem 2.1.17. *If n is finite cardinal and a is an infinite cardinal, then $n < a$.*

Theorem 2.1.18. *No finite set is equinumerous to a proper subset of itself. In particular, no two different natural numbers are equinumerous.*

Note that these properties were used by Peirce and Dedekind to define finite sets. On the other hand, only an infinite set can be equinumerous to a proper subset of itself.

Note that two sets X and Y have the same cardinality, i.e.,

$$|X| = |Y|,$$

if and only if there is a bijection of X onto Y.

If there exists a one-to-one function (injection) of X into Y, then we write

$$|X| \leq |Y|.$$

Furthermore, we write $|X| < |Y|$ to mean that $|X| \leq |Y|$ but $|X| \neq |Y|$. The relation on $\leq$ is clearly transitive.

Theorem 2.1.19. [Cantor Theorem] *For every set X, $|X| < |P(X)|$.*

I personally think that the most beautiful theorem in Set Theory is the Cantor-Bernstein-Schröder theorem. This theorem was first stated by Cantor and Schröder, but their proof was wrong. In 1898, Bernstein gave a correct proof of the theorem in his Ph.D. thesis. Nowadays the proof of the theorem is available in many Set Theory textbooks. Nevertheless, I am inclined to include the proof in my thesis.

Theorem 2.1.20. [Cantor-Bernstein-Schröder Theorem] *If A and B are sets, and the functions $f : A \to B$ and $g : B \to A$ are injections, then there exists a bijection from A onto B.*

Proof. First we define

$$S = \bigcup_{n=0}^{\infty} (g \circ f)^n (A \setminus g(B))$$

and

$$h(x) = \begin{cases} f(x), & \text{if } x \in S; \\ g^{-1}(x), & \text{if } x \notin S. \end{cases}$$

If $g(B) = A$, then g is an onto function (surjection). Since g is an injection, g is a bijection and the theorem holds. So we may assume that $g(B) \subsetneq A$. Note that $f(A \setminus g(B)) \subset B$, $g(f(A \setminus g(B))) \subset A$ and in general $(g \circ f)^n (A \setminus g(B)) \subseteq A$. Therefore

$$\begin{aligned} S &= (g \circ f)^0 (A \setminus g(B)) \cup (g \circ f)^1 (A \setminus g(B)) \cup (g \circ f)^2 (A \setminus g(B)) \cup \cdots \\ &= (A \setminus g(B)) \cup (g \circ f)(A \setminus g(B)) \cup (g \circ f)(g \circ f)(A \setminus g(B)) \cup \cdots \end{aligned}$$

is a subset of A.

Now we need to show that h is well-defined on A.

If $x \in S$, then $h(x) = f(x)$. If $x \notin S$, then $h(x) = g^{-1}(x)$. We do not know whether there is a $u \in B$ with $g(u) = x$. If there is, then $h(x) = u$ and h is well-defined. Note that $x \in A \setminus S$. Since $A \setminus g(B) \subseteq S$, $A \setminus S \subseteq g(B)$. Therefore if $x \notin S$, then $x = g(u)$ for some $u \in B$. Hence $h(x) = g^{-1}(x) = g^{-1}(g(u)) = u$ and h is well-defined.

Now it is left to show that h is a bijection from A onto B. We first prove that it is injective, that is, $h(x) = h(y)$ implies $x = y$. There are three cases to be considered.

Case(1) $x, y \in S$.

Then $h(x) = h(y)$ implies that $f(x) = f(y)$. Since f is injective, $x = y$.

Case(2) $x, y \notin S$.

Then $h(x) = h(y)$ implies that $g^{-1}(x) = g^{-1}(y)$. So $g(g^{-1}(x)) = g(g^{-1}(y))$ and $x = y$.

Case(3) $x \in S, y \notin S$ (or $x \notin S$, $y \in S$).

Then $h(x) = h(y)$ implies that $f(x) = g^{-1}(y)$. So $y = g(f(x)) = (g \circ f)(x)$. Let $x \in (g \circ f)^k(A \setminus g(B))$ for some integer $k \geq 0$. Then $y = g(f(x)) \in (g \circ f)^{k+1}(A \setminus g(B)) \subset S$, a contradiction. Hence Case(3) can never occur, and h is injective.

Now we prove that h is surjective, that is for every $y \in B$, there is an element $x \in A$ with $h(x) = y$. Let $y \in B$. Note that either $g(y) \in S$ or $g(y) \notin S$. If $g(y) \notin S$, then set $u = g(y)$. Since $u \notin S$, $h(u) = g^{-1}(u) = y$.

Suppose that $g(y) \in S$. Since

$$S = \bigcup_{n=0}^{\infty} (g \circ f)^n (A \setminus g(B))$$

there is a $x \in A \setminus g(B)$ and some integer $k \geq 0$ with $g(y) = (g \circ f)^k(x)$. If $k = 0$, then $g(y) \in A \setminus g(B)$, which is impossible for $g(y) \in g(B)$. Thus $k > 0$. Now $(g \circ f)^k(x) = (g \circ f)(g \circ f)^{k-1}(x) = (g \circ f)(z) = g(f(z))$ where $z = (g \circ f)^{k-1}(x)$. That is $g(y) = (g \circ f)(z) = g(f(z))$. Since g is injective, $f(z) = y$. Then since $z \in S$, so $h(z) = f(z) = y$ and hence, h is surjective. $\square$

2.2 Main Results

2.2.1 Motivation for Main Result 1

Firstly, recall the following result by Reid [21].

Theorem 1.3.3. *A torsion-free abelian group G can be written as the sum of two free abelian subgroups if and only if G is free or G has infinite rank.*

This means that a torsion-free abelian group of infinite rank can be realised as "almost free", by saying that, it is the sum of two free abelian subgroups. So, it is natural to ask whether one of the two free subgroups can be pure. The answer is affirmative. In fact, with stronger condition on the rank, both of the two free subgroups can be pure (see [2]).

Theorem 1.3.27. *For any torsion-free abelian group G of infinite rank κ, the following two statements are equivalent.*

(1) *G is the sum of two free subgroups, at least one of which is pure in G.*

(2) *G has a pure free subgroup of rank κ.*

Moreover, any subgroup as in (2) can serve as one of the subgroups as in (1).

Theorem 1.3.28. *For any torsion-free abelian group G of uncountable rank κ, the following two statements are equivalent.*

(1) *G is the sum of two pure free subgroups.*

(2) *G has a pure free subgroup of rank κ.*

Moreover, any subgroup as in (2) can serve as one of the subgroups in (1).

So we ask the following questions.

Question 2.2.1. Can the pure subgroup in (1) in Theorem 1.3.27 be replaced with basic subgroup?

Question 2.2.2. With stronger condition on the rank, can the two pure subgroups in (1) in Theorem 1.3.28 be replaced with two basic subgroups?

The answers are affirmative (see Theorem 2.2.5 and Corollary 2.2.6).

2.2.2 Main Result 1

Lemma 2.2.3. *Let G be a torsion-free abelian group of rank κ, and let E, B be subgroups of G. Suppose $G = E + B$, E is free abelian of rank κ, and G/B is divisible. Then the rank of $E \cap B$ is κ.*

Proof. Suppose $E \cap B$ is of rank $\mu < \kappa$. Being a subgroup of E, $E \cap B$ is freely generated by a set X of cardinality μ. Since E is free, fix a basis for it, and express all elements of X in terms of this basis. Since $\mu < \kappa$, fewer than κ basis elements occur in these expressions. Let E_1 be the subgroup of E generated by these basis elements and E_2 the subgroup generated by the rest of the basis for E. Then $E = E_1 \oplus E_2$, $E \cap B \subseteq E_1$, and E_2 is free abelian of rank κ.

Now we show that $(E_1 + B) \cap E_2 = \{0\}$. Let $e_1 + b = e_2 \in (E_1 + B) \cap E_2$, where $e_1 \in E_1$, $b \in B$, and $e_2 \in E_2$. Then $b = e_2 - e_1 \in B \cap E \subseteq E_1$. Therefore $e_2 \in E_1 \cap E_2 = \{0\}$, and hence $(E_1 + B) \cap E_2 = \{0\}$.

So $G = (E_1 + B) \oplus E_2$. Since G/B is divisible, we deduce that $G/(E_1 + B)$ is divisible. But then E_2 is divisible, a contradiction, for E_2 is free abelian and not zero.

Hence the rank of $E \cap B$ is κ. □

Corollary 2.2.4. *Let G be a torsion-free abelian group of rank κ, and E, B be subgroups of G. Suppose $G = E + B$, E is free abelian, and G/B is divisible. Then the rank of B is κ.*

Proof. If the rank of E is not κ, we can expand E to a free abelian group E' of rank κ. By Lemma 2.2.3, $E' \cap B$ is of rank κ. So B must be of rank κ. □

Theorem 2.2.5. *For any torsion-free abelian group G of uncountable rank κ, the following two statements are equivalent.*

(a) *G is the sum of a pure free subgroup and a basic subgroup.*

(b) *G has a pure free subgroup of rank κ.*

Moreover, any subgroup as in (b) *can serve as the pure free subgroup in* (a).

Proof. Since a basic subgroup in a torsion-free abelian group is a pure free subgroup, by Corollary 2.2.4, it is sufficient to show the implication from (b) to (a).

Since G has infinite rank κ and it is torsion-free, it has cardinality κ. So we can enumerate it as $G = \{g_\alpha \ : \ \alpha < \kappa\}$.

Let P be the set of all the primes. Consider the following set $I = \{\alpha \ : \ \alpha < \kappa\} \times P$. For each $i = (\alpha, p) \in I$, we set $g_i = g_\alpha$.

Note that the cardinality of I is κ. So there is a one-to-one correspondence ψ from the set $\{\alpha \ : \ \alpha < \kappa\}$ onto I. We may assume that $\psi(0) = (0, 2)$.

Let E be a pure free subgroup of G of rank κ. We shall define f_α recursively, so that the subgroup B generated by the set $\{f_\alpha \ : \ \alpha < \kappa\}$ is a pure free subgroup, $E + B = G$ and G/B is divisible.

Let Y be a basis of E. We may assume $g_0 = 0$. So we set $f_0 = e_0$ for some $e_0 \in Y$. Let q be a prime. Suppose $mf_0 \in qG$. Since E is pure and e_0 is an element in Y, we deduce that q divides m. Furthermore $f_0 = g_{\psi(0)} + e_0$.

Let $\beta < \kappa$. Suppose we have defined f_α for all $\alpha < \beta$, such that given any finite number of ordinals (say k), $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \beta$, and for any prime q, if $m_1 f_{\alpha_1} + \cdots + m_k f_{\alpha_k} \in qG$ then q divides m_i for $i = 1, \ldots, k$. We shall define f_β as follows:

Let F_β be the subgroup generated by $\{f_\alpha \ : \ \alpha < \beta\}$. Let q be a prime. Note that $(F_\beta + qG)/qG$ is a subspace of G/qG (view G/qG as vector space over $\mathbb{Z}/q\mathbb{Z}$). Furthermore, the set of cosets $\{f_\alpha + qG \ : \ \alpha < \beta\}$ can be chosen to be part of a basis for G/qG. Note that the cardinality of $(F_\beta + qG)/qG$ is at most $\max(\aleph_0, \beta)$ (here $\aleph_0$ denotes the cardinality of the set of natural numbers). Furthermore, the cardinality of $\bigcup_{q \in P}(\{q\} \times ((F_\beta + qG)/qG))$ is at most $\max(\aleph_0, \beta) < \kappa$ (for κ is uncountable).

Let $\psi(\beta) = (\gamma, p) \in I$. Then $g_{\psi(\beta)} = g_\gamma$. We shall distinguish two cases.

Case 1. Suppose $g_\gamma \notin (F_\beta + pG)$. We claim that we can find a $e_\beta \in Y$ such that $g_\gamma + pe_\beta \notin \bigcup_{q \in P}(F_\beta + qG)$. Suppose the contrary. Then for each $e \in Y$ there is a prime q_e such that $g_\gamma + pe \in (F_\beta + q_e G)$. So we may define a function $\phi : Y \to \bigcup_{q \in P}(\{q\} \times ((F_\beta + qG)/qG))$ by $\phi(e) = (q_e, g_\gamma + pe + q_e G)$. Now we show that ϕ is one-to-one. Suppose $\phi(e_1) = \phi(e_2)$. Then $q_{e_1} = q_{e_2}$ and $g_\gamma + pe_1 + q_{e_1} G = g_\gamma + pe_2 + q_{e_2} G$. This implies that $p(e_1 - e_2) \in q_{e_1} G \cap E = q_{e_1} E$ (for E is pure in G). Since E is free and $e_1, e_2 \in Y$, we deduce that either $e_1 = e_2$ or $q_{e_1} = p$. Suppose the latter holds. Then $g_\gamma \in (F_\beta + pG)$, a contradiction. Hence the former holds and ϕ is one-to-one. But then the cardinality of Y is less than or equal to the cardinality of $\bigcup_{q \in P}(\{q\} \times ((F_\beta + qG)/qG))$, a contradiction (for the cardinality of Y is κ). Hence there is $e_\beta \in Y$ such that $g_\gamma + pe_\beta \notin \bigcup_{q \in P}(F_\beta + qG)$. Set $f_\beta = g_\gamma + pe_\beta$.

Case 2. Suppose $g_\gamma \in (F_\beta + pG)$. Using a similar argument as in Case 1, we can find a $e_\beta \in Y$ such that $g_\gamma + e_\beta \notin \bigcup_{q \in P}(F_\beta + qG)$. Set $f_\beta = g_\gamma + e_\beta$.

In either case the following set $\{f_\alpha \ : \ \alpha < \beta + 1\} = \{f_\alpha \ : \ \alpha < \beta\} \cup \{f_\beta\}$ has the property that given any finite number of ordinals (say k), $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \beta + 1$, and for any prime q, if $m_1 f_{\alpha_1} + \cdots + m_k f_{\alpha_k} \in qG$ then q divides m_i for $i = 1, \ldots, k$.

Note that by construction the set $\{f_\alpha \ : \ \alpha < \kappa\}$ will have the property mentioned in the previous paragraph. So one can deduce from the property that the subgroup B generated by $\{f_\alpha \ : \ \alpha < \kappa\}$ is a pure free subgroup. It is left to show that G/B is divisible, i.e., $G = p^n G + B$ for all $p \in P$ and $n \in \mathbb{N}$ (here $\mathbb{N}$

denotes the set of natural numbers).

Suppose G/B is not divisible. Then there is a $g \in G \setminus \{0\}$ such that $g \in p^{n-1}G + B$ but $g \notin p^n G + B$, for some $p \in P$, $n \in \mathbb{N}$.

Let $g = p^{n-1}g' + b$ for some $g' \in G$ and $b \in B$. Let $g' = g_\alpha$ for some $\alpha < \kappa$. Then $g_{(\alpha,p)} = g_\alpha$. Let $\psi(\beta_1) = (\alpha, p)$. Note that $g_\alpha \notin (F_{\beta_1} + pG)$ (for otherwise $g \in p^n G + B$). By construction $f_{\beta_1} = g_\alpha + pe_{\beta_1}$ for some $e_{\beta_1} \in Y$. So $g_\alpha \in pG + B$. But then $g \in p^n G + B$, a contradiction. Hence G/B must be divisible.

This completes the proof of Theorem 2.2.5. □

Finally, Corollary 2.2.6 follows from Theorem 2.2.5 and Corollary 2.2.4.

Corollary 2.2.6. *For any torsion-free abelian group G of uncountable rank κ, the following two statements are equivalent.*

(a) *G is the sum of two basic subgroups.*

(b) *G has a basic subgroup of rank κ.*

Moreover, any subgroup as in (b) *can serve as a basic subgroup in* (a).

2.2.3 Motivation for Main Results 2

Firstly, recall the following result by Benabdallah and Irwin [1].

Theorem 1.3.29. *Let B be a basic subgroup of a p-group G without elements of infinite height. Then all B-high subgroups of G are direct sums of cyclic groups if and only if G is a direct sum of cyclic groups.*

Recall that "Disjoint" means that the intersection is $\{0\}$ (or it has trivial intersection), not $\varnothing$, as the latter is impossible for subgroups.

So, it is natural to ask whether there is an analogue of Theorem 1.3.29 for torsion-free abelian groups. A partial affirmative answer is given by Blass and Irwin [3].

Theorem 1.3.31. *Let G be a torsion-free abelian group such that*

(1) *G has a basic subgroup of infinite rank, and*

(2) *for every basic subgroup B of G, all B-high subgroups of G are free.*

Then G is free.

The answer is partial, in the sense that we require all B-high subgroups of G to be free for every basic subgroup B of G in Theorem 1.3.31, but in Theorem 1.3.29, we only require all B-high subgroups of G to be free for one basic subgroup B of G.

Recently, Blass and Shelah [4] constructed a non-free torsion-free abelian group G with a basic subgroup B such that all subgroups of G disjoint from B are free.

Theorem 1.3.35. *There exists an $\aleph_1$ separable torsion-free abelian group G of size $\aleph_1$, with a basic subgroup B of rank $\aleph_1$ such that all subgroups of G disjoint from B are free but G itself is not free abelian .*

The following question is suggested by the proof of Theorem 1.3.31.

Question 2.2.7. Let G be a torsion-free abelian group and $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$, be a descending chain of pure subgroups of G, such that for each i, every H_i-high subgroup of G is free abelian. When is G free abelian?

We shall give some sufficient conditions in which G is free abelian (see Theorem 2.2.11, Theorem 2.2.12 and Theorem 2.2.13).

2.2.4 Main Results 2

The main results in this section are Theorem 2.2.11, Theorem 2.2.12 and Theorem 2.2.13.

For each subgroup A of a torsion-free abelian group G, we define $C(A) = \{g \in G : mg \in A \text{ for some integer } m \neq 0\}$. Clearly, $C(A)$ is the minimal pure subgroup of G containing A. As always, we shall denote the set of natural numbers by $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the set of integers by $\mathbb{Z}$.

Recall the following result by Hill [11].

Theorem 1.3.10. *If the torsion-free abelian group G is the union of a countable chain of pure free subgroups, then G is free.*

Lemma 2.2.8. *Let G be a torsion-free abelian group and H be a subgroup of G. If A is a subgroup of G and $A \cap H = \{0\}$, then $C(A) \cap H = \{0\}$.*

Proof. Let $y \in C(A) \cap H$. Then $my \in A \cap H = \{0\}$ for some non-zero integer m. Since G is torsion-free, we conclude that $y = 0$. □

Lemma 2.2.9. *Let G be a torsion-free abelian group and H a subgroup of G. Every H-high subgroup of G is pure in G.*

Proof. Let B be an H-high subgroup of G. We shall show that $C(B) = B$. Since B is an B-high subgroup, it follows by Lemma 2.2.8 that $C(B) \cap H = \{0\}$. Thus $C(B) \subseteq B$. The equality $C(B) = B$ then follows as $B \subseteq C(B)$. □

Lemma 2.2.10. *Let G be a torsion-free abelian group and $\{H_i\}_{i\in\mathbb{N}}$ be a set of subgroups of G. Further suppose*

(a) $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$,

(b) *for all $i \in \mathbb{N}$, H_i/H_{i+1} is torsion-free (i.e. H_{i+1} is pure in H_i),*

(c) *for each $i \in \mathbb{N}$, all H_i-high subgroups of G are free, and*

(d) $\bigcap_{i\in\mathbb{N}} H_i = \{0\}$.

Then for each countable subset A of G, there exists an ascending chain of pure free subgroups of G, $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$, and integers $n_1 < n_2 < n_3 < \cdots$, such that $A \subseteq \bigcup_{i\in\mathbb{N}} M_i$, and $M_j \cap H_{n_j} = \{0\}$ for all $j \geq 1$.

Proof. Let M_0 be an H_1-high subgroup of G. By (c) and Lemma 2.2.9, M_0 is a pure free abelian subgroup of G.

Since A is countable, we can enumerate $A = \{a_1, a_2, a_3, \ldots\}$. If $a_1 \in M_0$, we set $M_1 = M_0$ and $n_1 = 1$. Suppose $a_1 \notin M_0$. Then $\langle a_1, M_0\rangle \cap H_1 \neq \{0\}$ and $m_1 a_1 + b_0 = h_1$ for some $b_0 \in M_0$, $h_1 \in H_1 \setminus \{0\}$, and non-zero integer m_1.

Since $\bigcap_{i\in\mathbb{N}} H_i = \{0\}$, there is an n_1 with $\langle h_1\rangle \cap H_{n_1} = \{0\}$. Furthermore $(M_0 \oplus \langle h_1\rangle) \cap H_{n_1} = \{0\}$ with $\langle h_1\rangle \cup H_{n_1} \subseteq H_1$. By Zorn's Lemma, there is an H_{n_1}-high subgroup of G, say M_1 containing $(M_0 \oplus \langle h_1\rangle)$. By (c) and Lemma 2.2.9, M_1 is a pure free abelian subgroup of G. Note that $a_1 \in M_1$ for $m_1 a_1 = h_1 - b_0 \in M_1$.

Now if $a_2 \in M_1$, set $M_2 = M_1$ and $n_2 = n_1 + 1$. Suppose $a_2 \notin M_1$. Then $\langle a_2, M_1 \rangle \cap H_{n_1} \neq \{0\}$. So $m_2 a_2 + b_1 = h_2$ for some $b_1 \in M_1$, $h_2 \in H_{n_1} \setminus \{0\}$ and non-zero integer m_2. Again from $\bigcap_{i \in \mathbb{N}} H_i = \{0\}$, we deduce that there is an $n_2 > n_1$ with $\langle h_2 \rangle \cap H_{n_2} = \{0\}$. Therefore $(M_1 \oplus \langle h_2 \rangle) \cap H_{n_2} = \{0\}$ with $\langle h_2 \rangle \cup H_{n_2} \subseteq H_{n_1}$. Again by Zorn's Lemma, (c) and Lemma 2.2.9, we conclude that there is a pure free abelian subgroup M_2 of G containing $(M_1 \oplus \langle h_2 \rangle)$ (M_2 is an H_{n_2}-high subgroup). Note that $a_2 \in M_2$.

Note that this process can be continued, that is we have a chain of pure free abelian subgroups of G, $M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$, and integers $n_1 < n_2 < n_3 < \cdots$, such that $a_i \in M_i$ and $M_i \cap H_{n_i} = \{0\}$ for all $i \geq 1$. So $A \subseteq \bigcup_{i \in \mathbb{N}} M_i$. □

Theorem 2.2.11. *Let G be a torsion-free abelian group and $\{H_i\}_{i \in \mathbb{N}}$ be a set of subgroups of G. Suppose all the hypotheses of Lemma 2.2.10 are satisfied. Then every countable subgroup of G is free.*

Proof. Let A be a countable subgroup of G. Then by Lemma 2.2.10, there exists an ascending chain of pure free subgroups of G, $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$, such that $A \subseteq \bigcup_{i \in \mathbb{N}} M_i$. Now $A = \bigcup_{i \in \mathbb{N}} (A \cap M_i)$, and $(A \cap M_1) \subseteq (A \cap M_2) \subseteq (A \cap M_3) \subseteq \cdots$, is an ascending chain of pure free subgroups of A. It then follows from Theorem 1.3.10 that A is free abelian. □

Note that if H_n is countable for some n, we can show that G is free abelian.

Theorem 2.2.12. *Let G be a torsion-free abelian group and $\{H_i\}_{i \in \mathbb{N}}$ be a set of subgroups of G. Suppose all the hypotheses of Lemma 2.2.10 are satisfied. Further suppose H_1 is countable. Then G is free abelian.*

Proof. By Lemma 2.2.10, there exists an ascending chain of pure free subgroups of H_1, $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$, and integers $2 \leq n_1 < n_2 < n_3 < \cdots$, such that $H_1 = \bigcup_{i \in \mathbb{N}} M_i$, and $M_j \cap H_{n_j} = \{0\}$ for all $j \geq 1$.

Let B_0 be an H_1-high subgroup of G. By Lemma 2.2.9 and the fact that all H_1-high subgroups of G are free, we deduce that B_0 is a pure free abelian subgroup of G. For each $j \geq 1$, $(B_0 \oplus M_j) \cap H_{n_j} = \{0\}$ (for M_j is a subgroup of H_1). Let $B_j = C(B_0 \oplus M_j)$, by Zorn's Lemma, there is an H_{n_j}-high subgroup of G containing B_j. So, B_j is a pure free abelian subgroup of G. Furthermore, $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$ is an ascending chain of pure free subgroups.

Now we shall show that $G = \bigcup_{i \in \mathbb{N}_0} B_i$. Assume, for a contradiction, that there is an element $g \in G \setminus (\bigcup_{i \in \mathbb{N}_0} B_i)$. Then $\langle B_0, g \rangle \cap H_1 \neq \{0\}$ (for B_0 is an H_1-high subgroup of G). This implies that $a + mg = h_1$ for some $a \in B_0$, $h_1 \in H_1 \setminus \{0\}$ and non-zero integer m. Since $H_1 = \bigcup_{i \in \mathbb{N}} M_i$, $h_1 \in M_r$ for some integer r. So $mg = h_1 - a \in \langle B_0, M_r \rangle \subseteq B_r$, and $g \in B_r$, a contradiction. Hence $G = \bigcup_{i \in \mathbb{N}_0} B_i$ and by Theorem 1.3.10, G is free abelian. $\square$

Now if we strengthen the condition (d) in Lemma 2.2.10, we can prove that G is free abelian, even without the countability condition on H_1.

Theorem 2.2.13. *Let G be a torsion-free abelian group and $\{H_i\}_{i \in \mathbb{N}}$ be a set of subgroups of G. Let W_1 be a maximal independent subset of H_1. Suppose*

(a) $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$,

(b) *for all $i \in \mathbb{N}$, H_i/H_{i+1} is torsion-free (i.e. H_{i+1} is pure in H_i),*

(c) *for each i, all H_i-high subgroups of G are free, and*

(d) *for each $w \in W_1$, there is an integer $n \geq 2$ such that for any finite subset $S \subseteq W_1 \setminus \{w\}$,*

$$mw + \sum_{s \in S} m_s s \in H_n, \ (m, m_s \in \mathbb{Z})$$

implies that $m = 0$.

Then G is free abelian.

Proof. For each $w \in W_1$, we define $u(w)$ to be the least positive integer such that (d) holds, that is if $mw + \sum_{s \in S} m_s s \in H_{u(w)}$ for a subset $S \subseteq W_1 \setminus \{w\}$, then $m = 0$. For each integer $n \geq 2$, set $X_n = \{w \in W_1 \ : \ u(w) = n\}$. Note that $\bigcup_{n \geq 2} X_n$ is a partition of W_1.

For each integer $i \geq 2$, set $B_i = \langle \bigcup_{2 \leq n \leq i} X_n \rangle$. Then, by (a) and (d) we deduce that $B_i \cap H_i = \{0\}$.

Let M_1 be an H_1-high subgroup of G. By (c) and Lemma 2.2.9, M_1 is a pure free abelian subgroup of G. Note that for each integer $i \geq 2$, $\langle M_1, B_i \rangle \cap H_i = \{0\}$. Set $M_i = C(\langle M_1, B_i \rangle)$. By Lemma 2.2.8, $M_i \cap H_i = \{0\}$. By Zorn's Lemma, there is an H_i-high subgroup of G containing M_i. By (c) and the fact that a subgroup of a free abelian group is free, we conclude that M_i is a pure free abelian subgroup of G. Furthermore $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$.

We shall show that $G = \bigcup_{i \in \mathbb{N}} M_i$. Assume, for a contradiction, that there is an element $g \in G \setminus (\bigcup_{i \in \mathbb{N}} M_i)$. Then $\langle M_1, g \rangle \cap H_1 \neq \{0\}$ (for M_1 is an H_1-high subgroup of G). This implies that $a + mg = h_1$ for some $a \in M_1$, $h_1 \in H_1$ and non-zero integer m. Since W_1 is a maximal independent subset of H_1, there is a non-zero integer m' such that $m'h_1 \in \langle W_1 \rangle$. So there is an integer $r \geq 2$ such that $m'h_1 \in B_r$. But then $m'mg = m'h_1 - m'a \in \langle M_1, B_r \rangle$, and thus $g \in M_r$, a contradiction. Hence $G = \bigcup_{i \in \mathbb{N}} M_i$ and by Theorem 1.3.10, G is free abelian. $\square$

2.3 Conclusion

At this moment of time, Problem 1.3.36, that is

'When is a torsion-free abelian group free abelian?'

still has no satisfactory answer. One of my future plans is to give a satisfactory answer to Problem 1.3.36. To do this, I guess one needs to find an 'efficient' algorithm to decide on whether a torsion-free abelian group is free abelian.

On the other hand, an easier problem is to answer Question 2.2.7. Up to now, I only manage to obtain sufficient conditions for G to be free abelian. The answer is not satisfactory yet. So, I guess I will be busy working on these problems in some years to come.