PRICING OF INTEREST RATE DERIVATIVES

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FACULTY OF SCIENCE
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ABSTRACT

A numerical method is proposed to find the time-t bond price of a zero-coupon bond with maturity time T under the Cox, Ingersoll and Ross (CIR) model described by a Lévy process. When the underlying distribution in the Lévy process is normal, the numerical results thus found for the bond prices are fairly close to the corresponding theoretical values. The similar numerical method is next applied to evaluate the bond price of a zero-coupon bond with maturity time T under the Chan, Karolyi, Longstaff and Sanders (CKLS) model described by a Lévy process. The numerical results obtained show that bond price decreases slightly when the parameter γ in the CKLS model increases, and the variation of the bond price is slight as the non-normality of the underlying distribution in the Lévy process varies. A method is also proposed for pricing the European call option with maturity T and strike price K written on a zero-coupon bond with maturity $S > T$. The numerical results thus found show that option price decreases as the parameter γ in the CKLS model increases, and the variation of the option price is slight when the non-normality of underlying distribution in the Lévy process becomes more severe. So far, the parameters in the interest rate models are assumed to be constants. The restriction on constant parameters is lifted by describing the parameters as ones which follow a multivariate non-normal distribution. Compared to the CKLS model with fixed parameters, the CKLS model with stochastic parameters is found to yield more reasonable prediction interval for the future interest rate.

ABSTRAK

Satu kaedah berangka dicadangkan untuk mencari harga pada masa t bagi bon kupon-sifar dengan masa matang T di bawah model Cox, Ingersoll and Ross (CIR) yang digambarkan oleh proses Lévy. Apabila taburan pendasar di dalam proses Levy adalah normal, keputusan berangka yang didapati bagi harga bon agak dekat dengan nilai teori. Kaedah berangka yang serupa digunakan seterusnya untuk menilai harga bon kupon-sifar dengan masa matang T di bawah model kadar faedah Chan, Karolyi, Longstaff dan Sanders (CKLS) yang digambarkan oleh proses Lévy. Keputusan berangka yang diperolehi menunjukkan bahawa harga bon berkurang sedikit apabila parameter γ dalam model CKLS meningkat, dan perubahan harga bon adalah sedikit apabila ketaknormalan yang berada di dalam taburan pendasar dalam proses Lévy diubah. Satu kaedah juga dicadangkan untuk menilai harga opsyen panggilan gaya Eropah dengan masa matang T dan harga laksana K yang berasaskan bon kupon-sifar dengan masa matang $S > T$. Keputusan berangka yang diperolehi menunjukkan bahawa harga opsyen berkurang apabila parameter γ dalam model CKLS meningkat, dan perubahan harga opsyen adalah sedikit sahaja apabila ketaknormalan yang berada di dalam taburan pendasar dalam proses Lévy menjadi lebih teruk. Setakat ini, parameter model kadar faedah dianggap sebagai pemalar. Kekangan ke atas parameter boleh dimansuhkan dengan menganggap bahawa parameter bertaburan tak normal multivariat. Berbanding dengan model CKLS yang mengandaikan parameter sebagai pemalar, model CKLS dengan parameter stokastik adalah didapati menghasilkan selang ramalan yang lebih munasabah untuk kadar faedah pada masa kelak.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Interest-rate-contingent claims such as bond options, caps, swaptions, captions and mortgage-backed securities are getting popular nowadays. The valuation of these instruments usually utilizes interest rate models.

The fluctuation of interest rates is affected by many factors, such as the general economic conditions, inflation, monetary policy by government, supply and demand for bonds and others. These factors are random and they lead to unpredictable future in financial market. Due to these factors, the fluctuation of interest rates is considered as a primary source of market risk. The market risk can be minimized if we understand the behavior and fluctuation of interest rates. Therefore, various types of models are implemented to model the fluctuation of interest rates.

One-factor interest rate models are a popular class of interest rate models. In general, the one-factor interest rate model can be described by the following stochastic differential equation (SDE):

$$dr = \mu(r,t)\,dt + \omega(r,t)\,dz,\ \ t \ge 0. \tag{1.1}$$

where $\mu(r,t)$ and $\omega(r,t)$ determine the behavior of the interest rate r at time t and dz is the increment of a standard Brownian motion. Examples of one-factor interest rate models are the Vasicek model (see Vasicek (1977)), the Cox, Ingersoll and Ross (CIR) model (see Cox et al. (1985)), the Ho-Lee model (see Ho and Lee (1986)), the Hull-White model (see Hull and White (1990)), the Chan, Karolyi, Longstaff and Sanders (CKLS) model (see Chan et al. (1992)) and others.

In our research works, we focus on CIR and CKLS models. The CIR model is chosen as it is one of the most widely accepted one-factor interest rate models in mathematical finance. The CIR model's tractability property in bond pricing and its interesting stochastic characteristics make the model quite popular. The purpose of choosing the CKLS model in our research work is that it covers many types of one-factor interest rate models.

The description of the Vasicek, CIR, Ho-Lee, Hull-White and CKLS models under the risk neutral measure *Q* are respectively given below.

The Vasicek model

Under the risk neutral measure *Q*, the Vasicek model follows the following SDE:

$$dr = a(m - r)dt + \sigma\, dz\,,\ t \geq 0, \tag{1.2}$$

where $r = r(t)$ is the interest rate at time *t*, *a* the drift factor, $m > 0$ the long term average rate for *r*, σ the volatility factor and dz is the increment of a standard Brownian motion. The drift factor, *a* governs the speed of mean reversion. The parameter *m* reflects the mean around which mean reversion occurs.

In mean reversion, if the interest rate r is above the long run mean *m*, then the coefficient of *a* performs a negative drift such that the interest rate is pulled down back to the long run mean *m*. Likewise, if the interest rate r is below the long run mean *m*, then the coefficient of *a* performs a positive drift such that the interest rate is pulled up back to the long run mean *m*. Thus, the coefficient of *a* is a speed of adjustment of interest rate towards *m* when it wanders away (see Zeytun and Gupta (2007)). Figure 1.1 shows the occurrence of the mean reversion.

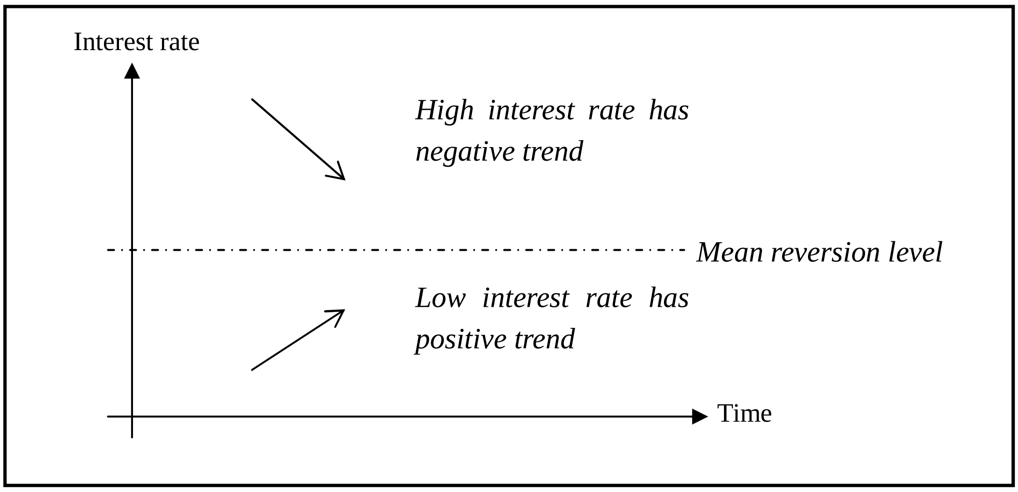

Figure 1.1 The occurrence of the mean reversion.

However, there are some drawbacks of the Vasicek model. The dependence on the one-factor interest rate limits the possible shapes of the yield curve where the theoretical yield curve does not correspond to the market yield curve. Another shortcoming is the yield curves of all maturities are perfectly correlated, which is an impractical assumption about the behavior of the yield curve. The most significant drawback is that the interest rate is theoretically possible to become negative which obviously does not make sense in financial market. This main shortcoming was then fixed in the CIR model.

The Cox, Ingersoll and Ross (CIR) model

Eight years after the Vasicek model was developed, Cox et al. (1985) introduced a model named as the CIR model to solve the drawback of the Vasicek model. Under the risk neutral measure Q, the CIR model follows the SDE given by

$$dr = a(m - r)dt + \sigma\sqrt{r}\,dz,\ t \geq 0. \tag{1.3}$$

In the CIR model, the term $\sqrt{r}$ is imposed and its standard deviation factor becomes $\sigma\sqrt{r}$. This standard deviation factor guarantees a non-negative interest rate and hence eliminates the main drawback of the Vasicek model.

The Ho-Lee model

Ho and Lee (see Ho and Lee (1986)) proposed the first model of the whole term structure of interest rates as a random process over time. They constructed the model in the form of a binomial tree of bond prices with two parameters, which are the short rate standard deviation and the market price of the risk of the short rate in discrete time. Under the risk neutral measure Q, the Ho-Lee model follows the SDE given by

$$dr = \theta(t)dt + \sigma\,dz,\ t \geq 0,$$

where $\theta(t)$ is a function of t and σ is a nonnegative constant. The variable $\theta(t)$ defines the average direction that r moves at time t. The main drawback of this model is that it does not exhibit the mean reversion as in the Vasicek model.

The Hull-White model

In 1990, Hull and White (see Hull and White (1990)) proposed a one-factor interest rate model that contains three time-varying parameters $\theta(t)$, $a(t)$ and $\sigma(t)$. By adding $\theta(t)$ to the process for interest rate *r*, and allowing the parameters *a* and σ to be functions of time *t*, Hull and White presented the extensions of the Vasicek and CIR models as follows:

Hull-White model (Extended Vasicek)

Under the risk neutral measure *Q*, the Hull-White model (extended Vasicek) follows the SDE given by

$$dr = [\theta(t) + a(t)(m - r)]\,dt + \sigma(t)\,dz,\ t \geq 0,$$

where $a(t)$ and $\sigma(t)$ are functions of *t*. As mentioned earlier, the Ho-Lee model does not deal with the mean reversion property, and by the Hull-White model (extended Vasicek), Hull and White incorporated the mean reversion property.

Hull-White model (Extended CIR)

In the CIR model (see Equation (1.3)), the parameters *a*, *m* and σ are assumed to be some constants. Hull and White extended the CIR model with the incorporations of $\theta(t)$, $a(t)$ and $\sigma(t)$. Under the risk neutral measure *Q*, the Hull-White model (extended CIR) follows the SDE given by

$$dr = [\theta(t) + a(t)(m - r)]\,dt + \sigma(t)\sqrt{r}\,dz,\ t \geq 0.$$

The Chan, Karolyi, Longstaff and Sanders (CKLS) model

In 1992, Chan et al. (1992) were the first to come out with the general one-factor interest rate model. The generalized model is known as the CKLS model. Under the risk neutral measure *Q*, the generalized CKLS interest rate model follows the SDE given by

$$dr = a(m-r)dt + \sigma r^{\gamma}\, dz,\ t \geq 0, \tag{1.4}$$

where γ is a positive constant that indicates the power of the interest rate at time *t*. Chan et al. (1992) showed that the value of γ is the most important parameter to discriminate the one-factor interest rate models that have been studied in finance. For instance, the CKLS model becomes the Vasicek model and the CIR model when the value of γ takes 0 and 0.5 respectively.

The one-factor interest rate models play a vital role in the pricing of bonds. In a very broad sense, a bond or conventionally known as a fixed-income security, which is a formal contract that pays interest known as coupon at fixed time intervals during the lifetime of the contract and principal at the maturity time *T*. The principal of a bond contract is also known as face value or par value which is paid when the contract expires at the maturity time *T*.

There are two main types of bonds, namely coupon bond and zero-coupon bond. A coupon bond is a type of bond which issues interest payment during the lifetime of the bond period. If there are no periodic coupon payments, the bond is known as a zero-coupon bond. In other words, an investor can buy a zero-coupon bond at a price cheaper than its face value for a maturity time *T*. The equation for finding bond price of a zero-coupon bond is given below.

Let $P(t,T)$ denote the bond price at time t of a zero-coupon bond that matures at time T. The bond price $P(t,T)$ can be expressed as an expectation E^Q under the risk neutral measure Q as shown below:

$$P(t,T) = E^Q\left[e^{-\int_t^T r(s)ds}\right], \tag{1.5}$$

where $r(s)$ is the interest rate at time s, $t \le s \le T$.

In the Vasicek model, the original derivation of the explicit formula for the bond price was based on solving the partial differential equation (PDE). Duffie and Kan (1996) provided a further characterization of this PDE. They showed that the bond price has an exponential affine form if some Ricatti equations have solutions to the required maturity. Alvarez (1998) considered the valuation of zero-coupon bonds with maturity T in the presence of an affine term structure, linear drift and affine diffusion coefficient. In the paper, he also showed that the term structure of interest rates can be represented explicitly in terms of the fundamental solutions of an associated ordinary second order linear differential equation with variable coefficients. Elliott and Van der Hoek (2001) proposed a new method of solving the problem studied by Duffie and Kan (1996). Their proposed method showed that the evaluation of the bond price under the Vasicek model is done by integrating the linear ordinary differential equation and the Ricatti equations are not needed. Chacko and Das (2002) introduced the exponential jump-extended Vasicek model, which is also called the Vasicek-EJ model. In their paper, they considered the bond pricing under their Vasicek-EJ model and derived an analytical solution to the bond price. The main advantage of the Vasicek-EJ model is its flexibility in allowing different distributions for the upward jumps and downward jumps. Mamon (2004) discussed three approaches in obtaining the closed-form solution of the Vasicek bond pricing problem, which are a derivation based solely on the distribution of the short rate process, a method of solving the PDE of the bond price and the consideration of the bond pricing problem within the Heath-Jarrow-Morton (HJM) framework in which the analytic solution follows directly from the short rate dynamics under the forward measure. Stehlikova and Sevcovic (2009) proposed an explicit formula for a higher order approximation for pricing zero-coupon bonds. The proposed approximation was tested numerically for a class of one-factor interest rate models and it showed a high accuracy for a reasonable range of time horizons. Honda et al. (2010) described the influence of non-Gaussianity and the dependency of innovations in zero-coupon bond pricing under the discretized version of Vasicek model.

In recent years, regime-switching models have become more and more important in different branches of modern financial economics. Elliott and Tak (2009) established a Markovian-modulated exponential affine bond price formula with coefficients given in terms of fundamental matrix solutions of linear matrix differential equations when the short rate process is described by a Markovian regime-switching CIR model or a Markovian regime-switching Hull-White model. Tak (2010) employed the concept of stochastic flows to derive an exponential affine form of the bond price when the short rate process is governed by a Markovian regime-switching jump-augmented Vasicek model. In the paper, a representation of the exponential affine form of the bond price in terms of fundamental matrix solutions of linear matrix differential equations was provided.

Recently, Lo (2013) applied the Lie-algebraic approach to solve bond pricing problem in one-factor interest rate models. He derived a general analytical closed-form bond pricing for four types of one-factor interest rate models, namely Vasicek model, CIR model, double square-root model and Ahn-Gao model. In his paper, he also incorporated time-varying model parameters into the derivation of the bond price formula.

There are many methods including the analytical and the numerical methods for evaluating the bond price. Despite the fact that analytical solutions are found for the bond price based on the Vasicek, CIR, Ho-Lee and Hull-White models, no analytical solutions are available for the bond price based on the CKLS model. Thus in order to evaluate the bond price based on the CKLS model, numerical approaches are required. The following are some numerical methods for evaluating the bond price of the zero-coupon bond when the CKLS model is described by a Brownian motion

Barone-Adesi et al. (1997) introduced a numerical approach call the Box method for valuing the bond price of a zero-coupon bond. In Barone-Adesi et al. (1999), the Box method was found to lead to more accurate bond prices in the case when the parameter γ in the CKLS model takes the value of 0.5 when compared with the traditional Crank Nicholson Scheme (see Crank and Nicolson (1947)). The Box method was applied later using historical interbank estimates of the CKLS model obtained for a few countries to calculate the bond prices (see Nowman and Sorwar (2001, 2003, 2005)). Sorwar et al. (2007) found that in specific cases of the CKLS model where analytical prices are available, the obtained bond prices using the Box method produced more accurate results than those found from the traditional Crank Nicolson method (see Crank and Nicolson (1947)). Choi and Wirjanto (2007) proposed an analytic approximation formula for pricing zero-coupon bond when the CKLS model is described by using a Brownian motion. Their bond price results showed a high accuracy level provided that the maturity time was not longer than 5 years. Tangman et al. (2011) developed another numerical approach for approximating prices under CKLS models described by using a Brownian motion. The new approach, known as the exact exponential time integration (ETI) method, gives a higher accuracy level in bond pricing when it is compared with the Box method.

Many researchers work on the interest rate models that are described by using a Brownian motion. In practice, it would be more suitable to use a Lévy process rather than a Brownian motion as the non-normal distribution in the Lévy process can better model the fat-tailed characteristic exhibited by the financial data. Hence, we are motivated to carry out research in finding the bond price when the one-factor interest rate models described by a Lévy process. The researches on bond price findings thus found have been reported in Khor and Pooi (2009) and Khor et al. (2010). In Khor and Pooi (2009), a numerical method was proposed for evaluating the bond price under the CIR model described by using a Lévy process. When the underlying distribution in the Lévy process is normal, the numerical result found for the bond price in the CIR model is fairly close to the corresponding theoretical value. In Khor et al. (2010), a numerical method was proposed for finding bond price of a zero-coupon bond with maturity T under the CKLS model described by using a Lévy process. The numerical results show that bond price tends to decrease slightly when the parameter γ in the CKLS model increases. The numerical methods in Khor and Pooi (2009) and Khor et al. (2010) are described respectively in Chapters 2 and 3.

Next consider the bond option. A bond option is a trading derivative contract whereby the underlying asset is a bond. The bond option contract allows the owner of a bond to buy or sell the bond at a fixed price, known as strike price, either on or prior to the expiry date of the option.

Basically, there are two types of options, namely call bond option and put bond option. A call bond option gives authority to an owner of a bond to buy the bond whereby a put bond option gives authority to an owner of a bond to sell the bond. Options can be either American option or European option. The difference between them is that an American option can be exercised before or at the expiration time of the bond contract, whereas a European option can only be exercised at the expiration time.

Under the risk neutral measure *Q*, the price of the call option with maturity *T* and strike price *K* written on a zero-coupon bond that matures at time *S* is given by

$$C = E^{Q}\left\{\exp\left[-\int_0^T r(s)ds\right]\left[B_T^S - K\right]^+\right\}, \tag{1.6}$$

where B_T^S denotes the bond price at time $t = T$ of a zero-coupon bond that matures at time $t = S$, $0 \le T \le S$ and $\left[B_T^S - K\right]^+ = \max\left\{\left[B_T^S - K\right], 0\right\}$ is the payoff function.

A number of research studies are carried out to investigate the call option price written on a zero-coupon bond when the underlying distribution is a normal distribution. Nowman and Sorwar (2003) used the Box method developed by Barone-Adesi et al. (1997) to value the call option prices based on the U.K. interbank rates for different maturities using the CKLS model. Nowman and Sorwar (2005) again applied the Box method to calculate the call option prices using the historical Euro-currency estimates of the CKLS model obtained for a few countries. Sorwar and Mozumder (2010) derived the Expanded Box method to price the call option. Tangman et al. (2011) proposed a numerical method, known as the exact exponential time integration (ETI), to calculate the bond option prices under the CKLS model. The computed results are comparable with those found from the Box method. Recently, Yang and Tak (2013) derived an integral representation for the pricing formula of an European option written on a zero-coupon bond under a Markov regime-switching Hull-White model. Their numerical results for the prices of bond option with different strike prices and expiry dates have motivated them to consider the regime-switching extension of the Hull-White model in pricing bond option.

In this thesis, we propose a numerical method for deriving the European call option price under the CKLS model described by a Lévy process. The method thus proposed has been reported in Khor et al. (2013). The description of this method will also be given in Chapter 4.

Apart from that, some research works also have been carried out on the one-factor interest rate models with the consideration of stochastic parameters. Hull and White (1988) proposed a Brownian motion model of which the volatility is modeled by another Brownian motion process. Andersen and Lund (1997) showed that the CIR model with an added stochastic volatility factor provides a good characterization of the short rate process. Vetzal (1997) provided empirical evidence which strongly indicates that the incorporating stochastic volatility significantly improves the ability of the models to fit movements in short term interest rates. Duan and Simonato (1999) proposed a unified state-space formulation for parameter estimation of exponential-affine term structure models. The proposed method uses an approximate linear Kalman filter which only requires specifying the conditional mean and variance of the system.

In this thesis, we consider the CKLS model of which the parameters a, m, σ and γ are all treated as random variables. The CKLS model with stochastic parameters is found to be a more reasonable model for predicting the future interest rate. The results on the CKLS model with stochastic parameters will be given in Chapter 5.

1.2 Layout of the thesis

The remaining of this thesis is organized as follows: Chapter 2 illustrates the methods for evaluating the bond price of a zero-coupon bond under the CIR model described by a Lévy process. Chapter 3 explains the methods for evaluating the bond price of a zero-coupon bond under the CKLS model described by a Lévy process. Chapter 4 describes the bond option pricing under the CKLS model described by a Lévy process. Chapter 5 gives the prediction of interest rates using the CKLS model with stochastic parameters. The thesis ends with some concluding remarks.

CHAPTER 2

BOND PRICE UNDER THE CIR MODEL

2.1 Introduction

In this chapter, the time-t bond price of a zero-coupon bond with maturity T under the CIR model described by a Lévy process is evaluated. Section 2.2 presents the proposed method based on moments, simulation method, and the analytical formula for evaluating the bond price under the CIR model. The key steps in the proposed method include (i) the discretization of the model; (ii) the application of recursive procedures to find the first four moments of the random variable V given by the integration of the short rate from t to T; (iii) the derivation of approximate distribution for V based on its first four moments; (iv) the evaluation of the expected value of e^{-V} to get the bond price. In Section 2.3, the numerical results based on the proposed method for the bond price in the CIR model are presented. These numerical results are fairly close to the corresponding theoretical values in the original CIR model. This means that non-normality in the distributions of the increments has very slight effect on the zero-coupon bond price. Section 2.4 gives some conclusions.

2.2 Methods for evaluating the bond price

This section describes three methods for evaluating the bond price of a zero-coupon bond under the CIR model. In this model, under the risk neutral measure *Q*, the instantaneous short rate follows the SDE.

$$dr = a(m-r)dt + \sigma\sqrt{r}\,dw,\ t \ge 0, \tag{2.1}$$

where *dw* is a Lévy process. The definition of Lévy process is as follows:

A stochastic process $X = (X_t)_{0\le t<\infty}$ with $X_0 = 0$ almost surely is a Lévy process if

(i) X has increments independent of the past:

For any $0 \le t_1 < t_2 < ... < t_n < \infty$, $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.

(ii) X has stationary increments:

For any $0 \le s < t < \infty$, $X_{t+s} - X_t$ does not depend on *t*.

(iii) X_t is continuous in probability:

$\lim_{t\to s} X_t = X_s$ where the limit is taken in probability.

Let Δt be a small increment in time *t*. The model for the interest rate r in Equation (2.1) may be approximated by the method of discretization. The discretized version of Equation (2.1) is

$$r_k = r_{k-1} + a(m - r_{k-1})\Delta t + \sigma\sqrt{r_{k-1}}\,w_k\sqrt{\Delta t},\quad k \ge 1. \tag{2.2}$$

In Equation (2.2), $r_k = r(k\Delta t)$, $w_1, w_2, ...$ are independent and identically distributed with mean 0 and variance 1.

In this thesis, we assume that w_k has a type of non-normal distribution called the quadratic-normal distribution. This non-normal distribution, as noted in Page 10 of

Chapter 1, can model the fat-tailed characteristic exhibited by the financial data. We note that after the change of the distribution of w_k from standard normal to quadratic-normal, the stationary property of the increments of the process is still preserved. The description of the quadratic-normal distribution is as follows:

Let μ and $\boldsymbol{\lambda} = (\lambda_1 \quad \lambda_2 \quad \lambda_3)^T$ be constants and consider the following transformation of the standard normal random variable *z* to the random variable $r^{(q)}$:

$$r^{(q)} = \begin{cases} \mu + \lambda_1 z + \lambda_2 \left(z^2 - \dfrac{(1+\lambda_3)}{2} \right) & , z \geq 0 \\ \mu + \lambda_1 z + \lambda_2 \left(\lambda_3 z^2 - \dfrac{(1+\lambda_3)}{2} \right) & , z < 0 \end{cases} \tag{2.3}$$

When $\boldsymbol{\lambda}$ is such that $r^{(q)}$ is a one-to-one function of *z*, the random variable $r^{(q)}$ is said to have a quadratic-normal distribution with parameters μ and $\boldsymbol{\lambda}$, we may write $r^{(q)} \sim QN(\mu, \boldsymbol{\lambda})$ (see Pooi (2003)). With suitable choice of the parameters, the quadratic-normal distribution will have narrow waist and fat tails. Thus this distribution is suitable for modeling the interest rate data which are known to exhibit narrow waist and fat tails characteristics.

2.2.1 Method based on moments

In the method based on moments, we first use Equation (2.2) to obtain the following results on the moments of r_k conditional on the value of r_{k-1}.

$$E(r_k \mid r_{k-1}) = c r_{k-1} + d \,, \tag{2.4}$$

$$E(r_k^2 \mid r_{k-1}) = e r_{k-1}^2 + f r_{k-1} + g \,, \tag{2.5}$$

$$E(r_k^3 \mid r_{k-1}) = p r_{k-1}^3 + q r_{k-1}^2 + s r_{k-1} + v \,, \tag{2.6}$$

$$E(r_k^4 \mid r_{k-1}) = \kappa r_{k-1}^4 + u r_{k-1}^3 + w r_{k-1}^2 + x r_{k-1} + y \,, \tag{2.7}$$

where

$c = 1 - a\Delta t$, $d = am\Delta t$, $e = (1 - a\Delta t)^2$, $f = 2am\Delta t(1 - a\Delta t) + \sigma^2 \Delta t$, $g = (am\Delta t)^2$,

$p = (1 - a\Delta t)^3$, $q = 3\Delta t(1 - a\Delta t)\left[am(1 - a\Delta t) + \sigma^2\right]$, $s = 3am\Delta t^2 \left[am(1 - a\Delta t) + \sigma^2\right]$,

$v = (am\Delta t)^3$, $\kappa = (1 - a\Delta t)^4$, $u = \Delta t(1 - a\Delta t)^2\left[4am(1 - a\Delta t) + 6\sigma^2\right]$,

$w = \Delta t^2\left[6(am)^2(1 - a\Delta t)^2 + 12am(1 - a\Delta t)\sigma^2 + E(z_k^4)\sigma^4\right]$ and

$x = 4(1 - a\Delta t)(am\Delta t)^3 + 6(am)^2\Delta t^3\sigma^2$ and $y = (am\Delta t)^4$.

The iterative formulas which are similar to those given by Equations (2.4) – (2.7) have been used in Ng et al. (2008) and Pooi et al. (2008) for finding the distribution of the interest rate r . Presently, we show that this type of iterative formulas may also be used to find the bond price.

Suppose we approximate the integral which appears in the right side of Equation (1.5) by

$$V = (r_1 + r_2 + ... + r_k)\Delta t . \qquad (2.8)$$

Furthermore, let $\{r_{k-1}\}$ denote the vector $(r_1, r_2, ..., r_{k-1})$ and perform the following iterative process.

Applying Equation (2.4) with $k = K$ to find the expected values of V conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$E(V \mid \{r_{K-1}\}) = \left[\sum_{i=1}^{K-2} r_i + (1 + c)r_{K-1} + d\right]\Delta t \quad ,$$

which is a linear function of $r_1, r_2, ..., r_{K-1}$. If the process of applying Equation (2.4) for $k = K - 1, K - 2, ..., 1$ is continued, we can finally obtain $E(V)$ in terms of *c* and *d*.

Applying Equations (2.4) and (2.5) with $k = K$ to find the expected value of V^2 conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$E\left(V^2 \middle| \{r_{K-1}\}\right) = \Bigg[\sum_{i=1}^{K-2} r_i^2 + (1+e+2c)\, r_{K-1}^2 + 2\sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i\, r_j + 2(1+c)\, r_{K-1} \sum_{i=1}^{K-2} r_i$$

$$+ 2d \sum_{i=1}^{K-2} r_i + (f+2d)\, r_{K-1} + g \Bigg] (\Delta t)^2 ,$$

which is a quadratic function of $r_1, r_2, ..., r_{K-1}$.Continuing the process of applying Equations (2.4) and (2.5) with $k = K-1, K-2, ..., 1$, we can finally obtain $E\left(V^2\right)$ in terms of *c*, *d*, *e*, *f* and *g*.

If we apply Equations (2.4), (2.5) and (2.6) with $k = K$ in finding the expected value of V^3 conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$E\left(V^3 \middle| \{r_{K-1}\}\right) = \Bigg\{ \sum_{i=1}^{K-2} r_i^3 + [1+p+3(c+e)]\, r_{K-1}^3 + 3\sum_{i=1}^{K-2} \sum_{\substack{j=1 \\ i \neq j}}^{K-2} r_i^2 r_j$$

$$+ 3(1+c)\, r_{K-1} \sum_{i=1}^{K-2} r_i^2 + 3(1+e+2c)\, r_{K-1}^2 \sum_{i=1}^{K-2} r_i + 3d \sum_{i=1}^{K-2} r_i^2$$

$$+ [q+3(d+f)]\, r_{K-1}^2 + 6\sum_{i=1}^{K-4} \sum_{j=i+1}^{K-3} \sum_{l=j+1}^{K-2} r_i r_j r_l$$

$$+ 6(1+c)\, r_{K-1} \sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i r_j + 6d \sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i r_j$$

$$+ 3(f+2d)\, r_{K-1} \sum_{i=1}^{K-2} r_i + 3g \sum_{i=1}^{K-2} r_i + (s+3g)\, r_{K-1} + v \Bigg\} (\Delta t)^3 ,$$

which is a cubic function of $r_1, r_2, ..., r_{K-1}$. Continuing the process of applying Equations (2.4), (2.5) and (2.6) with $k = K-1, K-2, ..., 1$, $E\left(V^3\right)$ in terms of *c*, *d*, *e*, *f*, *g*, *p*, *q*, *s* and *v* can be finally obtained.

Similarly if we apply Equations (2.4), (2.5), (2.6) and (2.7) with $k = K$ in finding the expected value of V^4 conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$
\begin{aligned}
E\left(V^4\middle|\{r_{K-1}\}\right)=\Bigg\{&\sum_{i=1}^{K-2} r_i^4 + (1+\kappa+4c+4p+6e)\,r_{K-1}^4 + 4\sum_{i=1}^{K-2}\sum_{\substack{j=1,\\ j\neq i}}^{K-2} r_i^3 r_j \\
&+(4+4c)\,r_{K-1}\sum_{i=1}^{K-2} r_i^3 + (4+4p+12c+12e)\,r_{K-1}^3\sum_{i=1}^{K-2} r_i \\
&+6\sum_{i=1}^{K-3}\sum_{j=i+1}^{K-2} r_i^2 r_j^2 + (6+6e+12c)\,r_{K-1}^2\sum_{i=1}^{K-2} r_i^2 \\
&+12\sum_{i=1}^{K-2}\sum_{\substack{j=1,\\ j\neq i}}^{K-4}\sum_{l=j+1}^{K-3} r_i^2 r_j r_l + (12+12c)\,r_{K-1}\sum_{i=1}^{K-2}\sum_{\substack{j=1,\\ j\neq i}}^{K-2} r_i^2 r_j \\
&+(12+12e+24c)\,r_{K-1}^2\sum_{i=1}^{K-3}\sum_{j=i+1}^{K-2} r_i r_j + 24\sum_{i=1}^{K-5}\sum_{j=i+1}^{K-4}\sum_{l=j+1}^{K-3}\sum_{m=l+1}^{K-2} r_i r_j r_l r_m \\
&+(24+24c)\,r_{K-1}\sum_{i=1}^{K-4}\sum_{j=i+1}^{K-3}\sum_{l=j+1}^{K-2} r_i r_j r_l + (4d)\sum_{i=1}^{K-2} r_i^3 \\
&+(u+4d+4q+6f)\,r_{K-1}^3 + (12d)\sum_{i=1}^{K-2}\sum_{\substack{j=1,\\ j\neq i}}^{K-2} r_i^2 r_j \\
&+(6f+12d)\,r_{K-1}\sum_{i=1}^{K-2} r_i^2 + (4q+12d+12f)\,r_{K-1}^2\sum_{i=1}^{K-2} r_i \\
&+(24d)\sum_{i=1}^{K-4}\sum_{j=i+1}^{K-3}\sum_{l=j+1}^{K-2} r_i r_j r_l + (12f+24d)\,r_{K-1}\sum_{i=1}^{K-3}\sum_{j=i+1}^{K-2} r_i r_j \\
&+(6g)\sum_{i=1}^{K-2} r_i^2 + (w+4s+6g)\,r_{K-1}^2 + (12g)\sum_{i=1}^{K-3}\sum_{j=i+1}^{K-2} r_i r_j \\
&+(4s+12g)\,r_{K-1}\sum_{i=1}^{K-2} r_i + (4v)\sum_{i=1}^{K-2} r_i + (x+4v)\,r_{K-1} + y\Bigg\}(\Delta t)^4,
\end{aligned}
$$

which is a quartic function of $r_1, r_2, ..., r_{K-1}$. Continuing the process of applying Equations (2.4), (2.5), (2.6) and (2.7) with $k = K-1, K-2, ..., 1$, we can finally obtain $E(V^4)$ in terms of *c*, *d*, *e*, *f*, *g*, *p*, *q*, *s*, *v*, κ, *u*, *w*, *x* and *y*.

After getting the first four moments of *V*, we may find the bond price using the following truncated series expansion of the exponential function (exponential expansion):

$$P(0,T) \approx 1 - E(V) + \frac{E(V^2)}{2} - \frac{E(V^3)}{6} + \frac{E(V^4)}{24}. \tag{2.9}$$

Alternatively, we may evaluate the bond price using the power-normal distribution described below.

Yeo and Johnson (2000) considered the following power transformation:

$$\tilde{\varepsilon} = \psi(\lambda^+, \lambda^-, z) = \begin{cases} \left[(z+1)^{\lambda^+} - 1\right] / \lambda^+ & (z \geq 0, \lambda^+ \neq 0) \\ \log(z+1) & (z \geq 0, \lambda^+ = 0) \\ -\left[(-z+1)^{\lambda^-} - 1\right] / \lambda^- & (z < 0, \lambda^- \neq 0) \\ -\log(-z+1) & (z < 0, \lambda^- = 0) \end{cases} \tag{2.10}$$

If z has the standard normal distribution, then $\tilde{\varepsilon}$ has a non-normal distribution which is derived by a type of power transformation of random variable with normal distribution. Then, we may refer to $\tilde{\varepsilon}$ as one which has a power-normal distribution.

Let

$$r^{(q)} = \mu^{(q)} + \sigma^{(q)} \frac{[\tilde{\varepsilon} - E(\tilde{\varepsilon})]}{[Var(\tilde{\varepsilon})]^{1/2}}. \tag{2.11}$$

The random variable $r^{(q)}$ is referred to as a random variable which has a power-normal distribution with parameters $\mu^{(q)}$, $\sigma^{(q)}$, λ^- and λ^+.

To find the bond price using the power-normal distribution, we first find the power-normal distribution with parameters $\mu^{(q)}$, $\sigma^{(q)}$, λ^- and λ^+ (see Equations (2.10) and (2.11)) such that the first four moments of $r^{(q)}$ given by Equation (2.11) are approximately equal to those of V, i.e.

$$E\left(r^{(q)l}\right) = E\left(V^l\right), \ \ 1 \leq l \leq 4.$$

The distribution of V may then be approximated by the power-normal distribution with parameters $\mu^{(q)}$, $\sigma^{(q)}$, λ^{-} and λ^{+} and the bond price $P(0,T) = E(e^{-V})$ may be evaluated by means of numerical integration.

2.2.2 Simulation method

Suppose $t = 0$ and the interval $[0,T]$ is divided into $K = T/\Delta t$ intervals of the same length Δt. In the simulation procedure, we first generate M values of $\mathbf{r} = (r_1, r_2, ..., r_K)$ using Equation (2.2). For the i-th generated value of **r**, we compute

$$P^{(i)} = \exp\left(-\sum_{k=1}^{K} r_k \Delta t\right). \tag{2.12}$$

The bond price estimated by means of simulation is then given by

$$P^{(s)}(0,T) = \frac{1}{M}\sum_{i=1}^{M} P^{(i)}. \tag{2.13}$$

2.2.3 Analytical formula for bond price under the CIR model

The bond price can be obtained directly from the following explicit expression for the bond price under the CIR model (see Cox et al. (1985)).

$$P(t,T) = A(t,T)\exp[-B(t,T)r_0] \tag{2.14}$$

where

$$A(t,T) = \left(\frac{2h\exp\left[\frac{1}{2}(a+h)(T-t)\right]}{2h+(a+h)[\exp(h(T-t))-1]}\right)^{\frac{2am}{\sigma^2}}, \quad B(t,T) = \frac{2[\exp((T-t)h)-1]}{2h+(a+h)[\exp(h(T-t))-1]},$$

and $h = (a^2 + 2\sigma^2)^{1/2}$.

2.3 Numerical results

For the random variable w_k which appears in Equation (2.2), its third moment $E\left(w_k^3\right)$ and fourth moment $E\left(w_k^4\right)$ are referred to as the measure of skewness and the measure of kurtosis of the underlying distribution respectively. We indicate $\overline{m}_3$ as $E\left(w_k^3\right)$ and $\overline{m}_4$ as $E\left(w_k^4\right)$. When $w_k \sim N(0,1)$, the values of $\overline{m}_3$ and $\overline{m}_4$ are respectively 0 and 3. When $\overline{m}_4$ is large, the distribution of w_k will have narrow waist and heavy tails.

Consider the CIR model of which the parameter vector (a, m, σ) assumes the value $(1.0,1.0,0.1)$ or $(0.5,0.08,0.01)$. For a variety of values of $(\overline{m}_3, \overline{m}_4)$, the first four moments of *V* obtained from the methods of based on moments and simulation are given in Tables 2.1 and 2.2, respectively.

Table 2.1 Moments of V when $(a, m, \sigma) = (1.0, 1.0, 0.1)$, $\Delta t = 1/365$, $r_0 = 0.08$, $T = 1$ and a variety of values of $(\overline{m}_3, \overline{m}_4)$ are used. (The values in bold form are the values based on simulation.)

$(\overline{m}_3, \overline{m}_4)$	$E(V)$	$E(V^2)$	$E(V^3)$	$E(V^4)$
$(0.0, 3.0)$	0.41957949	0.17654375	0.07449251	0.03152051
	0.41950345	**0.17647390**	**0.07444450**	**0.03149124**
$(0.0, 8.0)$	0.41957949	0.17654375	0.07449251	0.03152052
	0.41957949	**0.17604695**	**0.07386569**	**0.03099253**
$(0.5, 6.2)$	0.41957949	0.17654375	0.07449283	0.03152106
	0.41930091	**0.17625339**	**0.07427371**	**0.03137753**
$(2.0, 16.0)$	0.41957949	0.17654375	0.07449381	0.03152272
	0.41966568	**0.17657774**	**0.07449014**	**0.03150599**
$(3.0, 20.0)$	0.41957949	0.17654375	0.07449446	0.03152382
	0.42012692	**0.17698566**	**0.07476222**	**0.03166811**

Table 2.2 Moments of V when $(a,m,\sigma)=(0.5,0.08,0.01)$, $\Delta t = 1/365$, $r_0 = 0.05$, $T = 5$ and a variety of values of $(\overline{m}_3,\overline{m}_4)$ are used. (The values in bold form are the values based on simulation.)

$(\overline{m}_3,\overline{m}_4)$	$E(V)$	$E(V^2)$	$E(V^3)$	$E(V^4)$
$(0.0,3.0)$	0.34499212 **0.34504610**	0.11907993 **0.11911737**	0.04112331 **0.04114281**	0.01420881 **0.01421784**
$(0.0,8.0)$	0.34499212 **0.34494643**	0.11907993 **0.11904816**	0.04112331 **0.04110672**	0.01420881 **0.01420109**
$(0.5,6.2)$	0.34499212 **0.34493565**	0.11907993 **0.11903986**	0.04112332 **0.04110201**	0.01420882 **0.01419875**
$(2.0,16.0)$	0.34499212 **0.34497355**	0.11907993 **0.11906882**	0.04112334 **0.04111847**	0.01420884 **0.01420702**
$(3.0,20.0)$	0.34499212 **0.34499419**	0.11907993 **0.11908314**	0.04112336 **0.04112593**	0.01420886 **0.01421046**

Tables 2.1 and 2.2 show that the moments of V based on the iterative procedure in Section 2.2.1 agree fairly well with those found by using simulation. The results based on simulation vary compared to the others. However, the variation is slight, being noticeable only at about the fifth decimal point. As the simulation is used basically to check the validity of the results based on the iterative procedure, we do not choose to explore the small variance exhibited by the simulated results. Besides, the moments hardly vary when we vary the value of $(\overline{m}_3,\overline{m}_4)$. The reason behind this appears to be quite complicated. However, the fact that V in Equation (2.8) takes the form of a large number of random variables could be the main reason behind the small effects of $\overline{m}_3$ and $\overline{m}_4$.

Table 2.3 shows the parameters of the power-normal distribution which has about the same values of the first four moments of V obtained from the method given in Section 2.2.1.

Table 2.3 Parameters of the power-normal distribution when a variety of values of $(\overline{m}_3, \overline{m}_4)$ are used.

$(\overline{m}_3, \overline{m}_4)$	$\mu^{(q)}$	$\sigma^{(q)}$	λ^-	λ^+
$(0.0, 3.0)$	0.34499212	0.80776956	0.4	0.5
$(0.0, 8.0)$	0.34499212	0.759722681	0.4	0.5
$(0.5, 6.2)$	0.34499212	0.878889077	0.7	0.8
$(2.0, 16.0)$	0.34499212	0.710188104	0.2	0.4
$(3.0, 20.0)$	0.34499212	0.817507843	0.5	0.7

Apart from using the methods described in Section 2.2 to find the price of the bond, we also use a method known as the exact exponential time integration (ETI) method. The ETI is a numerical approach developed by Tangman et al. (2011) for approximating bond prices under the CIR model described by using a standard Brownian motion. We choose to compare the bond price which is obtained from our proposed method with the one obtained from the ETI as the ETI gives a high accuracy value of bond price when the underlying distribution follows a standard Brownian motion in the CIR model. Table 2.4 shows the computed bond prices from the methods of exponential expansion, power-normal approximation, ETI, analytical formula and simulation for $(a,m,\sigma)=(0.5,0.08,0.01)$.

Table 2.4 Bond price when $(a,m,\sigma)=(0.5,0.08,0.01)$, $\overline{m}_3=0$, $\overline{m}_4=3$, $\Delta t=1/365$, $r_0=0.05$ and $T=5$.

Method	Bond Price
Exponential expansion	0.708333
Power-normal approximation	0.708294
ETI	0.708294
Analytical formula	0.708294
Simulation	0.708253

Table 2.4 shows that the bond price given by the power-normal distribution is comparable to those based on ETI and analytical formula. The performance of power-normal approximation depends on whether the first four moments of *V* are sufficient for specifying the distribution of *V*. Table 2.4 shows that for the given example, the first four moments of *V* are quite sufficient for specifying the distribution of *V*.

For other values of $(\overline{m}_3,\overline{m}_4)$, we can likewise find the first four moments of *V*. It is found that the first four moments have extremely small variation as we vary the value of $(\overline{m}_3,\overline{m}_4)$. Based on the first four moments, we can next evaluate the bond price. Tables 2.5 and 2.6 display the bond prices under the CIR model described by using a Lévy process.

Table 2.5 Bond prices when $(a, m, \sigma) = (1.0, 1.0, 0.1)$, $\Delta t = 1/365$, $r_0 = 0.08$, $T = 1$ and a variety of values of $(\overline{m}_3, \overline{m}_4)$ are used. (The value of M used in the simulation is 10,000.)

Method	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 3.0$	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 8.0$	$\overline{m}_3 = 0.5$ $\overline{m}_4 = 6.2$	$\overline{m}_3 = 2.0$ $\overline{m}_4 = 16.0$	$\overline{m}_3 = 3.0$ $\overline{m}_4 = 20.0$
Exponential expansion	0.658332	0.658332	0.658332	0.658332	0.658332
Power-normal approximation	0.658229	0.658229	0.658229	0.658229	0.658229
Simulation	0.657588	0.658067	0.656193	0.659628	0.655226

Table 2.6 Bond prices when $(a, m, \sigma) = (0.5, 0.08, 0.01)$, $\Delta t = 1/365$, $r_0 = 0.05$, $T = 5$ and a variety of values of $(\overline{m}_3, \overline{m}_4)$ are used. (The value of M used in the simulation is 10,000.)

Method	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 3.0$	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 8.0$	$\overline{m}_3 = 0.5$ $\overline{m}_4 = 6.2$	$\overline{m}_3 = 2.0$ $\overline{m}_4 = 16.0$	$\overline{m}_3 = 3.0$ $\overline{m}_4 = 20.0$
Exponential expansion	0.708333	0.708333	0.708333	0.708333	0.708333
Power-normal approximation	0.708294	0.708294	0.708294	0.708294	0.708294
Simulation	0.708253	0.708279	0.708286	0.708261	0.708246

Tables 2.5 and 2.6 show that the effect of non-normality on bond price is extremely small. The small effects of non-normality again may probably be due to the fact that V in Equation (2.8) takes the form of a large number of random variables.

2.4 CONCLUSION

When the underlying distribution in the CIR model is non-normal, it is possible to find the first four moments of the integral *V* of the interest rate function. From the first four moments of *V*, we can approximate the bond price by using truncated exponential expansion and power-normal distribution.

For other more general one-factor models in which the interest rate r_k at time $t = k\Delta t$ depends on r_{k-1}, it is possible that the first four moments of r_k can also be expressed in terms of r_{k-1} as has been done in Equations (2.4) – (2.7). The method given in Section 2.2.1 may then be applied to find the first four moments of *V* from which an approximate value for the bond price can be obtained.

CHAPTER 3

BOND PRICE UNDER THE CKLS MODEL DESCRIBED BY A LÉVY PROCESS

3.1 Introduction

In this chapter, a numerical method similar to that described in Section 2.2.1 is applied to evaluate the bond price of a zero-coupon bond under the CKLS model described by a Lévy process. Section 3.2 gives the description of the proposed method for finding the bond price. Section 3.3 displays some numerical results on bond price. The results show that the bond price tends to decrease slightly when the parameter γ in the CKLS model increases, and the variation of the bond price is slight as the non-normality of the underlying distribution in the Lévy process is varied. Some conclusions given in Section 3.4 end this chapter.

3.2 Methods for evaluating the bond price

This section presents a numerical method for evaluating the bond price of a zero-coupon bond under the CKLS model described by a Lévy process:

$$dr = a(m-r)dt + \sigma\, r^{\gamma}\, dw,\ \ t \geq 0, \tag{3.1}$$

where dw is the Lévy process.

Let Δt be a small increment in time *t*. The CKLS interest rate model in Equation (3.1) may be discretized to

$$r_k = r_{k-1} + a(m - r_{k-1})\Delta t + \sigma r_{k-1}^{\gamma} w_k \sqrt{\Delta t},\ k \geq 1, \tag{3.2}$$

where r_k is the spot rate at $t = k\Delta t$, γ a positive constant which indicates the power of the interest rate at time *t*, $w_1, w_2, \ldots$ are independent, and w_k has a quadratic-normal distribution with parameters 0 and $\bar{\lambda}$, and $\bar{\lambda}$ is such that $E\left(w_k^2\right) = 1$.

3.2.1 Method based on moments

To evaluate the bond price $P(0,T)$ (see Equation (1.5)), we use a method adapted from the procedure based on moments given in Section 2.2.1.

Firstly, the discretized version of the CKLS model in Equation (3.2) is used to obtain the following results on the moments of r_k conditional on the value of r_{k-1}.

$$E(r_k \mid r_{k-1}) = (1 - a\Delta t) r_{k-1} + am\Delta t, \tag{3.3}$$

$$E\left(r_k^2 \mid r_{k-1}\right) = (1 - a\Delta t)^2 r_{k-1}^2 + 2am\Delta t(1 - a\Delta t) r_{k-1} + \sigma^2 \Delta t\, r_{k-1}^{2\gamma} + (am\Delta t)^2, \tag{3.4}$$

$$\begin{aligned} E\left(r_k^3 \mid r_{k-1}\right) = {} & (1 - a\Delta t)^3 r_{k-1}^3 + 3am\Delta t(1 - a\Delta t)^2 r_{k-1}^2 + 3(am\Delta t)^2 (1 - a\Delta t) r_{k-1} \\ & + 3am\sigma^2 \Delta t^2 r_{k-1}^{2\gamma} + 3\sigma^2 \Delta t(1 - a\Delta t) r_{k-1}^{2\gamma+1} \\ & + \overline{m}_3 \sigma^3 \Delta t^{3/2} r_{k-1}^{3\gamma} + (am\Delta t)^3, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} E(r_k^4 \mid r_{k-1}) = {} & (1 - a\Delta t)^4 r_{k-1}^4 + 4am\Delta t(1 - a\Delta t)^3 r_{k-1}^3 \\ & + 6(am\Delta t)^2 (1 - a\Delta t)^2 r_{k-1}^2 + 4(am\Delta t)^3 (1 - a\Delta t) r_{k-1} \\ & + 6(am\Delta t)^2 \Delta t^3 r_{k-1}^{2\gamma} + 12am\sigma^2 \Delta t^2 (1 - a\Delta t) r_{k-1}^{2\gamma+1} \\ & + 6\sigma^2 \Delta t(1 - a\Delta t)^2 r_{k-1}^{2\gamma+2} + 4am\sigma^3 \overline{m}_3 \Delta t^{5/2} r_{k-1}^{3\gamma} \end{aligned} \tag{3.6}$$

$$+4\sigma^3\overline{m}_3(1-a\Delta t)\Delta t^{3/2} r_{k-1}^{3\gamma+1} + \sigma^4\overline{m}_4\Delta t^2 r_{k-1}^{4\gamma} + (am\Delta t)^4,$$

where $\overline{m}_3$ and $\overline{m}_4$, as defined in Section 2.3, are respectively the measure of skewness and kurtosis of the random variable w_k.

Next, a polynomial $P_j(r_{k-1}) = c_{j0} + c_{j1}r_{k-1} + c_{j2}r_{k-1}^2 + \cdots + c_{jj}r_{k-1}^j$ of degree j $(2 \le j \le 4)$ is found such that

$$P_j(x_{jl}) = E\left(r_k^j \mid r_{k-1} = x_{jl}\right),\ 1 \le l \le j+1,$$

where the x_{jl} are given in Table 3.1.

Table 3.1 Value of x_{jl}.

j	l				
	1	2	3	4	5
2	0.01	0.50	0.99		
3	0.01	0.25	0.75	0.99	
4	0.01	0.25	0.50	0.75	0.99

The values in Table 3.1 are chosen such that the polynomials would give good approximations when the values of the interest rate x_{jl} lie in the interval $[0,1]$.

The expected value $E\left(r_k^j \mid r_{k-1}\right)$ is then approximated by the polynomial $P_j(r_{k-1})$. The method given in Khor and Pooi (2009) may be applied to find the first four moments of the random variable V given in Equation (2.8). An outline of the method in Khor and Pooi (2009) is given below.

Let $\{r_{k-1}\}$ denote the vector $(r_1, r_2, ..., r_{k-1})$.

Applying Equation (3.3) with $k = K$ to find the expected value of V conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$E\left(V \mid \{r_{K-1}\}\right)=\left[\sum_{i=1}^{K-2} r_i+\left(1+c_{11}\right) r_{K-1}+c_{10}\right] \Delta t \ ,$$

which is a linear function of $r_1, r_2, ..., r_{K-1}$. If the process of applying Equation (3.3) for $k = K-1, K-2, ..., 1$ is continued, the equation of $E(V)$ in terms of c_{11} and c_{10} can finally be obtained.

Applying Equations (3.3) and (3.4) with $k = K$ to find the expected value of V^2 conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$E\left(V^2 \middle| \{r_{K-1}\}\right)=\left[\sum_{i=1}^{K-2} r_i^2+\left(1+c_{22}+2c_{11}\right) r_{K-1}^2+2\sum_{i=1}^{K-3}\sum_{j=i+1}^{K-2} r_i r_j+2\left(1+c_{11}\right) r_{K-1}\sum_{i=1}^{K-2} r_i\right.$$
$$\left.+2c_{10}\sum_{i=1}^{K-2} r_i+\left(c_{21}+2c_{10}\right) r_{K-1}+c_{20}\right](\Delta t)^2 ,$$

which is a quadratic function of $r_1, r_2, ..., r_{K-1}$. Continuing the process of applying Equations (3.3) and (3.4) with $k = K-1, K-2, ..., 1$, the equation of $E(V^2)$ in terms of $c_{11}, c_{10}, c_{22}, c_{21}$ and c_{20} can finally be obtained.

By applying Equations (3.3), (3.4) and (3.5) with $k = K$ to find the expected value of V^3 conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$E\left(V^3 \middle| \{r_{K-1}\}\right)=\left\{\sum_{i=1}^{K-2} r_i^3+\left[1+c_{33}+3\left(c_{11}+c_{22}\right)\right] r_{K-1}^3+3\sum_{i=1}^{K-2}\sum_{j=1, i\neq j}^{K-2} r_i^2 r_j\right.$$
$$+3\left(1+c_{11}\right) r_{K-1}\sum_{i=1}^{K-2} r_i^2+3\left(1+c_{22}+2c_{11}\right) r_{K-1}^2\sum_{i=1}^{K-2} r_i+3c_{10}\sum_{i=1}^{K-2} r_i^2$$
$$+\left[c_{32}+3\left(c_{10}+c_{21}\right)\right] r_{K-1}^2+6\sum_{i=1}^{K-4}\sum_{j=i+1}^{K-3}\sum_{l=j+1}^{K-2} r_i r_j r_l$$
$$+6\left(1+c_{11}\right) r_{K-1}\sum_{i=1}^{K-3}\sum_{j=i+1}^{K-2} r_i r_j+6c_{10}\sum_{i=1}^{K-3}\sum_{j=i+1}^{K-2} r_i r_j+3\left(c_{21}+2c_{10}\right) r_{K-1}\sum_{i=1}^{K-2} r_i$$
$$\left.+3c_{20}\sum_{i=1}^{K-2} r_i+\left(c_{31}+3c_{20}\right) r_{K-1}+c_{30}\right\}(\Delta t)^3 ,$$

which is a cubic function of $r_1, r_2, ..., r_{K-1}$. Continuing the process of applying Equations (3.3), (3.4) and (3.5) with $k = K-1, K-2, ..., 1$, the equation of $E\left(V^3\right)$ in terms of $c_{11}, c_{10}, c_{22}, c_{21}, c_{20}, c_{33}, c_{32}, c_{31}$ and c_{30} can finally be obtained.

Similarly by applying Equations (3.3), (3.4), (3.5) and (3.6) with $k = K$ to find the expected value of V^4 conditional on the values of $r_1, r_2, ..., r_{K-1}$, we get

$$
\begin{aligned}
E\left(V^4 \middle| \{r_{K-1}\}\right) = \Bigg\{ & \sum_{i=1}^{K-2} r_i^4 + \left(1 + c_{44} + 4c_{11} + 4c_{33} + 6c_{22}\right) r_{K-1}^4 \\
& + 4\sum_{i=1}^{K-2} \sum_{\substack{j=1,\\ j\neq i}}^{K-2} r_i^3 r_j + \left(4 + 4c_{11}\right) r_{K-1} \sum_{i=1}^{K-2} r_i^3 \\
& + \left(4 + 4c_{33} + 12c_{11} + 12c_{22}\right) r_{K-1}^3 \sum_{i=1}^{K-2} r_i + 6\sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i^2 r_j^2 \\
& + \left(6 + 6c_{22} + 12c_{11}\right) r_{K-1}^2 \sum_{i=1}^{K-2} r_i^2 + 12\sum_{i=1}^{K-2} \sum_{\substack{j=1,\\ j\neq i}}^{K-4} \sum_{l=j+1}^{K-3} r_i^2 r_j r_l \\
& + \left(12 + 12c_{11}\right) r_{K-1} \sum_{i=1}^{K-2} \sum_{\substack{j=1,\\ j\neq i}}^{K-2} r_i^2 r_j + \left(12 + 12c_{22} + 24c_{11}\right) r_{K-1}^2 \sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i r_j \\
& + 24\sum_{i=1}^{K-5} \sum_{j=i+1}^{K-4} \sum_{l=j+1}^{K-3} \sum_{m=l+1}^{K-2} r_i r_j r_l r_m + \left(24 + 24c_{11}\right) r_{K-1} \sum_{i=1}^{K-4} \sum_{j=i+1}^{K-3} \sum_{l=j+1}^{K-2} r_i r_j r_l \\
& + \left(4c_{10}\right) \sum_{i=1}^{K-2} r_i^3 + \left(c_{43} + 4c_{10} + 4c_{32} + 6c_{21}\right) r_{K-1}^3 + \left(12c_{10}\right) \sum_{i=1}^{K-2} \sum_{\substack{j=1,\\ j\neq i}}^{K-2} r_i^2 r_j \\
& + \left(6c_{21} + 12c_{10}\right) r_{K-1} \sum_{i=1}^{K-2} r_i^2 + \left(4c_{32} + 12c_{10} + 12c_{21}\right) r_{K-1}^2 \sum_{i=1}^{K-2} r_i \\
& + \left(24c_{10}\right) \sum_{i=1}^{K-4} \sum_{j=i+1}^{K-3} \sum_{l=j+1}^{K-2} r_i r_j r_l + \left(12c_{21} + 24c_{10}\right) r_{K-1} \sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i r_j \\
& + \left(6c_{20}\right) \sum_{i=1}^{K-2} r_i^2 + \left(12c_{21} + 24c_{10}\right) r_{K-1} \sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i r_j \\
& + \left(6c_{20}\right) \sum_{i=1}^{K-2} r_i^2 + \left(c_{42} + 4c_{31} + 6c_{20}\right) r_{K-1}^2 + \left(12c_{20}\right) \sum_{i=1}^{K-3} \sum_{j=i+1}^{K-2} r_i r_j \\
& + \left(4c_{31} + 12c_{20}\right) r_{K-1} \sum_{i=1}^{K-2} r_i + \left(4c_{30}\right) \sum_{i=1}^{K-2} r_i
\end{aligned}
$$

$$+\left(c_{41}+4c_{30}\right)r_{K-1}+c_{40}\}(\Delta t)^4,$$

which is a quartic function of $r_1, r_2, ..., r_{K-1}$. Continuing the process of applying Equations (3.3), (3.4), (3.5) and (3.6) with $k = K-1, K-2, ..., 1$, the equation of $E\left(V^4\right)$ in terms of c_{11}, c_{10}, c_{22}, c_{21}, c_{20}, c_{33}, c_{32}, c_{31}, c_{30}, c_{44}, c_{43}, c_{42}, c_{41} and c_{40} can be obtained.

After getting the first four moments of *V*, the bond price can be evaluated approximately by using either a truncated series expansion of the exponential function (exponential expansion) as shown in Equation (2.9) or numerical integration of $E\left(e^{-V}\right)$ after obtaining a power-normal distribution for *V*.

3.2.2 Simulation method

In the simulation method, the procedure of finding the bond price of a zero-coupon bond under the CKLS model described by a Lévy process is similar to the one explained in Section 2.2.1. The generation of *M* values of $\mathbf{r} = \left(r_1, r_2, ..., r_K\right)$ is now performed using Equation (3.2) instead of Equation (2.2). For the *i*-th generated value of $\mathbf{r}$, the value of $P^{(i)}$ in Equation (2.12) is computed and the estimated bond price is then obtained using the Equation (2.13).

3.3 Numerical results

This section presents some numerical values for the bond prices when the interest rate follows the CKLS model described by a Lévy process. The parameters of the CKLS model and the maturity time *T* are chosen to be

$(a, m, \sigma, \Delta t, r_0, T) = (1.0, 1.0, 0.1, 1/365, 0.08, 1), (0.5, 0.08, 0.01, 1/365, 0.05, 5)$.

Consider initially the situation where the random variable w_k in Equation (3.2) has a standard normal distribution, in which case $\overline{m}_3 = 0.0$ and $\overline{m}_4 = 3.0$.

Table 3.2 shows the results of bond price based on four methods when $\gamma = 0.0, 0.5, 1.5, 2.5$ and $w_k \sim N(0,1)$.

Table 3.2 Bond prices for $(a, m, \sigma, \Delta t, r_0, T) = (0.5, 0.08, 0.01, 1/365, 0.05, 5)$ when $\overline{m}_3 = 0.0$ and $\overline{m}_4 = 3.0$ are used.

γ	Exponential expansion	Power-normal approximation	ETI	Analytical formula
0.5	0.708333	0.708294	0.708294	0.708294
1.0	0.708313	0.708274	0.708275	-
1.5	0.708300	0.708273	0.708273	-
2.5	0.708287	0.708272	0.708273	-

[ETI - Exact Exponential Time Integration]

Table 3.2 shows that the bond prices obtained from the four methods are comparable. When $\gamma = 0.5$, we see that the bond price based on the power-normal approximation agrees well with those found from the ETI and analytical formula. On the other hand, the bond price obtained from the method of exponential expansion is only accurate up to the third decimal place when compares to the ETI and analytical formula.

When the distribution is non-normal, the bond price could not be evaluated using ETI or analytical formula. As a result, the proposed numerical methods are used to evaluate the bond price. Tables 3.3(a) to 3.4(b) display the bond prices for two different sets of value of $(a, m, \sigma, \Delta t, r_0, T)$ when different values of γ and $(\overline{m}_3, \overline{m}_4)$ are used.

Table 3.3(a) Bond prices for $(a, m, \sigma, \Delta t, r_0, T) = (1.0, 1.0, 0.1, 1/365, 0.08, 1)$ when a variety of values of $\gamma = 0.0, 0.1, \ldots, 0.6$ and $(\overline{m}_3, \overline{m}_4)$ are used. (Values not in parentheses are found by using exponential expansion (see Equation (2.9)). Values in parentheses are found by using power-normal approximation. Values in bold form are found by using simulation.)

γ	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 3.0$	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 8.0$	$\overline{m}_3 = 0.5$ $\overline{m}_4 = 6.2$	$\overline{m}_3 = 2.0$ $\overline{m}_4 = 16.0$	$\overline{m}_3 = 3.0$ $\overline{m}_4 = 20.0$
0.0	0.658729 (0.658620) **0.658384**	0.658729 (0.658620) **0.658671**	0.658729 (0.658620) **0.657961**	0.658729 (0.658619) **0.657605**	0.658729 (0.658619) **0.656624**
0.1	0.658556 (0.658450) **0.657766**	0.658556 (0.658450) **0.658913**	0.658556 (0.658450) **0.658401**	0.658556 (0.658450) **0.656975**	0.658556 (0.658449) **0.657280**
0.2	0.658464 (0.658359) **0.658461**	0.658464 (0.658359) **0.658407**	0.658464 (0.658359) **0.658194**	0.658464 (0.658359) **0.657920**	0.658464 (0.658359) **0.656243**
0.3	0.658406 (0.658303) **0.658178**	0.658406 (0.658303) **0.658216**	0.658406 (0.658303) **0.658064**	0.658406 (0.658302) **0.657600**	0.658406 (0.658303) **0.656933**
0.4	0.658365 (0.658262) **0.658235**	0.658365 (0.658262) **0.658278**	0.658365 (0.658262) **0.658139**	0.658365 (0.658262) **0.657287**	0.658365 (0.658262) **0.656798**
0.5	0.658332 (0.658229) **0.657875**	0.658332 (0.658229) **0.658531**	0.658332 (0.658229) **0.657802**	0.658332 (0.658229) **0.657970**	0.658332 (0.658229) **0.657119**
0.6	0.658304 (0.658202) **0.658064**	0.658304 (0.658202) **0.658246**	0.658304 (0.658202) **0.657765**	0.658304 (0.658202) **0.657874**	0.658304 (0.658202) **0.657413**

Table 3.3(b) Bond prices for $(a, m, \sigma, \Delta t, r_0, T) = (1.0, 1.0, 0.1, 1/365, 0.08, 1)$ when a variety of values of $\gamma = 0.7, 0.8, 0.9, 1.0, 1.5, 2.5$ and $(\overline{m}_3, \overline{m}_4)$ are used. (Values not in parentheses are found by using exponential expansion (see Equation (2.9)). Values in parentheses are found by using power-normal approximation. Values in bold form are found by using simulation.)

γ	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 3.0$	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 8.0$	$\overline{m}_3 = 0.5$ $\overline{m}_4 = 6.2$	$\overline{m}_3 = 2.0$ $\overline{m}_4 = 16.0$	$\overline{m}_3 = 3.0$ $\overline{m}_4 = 20.0$
0.7	0.658281 (0.658178) **0.658000**	0.658281 (0.658178) **0.658312**	0.658281 (0.658166) **0.657895**	0.658281 (0.658178) **0.657620**	0.658281 (0.658178) **0.657184**
0.8	0.658260 (0.658158) **0.658039**	0.658260 (0.658158) **0.658407**	0.658260 (0.658158) **0.657814**	0.658260 (0.658158) **0.657596**	0.658260 (0.658158) **0.657375**
0.9	0.658242 (0.658140) **0.657773**	0.658242 (0.658140) **0.657833**	0.658242 (0.658140) **0.657912**	0.658242 (0.658140) **0.657769**	0.658242 (0.658140) **0.657605**
1.0	0.658226 (0.658125) **0.657858**	0.658226 (0.658125) **0.658268**	0.658226 (0.658125) **0.657622**	0.658226 (0.658125) **0.657550**	0.658226 (0.658125) **0.657504**
1.5	0.658172 (0.658071) **0.658019**	0.658172 (0.658071) **0.657918**	0.658172 (0.658071) **0.658014**	0.658172 (0.658071) **0.657846**	0.658172 (0.658071) **0.657708**
2.5	0.658131 (0.658055) **0.657867**	0.658131 (0.658055) **0.657910**	0.658131 (0.658055) **0.657938**	0.658131 (0.658055) **0.657906**	0.658131 (0.658055) **0.657872**

Table 3.4(a) Bond prices for $(a, m, \sigma, \Delta t, r_0, T) = (0.5, 0.08, 0.01, 1/365, 0.05, 5)$ when a variety of values of $\gamma = 0.0, 0.1, ..., 0.6$ and $(\overline{m}_3, \overline{m}_4)$ are used. (Values not in parentheses are found by using exponential expansion (see Equation (2.9)). Values in parentheses are found by using power-normal approximation. Values in bold form are found by using simulation.)

γ	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 3.0$	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 8.0$	$\overline{m}_3 = 0.5$ $\overline{m}_4 = 6.2$	$\overline{m}_3 = 2.0$ $\overline{m}_4 = 16.0$	$\overline{m}_3 = 3.0$ $\overline{m}_4 = 20.0$
0.0	0.708643 (0.708602) **0.708516**	0.708643 (0.708602) **0.704775**	0.708643 (0.708602) **0.709235**	0.708643 (0.708602) **0.708525**	0.708643 (0.708602) **0.708567**
0.1	0.708461 (0.708423) **0.708547**	0.708461 (0.708423) **0.708463**	0.708461 (0.708423) **0.708561**	0.708461 (0.708423) **0.708444**	0.708461 (0.708423) **0.708383**
0.2	0.708387 (0.708350) **0.7083242**	0.708387 (0.708350) **0.709200**	0.708387 (0.708350) **0.708543**	0.708387 (0.708350) **0.709255**	0.708387 (0.708350) **0.709581**
0.3	0.708356 (0.708318) **0.708159**	0.708356 (0.708318) **0.708341**	0.708356 (0.708318) **0.708474**	0.708356 (0.708318) **0.708638**	0.708356 (0.708318) **0.709390**
0.4	0.708341 (0.708303) **0.708287**	0.708341 (0.708303) **0.708298**	0.708341 (0.708303) **0.708319**	0.708341 (0.708303) **0.708548**	0.708341 (0.708303) **0.709265**
0.5	0.708333 (0.708294) **0.708209**	0.708333 (0.708294) **0.708279**	0.708333 (0.708294) **0.708286**	0.708333 (0.708294) **0.708261**	0.708333 (0.708294) **0.708246**
0.6	0.708327 (0.708290) **0.708220**	0.708327 (0.708290) **0.708272**	0.708327 (0.708290) **0.708275**	0.708327 (0.708290) **0.708536**	0.708327 (0.708290) **0.708246**

Table 3.4(b) Bond prices for $(a, m, \sigma, \Delta t, r_0, T) = (0.5, 0.08, 0.01, 1/365, 0.05, 5)$ when a variety of values of $\gamma = 0.7, 0.8, 0.9, 1.0, 1.5, 2.5$ and $(\overline{m}_3, \overline{m}_4)$ are used. (Values not in parentheses are found by using exponential expansion (see Equation (2.9)). Values in parentheses are found by using power-normal approximation. Values in bold form are found by using simulation.)

γ	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 3.0$	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 8.0$	$\overline{m}_3 = 0.5$ $\overline{m}_4 = 6.2$	$\overline{m}_3 = 2.0$ $\overline{m}_4 = 16.0$	$\overline{m}_3 = 3.0$ $\overline{m}_4 = 20.0$
0.7	0.708323 (0.708350) **0.7082204**	0.708323 (0.708285) **0.708264**	0.708323 (0.708285) **0.708267**	0.708323 (0.708286) **0.708289**	0.708323 (0.708285) **0.708254**
0.8	0.708319 (0.708276) **0.708228**	0.708319 (0.708277) **0.708240**	0.708319 (0.708277) **0.708368**	0.708319 (0.708266) **0.708357**	0.708319 (0.708277) **0.708292**
0.9	0.708316 (0.708285) **0.708208**	0.708316 (0.7082898) **0.708258**	0.708316 (0.708288) **0.708301**	0.708316 (0.708303) **0.708309**	0.708314 (0.708294) **0.708259**
1.0	0.708313 (0.708274) **0.708212**	0.708313 (0.708274) **0.708234**	0.708313 (0.708274) **0.707348**	0.708313 (0.708274) **0.707841**	0.708313 (0.708274) **0.708255**
1.5	0.708300 (0.708273) **0.708226**	0.708300 (0.708282) **0.708262**	0.708300 (0.708311) **0.708283**	0.708300 (0.708291) **0.708258**	0.708300 (0.708270) **0.708260**
2.5	0.708287 (0.708272) **0.708225**	0.708287 (0.708273) **0.708229**	0.708287 (0.708276) **0.708231**	0.708287 (0.708271) **0.708226**	0.708287 (0.708300) **0.708229**

Tables 3.3(a) to 3.4(b) show that bond price tends to decrease slightly when γ increases and the effect of non-normality on bond price is very small. It would be interesting to investigate further the effects of γ using more examples of the parameters of CKLS model.

3.4 CONCLUSION

When the distributions of the increments in the CKLS model are non-normal, we approximate the expected first four moments of the interest rate by polynomial functions. From the polynomial approximations for the first four moments, we can find the bond price under the CKLS model described by a Lévy process. When the value of γ takes 0.5 in the CKLS model and the underlying distribution in the Lévy process is normal, the obtained bond price is found to be close to those based on the analytical formula of the CIR model and the ETI method.

It should be possible to improve the precision of the bond price found by the proposed numerical method if we increase the degrees of the polynomials used to approximate the expected moments of the interest rate.

The formulas used in proposed numerical method are all basically iterative in nature. The iterative characteristic makes it convenient to carry out the related computation in a computer. Furthermore, the computation time required is found to be fairly short. Further comparisons of computation times among the above three methods are an interesting topic for future research.

CHAPTER 4

BOND OPTION PRICING UNDER THE CKLS MODEL

4.1 Introduction

Consider the European call option written on a zero-coupon bond. Suppose the call option has maturity T and strike price K while the bond has maturity $S > T$. In this chapter, the evaluation of the call option price under the CKLS model is described. Section 4.2 illustrates the numerical method for evaluating the call option price under the CKLS model in which the increment of the short rate over a time interval of length dt, apart from being independent and stationary, is having the quadratic-normal distribution with mean zero and variance dt. The key steps in the numerical procedure include (i) the discretization of the CKLS model; (ii) the quadratic approximation of the time-T bond price as a function of the short rate $r(T)$ at time T; and (iii) the application of recursive formulas to find the moments of $r(t+dt)$ given the value of $r(t)$. The simulation method and analytical formula for evaluating the price of a European call option written on a zero-coupon bond are described as well in Section 4.2. The numerical results in Section 4.3 show that the option price decreases as the parameter γ in the CKLS model increases, and the variation of the option price is slight when the distribution of the increment departs from the normal distribution. Section 4.4 ends this chapter with some conclusions.

4.2 Methods for evaluating the call option price written on a zero-coupon bond under the CKLS model

The discretized version of the CKLS model described by a Lévy process with quadratic-normal increments is shown below:

$$r_k = r_{k-1} + a\left(m - r_{k-1}\right)\Delta t + \sigma r_{k-1}^{\gamma} w_k \sqrt{\Delta t},\ k \geq 1.$$

(see also Equation (3.2))

Suppose the interval $[0,T]$ is divided into N_c intervals each of length Δt and the interval $[T,S]$ into N_b intervals each also of length Δt (see Figure 4.1). In Figure 4.1, u_k and r_k denote the interest rates at a time $k\Delta t$ units ahead when $t=0$ and $t=T$ are taken respectively as the original time point.

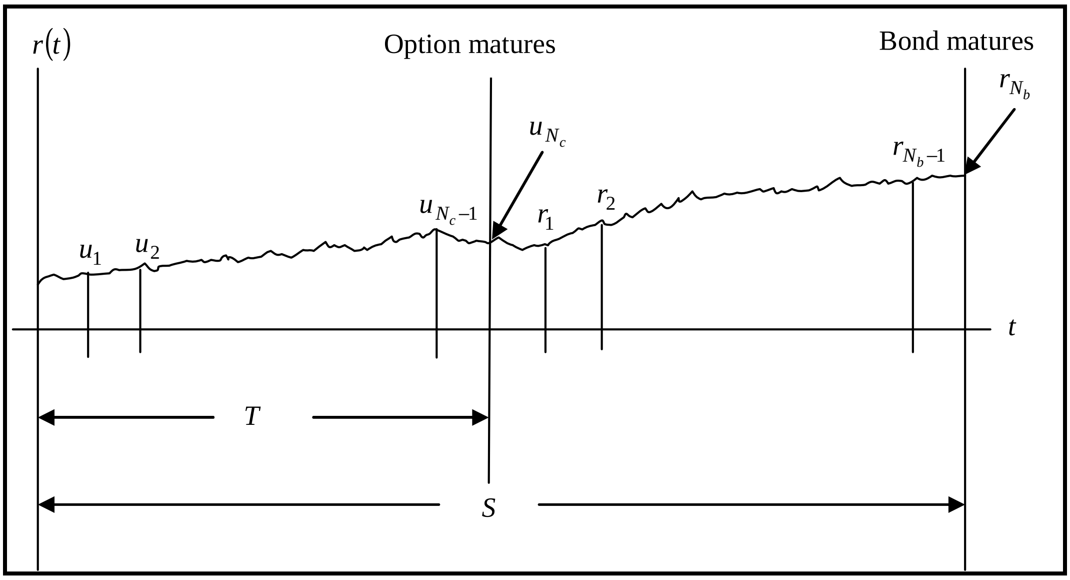

Figure 4.1 Short rate *r* and times of the maturity of bond and option.

4.2.1 Numerical method

Recall that in Section 3.2.1, the polynomial approximations of the first four moments of r_k in Equation (3.2) conditioned on the given value of r_{k-1} are given as

$$E(r_k \mid r_{k-1}) = c_{10} + c_{11} r_{k-1}, \tag{4.1}$$

$$E(r_k^2 \mid r_{k-1}) = c_{20} + c_{21} r_{k-1} + c_{22} r_{k-1}^2, \tag{4.2}$$

$$E(r_k^3 \mid r_{k-1}) = c_{30} + c_{31} r_{k-1} + c_{32} r_{k-1}^2 + c_{33} r_{k-1}^3, \tag{4.3}$$

$$E(r_k^4 \mid r_{k-1}) = c_{40} + c_{41} r_{k-1} + c_{42} r_{k-1}^2 + c_{43} r_{k-1}^3 + c_{44} r_{k-1}^4, \tag{4.4}$$

where $c_{10}, c_{11}, ..., c_{44}$ are some constants.

The polynomial approximations for the first four moments of r_k conditioned on the value of r_{k-1} are now used to find the bond price B_T^S. Under the risk neutral measure *Q*, the bond price B_T^S at time $t = T$ of a zero-coupon bond that matures at time $t = S$, $0 \le T \le S$ is given by

$$B_T^S = E^Q\left(e^{-\int_T^S r(s)ds} \right). \tag{4.5}$$

Let $V_b = (r_1 + r_2 + ... + r_{N_b})\Delta t.$ Applying Equation (4.1) with $k = N_b, N_b - 1, ..., 1$, $E(V_b)$ can be found in terms of a linear function of c_{10} and c_{11}. Applying Equations (4.1) and (4.2) with $k = N_b, N_b - 1, ..., 1$, we can get $E(V_b^2)$ in terms of a quadratic function of c_{10}, c_{11}, c_{20}, c_{21} and c_{22}.

Applying Equations (4.1), (4.2) and (4.3), $E(V_b^3)$ is found in terms of a cubic function of c_{10}, c_{11}, c_{20}, c_{21}, c_{22}, c_{30}, c_{31}, c_{32} and c_{33}. Applying Equations (4.1), (4.2), (4.3) and (4.4), we get $E(V_b^4)$ in terms of a quartic function of c_{10}, c_{11}, c_{20}, c_{21}, c_{22}, c_{30}, c_{31}, c_{32}, c_{33}, c_{40}, c_{41}, c_{42}, c_{43} and c_{44}.

By using Equation (4.5), the following bond price can then be obtained for a given value of r_0:

$$B_T^S \approx 1 - E(V_b) + \frac{E(V_b^2)}{2} - \frac{E(V_b^3)}{6} + \frac{E(V_b^4)}{24}.$$

After computing B_T^S for a number of selected values of r_0 in $(0,1)$, we then fit a quadratic function to B_T^S:

$$B_T^S \approx g_0 + g_1 r_0 + g_2 r_0^2. \tag{4.6}$$

We note that the value of r_0 coincides with the value u_{N_c} which appears in the vector $\mathbf{V}_{N_c} = (u_1, u_2, \ldots, u_{N_c})\Delta t$. Thus from Equations (1.6) and (4.6), we get

$$C = E(\phi) \tag{4.7}$$

$$= E_{u_1|u_0} E_{u_2|u_1} E_{u_3|u_2} \ldots E_{u_{N_c}|u_{N_c-1}}(\phi),$$

where

$$\phi = \left\{ \exp\left[-\int_0^T r(s)ds \right] \left[\left(g_0 + g_1 u_{N_c} + g_2 u_{N_c}^2 \right) - K \right]^+ \right\}$$
$$= \left(1 - V_c + \frac{V_c^2}{2} - \frac{V_c^3}{6} + \frac{V_c^4}{24} \right) \left[\left(g_0 + g_1 u_{N_c} + g_2 u_{N_c}^2 \right) - K \right]^+,$$

with $V_c = (u_1 + u_2 + \ldots + u_{N_c})\Delta t$.

Next, let

$$F_{u_{N_c-1}} = E_{u_{N_c}|u_{N_c-1}}(\phi), \tag{4.8}$$

$$F_{u_{N_c-2}} = E_{u_{N_c-1}|u_{N_c-2}} E_{u_{N_c}|u_{N_c-1}}(\phi), \tag{4.9}$$

$$\vdots$$

$$F_{u_1} = E_{u_2|u_1} E_{u_3|u_2} \ldots E_{u_{N_c}|u_{N_c-1}}(\phi). \tag{4.10}$$

To find $F_{u_{N_c-1}}$, we first use numerical integration to compute

$$E_{\alpha,u_{N_c-1}} = E_{u_{N_c}|u_{N_c-1}}\left\{ u_{N_c}^{\alpha}\left[\left(g_0 + g_1 u_{N_c} + g_2 u_{N_c}^2\right) - K\right]\right\}, \quad \alpha = 0, 1, 2, 3, 4,$$

and express each $E_{\alpha,u_{N_c-1}}$ as a low degree polynomial function of u_{N_c-1}:

$$E_{\alpha,u_{N_c-1}} = \tau_{\alpha 0} + \tau_{\alpha 1} u_{N_c-1} + \tau_{\alpha 2} u_{N_c-1}^2, \qquad \alpha = 0. \tag{4.11}$$

$$E_{\alpha,u_{N_c-1}} = \tau_{\alpha 0} + \tau_{\alpha 1} u_{N_c-1} + \tau_{\alpha 2} u_{N_c-1}^2 + \tau_{\alpha 3} u_{N_c-1}^3, \qquad \alpha = 1,2. \tag{4.12}$$

$$E_{\alpha,u_{N_c-1}} = \tau_{\alpha 0} + \tau_{\alpha 1} u_{N_c-1} + \tau_{\alpha 2} u_{N_c-1}^2 + \tau_{\alpha 3} u_{N_c-1}^3 + \tau_{\alpha 4} u_{N_c-1}^4, \quad \alpha = 3,4. \tag{4.13}$$

By using Equations (4.11), (4.12) and (4.13), we can find $F_{u_{N_c-1}}$ approximately (see Equation (4.8)) and express $F_{u_{N_c-1}}$ as a low degree polynomial function $F_{u_{N_c-1}}$ of u_{N_c-1}.

We next use Equations (4.1), (4.2), (4.3) and (4.4) to compute $F_{u_{N_c-2}}$ given by Equation (4.9):

$$F_{u_{N_c-2}} = E_{u_{N_c-1}|u_{N_c-2}} F_{u_{N_c-1}}.$$

Similarly by using the iterative formulas in Equations (4.1), (4.2), (4.3) and (4.4), $F_{u_{N_c-3}}, \ldots, F_{u_2}, F_{u_1}$ and C can also be found.

4.2.2 Simulation method

Firstly, we use Equation (3.2) to generate M_c values of $\mathbf{V}_c = \left(u_1, u_2, ..., u_{N_c}\right)\Delta t$. For each generated value of $\mathbf{V}_c$, we again use Equation (3.2) to generate M_b values of $\mathbf{V}_b = \left(r_1, r_2, ..., r_{N_b}\right)\Delta t$.

Denote the i-th generated value of $\mathbf{V}_c$ by $\mathbf{V}_{c_i} = \left(u_{i1}, u_{i2}, ..., u_{iN_c}\right)\Delta t$ and the j-th generated of $\mathbf{V}_b$ by $\mathbf{V}_{b_j} = \left(r_{j1}, r_{j2}, ..., r_{jN_b}\right)\Delta t$. For the i-th generated value $\mathbf{V}_{c_i}$, the following bond price, $p^{(i)}$ and payoff, $q^{(i)}$ are computed.

$$p^{(i)} = \frac{1}{M_b}\sum_{j=1}^{M_b} \exp\left(-V_{b_j}\right) \text{ and } q^{(i)} = \left\{p^{(i)} - K\right\}^+ ,$$

where $V_{b_j} = \left(r_{j1} + r_{j2} + \ldots + r_{jN_b}\right)\Delta t$.

The price C of the European call option written on a zero-coupon bond can be estimated using the following expression:

$$C = \frac{1}{M_c}\sum_{i=1}^{M_c} \left[\exp\left(-V_{c_i}\right)\right] q^{(i)} ,$$

where $V_{c_i} = \left(u_{i1} + u_{i2} + \ldots + u_{iN_c}\right)\Delta t$.

4.2.3 Analytical formula for the call option price written on a zero-coupon bond under the CIR model

When the increment dw (see Equation (3.1)) in the CIR model is having the normal distribution with mean zero and variance *dt*, the analytical result for the price of the European call option written on a zero coupon bond was derived in Cox, Ingersoll and Ross (see Cox et al. (1985)). The explicit expression for the price is given below:

$$ZBC(t,T,S,K) = P(t,S)\chi^2\left(2\bar{r}\left[\rho + \varphi + B(T,S)\right]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h(T-t)\}}{\rho + \varphi + B(T,S)}\right) -$$

$$KP(t,T)\chi^2\left(2\bar{r}\left[\rho + \varphi\right]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h(T-t)\}}{\rho + \varphi}\right),$$

where

$$P(t,U) = A(t,U)\exp\left[-B(t,U)r_0\right] \ , \ U = T \text{ or } S ,$$

$$A(t,U)\left(\frac{2h\exp\left[\frac{1}{2}(\kappa + h)(U-t)\right]}{2h + (\kappa + h)\left[\exp(h(U-t)) - 1\right]}\right)^{\frac{2\kappa\theta}{\sigma^2}} ,$$

$$B(t,U)=\frac{2[\exp((U-t)h)-1]}{2h+(\kappa+h)[\exp(h(U-t))-1]},$$

$$\bar{r}=\bar{r}(S-T)=\frac{\ln\left[A(T,S)\big/K\right]}{B(T,S)},$$

$$\rho=\rho(T-t)=\frac{2h}{\sigma^2[\exp\{h(T-t)\}-1]},$$

$$\varphi=\frac{\kappa+h}{\sigma^2},$$

$$h=\sqrt{\kappa^2+2\sigma^2},$$

$\theta=m$ which appears in Equation (1.3).

The function $\chi^2(x;\kappa,\lambda)$ is a non-central chi-squared cumulative distribution function with κ degree of freedom and non-centrality parameter λ. In Section 4.3, the numerical results based on the above formula will be compared with those obtained by using the proposed numerical method and the simulation method.

4.3 Numerical results

This section presents some numerical values for the price of the European call option written on a zero-coupon bond when the interest rate follows the CKLS model with quadratic-normal increments.

Consider the example when $(a,m,\sigma,\gamma)=(1.0,1.0,1.0,0.5)$, $r_0=0.1$, $\overline{m}_3=0$, $\overline{m}_4=3.0$ and strike price K = 0.4. For a given value of the time T of maturity of the bond option, two sets of value of $(k,\Delta t)$ are chosen. For each chosen value of $(k,\Delta t)$, the bond option price is evaluated by using the numerical method in Section 4.2.1.

Treating the bond option price as a function of Δt, a linear extrapolation procedure is then applied to obtain the price when $\Delta t = 0$. A similar extrapolation procedure is also applied to the price based on simulation. The extrapolated prices and the price based on analytical formula are shown in Table 4.1.

Table 4.1 shows that the option prices found by using the numerical method in Section 4.2.1 and the simulation method in Section 4.2.2 agree fairly well with those found from the analytical formula.

Table 4.1 Call option price under the CIR model when $(\overline{m}_3, \overline{m}_4) = (0, 3)$.

$[(a, m, \sigma, r_0) = (1.0, 1.0, 1.0, 0.1), K = 0.4]$

T	$(k, \Delta t)$		Extrapolated (Numerical method)	Extrapolated (Simulation)	Analytical formula
$50/365$	$(50, 1/365)$	$(40, 1/292)$	0.22202743	0.21734559	0.22020237
$200/365$	$(200, 1/365)$	$(150, 4/1095)$	0.13439033	0.12651932	0.13071689
1	$(365, 1/365)$	$(100, 1/100)$	0.08315497	0.08100513	0.08228832

When the random variable w_k and the value of γ in Equation (3.2) are respectively having quadratic-normal distribution and taking 0.5, the option prices obtained from simulation and numerical method for the CIR model are shown in Table 4.2.

Table 4.2 Call option prices when $\gamma = 0.5$ and various values of $(\overline{m}_3, \overline{m}_4)$ are used. $[(a, m, \sigma, \Delta t, r_0, T) = (1.0, 1.0, 1.0, 1/365, 0.1, 1), K = 0.4]$

$\overline{m}_3$	$\overline{m}_4$	Option Price	
		Simulation	Numerical method
0.0	2.6	0.08222548	0.08321126
0.0	8.0	0.08085196	0.08320795
0.5	6.2	0.08106944	0.08307427
2.0	16.0	0.08208495	0.08266137
3.0	20.0	0.08208171	0.08238580

In Table 4.2, we observe that when w_k has the quadratic-normal distribution, the option prices found by using the numerical method agree fairly well with the simulation results.

Table 4.3 displays the option prices when $(a, m, \sigma, \Delta t, r_0, T) = (0.5, 0.08, 0.01, 1/365, 0.05, 5)$ and γ is varied from 0.0 to 2.5.

Table 4.3 Call option prices when $(a, m, \sigma, \Delta t, r_0, T) = (0.5, 0.08, 0.01, 1/365, 0.05, 5)$.

(Values in bold form are found by using simulation.)

γ	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 3.0$	$\overline{m}_3 = 0.0$ $\overline{m}_4 = 8.0$	$\overline{m}_3 = 0.5$ $\overline{m}_4 = 6.2$	$\overline{m}_3 = 2.0$ $\overline{m}_4 = 16.0$	$\overline{m}_3 = 3.0$ $\overline{m}_4 = 20.0$
0.0	0.19414593	0.19414593	0.19414591	0.19414585	0.19414581
	0.19412724	**0.19482926**	**0.19283685**	**0.19296948**	**0.19167305**
0.1	0.19406680	0.19407512	0.19406680	0.19406677	0.19406676
	0.19394658	**0.19418128**	**0.19459041**	**0.19402156**	**0.19305651**
0.2	0.19403468	0.19404221	0.19403468	0.19403467	0.19403466
	0.19362533	**0.19409710**	**0.19493375**	**0.19449029**	**0.19325086**
0.3	0.19402105	0.19402565	0.19402105	0.19402104	0.19402104
	0.19368910	**0.19388792**	**0.19450377**	**0.19500121**	**0.19322113**
0.4	0.19401467	0.19401655	0.19401467	0.19401467	0.19401467
	0.19368012	**0.19387504**	**0.19438058**	**0.19463014**	**0.19329431**
0.5	0.19401113	0.19401113	0.19401113	0.19401113	0.19401113
	0.19361511	**0.19388729**	**0.19367042**	**0.19347181**	**0.19317296**
0.6	0.19400872	0.19400769	0.19400872	0.19400872	0.19400872
	0.19363350	**0.19356810**	**0.19408560**	**0.19409402**	**0.19302674**
0.7	0.19400678	0.19400541	0.19400678	0.19400678	0.19400678
	0.19365208	**0.19357328**	**0.19391341**	**0.19401571**	**0.19319893**
0.8	0.19400507	0.19400386	0.19400507	0.19400507	0.19400507
	0.19363338	**0.19359546**	**0.19387127**	**0.19418270**	**0.19324893**
0.9	0.19400351	0.19400280	0.19400351	0.19400351	0.19400351
	0.19362739	**0.19360623**	**0.19383953**	**0.19398982**	**0.19336284**
1.0	0.19400206	0.19400206	0.19400206	0.19400206	0.19400206
	0.19360182	**0.19365199**	**0.19353956**	**0.19354853**	**0.19343928**
1.5	0.19399626	0.19400060	0.19399626	0.19399626	0.19399626
	0.19360901	**0.19360983**	**0.19363808**	**0.19361205**	**0.19354709**
2.5	0.19398990	0.19399948	0.19398990	0.19398989	0.19398989
	0.19360729	**0.19360775**	**0.19360951**	**0.19360792**	**0.19360224**

From Table 4.3, we see that the option price decreases slightly only when γ increases, but the variation of $\overline{m}_3$ and $\overline{m}_4$ does not have much influence on the prices.

The change of $(\overline{m}_3, \overline{m}_4)$ from $(0, 3)$ to other values yields more realistic interest rate models. It happens that these changes of $(\overline{m}_3, \overline{m}_4)$ do not have much influence on the call option prices. However, for other interest rate derivatives, we cannot as yet rule out the possibility that $(\overline{m}_3, \overline{m}_4)$ may have an effect on their prices.

4.4 CONCLUSION

The major characteristics of the proposed numerical method for evaluating the bond option price are the approximations of the CKLS model by its discretized version and the approximation of the price of the underlying bond by a quadratic function. The approximation of the bond option price may be improved by using an extrapolation on the bond option price treated as a function of the step size Δt, and a polynomial of degree higher than two for the price of the underlying bond.

CHAPTER 5

PREDICTION OF INTEREST RATE USING CKLS MODEL WITH STOCHASTIC PARAMETERS

5.1 Introduction

In the previous chapters, we use the CKLS model with fixed parameters to model the interest rates. In this chapter, we use instead the CKLS model with stochastic parameters to forecast the future interest rates.

Section 5.2 recapitulates the CKLS model that has been discussed in Section 1.1. In Section 5.3, the data used in this chapter are described. Before introducing the method for forecasting the *d*-week ahead interest rate using the CKLS model with stochastic parameters, we introduce in the next three sections the distributions which are important in the implementation of the method. In Section 5.4, a multivariate non-normal distribution called the multivariate power-normal distribution for the random vector $\mathbf{y}$ of k variables is introduced. In the same section, we also introduce the method for computing the conditional probability density function (pdf) of the last variable in the random vector $\mathbf{y}$ when the values of the initial $k-1$ variables in $\mathbf{y}$ are given. Furthermore in Sections 5.5 and 5.6, we introduce respectively the methods given in Pooi (2012) for estimating the parameters of the multivariate power-normal distribution, and computing the conditional joint pdf of a number of variables in the random vector when the remaining variables in the random vector are given.

The methods for forecasting the *d*-week ahead interest rate using the CKLS model with fixed and stochastic parameters are described in Section 5.7. Some numerical results are shown in Section 5.8. Section 5.9 gives some conclusions.

5.2 Model examined

The CKLS model under the objective measure Q_0 follows the following stochastic differential equation:

$$dr(t) = [am - (a + \lambda\sigma) r(t)]dt + \sigma[r(t)]^{\gamma} dW^0(t) \ . \tag{5.1}$$

The discrete version of Equation (5.1) is

$$\begin{aligned} r_k &= r_{k-1} + am\Delta t - (a + \lambda\sigma) r_{k-1}\Delta t + \sigma[r_{k-1}]^{\gamma} W_k^0 \sqrt{\Delta t} \\ &= \beta_0 + \beta_1 r_{k-1} + \sigma(r_{k-1})^{\gamma} W_k^0 \sqrt{\Delta t} \end{aligned} ,$$

where $\beta_0 = am\Delta t$, $\beta_1 = 1 - (a + \lambda\sigma)\Delta t$, λ is a new parameter, contributing to the market price of risk and W_k^0 is assumed to have a quadratic-normal distribution with parameters 0 and $\bar{\boldsymbol{\lambda}}$, and $\bar{\boldsymbol{\lambda}}$ is such that $E\left[(w_k^0)^2\right] = 1$.

If we treat the parameters β_0, β_1, σ and γ as correlated random variables, then the resulting model may be referred to as the CKLS model with stochastic parameters β_0, β_1, σ and γ.

In this chapter, the interest rate at the end of a period of length *d* units ahead is forecasted using respectively the CKLS models with fixed and stochastic parameters.

5.3 Data

To illustrate the forecasting of the interest rate based on the CKLS models with respectively fixed and stochastic parameters, we use the weekly 6-month Treasury Bill Interest Rates (T-Bill) taken from the Board of Governors of the Federal Reserve System via the link http:research.stlouisfed.org/fred2/categories/116 under the file name WTB6MS.xls. The data cover the period from December 12, 1958 to January 25, 2008. The total number of data points is $N = 2564$ and the length of a one-week period is given by $\Delta t = 7/365$. Some of the T-Bill Interest Rates are shown in Table 5.1. Figure 5.1 shows the plot of r_k given in WTB6MS.xls against *k*.

Table 5.1 T-Bill interest rates.

week *k*	r_k
1	0.0307
2	0.0304
3	0.0297
4	0.0292
⋮	⋮
⋮	⋮
2564	0.0232

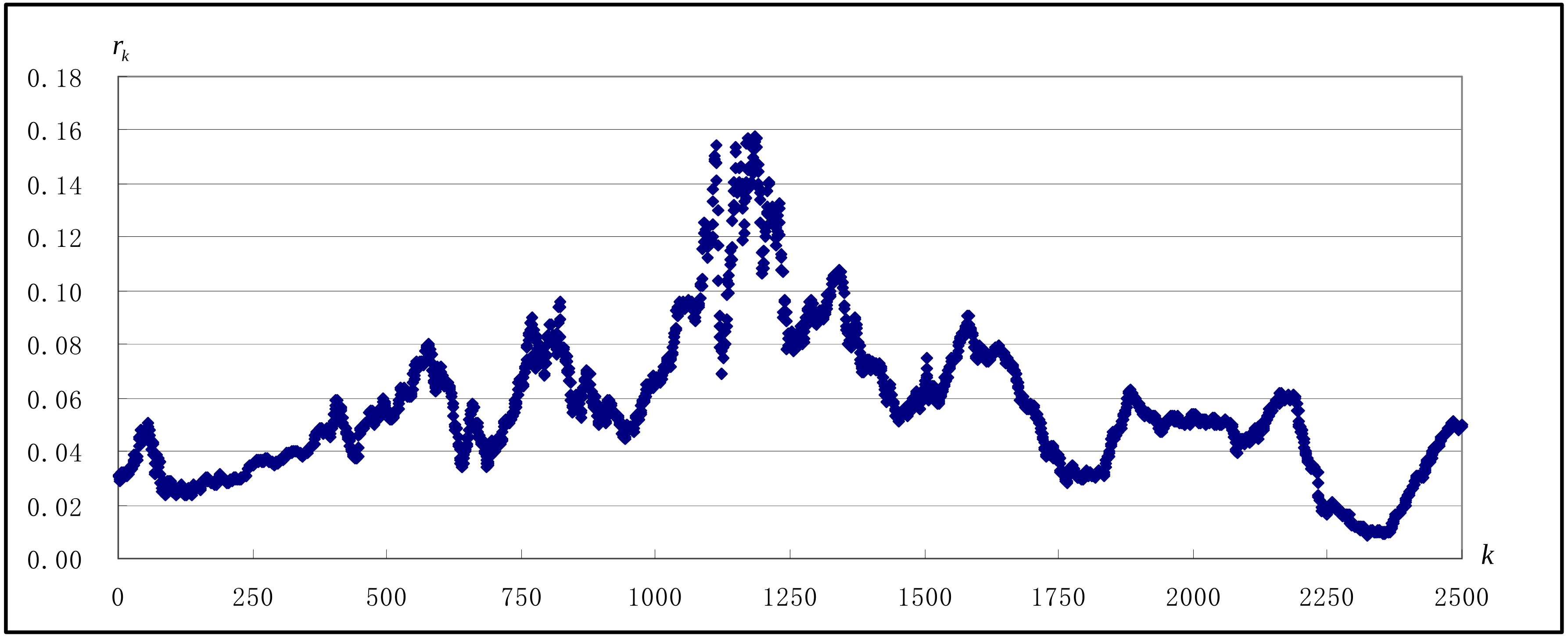

Figure 5.1 Plot of r_k against k.

5.4 Conditional pdf derived from multivariate power-normal distribution

Consider the univariate power-normal distribution described in Section 2.2.1. In what follows, we use the univariate power-normal distribution to obtain the multivariate power-normal distribution (see Pooi (2012)).

First let $\mathbf{y}$ be a vector consisting of *k* correlated random variables. The vector $\mathbf{y}$ is said to have a *k*-dimensional power-normal distribution with parameters $\boldsymbol{\mu}, \mathbf{H}, \lambda_i^-, \lambda_i^+, \sigma_i,\ 1 \le i \le k$ if

$$\mathbf{y} = \boldsymbol{\mu} + \mathbf{H}\boldsymbol{\varepsilon}, \tag{5.2}$$

where $\boldsymbol{\mu} = E(\mathbf{y})$, $\mathbf{H}$ is an orthogonal matrix, $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{\boldsymbol{k}}$ are uncorrelated,

$$\varepsilon_i = \sigma_i \left[\tilde{\varepsilon}_i - E(\tilde{\varepsilon}_i)\right] / \left[\mathrm{var}(\tilde{\varepsilon}_i)\right]^{1/2}, \tag{5.3}$$

$\sigma_i > 0$ is a constant, and $\tilde{\varepsilon}_i$ has a power-normal distribution with parameters λ_i^- and λ_i^+.

When the values of $y_1, y_2, ..., y_{k-1}$ are given, we may use the following method given in Pooi (2012) to find an approximation for the conditional pdf of y_k:

(1) Select a large integer $N_p > 0$ and compute $y_k^{(i_p)} = y_k^- + (i_p - 1)h,\ 1 \le i_p \le N_p$, where y_k^- and y_k^+ are such that $P(y_k^- < y_k \le y_k^+)$ is close to 1, and $h = (y_k^+ - y_k^-)/N_p$.

(2) Form the vector $\mathbf{y}^{(i_p)} = \left[y_1, y_2, ..., y_{k-1}, y_k^{(i_p)}\right]^T$ and find the value of $\boldsymbol{\varepsilon}^{(i_p)}$ such that

$$\mathbf{y}^{(i_p)} = \boldsymbol{\mu} + \mathbf{H}\boldsymbol{\varepsilon}^{(i_p)}.$$

(3) Replace (λ^-, λ^+) in Equation (2.10) by $(\lambda_i^-, \lambda_i^+)$ and find *z* such that

$\varepsilon_i = \varepsilon_i^{(i_p)}$. Let the answer of z be denoted by $z_i^{(i_p)}$.

(4) Compute

$$f_{i_p} = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(z_i^{(i_p)}\right)^2\right] \Bigg/ \left| \left[\frac{d\varepsilon_i}{dz_i}\right]_{z_i = z_i^{(i_p)}} \right|.$$

Then the conditional pdf (evaluated at $y_k^{(i_p)}$) of y_k can be estimated by $f_{i_p} \Big/ \sum_{i_p=1}^{N_p} \left(f_{i_p} h\right)$.

5.5 Estimation of the parameters of the multivariate power-normal distribution

Suppose $\tilde{\mathbf{r}}$ is a $(l+1)\times 1$ random vector of which the n_1-th observed value is denoted by $\tilde{\mathbf{r}}^{(n_1)}, 1 \le n_1 \le N_1$. The following is a procedure to fit an $(l+1)$–dimensional multivariate power-normal distribution to $\tilde{\mathbf{r}}$:

(1) Compute $\bar{r}_i = \frac{1}{N_1}\sum_{n_1=1}^{N_1} \tilde{r}_i^{(n_1)}$ and $m_{ij}^{(k_1,k_2)} = \frac{1}{N_1}\sum_{n_1=1}^{N_1} \left(\tilde{r}_i^{(n_1)} - \bar{r}_i\right)^{k_1} \left(\tilde{r}_j^{(n_1)} - \bar{r}_j\right)^{k_2}$, $1 \le i, j \le l+1;\ \ 0 \le k_1, k_2 \le 1$.

(2) Compute the $l+1$ eigenvectors of the variance-covariance matrix $\left\{m_{ij}^{(1,1)} - m_{ij}^{(1,0)} m_{ij}^{(0,1)}\right\}$ and form the matrix $\mathbf{H}_1$ of which the i-th column is the i–th eigenvector.

(3) Compute $\mathbf{s}^{(n_1)} = \mathbf{H}_1^T \left(\tilde{\mathbf{r}}^{(n_1)} - \bar{\mathbf{r}}\right)$.

(4) Compute $m_i^{(k)} = \frac{1}{N_1}\sum_{n_1=1}^{N_1} [s_i^{(n_1)}]^k$, $1 \le i \le l+1;\ \ k = 2, 3, 4$.

(5) Find $\left(\lambda_i^+, \lambda_i^-\right)$ and σ_i such that $E(\varepsilon_i^k) = m_i^{(k)}$, where ε_i is defined in Equation (5.3) and $1 \le i \le l+1$; $k = 2, 3, 4$.

Then $\tilde{\mathbf{r}} = \bar{\mathbf{r}} + \mathbf{H}_1\mathbf{s}$ of which $s_i \cong \sigma_i \left[\tilde{\varepsilon}_i - E\left(\tilde{\varepsilon}_i\right)\right] \big/ \left[\mathrm{var}\left(\tilde{\varepsilon}_i\right)\right]^{1/2}$ and $\tilde{\varepsilon}_i$ has a power-normal distribution with parameters λ_i^- and λ_i^+.

5.6 Conditional pdf of a number of variables in random vector $\mathbf{y}$

Suppose $\tilde{\mathbf{x}}$ is an $(m+1)n_c \times 1$ random vector of which the n_2-th observed value is $\tilde{\mathbf{x}}^{(n_2)} = \left[\tilde{\mathbf{x}}_{n_2-m}, \ldots, \tilde{\mathbf{x}}_{n_2-1}, \tilde{\mathbf{x}}_{n_2}\right]$ and $\tilde{\mathbf{x}}_i$ is an $n_c \times 1$ vector, $1 \le n_2 \le N_2$. The following is a procedure to estimate the conditional pdf of the last n_c components of $\tilde{\mathbf{x}}$ when the values of the initial mn_c components are given by $\left[\tilde{\mathbf{x}}^*_{n_2-m}, \ldots, \tilde{\mathbf{x}}^*_{n_2-2}, \tilde{\mathbf{x}}^*_{n_2-1}\right]$.

(1) Obtain $\tilde{\mathbf{x}}^{(n_2)}_{[q]}$ as a vector consisting of the initial mn_c components of $\tilde{\mathbf{x}}^{(n_2)}$ and q initial components of $\tilde{\mathbf{x}}_{n_2}$, $1 \le q \le n_c$.

The value $\tilde{\mathbf{x}}^{(n_2)}_{[q]}$ may be viewed as the n_2-th observed value of a certain vector $\tilde{\mathbf{x}}_{[q]}$ consisting of $mn_c + q$ random variables.

(2) For $q = 1, 2, \ldots, n_c$, apply the method in Section 5.5 to find a multivariate power-normal distribution for $\tilde{\mathbf{x}}_{[q]}$ and use the resulting distribution to generate a value x_q^* for the $(mn_c + q)^{\text{th}}$ component of $\tilde{\mathbf{x}}$ when $\tilde{\mathbf{x}}^*_{n_2-m}, \ldots, \tilde{\mathbf{x}}^*_{n_2-1}, x_1^*, x_2^*, \ldots, x_{q-1}^*$ are given.

(3) Repeat (2) to generate $m*$ values of $\left(x_1^*, x_2^*, \ldots, x_{n_c}^*\right)$ and use the method in Section 5.5 to find a multivariate power-normal distribution as an estimate of the conditional distribution of the last n_2 components of $\tilde{\mathbf{x}}$.

5.7 Methods for forecasting *d*-week ahead interest rate using the CKLS models with respectively fixed and stochastic parameters

From the T-Bill Interest Rates in Section 5.3, we form Table 5.2 which shows the interest rates in the present and next weeks.

Table 5.2 Interest rates in the present and next weeks.

week k	r_k	r_{k+1}
1	0.0307	0.0304
2	0.0304	0.0297
3	0.0297	0.0292
4	0.0292	0.0298
5	0.0298	0.0305
$\vdots$	$\vdots$	$\vdots$
2562	0.0311	0.0291
2563	0.0291	0.0232

Next, we define the *j*-th window of size n as the set consisting of rows j to $j+(n-1)$ in Table 5.2. If we wish to investigate the procedure for forecasting the *d*-week ahead interest rate based on the data given in a window, then the total number of windows that we can form is $N_w = N - d - n$. For the *j*-th window $(1 \le j \le N_w)$, the four parameters of the CKLS model are estimated using the least squares method and the resulting estimated parameters are denoted as $\hat{\beta}_0^{(j)}, \hat{\beta}_1^{(j)}, \hat{\sigma}^{(j)}$ and $\hat{\gamma}^{(j)}$. An outline of the least squares method is given below.

Initially, for the *j*-th window, we regress r_{k+1} on r_k and obtain the following line of best fit:

$$\hat{r}_{k+1} = \hat{\beta}_1^{(j)} r_k + \hat{\beta}_0^{(j)}.$$

We next obtain the residuals

$$e_{k+1}^{(j)} = r_{k+1} - \hat{\beta}_0^{(j)} - \hat{\beta}_1^{(j)}\, r_k\,,$$

and regress $\left|e_{k+1}^{(j)}\right|$ on r_k to obtain the best power function which fits the points $\left(r_k, \left|e_{k+1}^{(j)}\right|\right)$, $j \le k \le j+n-1$:

$$\left|\hat{e}_{k+1}^{(j)}\right| = \hat{\sigma}^{(j)} \sqrt{\Delta t}\; r_k^{\hat{\gamma}^{(j)}}\,.$$

The above estimates are basically based on the method of least squares. For these estimates, their characteristics in terms of, for example, consistency and efficiency are clearly of interest to us. The investigation of these characteristics should be a good topic for future research.

Suppose by using the CKLS model with the parameters $\hat{\beta}_0^{(j)}, \hat{\beta}_1^{(j)}$, $\hat{\sigma}^{(j)}$ and $\hat{\gamma}^{(j)}$, we predict the value r_{j+n+1} of the interest rate in the next week when the value r_{j+n} of the interest rate in the present week is given. As the parameters $\hat{\beta}_0^{(j)}, \hat{\beta}_1^{(j)}$, $\hat{\sigma}^{(j)}$ and $\hat{\gamma}^{(j)}$ change when j increases, we may describe the variation of $\boldsymbol{\varphi}^{(j)} = \left[\hat{\beta}_0^{(j)}, \hat{\beta}_1^{(j)}, \hat{\sigma}^{(j)}, \hat{\gamma}^{(j)}\right]^T$ by means of a 4-dimensional multivariate power-normal distribution.

To find a 4-dimensional multivariate distribution for $\boldsymbol{\varphi}^{(j)}$, we first assume that $\boldsymbol{\varphi}^{(j)}$ will depend on $\boldsymbol{\varphi}^{(j-m)}, \boldsymbol{\varphi}^{(j-m+1)}, \ldots,$ $\boldsymbol{\varphi}^{(j-1)}, r_{j+n}$ where $m \ge 1$ is a chosen integer and $j \ge m+1$. We next use the method in Pooi (2012) to find a $[4(m+1)+1]$-dimensional multivariate power-normal distribution to the random vector $\tilde{\mathbf{r}}$ of which the n_1 - th observed value is $\tilde{\mathbf{r}}^{(n_1)} = \left[\boldsymbol{\varphi}^{(n_1)}, \boldsymbol{\varphi}^{(n_1+1)}, \ldots, \boldsymbol{\varphi}^{(n_1+m-1)}, r_{n_1+m+n}, \boldsymbol{\varphi}^{(n_1+m)}\right]^T$ $, 1 \le n_1 \le N_w - m$. A description of the multivariate power-normal distribution is given in Section 5.4 and the estimation of the distribution is given in Section 5.5.

From the above $[4(m+1)+1]$ -dimensional multivariate power-normal distribution, we use the method in Pooi (2012) to find a conditional distribution for the last 4 variables in $\tilde{\mathbf{r}}$ when the values of the initial $(4m+1)$ variables are given by

$$\left[\boldsymbol{\varphi}^{(j-m)}, \boldsymbol{\varphi}^{(j-m+1)}, \ldots, \boldsymbol{\varphi}^{(j-1)}, r_{j+n}\right]^T .$$

The resulting conditional distribution will then be the required 4-dimensional multivariate distribution for $\boldsymbol{\varphi}^{(j)}$. An outline of the computation of the above conditional distribution is given in Section 5.6.

From the 4-dimensional multivariate power-normal distribution of $\boldsymbol{\varphi}^{(j)}$, the d-week ahead interest rate, r_{j+n+d}, is then generated using the CKLS model with stochastic parameters.

Let $J = j+n+d$, and r_J^+ and r_J^* the d-week ahead interest rates generated using the CKLS models with fixed and stochastic parameters respectively. Sections 5.7.1 and 5.7.2 explain respectively the procedures of generating r_J^+ and r_J^*.

The computed numerical results for r_J^+ and r_J^* are displayed and analyzed in Section 5.8.

5.7.1 The CKLS model with fixed parameters

The value of r_{j+n+l} is obtained using the following discretized version of the CKLS model with the fixed parameter vector $\boldsymbol{\varphi}^{(j)} = \left[\hat{\beta}_0^{(j)}, \hat{\beta}_1^{(j)}, \hat{\sigma}^{(j)}, \hat{\gamma}^{(j)}\right]^T$:

$$r_{j+n+l} = \hat{\beta}_0^{(j)} + \hat{\beta}_1^{(j)} r_{j+n+(l-1)} + \hat{\sigma}^{(j)}\left[r_{j+n+(l-1)}\right]^{\hat{\gamma}^{(j)}} w_{j+n+l}\sqrt{\Delta t}, \quad 1 \le l \le d .$$

When $l = d$, the value of r_{j+n+l} is then the d-week ahead interest rate r_J^+.

Suppose we repeat the generation of r_J^+ M times. We next rank the M generated values in an ascending order and obtain L^+ and R^+ as respectively the estimated $100(\alpha/2)\%$ and $100(1-\alpha/2)\%$ points of the distribution of r_J^+. The interval $\left(L^+, R^+\right)$ may then be taken to be a prediction interval for r_J.

The coverage probability of the prediction interval for r_J may be estimated by using the proportion of times (among the N_w times) the observed value of the *d*-week ahead interest rate falls inside the prediction interval.

The expected length of the prediction interval on the other hand may be estimated by using the average value (over N_w values) of $R^+ - L^+$.

The numerical results for the coverage probability and expected length of the prediction interval for r_J are given in Section 5.8.

5.7.2 The CKLS model with stochastic parameters

The value of r_{j+n+1} is obtained using the following discretized version of the CKLS model with the parameter $\boldsymbol{\varphi}^{(j)} = \left[\hat{\beta}_0^{(j)}, \hat{\beta}_1^{(j)}, \hat{\sigma}^{(j)}, \hat{\gamma}^{(j)}\right]^T$ which is generated from the conditional distribution of $\tilde{\boldsymbol{\varphi}}^{(j)}$ when $\left[\boldsymbol{\varphi}^{(j-m)}, \boldsymbol{\varphi}^{(j-m+1)}, \ldots, \boldsymbol{\varphi}^{(j-1)}, r_{j+n}\right]^T$ is given:

$$r_{j+n+1} = \hat{\beta}_0^{(j)} + \hat{\beta}_1^{(j)} r_{j+n} + \hat{\sigma}^{(j)}\left[r_{j+n}\right]^{\hat{\gamma}^{(j)}} w_{j+n+1}\sqrt{\Delta t}.$$

The value of r_{j+n+2} is next obtained from the CKLS model of which the parameter $\tilde{\boldsymbol{\varphi}}^{(j+1)}$ is generated from the conditional distribution of $\tilde{\boldsymbol{\varphi}}^{(j+1)}$ when $\left[\boldsymbol{\varphi}^{(j-m+1)}, \ldots, \boldsymbol{\varphi}^{(j-1)}, \tilde{\boldsymbol{\varphi}}^{(j)}, r_{j+n+1}\right]^T$ is given.

The process to generate r_{j+n+l} for $l = 3, 4, \ldots, d$ is as follows. We first define

$$\boldsymbol{\varphi}^{*(j^*)} = \begin{cases} \boldsymbol{\varphi}^{(j^*)} & \text{if} \quad j^* \le j-1 \\ \tilde{\boldsymbol{\varphi}}^{(j^*)} & \text{if} \quad j^* \ge j \end{cases}.$$

We next generate the parameter $\tilde{\boldsymbol{\varphi}}^{(j+l-1)}$ from its conditional distribution when

$\left[\boldsymbol{\varphi}^{*(j-m+l-1)},\ldots,\boldsymbol{\varphi}^{*(j+l-3)},\boldsymbol{\varphi}^{*(j+l-2)},r_{j+n+l-1}\right]^T$ is given, and then generate the value of r_{j+n+l} using the CKLS model with the generated parameter $\tilde{\boldsymbol{\varphi}}^{(j+l-1)}$. When $l = d$, the value of r_{j+n+l} is then the *d*-week ahead interest rate r_J^*.

Next, we generate *M* values of r_J^* and arrange the *M* generated values in an ascending order. From the ranked values of r_J^*, we estimate the $100(\alpha/2)\%$ and $100(1-\alpha/2)\%$ points of the distribution of r_J^*. Let the estimates obtained be denoted by L^* and R^* respectively. The interval (L^*, R^*) may then be treated as a nominally $100(1-\alpha)\%$ prediction interval for r_J.

From the N_w values of (L^*, R^*), we estimate the coverage probability and expected length of the prediction interval.

In Section 5.8, the coverage probability and expected length of (L^*, R^*) for r_J are compared with those found from the CKLS model with fixed parameters.

5.8 Numerical results

Let $n = 200$, $d = 20$, $m = 2$, $M = 120$ and $\alpha = 0.05$.

Figures 5.2 and 5.3 display respectively the nominally 95% prediction interval for the 20-week ahead interest rate, r^+_{j+220} and r^*_{j+220} found respectively using the CKLS model with fixed and stochastic parameters.

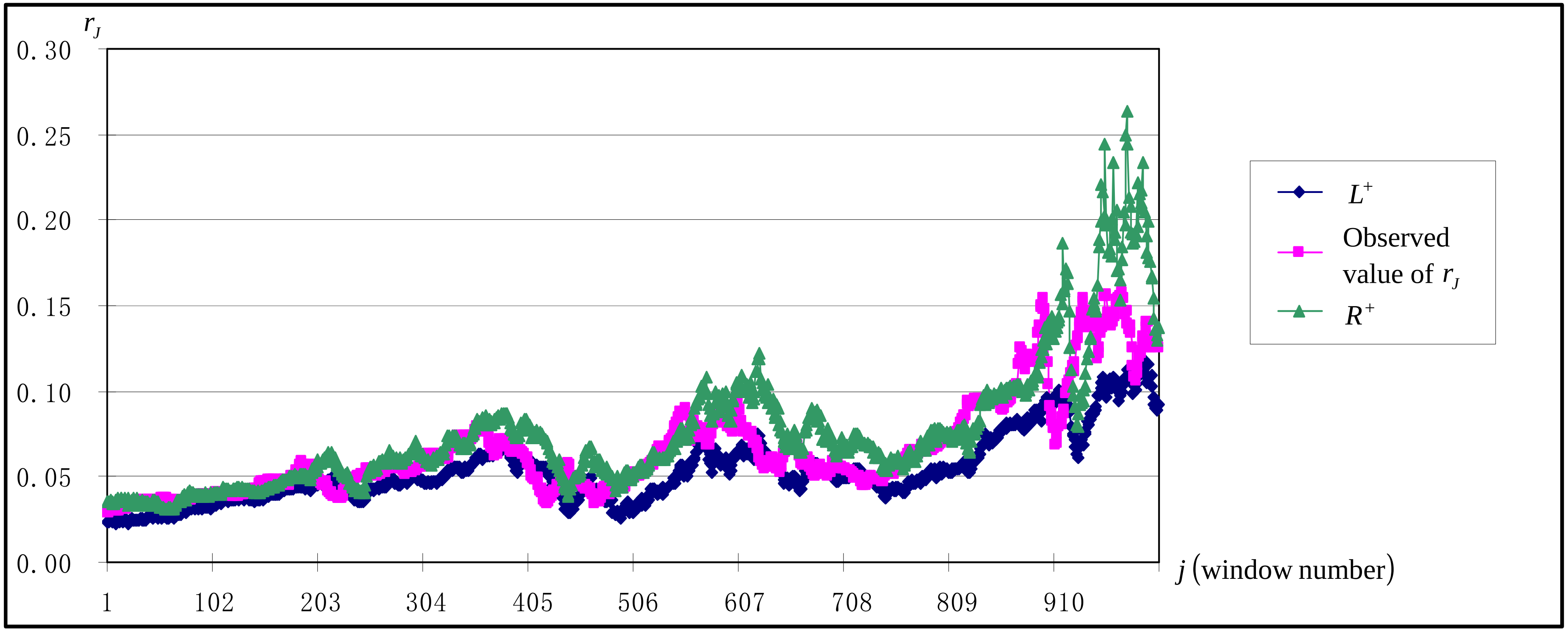

Figure 5.2 The nominally 95% prediction interval (L^+, R^+) for r_J.

$$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d]$$

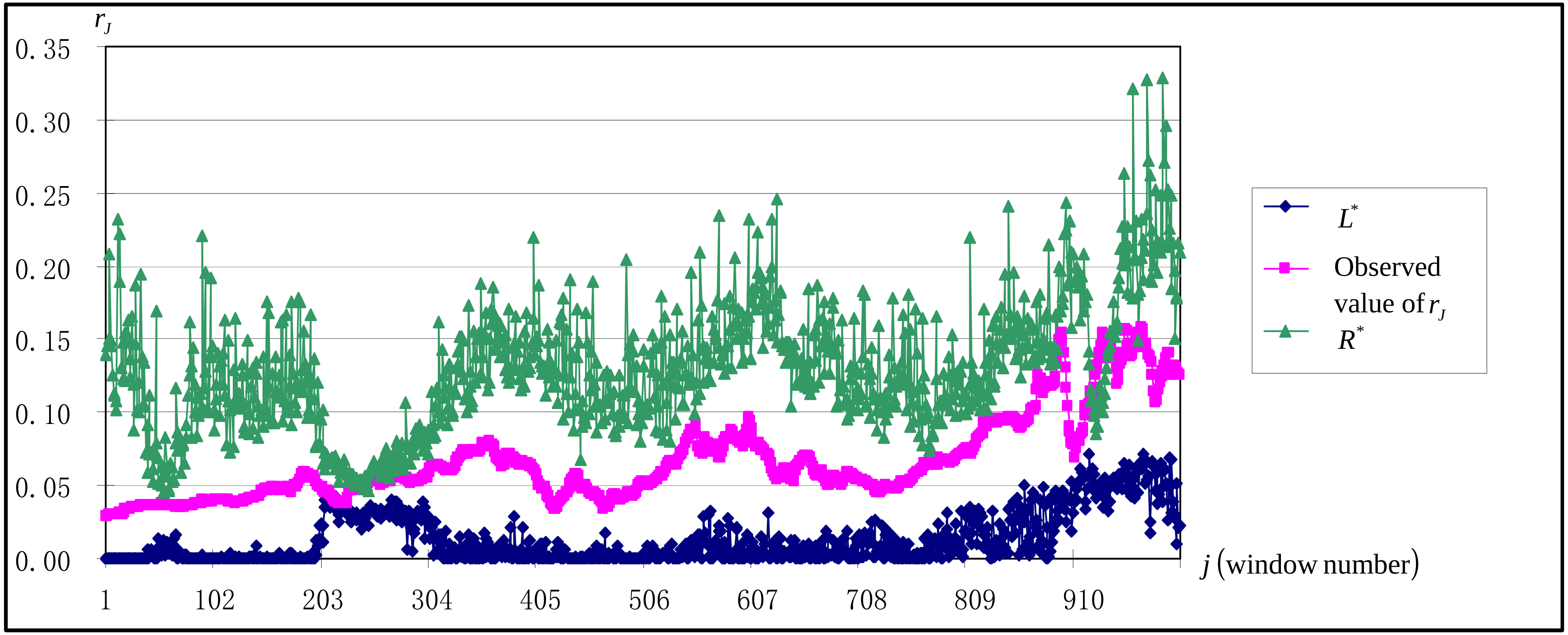

Figure 5.3 The nominally 95% prediction interval (L^*, R^*) for r_J.

$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d]$

The estimates of the coverage probability expected length of the prediction interval for r_J in Figure 5.2 are respectively 0.587 and 0.023, while the estimates of the coverage probability and expected length of the prediction interval for r_J in Figure 5.3 are respectively 0.972 and 0.116.

Compared to the estimate of the coverage probability of the nominally 95% prediction interval for r_J found by using the CKLS model with fixed parameters (see Figure 5.2), we can see clearly that the estimated coverage probability of the nominally 95% prediction interval for r_J found by using the CKLS model with stochastic parameters (see Figure 5.3) is very much closer to the target value 0.95.

On the other hand, the estimate of the expected length of the nominally 95% prediction interval for r_J found by using the CKLS model with stochastic parameters (see Figure 5.3) is longer than the one found by using the CKLS model with fixed parameters (see Figure 5.2). This is not surprising because in order to have larger coverage probability, the length of the prediction interval needs to be made longer.

Figures 5.4 to 5.11 provide the values of L^+, R^+, L^* and R^* for selected values of j.

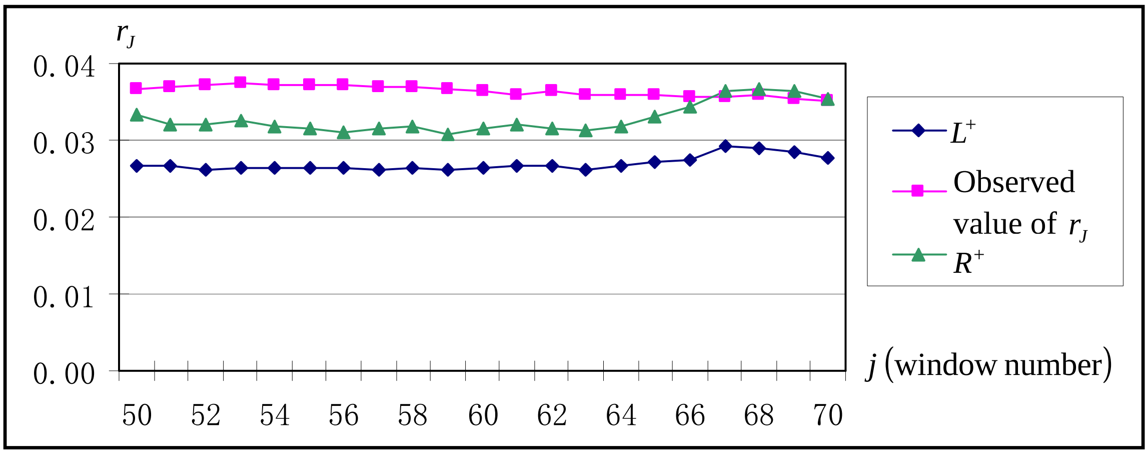

Figure 5.4 The nominally 95% prediction interval $\left(L^+, R^+\right)$ for r_J.

$$\left[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 50,51,\ldots,70\right]$$

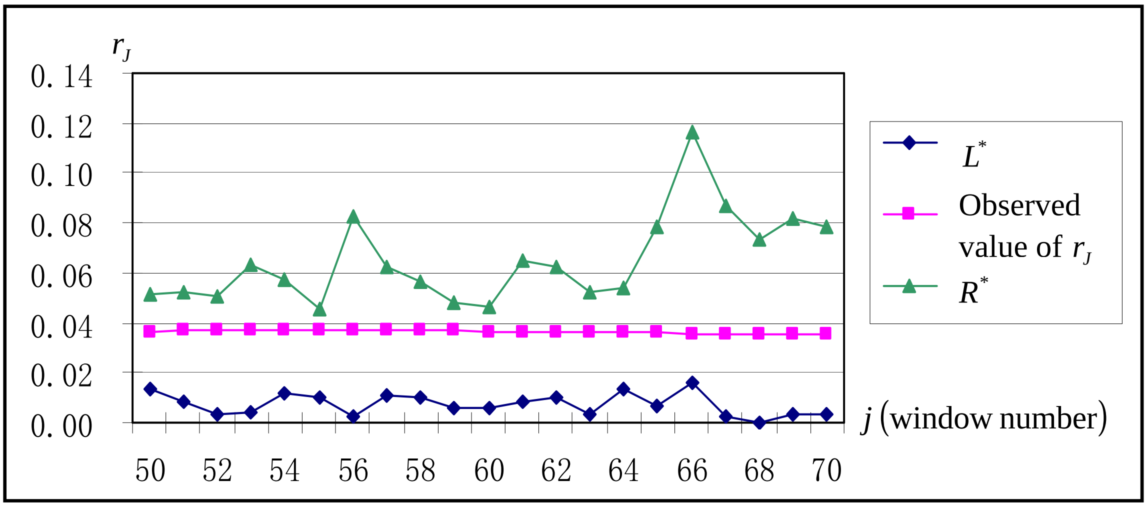

Figure 5.5 The nominally 95% prediction interval $\left(L^*, R^*\right)$ for r_J.

$$\left[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 50,51,\ldots,70\right]$$

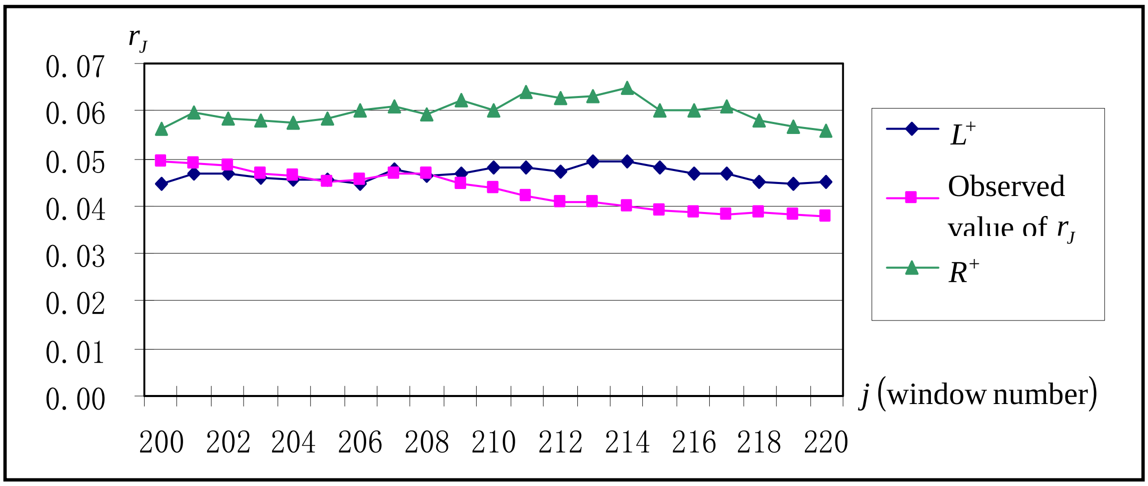

Figure 5.6 The nominally 95% prediction interval (L^+, R^+) for r_J.

$$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 200,201,...,220]$$

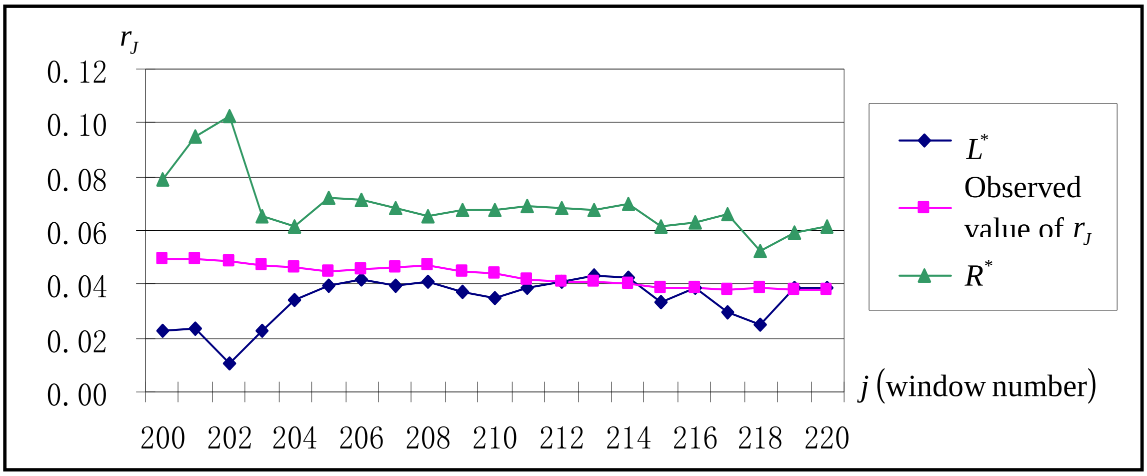

Figure 5.7 The nominally 95% prediction interval (L^*, R^*) for r_J.

$$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 200,201,...,220]$$

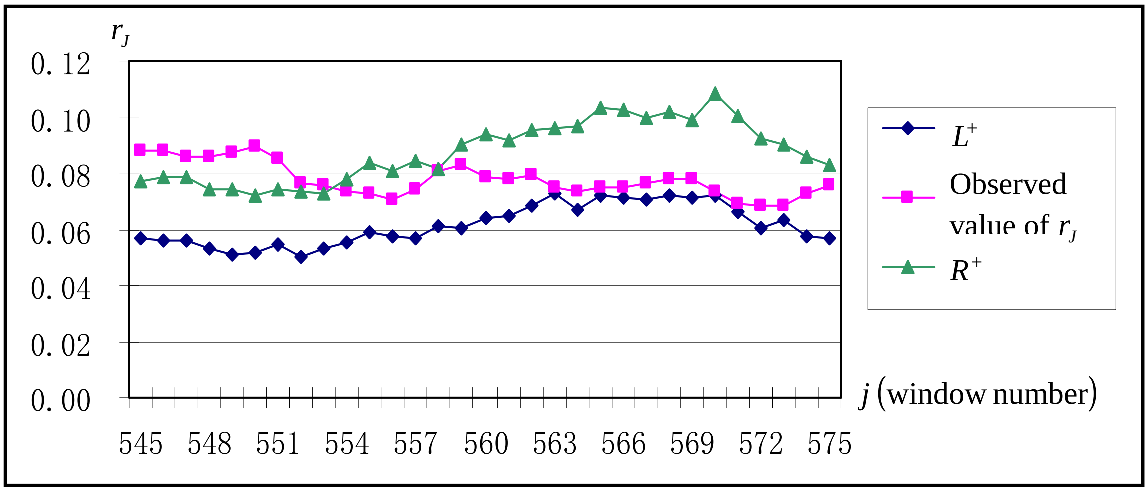

Figure 5.8 The nominally 95% prediction interval (L^+, R^+) for r_J.

$$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 545,546,...,575]$$

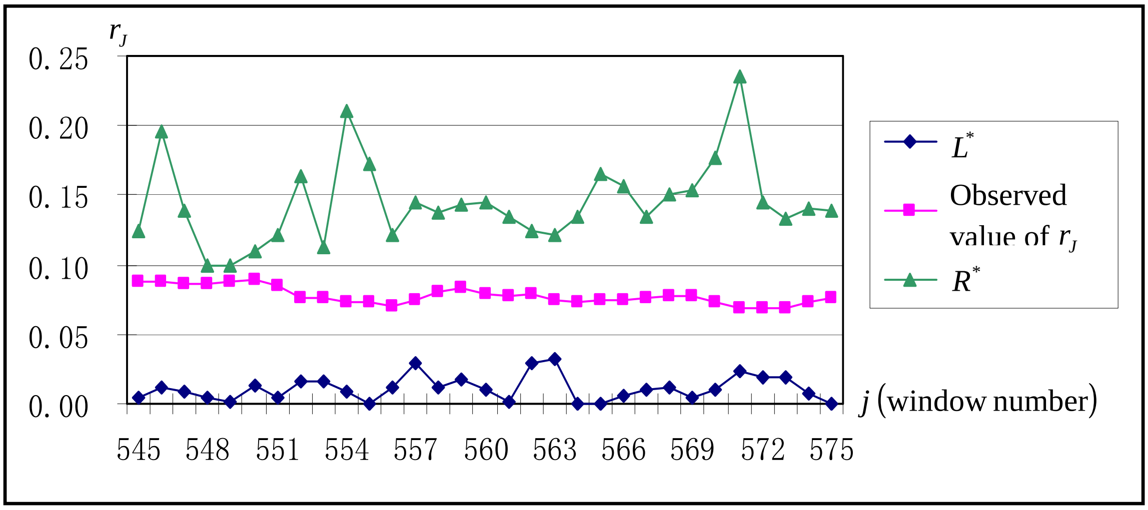

Figure 5.9 The nominally 95% prediction interval (L^*, R^*) for r_J.

$$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 545,546,...,575]$$

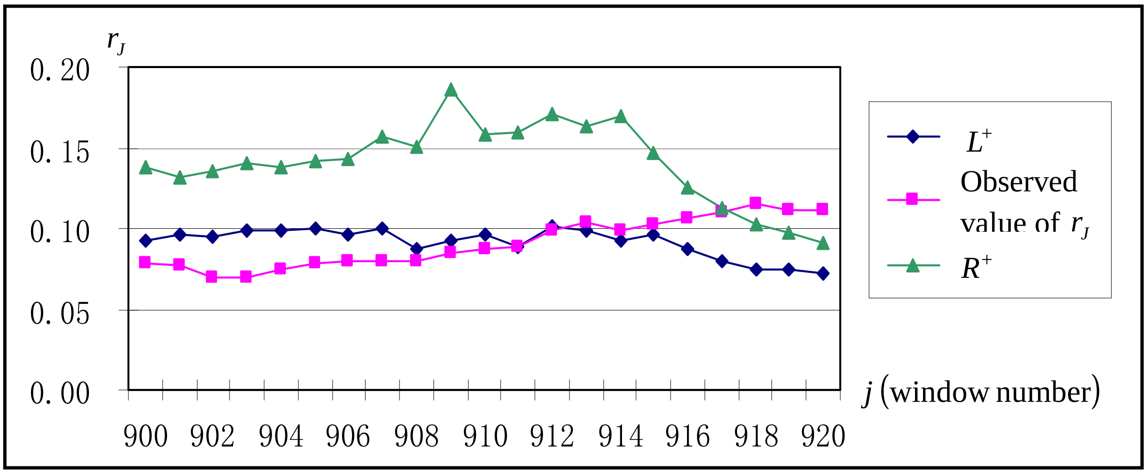

Figure 5.10 The nominally 95% prediction interval (L^+, R^+) for r_J.

$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 900, 901, \ldots, 920]$

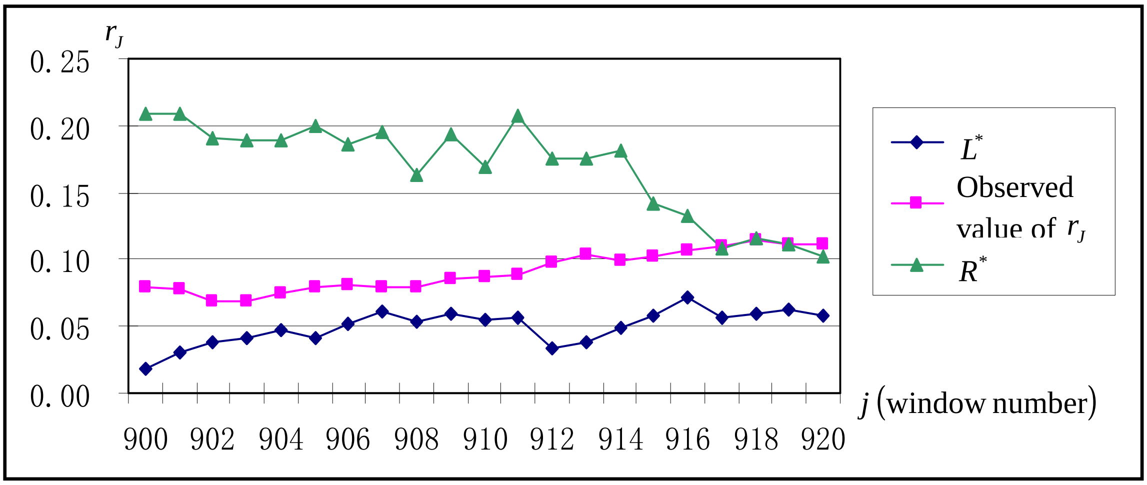

Figure 5.11 The nominally 95% prediction interval (L^*, R^*) for r_J.

$[n = 200, d = 20, m = 2, M = 120, \alpha = 0.05, J = j + n + d, j = 900, 901, \ldots, 920]$

Figures 5.4 to 5.11 show the instances when the prediction interval $\left(L^{+}, R^{+}\right)$ fails to cover the observed future interest rate r_J while the prediction interval $\left(L^{*}, R^{*}\right)$ succeeds in covering the observed r_J.

5.9 CONCLUSION

The prediction interval for the 20-week ahead interest rate using the CKLS model with fixed parameters is unable to cover the observed value of the 20-week ahead interest rate with a coverage probability which is close to the target value 0.95. This is because the parameters of the CKLS model have significant variation over the 20-week period. On the other hand, the CKLS model with stochastic parameters takes into account the possible variation of the parameters, and with this feature, it helps to yield a wider prediction interval with a satisfactory level of coverage probability.

The conditional distribution of the parameters (classified as Type 1 parameters in Pooi (2012)) of the CKLS model has been obtained from a $[4(m+1)+1]$-dimensional multivariate power-normal distribution of which the parameters (classified as Type 2 parameters in Pooi (2012)) do not vary with time. Future research may be carried out to investigate the stationarity of Type 2 parameters.

CONCLUDING REMARKS

The main contributions of this thesis are the evaluation of the prices of bond and bond option under the CIR and CKLS models of which the underlying distribution is quadratic-normal, and the prediction of future interest rate using the CKLS model with stochastic parameters.

The methods introduced make use of the approximation of the CKLS model by its discretized version, and the approximation of the expected moments of the interest rate and the bond price by low degree polynomials. The above approximations may be improved by using an extrapolation procedure and the polynomials of higher degrees.

The approximate methods introduced in this thesis may be applicable to other more general interest rate models like the multi-factor interest rate models, the models which describe the dependence of the future interest rate on the present interest rate and several other past interest rates, and also the dynamic interest rate models.

Although the prediction interval based on the CKLS model with stochastic parameters has approximately the required coverage probability, its expected length is very much longer than that of the prediction interval based on the model with fixed parameters. To find a prediction interval which exhibits smaller expected length apart from having the required coverage probability, it is likely that the CKLS model has to be changed to a model in which the future interest rate depends not only on the present interest rate, but also on several past interest rates and some important macro-economic factors. Furthermore, it could be that the description of the parameters of the model by a fixed set of Type 2 parameters may not be adequate for long range prediction. The dynamic modeling of Type 2 parameters offers a very challenging research problem.

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