# SOME PROPERTIES ASSOCIATED WITH CERTAIN SUBCLASSES OF UNIVALENT AND MULTIVALENT FUNCTIONS

**WAN SABHI SALMI BT WAN HASSAN**

**FACULTY OF SCIENCE**

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WAN SABHI SALMI BT WAN HASSAN

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# ABSTRACT


This thesis investigates properties of certain analytic functions; in particular, functions which are univalent and multivalent in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the class of all normalised analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Interest is focused at several subclasses of $\mathcal{A}$. Functions belonging to these subclasses are defined via some differential operator; namely the Sălăgean and Al-Oboudi operator. These classes formed are subclasses of $\mathcal{S}$, the class of univalent functions.

Let $f \in \mathcal{B}_n(\alpha)$ for $\alpha > 0$ and $n = 0, 1, 2, \ldots$ be defined by

$$Re \frac{D^n f(z)^\alpha}{z^\alpha} > \alpha.$$

where $D^n$ denote the Sălăgean operator.

For functions $f \in \mathcal{B}_n(\alpha)$, we obtain estimates for the second, third and fourth coefficients of the inverse functions. Further, we investigate similar coefficient problems for functions in the $\mathcal{B}_n^\lambda(\alpha)$, an extension of the above class defined via the Al-Oboudi operator. In addition, these are then applied to obtain the Fekete-Szegö inequalities.

Next, besides functions of the above normalised form, the thesis also looks at functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n},$$

where $p$ a fixed positive integer. For functions of this form, we denote $\mathcal{A}_p$ as the class consisting of such functions. For such class we investigate sharp lower bounds on

the real part of the quotients between the normalised functions and their sequence of partial sums for convex and starlike functions as well as their related classes, the uniformly convex and parabolic starlike functions which satisfy certain conditions.

Finally, for function $f(z) \in \mathcal{A}_p$ which are analytic in $\mathcal{U}$, results on the preservation of two integral operators $I_p^\sigma f(z)$ and $J_\beta^\sigma f(z)$ given by

$$I_p^\sigma f(z) = \frac{(p+1)^\sigma}{z^p \Gamma(\sigma)} \int_0^z t^{p-1} \left( log\ \frac{z}{t} \right)^{\sigma-1} f(t)\ dt \quad (\sigma > 0).$$

$$J_\beta^\sigma f(z) = \binom{p+\sigma+\beta-1}{p+\beta-1} \frac{\sigma}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\sigma-1} t^{\beta-1} f(t)\ dt \quad (\sigma > 0,\ \beta > -1),$$

are established for some of the classes of functions defined using the Hadamard product.

## ABSTRAK

Tesis ini mengkaji sifat-sifat fungsi analisis; khususnya, fungsi-fungsi yang univalen dan multivalen dalam unit cakera $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Andaikan $\mathcal{A}$ mewakili kelas bagi semua fungsi analisis ternormal dalam bentuk:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Kajian ini tertumpu kepada beberapa subkelas $\mathcal{A}$, fungsi yang dimiliki oleh subkelas ini ditakrif melalui beberapa pengoperasian pembezaan; iaitu pengoperasi Sălăgean dan Al-Oboudi. Kelas-kelas yang dibentuk adalah subkelas $\mathcal{S}$, iaitu kelas daripada fungsi univalen.

Andaikan $f \in \mathcal{B}_n(\alpha)$ bagi $\alpha > 0$ dan $n = 0, 1, 2, \ldots$ yang ditakrifkan sebagai

$$Re\ \frac{D^n f(z)^\alpha}{z^\alpha} > \alpha.$$

dengan $D^n$ mewakili pengoperasi Sălăgean.

Bagi fungsi $f \in \mathcal{B}_n(\alpha)$, anggaran bagi pekali kedua, ketiga dan keempat bagi fungsi songsang diperoleh. Seterusnya, dengan cara yang serupa masalah pekali bagi fungsi $\mathcal{B}_n^\lambda(\alpha)$ yang merupakan lanjutan bagi kelas di atas yang ditakrif melalui pengoperasi Al-Oboudi dikaji. Kemudian ia digunakan untuk mendapatkan ketaksamaan Fekete-Szegö.

Seterusnya, selain fungsi-fungsi bentuk ternormal di atas, tesis ini juga mengkaji fungsi berbentuk

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n},$$

dengan $p$ adalah integer positif tetap dan mewakilkan $\mathcal{A}_p$ sebagai kelas yang mengandungi fungsi-fungsi tersebut. Bagi kelas tersebut, batas-batas bawah tepat pada

bahagian nyata hasil bahagi antara fungsi-fungsi ternormal dan jujukan jumlah separa bagi fungsi cembung dan fungsi bak-bintang serta kelas-kelas yang berkaitan, fungsi seragam cembung dan fungsi bak-bintang parabola yang memenuhi syarat-syarat tertentu diperolehi.

Akhir sekali, bagi fungsi $f \in \mathcal{A}_p$ yang analisis dalam $\mathcal{U}$, hasil pengawetan bagi dua pengoperasian kamiran $I_p^\sigma f(z)$ dan $J_\beta^\sigma f(z)$ seperti

$$\begin{aligned} I_p^\sigma f(z) &= \frac{(p+1)^\sigma}{z^p\Gamma(\sigma)}\int_0^z t^{p-1}\left(log\,\frac{z}{t}\right)^{\sigma-1} f(t)\,dt \quad (\sigma>0). \\ J_\beta^\sigma f(z) &= \binom{p+\sigma+\beta-1}{p+\beta-1}\frac{\sigma}{z^\beta}\int_0^z \left(1-\frac{t}{z}\right)^{\sigma-1} t^{\beta-1}f(t)\,dt \quad (\sigma>0,\ \beta>-1), \end{aligned}$$

diperkenalkan bagi beberapa kelas fungsi yang ditakrif menggunakan hasildarap Hadamard.

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## PUBLICATIONS AND PRESENTATIONS

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W. S. S. Wan Hassan & Suzeini Abdul Halim. (2011). Partial sums of $p$-valent analytic functions, *Procedings of the 5th International Conference on Research and Education in Mathematics*, Institut Teknologi Bandung, Indonesia.

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# CONTENTS

## SYMBOLS

| | |
|---|---|
| $\mathcal{A} := \mathcal{A}_1$ | Class of analytic functions of the form |
| | $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U})$ |
| $\mathcal{A}_p$ | Class of all $p$-valent analytic functions of the form |
| | $f(z) = z^p + \sum_{n=2}^{\infty} a_{p+n} z^{p+n}, \quad (z \in \mathcal{U})$ |
| $\mathbb{C}$ | Complex plane |
| $\mathcal{CV}$ | Class of convex functions in $\mathcal{U}$ |
| $\mathcal{CV}(\alpha)$ | Class of convex functions of order $\alpha$ in $\mathcal{U}$ |
| $\mathcal{CV}_p(\alpha)$ | Class of $p$-valent convex functions of order $\alpha$ in $\mathcal{U}$ |
| $\mathcal{CCV}$ | Class of close-to-convex functions in $\mathcal{U}$ |
| $\mathcal{CCV}(\alpha)$ | Class of close-to-convex functions of order $\alpha$ in $\mathcal{U}$ |
| $\mathcal{CCV}_p(\alpha)$ | Class of $p$-valent close-to-convex functions of order $\alpha$ in $\mathcal{U}$ |
| $\mathcal{D}$ | Domain |
| $\mathcal{H}(\mathcal{U})$ | Class of analytic functions in $\mathcal{U}$ |
| $\mathcal{PST}$ | Class of parabolic starlike functions in $\mathcal{U}$ |
| $\mathcal{P}(\alpha)$ | Class of bounded turning functions |
| $\mathcal{S}$ | Class of all normalized univalent functions of the form |
| | $f(z) = z + a_2 z^2 + \ldots, \quad (z \in \mathcal{U})$ |
| $\mathcal{ST}$ | Class of starlike functions in $\mathcal{U}$ |
| $\mathcal{ST}(\alpha)$ | Class of starlike functions of order $\alpha$ in $\mathcal{U}$ |
| $\mathcal{ST}_p(\alpha)$ | Class of $p$-valent starlike functions of order $\alpha$ in $\mathcal{U}$ |
| $\mathcal{U}$ | Open unit disc $\{z \in \mathbb{C} : \|z\| < 1\}$ |
| $\mathcal{U}_r$ | Open disc of radius r, $\{z \in \mathbb{C} : \|z\| < r\}$ |
| $\mathcal{UST}$ | Class of uniformly starlike functions in $\mathcal{U}$ |
| $\mathcal{UCV}$ | Class of uniformly convex functions in $\mathcal{U}$ |

# CHAPTER 1

## PRELIMINARIES

In this thesis we are mainly interested in univalent functions that are also analytic in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For this Chapter, basic results and background materials concerning the theory of the univalent and multivalent functions are stated. More details about those concepts can be found in Duren (1983) and Goodman (1983).

### 1.1 Scope of the thesis

This Chapter presents the basic concepts and definitions concerning the theory of starlike and convex functions as well as the terminologies. These materials will be required in the subsequent chapters of the thesis.

In next Chapter, we considered a new class of functions using the Al-Oboudi operator. The class proposed is an extension of the class $\mathcal{B}_n(\alpha)$ first introduced in Abdul Halim (2003) which incorporate the Sălăgean operator instead of the Al-Oboudi. We further the research in investigate the various coefficient properties for both the $\mathcal{B}_n(\alpha)$ and $\mathcal{B}_n^\lambda(\alpha)$ classes of functions. As a special case of our result, we obtain the Fekete-Szegö inequality for a class of functions defined through differential operator.

In Chapter 3, results on preservation of some integral operators for functions belonging to the class $\mathcal{ST}_{p,g}(\beta)$, $\mathcal{CV}_{p,g}(\beta)$ and $\mathcal{CCV}_{p,g}(\beta)$ are discussed. These classes are defined using certain characterization and convolution.

In Chapter 4, we obtain sharp lower bounds on the real part of the quotients between the normalized functions and their sequence of partial sums for certain subclasses of $\mathcal{S}$. In particular, the convex and starlike functions as well as their related classes, the *uniformly convex* and *parabolic starlike functions* which satisfy certain conditions. We look at the lower bounds for Re $\{f(z)/f_k(z)\}$, Re $\{f_k(z)/f(z)\}$, Re $\{f'(z)/f'_k(z)\}$, and Re $\{f'_k(z)/f'(z)\}$ where $f_k$ represent the partial sums.

### 1.2 Introduction

The theory of complex analysis was founded in the middle of the 19th century, is one of the classical branches in mathematical fields, which traditionally known as the theory of functions of a complex variable. In the 20th century, many important mathematicians like Euler, Gauss, Riemann, Cauchy, and Weierstrass that associate with complex analysis, which particularly involved the analytic functions of complex variables.

In mathematics, the main interests and central objects of study in complex analysis are the *analytic* functions that also known as holomorphic functions. An analytic function is said to be *univalent* on a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{C}$, if it provides a one-to-one mapping onto its image, $f(\mathcal{D})$. Both functions are defined as follows:

**Definition 1.2.1.** *Duren (1983). A function $f$ is said to be analytic at $z_0 \in \mathcal{D}$ if it has derivative at every point of some neighborhood of $z_0$, and so $f$ is analytic in $\mathcal{D}$*

*if it has a derivative at every point of $\mathcal{D}$.*

**Definition 1.2.2.** *Goodman (1983). A function $f(z)$ is said to be univalent in a domain $\mathcal{D}$ if the conditions*

$$f(z_1) = f(z_2), \qquad z_1 \in \mathcal{D}, \qquad z_2 \in \mathcal{D},$$

*imply that $z_1 = z_2$.*

The first study of univalent functions is perhaps due to a paper of Koebe (1907). In this 100 years, the theory of univalent functions has developed considerably. Many more papers and books have been published regarding the univalent functions theory.

Next, we describe some important elementary properties and theorem of analytic and univalent functions which are defined via some geometric condition.

The Riemann Mapping theorem is an important theorem in geometric function theory. It states that every simply connected domain which is not the whole complex plane can be mapped conformally onto the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

**Theorem 1.2.1.** *(Riemann Mapping Theorem), Duren (1983). Let $\mathcal{D}$ be a simply connected domain which is a proper subset of the complex plane. Let $\zeta$ be a given point in $\mathcal{D}$. Then there is a unique univalent analytic function $f$ which maps $\mathcal{D}$ onto the unit disc $\mathcal{U}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

For an analytic function $g$ in $\mathcal{U}$, it has a Maclaurin expansion

$$g(z) = b_0 + b_1 z + b_2 z^2 + ... = \sum_{n=0}^{\infty} b_n z^n,$$

that is convergent in $\mathcal{U}$. We observe that if $g$ is univalent in $\mathcal{U}$, then so is the function $g(z) - b_0$. Since $g$ is univalent, then $b_1 = g'(0) \neq 0$ and hence we may subtract $b_0$ and divide by $b_1$ and consider the analytic function $f(z) = (g(z) - b_0)/b_1$, which is normalized by $f(0) = f'(0) - 1 = 0$ and has expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathcal{U}). \tag{1.1}$$

Denote by $\mathcal{H}$ the class of functions, which are analytic in the open unit disc $\mathcal{U}$ and the subclass of $\mathcal{H}$ that consists functions of the form (1.1) is denoted by $\mathcal{A}$. The subclass of $\mathcal{A}$ consisting of functions, which are univalent in the open unit disk $\mathcal{U}$ is denoted by $\mathcal{S}$. The leading example of a function of class $\mathcal{S}$ is the Koebe function which is given by

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} n z^n,$$

which maps $\mathcal{U}$ onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$. In an intuitive sense this function is the *largest* function in $\mathcal{S}$, because it is impossible to add to the image domain any open set of points without destroying univalence.

The Koebe function plays a very important role in the study of $\mathcal{S}$. It is often the extremal functions for various problems in $\mathcal{S}$. In 1916, Bieberbach (1916) proved the following theorem for functions in $\mathcal{S}$.

**Theorem 1.2.2.** *(Bieberbach Theorem), Goodman (1983). If* $f \in \mathcal{S}$*, then* $|a_2| \leq 2$ *with equality if and only if* $f$ *is a rotation of the Koebe function.*

In the same paper, he mentioned "$|a_n| \leq n$ is generally valid". This statement is known as the Bieberbach conjecture. Löewner (1923) and Garabedian (1955) proved the Bieberbach conjecture, respectively for the cases $n = 3$ and $n = 4$. Much later in 1985, de Branges (1985) proved the Bieberbach conjecture for all coefficients with the help of the hypergeometric functions.

Next, we describe some of interesting subclasses of univalent functions, which are defined in geometrical and analytic characterizations.

## 1.3 Subclasses of Univalent Functions

One important problem in the field of univalent function is to study certain geometric properties of the image domain. Several authors have considered other classes such as *convex* and *starlike* functions.

**Definition 1.3.1.** *Goodman (1983). A set* $\mathcal{D}$ *in the plane is said to be starlike with respect to* $w_0$ *an interior point of* $\mathcal{D}$ *if for each ray with initial point* $w_0$ *intersects the interior of* $\mathcal{D}$ *in a set that is either a line segment or a ray. If a function* $f$ *maps* $\mathcal{U}$ *onto starlike domain with respect to* $w_0$*, then we say that* $f$ *is a starlike with respect to* $w_0$*. In the special case that* $w_0 = 0$*, we say that* $f$ *is a starlike function.*

**Definition 1.3.2.** *Goodman (1983). A set $\mathcal{D}$ in the plane is called convex if for every pair of points $w_1$ and $w_2$ in the interior of $\mathcal{D}$, the line segment joining $w_1$ and $w_2$ is also in $\mathcal{D}$. If a function $f$ maps $\mathcal{U}$ onto a convex domain, then $f$ is called a convex function in $\mathcal{U}$.*

The class of all functions of $\mathcal{S}$ which are starlike in $\mathcal{U}$ is denoted by $\mathcal{ST}$. Nevanlinna (1921) showed that $\mathcal{ST} \subset \mathcal{S}$ and that a necessary and sufficient condition for $f \in \mathcal{ST}$ is given by

**Theorem 1.3.1.** *Duren (1983). A function $f \in \mathcal{ST}$ if and only if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

The class of all functions of $\mathcal{S}$ which are convex in $\mathcal{U}$ is denoted by $\mathcal{CV}$. Löewner (1917) showed that $\mathcal{CV} \subset \mathcal{S}$ and that a necessary and sufficient condition for $f \in \mathcal{CV}$ is given by the following:

**Theorem 1.3.2.** *Duren (1983). A function $f \in \mathcal{CV}$ if and only if*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

In 1915, Alexander (1915) discovered the beautiful relationship between convex and starlike functions, that has a very simple proof based on the characterization of both functions in the unit disc.

**Theorem 1.3.3.** *(Alexander's Theorem), Goodman (1983). A function* $f \in \mathcal{CV}$ *if and only if* $zf'(z) \in \mathcal{ST}$.

Although the Bierberbach conjecture was quite recently (1984) solved for class $\mathcal{S}$, this was not the case for $\mathcal{ST}$ and $\mathcal{CV}$. In 1921, Nevanlinna (1921) proved the conjecture for $\mathcal{ST}$ and Reade (1954) showed that it also holds for $\mathcal{CV}$. Much earlier, Löewner (1917) proved that if $f \in \mathcal{CV}$, then $|a_n| \leq 1$ for $n \geq 1$.

The notion of convexity and starlikeness have been extended in many ways. In 1936, Robertson (1936) generalized the classes $\mathcal{ST}$ and $\mathcal{CV}$ in the simplest way by introducing the concept of functions starlike and convex of order $\alpha$ for $0 \leq \alpha < 1$.

**Definition 1.3.3.** *Goodman (1983). A function* $f \in \mathcal{A}$ *is said to be in the class of starlike functions of order* $\alpha$ *denoted by* $\mathcal{ST}(\alpha)$*, if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (0 \leq \alpha < 1,\ z \in \mathcal{U}). \tag{1.2}$$

*A function* $f \in \mathcal{A}$ *is in the class of convex functions of order* $\alpha$ *denoted by* $\mathcal{CV}(\alpha)$*, if*

$$Re\left\{1 + \frac{zf"(z)}{f'(z)}\right\} > \alpha, \quad (0 \leq \alpha < 1,\ z \in \mathcal{U}). \tag{1.3}$$

Note that if $\alpha < 0$, then a function in either of these classes may fail to be univalent, for example $f(z) = \frac{z}{(1-z)^{\lambda}}$, when $\lambda > 2$, $f(z)$ has the property $Re\frac{zf'(z)}{f(z)} > 1 - \frac{\lambda}{2}$, but it is not univalent in $\mathcal{U}$ as $f'(1/(1-\lambda)) = 0$. On the other hand, if $\alpha \geq 1$, the classes are empty set, since the inequalities (1.2) and (1.3) will not be satisfied at $z = 0$. Evidently $\mathcal{ST}(0) \equiv \mathcal{ST}$ and $\mathcal{CV}(0) \equiv \mathcal{CV}$. In addition, $f(z) \in \mathcal{CV}(\alpha)$ if and only if $zf'(z) \in \mathcal{ST}(\alpha)$ for $0 \leq \alpha < 1$. There is an extensive literature concerning $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$. See eg. Goel (1974), Jack (1971), Pinchuk (1968), Schild *et. al.* (1965).

Let $\mathcal{U}_r$ be the set of $|z| < r < 1$. Ford (1935) gave more general properties of $\mathcal{U}$ which hold in the subregions $\mathcal{U}_r$. In other words, if $f \in \mathcal{S}$ is starlike or convex, then $f(\mathcal{U}_r)$ is also a starlike or a convex domain.

**Theorem 1.3.4.** *(Ford's Theorem), Goodman (1983). Let $f$ be in $\mathcal{S}$. If $f(\mathcal{U})$ is a convex domain, then for each positive $r < 1$, $f(\mathcal{U}_r)$ is also a convex domain. If $f(\mathcal{U})$ is starlike with respect to the origin, then for each positive $r < 1$, $f(\mathcal{U}_r)$ is also starlike with respect to the origin.*

However the above theorem of geometric property does not hold in general for circle whose centers are not at the origin. This motivated Goodman to introduce a new class of normalized functions analytic and univalent in the unit disc. In Goodman (1991a) and Goodman (1991b), functions said to be *uniformly starlike* and *uniformly convex* are extensively discussed. The corresponding "uniform classes" are defined in the following way, by their geometrical mapping properties.

**Definition 1.3.4.** *Goodman (1991a). A function $f$ is said to be uniformly starlike in $\mathcal{U}$ if $f$ is starlike and has the property that, for every circular arc $\gamma$ contained in $\mathcal{U}$, with center $\xi$ also in $\mathcal{U}$, the arc $f(\gamma)$ is starlike with respect to $f(\xi)$. We let $\mathcal{UST}$*

*denote the class of all such functions.*

**Theorem 1.3.5.** *Goodman (1991a). Let $f$ have the form (1.1). Then $f \in \mathcal{UST}$ if and only if*

$$Re\left\{\frac{f(z)-f(\xi)}{(z-\xi)f'(z)}\right\} \geq 0, \quad (z,\xi \in \mathcal{U}). \tag{1.4}$$

**Definition 1.3.5.** *Goodman (1991b). A function $f$ is said to be uniformly convex in $\mathcal{U}$ if $f$ is a convex function and has the property that, for every circular arc $\gamma$ contained in $\mathcal{U}$, with center $\xi$ also in $\mathcal{U}$, the image arc $f(\gamma)$ is a convex arc. We let $\mathcal{UCV}$ denote the class of all such functions.*

**Theorem 1.3.6.** *Goodman (1991b). Let $f(z)$ have the form (1.1). Then $f \in \mathcal{UCV}$ if and only if*

$$Re\left\{1+(z-\xi)\frac{f''(z)}{f'(z)}\right\} \geq 0, \quad (z,\xi \in \mathcal{U}). \tag{1.5}$$

Note that by taking $\xi = 0$ in (1.4) and (1.5) we will get class $\mathcal{ST}$ and $\mathcal{CV}$, respectively. These classes have been studied extensively by Rönning (1993a) and independently by Ma and Minda (1992/1993) where they have proved the following one variable characterization for functions in $\mathcal{UCV}$.

**Theorem 1.3.7.** *Rönning (1993a). A function $f$ of the form (1.1) is in $\mathcal{UCV}$ if and only if*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathcal{U}).$$

However, we know that by Alexander's theorem stating that $f \in \mathcal{CV} \Leftrightarrow zf'(z) \in \mathcal{ST}$ provides a bridge between these two classes. Goodman (1991a) gave examples that demonstrated the Alexander's relation does not hold between the classes $\mathcal{UCV}$ and $\mathcal{UST}$. In Rönning (1993a), he introduced the class of parabolic starlike functions $\mathcal{PST}$ such that $f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{PST}$. It is established that

**Theorem 1.3.8.** *Rönning (1993a). A function $f$ is in $\mathcal{PST}$ if and only if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathcal{U}).$$

Later he proved (see Rönning (1993b)) that neither $\mathcal{PST} \not\subset \mathcal{UST}$ nor $\mathcal{UST} \not\subset \mathcal{PST}$. In Rönning (1991/1995) he further generalized the classes $\mathcal{UCV}$ and $\mathcal{PST}$ by introducing a parameter $\alpha$ in the following way.

**Theorem 1.3.9.** *Rönning (1991/1995). A function $f \in \mathcal{A}$ is in $\mathcal{PST}(\alpha)$ if it satisfies the analytic characterization*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} - \alpha \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (0 \leq \alpha \leq 1,\ z \in \mathcal{U}),$$

*and $f \in \mathcal{UCV}(\alpha)$, the class of uniformly convex functions of order $\alpha$, if it satisfies*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \alpha \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad (0 \leq \alpha \leq 1,\ z \in \mathcal{U}).$$

Other authors have also seek to develop a more general class. One such case is Kanas and Wisniowska (1998) which introduced the class of *k-uniformly convex* *functions*. Kaplan (1952) introduced an interesting subclass $\mathcal{A}$ which contains $\mathcal{ST}$ and has a simple geometric characterization.

**Theorem 1.3.10.** *Duren (1983). A function $f$ analytic in the unit disc is said to be close-to-convex if for $z \in \mathcal{U}$, there exists a convex function $g$ such that*

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0.$$

We denote the class of *close-to-convex* functions by $\mathcal{CCV}$. Note that, every starlike function is close-to-convex. Indeed, each $f \in \mathcal{ST}$ has the form $f(z) = zg'(z)$ for some $g \in \mathcal{CV}$, and

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} = Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Clearly, these are summarized by the chain of proper inclusions

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CCV} \subset \mathcal{S}.$$

Every close-to-convex function is univalent. In Noshiro (1934/1935) and Warchawski (1935), they obtained a simple but interesting criterion for univalence of analytic functions. The criterion is due to Noshiro and Warschawski.

**Theorem 1.3.11.** *(Noshiro-Warschawski Theorem), Noshiro (1934/1935). If $f$ is analytic in a convex domain $\mathcal{D}$ and $Re\{f'(z)\} > 0$ there, then $f$ is univalent in $\mathcal{D}$.*

A function $f$ is called *close-to-convex of order* $\alpha$, $\alpha \in [0,1)$ if $f(\mathcal{D})$ is accessible of order $\alpha$. This class is denoted by $CCV(\alpha)$. We shall give an analytical characterization for $f$ to be *close-to-convex of order* $\alpha$,

**Theorem 1.3.12.** *Libera (1964). A function $f \in \mathcal{A}$ is said to be close-to-convex of order $\alpha$, for all $z \in \mathcal{U}$, if there is a function $g \in \mathcal{ST}$ such that*

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha, \quad (0 \leq \alpha < 1).$$

For $0 \leq \alpha < 1$, a function $f$ of the form (1.1) is said to be in the class of *bounded turning* denoted by $\mathcal{P}(\alpha)$, if it satisfies $Re\ \{f'(z)\} > \alpha$. (see Goodman (1983)). By the Noshiro-Warschowski theorem the functions in $\mathcal{P}(\alpha)$ are univalent and also *close-to-convex* in $\mathcal{U}$.

We then discuss the multivalence of analytic functions, which will allow us to compute the various properties and characteristics for certain subclasses.

## 1.4 Subclasses of Multivalent Functions

The class of multivalent functions is an important one in complex analysis. Hayman (1958) introduced and generalized the univalent functions by proving corresponding results for *p-valent* functions (*multivalent of order* $p$).

**Definition 1.4.1.** *Hayman (1958). A function $f$ is p-valent if for each $w_0$ (infinity included), the equation $f(z) = w_0$ has at most $p$ roots in $\mathcal{U}$, where the roots* *are counted with their multiplicities, and if there is some $w_1$ such that the equation $f(z) = w_1$ has exactly $p$ roots in $\mathcal{U}$.*

Let $\mathcal{A}_p$ denote the class of all analytic functions $f$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N} := \{1, 2, \dots\})$$

that are $p$-valent in the open unit disc $\mathcal{U}$, and for $p = 1$, let $\mathcal{A}_1 := \mathcal{A}$.

The class of $p$-valent functions has been widely studied. In fact, Patil and Thakare (1983), Owa (1985) and Aouf (1988) studied the subclasses of *p-valent functions of order* $\alpha$ which are an extension of the familiar subclasses were studied earlier by Goodman (1950) and Livingston (1965). Evidently $\mathcal{ST}_1(\alpha) = \mathcal{ST}(\alpha)$, $\mathcal{CV}_1(\alpha) = \mathcal{CV}(\alpha)$ and $\mathcal{CCV}_1(\alpha) = \mathcal{CCV}(\alpha)$.

**Definition 1.4.2.** *Patil (1983). A function $f \in \mathcal{A}_p$ is said to belong to the class of p-valent starlike functions of order $\alpha$ in $\mathcal{U}$, and is denoted by $\mathcal{ST}_p(\alpha)$ if it satisfies*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (0 \le \alpha < p, \; z \in \mathcal{U}).$$

**Definition 1.4.3.** *Owa (1985). A function $f \in \mathcal{A}_p$ is said to belong to the class of p-valent convex functions of order $\alpha$ in $\mathcal{U}$, and is denoted by $\mathcal{CV}_p(\alpha)$ if it satisfies*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (0 \le \alpha < p, \; z \in \mathcal{U}).$$

**Definition 1.4.4.** *Aouf (1988). A function $f \in \mathcal{A}_p$ is said to belong to the class of p-valent close-to-convex of order $\alpha$, and is denoted by $\mathcal{CCV}_p(\alpha)$, if there exists a* *function* $g(z) \in \mathcal{ST}_p(\alpha)$ *such that*

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha, \qquad (0 \leq \alpha < p,\ z \in \mathcal{U}).$$

A result analogues to Alexander' theorem (1.3.3) was obtained by Ali *et. al.* (2009).

**Theorem 1.4.1.** *Ali (2009). The function* $f$ *belongs to* $\mathcal{CV}_p(\alpha)$ *if and only if* $\frac{zf'(z)}{p} \in \mathcal{ST}_p(\alpha)$.

For functions $f, g \in \mathcal{A}_p$, the Hadamard product (or convolution) of $f$ and $g$ is the functions $(f * g)(z)$ defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}$$

where $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ and $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$.

Inspired by Shamani *et. al.* (2009) idea, we introduced some similar classes that involve Hadamard product in determining preserving properties for certain integral operators by using the convex hull method.

**Definition 1.4.5.** *The class* $\mathcal{ST}_{p,g}(\beta)$ *consists of functions* $f \in \mathcal{A}_p$ *where* $g \in \mathcal{A}_p$ *is a fixed function satisfying* $\frac{(g*f)(z)}{z^p} \neq 0$ *and*

$$Re\ \frac{1}{p}\left\{\frac{z(g * f)'(z)}{(g * f)(z)}\right\} > \beta, \quad (0 \leq \beta < 1, z \in \mathcal{U}). \tag{1.6}$$

*Similarly, $\mathcal{CV}_{p,g}(\beta)$ is the class of functions $f \in \mathcal{A}_p$ where $g \in \mathcal{A}_p$ is a fixed function satisfying $\frac{(g*f)'(z)}{z^{p-1}} \neq 0$ and*

$$Re\ \frac{1}{p}\left\{1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right\} > \beta, \quad (0 \leq \beta < 1, z \in \mathcal{U}). \tag{1.7}$$

**Definition 1.4.6.** *The class $\mathcal{CCV}_{p,g}(\beta)$ consists of functions $f \in \mathcal{A}_p$ satisfying $\frac{(g*\psi)(z)}{z^p} \neq 0$ and*

$$Re\ \frac{1}{p}\left\{\frac{z(g*f)'(z)}{(g*\psi)(z)}\right\} > \beta, \quad (0 \leq \beta < 1, z \in \mathcal{U})$$

*for some $\psi \in \mathcal{ST}_{p,g}(\beta)$.*

Note that by taking $g(z) = z^p/(1-z)$ in (1.6) and (1.7), then $\mathcal{ST}_{p,g}(\beta) = \mathcal{ST}_p(\alpha)$ and $\mathcal{CV}_{p,g}(\beta) = \mathcal{CV}_p(\alpha)$. After a brief review of analytic functions, we will then discuss some of properties of the inverse functions.

## 1.5 Inverse functions

Inverse functions are very important in mathematics area as well as in many applied areas of science. In this thesis, we only focused on finding the coefficient estimates for the inverse function. The inverse of a function $f \in \mathcal{S}$ of the form (1.1) has a series expansion

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n. \tag{1.8}$$

Lőwner (1923) showed that for $f \in \mathcal{A}$, and $F$ given by (1.8) the coefficients of $F$ are bounded as below:

$$|A_n| \leq \frac{(2n)!}{n!(n+1)!}, \quad k = 2, 3, 4, ...$$

Equality is attained for the Koebe function $k(z) = \frac{z}{(1-z)^2}$ and its inverse

$$K(w) = \frac{1 - 2w - \sqrt{1-4w}}{2w}.$$

In Krzyz *et. al.* (1979), the authors found sharp coefficient estimates for inverse of functions in the class $\mathcal{ST}$ and their work has been extended in Kapoor and Mishra (2007). Libera and Zlotkiewicz (1982) obtained sharp lower bounds on the coefficients of inverse functions for $f \in \mathcal{CV}$ where $|A_n| \leq 1$ for $n = 2, 3, 4, 5, 6$ and 7. For other interesting developments on sharp coefficient estimate of inverses in connection with various subclasses of univalent functions, the reader can refer to Schober (1977), Ali (2003) and Ma (1990).

In particular, consider functions $f' \in P$ where $P$ consists of functions $p(z)$ of the form $1 + \sum_{n=1}^{\infty} c_n z^n$ and satisfies $Re\ p(z) > 0$ for $z \in \mathcal{U}$, the authors in Libera (1983) established $|A_2| \leq 1, |A_3| \leq \frac{4}{3}, |A_4| \leq \frac{13}{6}, |A_5| \leq \frac{59}{15}$ and $|A_6| \leq \frac{344}{45}$ and for other $n$'s

$$|A_n| \leq \frac{1}{\pi n} \int_0^{\pi} \frac{d\theta}{|1 + 2e^{-i\theta}\ log(1 - e^{i\theta})|^n} < \frac{1}{n\alpha^n}.$$

where $A_n$ are the coefficients of $F = f^{-1}$.

At the same time, another interesting approach consider by authors is to look at the normalised analytic univalent functions defined by operators noted as differential or integral operators.

## 1.6 Differential Operators

Generally, a differential operator is an operator involving differentiation and/or multiplication by other functions, that transforms a functions into another functions.

Many articles discuss on operators and new generalizations of various authors. Perhaps, Ruscheweyh (1975) was the pioneer in the differential operator who introduced it in 1975. It is then followed by the Sălăgean (1983) giving another version of differential operator.

**Definition 1.6.1.** *Ruscheweyh (1975). For a function $f \in \mathcal{A}$ and $n \in \mathbb{N} \cup \{0\}$, the Ruscheweyh differential operator, $R^n f$ defined by $R^n : \mathcal{A} \to \mathcal{A}$*

$$\begin{aligned} R^0 f(z) &= f(z) \\ (n+1)R^{n+1} f(z) &= z(R^n f(z))' + nR^n f(z), \quad (z \in \mathcal{U}). \end{aligned}$$

**Definition 1.6.2.** *Sălăgean (1983). For a function $f \in \mathcal{A}$, we define the Sălăgean differential operator, $D^n f$ defined by*

$$\begin{aligned} D^0 f(z) &= f(z) \\ D' f(z) &= Df(z) = zf'(z) \\ D^n f(z) &= D(D^{n-1} f(z)) = z(D^{n-1} f(z))' \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}). \end{aligned}$$

These two operators were used to study different properties and problems involving subclasses of univalent functions. After these operators are introduced, there have been numerous operators being formed, generalised and defined. (See Ibrahim (2008) and Lin (1998)). In 2004, Al-Oboudi generalized Sălăgean operator.

**Definition 1.6.3.** *Al-Oboudi (2004). For a function $f \in \mathcal{A}$, Al-Oboudi introduced the following operator:*

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \quad (\lambda \geq 0) \\ D_\lambda^n f(z) &= D_\lambda(D_\lambda^{n-1} f(z)), \qquad (n \in \mathcal{N} = \{1, 2, 3, ...\}). \end{aligned}$$

Their elementary properties are important for further research and have been discussed and studied by many researchers. Abdul Halim (1992) she introduced a new class, $\mathcal{B}_n(\alpha)$ that involve Sălăgean's operators and show that the class is analytic, normalized and univalent functions in $\mathcal{U}$.

**Definition 1.6.4.** *Abdul Halim (1992). For $\alpha > 0$ and $n = 0, 1, 2, \ldots,$ a function $f$ normalised by (1.1) belongs to $\mathcal{B}_n(\alpha)$ if and only if,*

$$Re\ \frac{D^n[f(z)]^\alpha}{z^\alpha} > 0, \qquad (z \in \mathcal{U}).$$

*where $D^n$ denotes the Sălăgean's differential operator.*

For $n = 1$, $\mathcal{B}_1(\alpha)$ denotes the class of Bazilević functions with logarithmic growth studied by others Babalola (2006), Thomas (1968) and Kim (2009). The class $\mathcal{B}_0(\alpha)$

was initiated by Yamaguchi (1966). For $\mathcal{B}_1(\alpha)$, Singh (1973) gave sharp estimates for the modulus of the coefficients $a_2, a_2, a_3$ of $f$. The results were extended by Abdul Halim (2003), to the class $\mathcal{B}_n(\alpha)$.

Later we derive some integral operators and preliminary results on the class defined by integral operators. Here we give a brief survey of these operators.

## 1.7 Integral Operators

The study of the integral operators has been rapidly investigated by many authors in the field of univalent functions. Recently, various integral operators have been introduced for certain classes of analytic univalent functions and their properties is one of the hot areas of current ongoing research in the geometric function theory. The first integral transform defined a subclass of $\mathcal{S}$ was introduced by Alexander in 1915.

**Definition 1.7.1.** *Alexander (1915). For a function $f \in \mathcal{A}$, Alexander introduced an integral operator as follows*

$$F(z) = \int_0^z \frac{f(t)}{t}\, dt, \qquad (z \in \mathcal{U}).$$

Alexander showed that the operator $F(z)$ maps $\mathcal{ST}$ onto $\mathcal{CV}$. In 1960, Biernacki falsely claimed that $F(z)$ is in $\mathcal{S}$ whenever $f \in \mathcal{S}$. Three years later this error was noticed by Krzyz and Lewandowski in Goodman (1983). Nevertheless, Biernackis consideration of the integral transform gave rise to the study of the integral operator.

**Definition 1.7.2.** *Biernacki (1960). For a function $f \in \mathcal{A}$, we define the integral operator $F_\alpha$ as follows*

$$F_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha \, dt \qquad (0 \leq \alpha < 1, \; z \in \mathcal{U}).$$

$F_\alpha(z)$ is known as integral of the *first type* and since then many papers have appeared concerning the operator $F_\alpha(z)$. Later in 1965, Libera introduced an integral operator. There are many author investigate some interesting characterization theorems involving the generalized Libera integral operator (see e.g., Libera (1965), Li (1997) and Oros (2006)).

**Definition 1.7.3.** *Libera (1965). For a function $f \in \mathcal{A}$, Libera introduced that the operator*

$$I(f(z)) = \frac{2}{z}\int_0^z f(t) \; dt, \qquad (z \in \mathcal{U}).$$

It is well known that if $f(z)$ is convex, starlike, or close-to-convex in $\mathcal{U}$, then the Libera integral operator $I(f(z))$ has the same property. In 1969, Bernardi gave a more general operator, $L_c f(z)$ and studied its properties. Some of other works on the Bernardi operator include Owa (1986) and Goa (2005) and references therein.

**Definition 1.7.4.** *Bernardi (1969). For a function $f \in \mathcal{A}$, the generalized Bernardi-Libera-Livingston integral operator, $L_c(f)$ is defined as follows*

$$L_c f(z) \;\; = \;\; \frac{c+1}{z^c}\int_0^z t^{c-1} f(t) \; dt, \qquad (c \in \mathbb{N}, \; z \in \mathcal{U}).$$

He also showed that the classes $\mathcal{ST}$ and $\mathcal{CV}$ are closed under this operator, i.e., the generalized Bernardi operator maps the classes of $\mathcal{ST}$ and $\mathcal{CV}$ onto the classes of $\mathcal{ST}$ and $\mathcal{CV}$ respectively. In 1993 Jung *et. al.* introduced the following one-parameter families of integral operators, $I^{\sigma}$ and then further investigated by Uralegaddi and Somanatha (1995), Li (1999) and Liu (2002).

**Definition 1.7.5.** *Jung (1993). For a function $f \in \mathcal{A}$, we define the integral operator $I^{\sigma}$ by*

$$I^{\sigma}f(z) = \frac{2^{\sigma}}{z\Gamma(\sigma)}\int_0^z \left(log\frac{z}{t}\right)^{\sigma-1} f(t)\ dt, \quad (\sigma > 0,\ z \in \mathcal{U}).$$

The operator $I^{\sigma}$ is closely related to the multiplier transformations investigated by Flett (1972) and Kim *et. al.* (1994). Recently, many authors have introduced and studied generalized integral operator of multivalent functions such as Bernardi-Libera-Livingston and Jung-Kim-Srivastava integral operator (see Öznur (2007), Goyal (2009) and Saitoh *et. al.* (1992)).

**Definition 1.7.6.** *Reddy (1982). For a function $f \in \mathcal{A}_p$, we define the integral operator $L_{p,c}$ as follows*

$$L_{p,c}f(z) \quad = \quad \frac{c+p}{z^c}\int_0^z f(t)t^{c-1}\ dt. \quad (c+p > 0,\ p \in \mathbb{N},\ z \in \mathcal{U}).$$

**Definition 1.7.7.** *Shams (2006). For a function $f(z) \in \mathcal{A}_p$, we define the integral operator $I_p^{\sigma}$ by*

$$I_p^{\sigma}f(z) \quad = \quad \frac{(p+1)^{\sigma}}{z^p\Gamma(\sigma)}\int_0^z t^{p-1}\left(log\ \frac{z}{t}\right)^{\sigma-1} f(t)\ dt. \quad (\sigma > 0,\ p \in \mathbb{N},\ z \in \mathcal{U}).$$

Many subclasses of analytic functions defined by the $p$-modified Jung-Kim-Srivastava integral operator were studied earlier by Shams *et. al.*, Liu (2004) and Patel and Mohanty (2003). Motivated essentially by the Jung-Kim-Srivastava integral operator, Liu and Owa introduced and investigated the following integral operator:

**Definition 1.7.8.** *Liu (2003). For a function $f(z) \in \mathcal{A}_p$, we define the integral operator $J_\beta^\sigma$ as follows*

$$J_\beta^\sigma f(z) \quad = \quad \binom{p+\sigma+\beta-1}{p+\beta-1} \frac{\sigma}{z^\beta} \int_0^z \left(1-\frac{t}{z}\right)^{\sigma-1} t^{\beta-1} f(t)\, dt \ (\sigma > 0,\ \beta > -1).$$

Jahangiri and Farahmand (2003) studied the partial sums of the Libera integral operator for functions of bounded turning, $\mathcal{P}(\alpha)$. It is proved that the partial sums of the operator is also of bounded turning under certain conditions. The result has been extended by Babalola (2007), to a more general class of functions involving the Salagean operator. Recently, Darus and Ibrahim (2010), obtained certain conditions under which the partial sums of the Jung-Kim-Srivastava integral operators are preserved for functions of bounded turning.

# CHAPTER 2

## DIFFERENTIAL OPERATORS

### 2.1 Introduction

Let $\mathcal{A}$ denote be the class of functions $f$ which are analytic in the open disc $\mathcal{U} = \{z : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2.1}$$

The subclass of $\mathcal{A}$ consisting of functions, which are univalent in the open unit disc $\mathcal{U}$ is denoted by $\mathcal{S}$. For $\alpha > 0$ and $n = 0, 1, 2, \ldots,$ a function $f$ normalised by (2.1) belongs to $\mathcal{B}_n(\alpha)$ if and only if,

$$Re\, \frac{D^n[f(z)]^\alpha}{z^\alpha} > 0, \qquad (z \in \mathcal{U}),$$

where $D^n$ denotes the Sălăgean's differential operator, Sălăgean (1983) with

$$\begin{aligned} D_1^0 f(z) &= f(z), \\ D_1^n f(z) &= D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'. \end{aligned}$$

For $f \in \mathcal{A}$, Al-Oboudi (2004) introduced generalized operator:

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \quad (\lambda \geq 0) \tag{2.2}$$

$$D_\lambda^n f(z) = D_\lambda(D_\lambda^{n-1} f(z)), \qquad (n \in \mathcal{N} = \{1, 2, 3, \ldots\}). \tag{2.3}$$

For $f$ given by (2.1), using (2.2) and (2.3) we can deduce that

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 - \lambda + \lambda k]^n \, a_k z^k, \qquad (n \in \mathcal{N} \cup \{0\}),$$

with $D_\lambda^n f(0) = 0$.

Using the Al-Oboudi operator, $D_\lambda^n$, the class $\mathcal{B}_n^\lambda(\alpha)$ is defined as follows:

$$Re \; \frac{D_\lambda^n [f(z)]^\alpha}{[1 - \lambda + \lambda\alpha]^n z^\alpha} > 0, \quad (\alpha > 0, n = 0, 1, 2, \dots)$$

where

$$\begin{aligned} D_\lambda^0 f(z)^\alpha &= f(z)^\alpha \\ D_\lambda^1 f(z)^\alpha &= (1 - \lambda) f(z)^\alpha + \lambda z (f(z)^\alpha)' = D_\lambda f(z)^\alpha, \qquad \lambda \geq 0 && (2.4) \\ D_\lambda^n f(z)^\alpha &= D_\lambda (D^{n-1} f(z)^\alpha). && (2.5) \end{aligned}$$

For $f$ of the form (2.1) and $\alpha > 0$, write

$$f(z)^\alpha = \left( z + \sum_{k=2}^{\infty} a_k z^k \right)^\alpha = z^\alpha + \sum_{k=2}^{\infty} H_k(\alpha) z^{\alpha+k-1}$$

which are analytic in the open disc $\mathcal{U}$. Then from (2.4) and (2.5),

$$D_\lambda^n f(z)^\alpha = [1 - \lambda + \lambda\alpha]^n z^\alpha + \sum_{k=2}^{\infty} [1 - \lambda + \lambda(\alpha + k - 1)]^n \, H_k(\alpha) z^{\alpha+k-1}. \quad (2.6)$$

Using the binomial expansion, it can also be established that

$$\begin{aligned} H_2(\alpha) &= \alpha a_2 \\ H_3(\alpha) &= \alpha a_3 + \frac{\alpha(\alpha-1)}{2} a_2^2 \\ H_4(\alpha) &= \alpha a_4 + \alpha(\alpha-1) a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} a_2^3. \end{aligned}$$

In proving our results, we shall need the following lemmas.

**Lemma 2.1.1.** *If $q(z)$ is analytic in $\mathcal{U}$, $q(0) = 1$ and $Re\ q(z) > \frac{1}{2}$, $z \in \mathcal{U}$, then for any function $h(z)$ analytic in $\mathcal{U}$, the convolution function $q * h$ takes its values in the convex hull of $h(\mathcal{U})$.*

The assertion of Lemma 2.1.1 follows by using the Herglotz representation for $p$. The next lemma is due to Fejér (1925). An infinite sequence $\{\lambda_n\}_{n=0}^{\infty}$ of nonnegative numbers is said to be a convex null sequence if $\lambda_n \to 0$ as $n \to \infty$ and

$$\lambda_0 - \lambda_1 \geq \lambda_1 - \lambda_2 \geq \cdots \geq \lambda_n - \lambda_{n+1} \geq \cdots \geq 0.$$

**Lemma 2.1.2.** *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a convex null sequence. Then the function $p(z) = \frac{\lambda_0}{2} + \sum_{n=1}^{\infty} \lambda_n z^n$, $z \in \mathcal{U}$, is analytic in $\mathcal{U}$ with $Re\ p(z) > 0$ in $\mathcal{U}$.*

**Remark 2.1.1.** *It is obvious that if $\{\lambda_n\}_{n=0}^{\infty}$ is a convex null sequence, then by the above Lemma, $Re\ p(z) > \frac{1}{2}$.*

Another well known result on the class of Caratheodory functions are the following lemmas.

**Lemma 2.1.3.** *Abdul Halim (2003). Let $p \in P$ and let it be of the form $p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$. Then*

**(i)** $|c_i| \leq 2, \quad \forall i \geq 1$

**(ii)** $|c_2 - \mu c_1^2| \leq 2\ max\{1, |1 - 2\mu|\}$ *for* $\mu \in \mathcal{C}$.

**Lemma 2.1.4.** *Nehari (1957). If the functions* $1 + \sum_{v=1}^{\infty} b_v z^v$ *and* $1 + \sum_{v=1}^{\infty} c_v z^v$ *belong to* $P$*, then the same is true for the function* $1 + (1/2)\sum_{v=1}^{\infty} b_v c_v z^v$.

**Lemma 2.1.5.** *Nehari (1957). Let* $h(z) = 1 + h_1 z + h_2 z^2 + \dots$ *and let* $1 + G(z) = 1 + g_1 z + g_2 z^2 + \dots$ *be functions in* $P$*. Set* $\gamma_0 = 1$ *and for* $v \geq 1$,

$$\gamma_v = \frac{1}{2^v}\left[1 + \frac{1}{2}\sum_{\mu=1}^{v}\binom{v}{\mu} h_\mu\right]. \tag{2.7}$$

*If* $D_k$ *is defined by*

$$\sum_{v=1}^{\infty}(-1)^{v+1}\gamma_{v-1}G^v(z) = \sum_{k=1}^{\infty} D_k z^k \tag{2.8}$$

*then* $|D_k| \leq 2$.

**Lemma 2.1.6.** *Libera (1982). If* $p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i \in P$ *and* $[p(z)]^{-1} = 1 + p_1 z + p_2 z^2 + \dots$ *then*

$$p_1 = c_1^2 - c_2$$

$$p_2 = c_3 - 2c_1c_2 + c_1^3$$

*and* $|p_n| \leq 2$ *for* $n = 1, 2, \dots, 6$.

The aim of this chapter is to investigate further, inclusion relation, coefficient estimates for functions as well as its inverse that involved Sălăgean and Al-Oboudi differential operator.

## 2.2 Sălăgean differential operator

We consider the problem on finding sharp lower bound for the coefficient and obtain Fekete-Szegö inequality of the inverse function in $B_n(\alpha)$ that involved Sălăgean differential operator.

**Theorem 2.2.1.** *For $f \in \mathcal{B}_n(\alpha)$ where $\alpha > 0, n \geq 1$, and $F(w) = f^{-1}(w) = w + \sum_{k=2}^{\infty} A_k w^k$, the following bounds are sharp.*

$$(i)\ |A_2| \leq \frac{2\alpha^{n-1}}{(1+\alpha)^n}$$

$$(ii)\ |A_3| \leq \begin{cases} \frac{2\alpha^{n-1}}{(2+\alpha)^n}(2\kappa - 1) & \textit{for } \kappa \geq 1, \\ \frac{2\alpha^{n-1}}{(2+\alpha)^n} & \textit{for } 0 < \kappa \leq 1. \end{cases}$$

$$(iii)\ |A_4| \leq \alpha^{n-1} \begin{cases} \frac{2}{(3+\alpha)^n} & \textit{for } \nu = 0, \\ \frac{10}{(3+\alpha)^n} - \frac{4(4+\alpha)\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n} & \textit{for } 0 \leq \mu \leq \nu, \\ \frac{4(4+\alpha)\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n} - \frac{6}{(3+\alpha)^n} & \textit{for } \nu \leq \mu \leq 0. \\ \frac{2}{(3+\alpha)^n} + \phi & \textit{for } 0 \leq \nu \leq \mu \textit{ or } \nu < 0, \mu > 0. \\ \frac{2}{(3+\alpha)^n} - \phi & \textit{for } \mu \leq \nu \leq 0 \textit{ or } \nu > 0, \mu < 0. \end{cases}$$

where

$$2\kappa = \frac{\alpha^{n-1}(3+\alpha)(2+\alpha)^n}{(1+\alpha)^{2n}} > 0,$$

$$\phi = \frac{4(4+\alpha)\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n} - \frac{8(4+\alpha)(2+\alpha)\alpha^{2n-2}}{3(1+\alpha)^{3n}},$$

$$\mu(\alpha, n) = \frac{1}{(3+\alpha)^n} - \frac{(4+\alpha)(2+\alpha)\alpha^{2n-2}}{3(1+\alpha)^{3n}}, \tag{2.9}$$

$$\nu(\alpha, n) = \frac{2}{(3+\alpha)^n} - \frac{(4+\alpha)\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n}. \tag{2.10}$$

*Proof* :Since $f \in \mathcal{B}_n(\alpha)$, $\exists\ p(z) \in P$ such that if $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, the following is true.

$$a_2 = \frac{\alpha^{n-1}c_1}{(1+\alpha)^n} \tag{2.11}$$

$$a_3 = \frac{\alpha^{n-1}c_2}{(2+\alpha)^n} + \frac{(1-\alpha)a_2^2}{2} \tag{2.12}$$

$$a_4 = \frac{\alpha^{n-1}c_3}{(3+\alpha)^n} + \frac{(1-\alpha)(\alpha-2)a_2^3}{6} + (1-\alpha)a_3a_2. \tag{2.13}$$

Let $F(w) = f^{-1}(w) = w + \sum_{k=2}^{\infty} A_k w^k$. Since $f(w + \sum_{k=2}^{\infty} A_k w^k) = w$, we then have,

$$(w + A_2w^2 + A_3w^3 + ...) + a_2(w + A_2w^2 + A_3w^3 + ...)^2$$
$$+ a_3(w + A_2w^2 + A_3w^3 + ...)^3 + a_4(w + A_2w^2 + A_3w^3 + ...)^4 + ... = w.$$

Equating coefficients, we obtain

$$A_2 + a_2 = 0$$

$$A_3 + 2a_2A_2 + a_3 = 0$$

$$A_4 + a_2(2A_3 + A_2^2) + 3a_3A_2 + a_4 = 0.$$

Follows trivially from equation (2.11) and Lemma 2.1.3(i) this complete the proof of Theorem 2.2.1(i). From equation (2.12) and application of Lemma 2.1.3(ii), we can write,

$$\begin{aligned} |A_3| &= |2a_2^2 - a_3| \\ &= \left| \frac{(3+\alpha)}{2} \left( \frac{\alpha^{n-1}c_1}{(1+\alpha)^n} \right)^2 - \frac{\alpha^{n-1}c_2}{(2+\alpha)^n} \right| \\ &= \left| \frac{-\alpha^{n-1}}{(2+\alpha)^n}(c_2 - \kappa c_1^2) \right| \\ &\leq \frac{2\alpha^{n-1}}{(2+\alpha)^n} \max\{1, |1-2\kappa|\}. \end{aligned}$$

$$\textit{where} \quad 2\kappa = \frac{\alpha^{n-1}(3+\alpha)(2+\alpha)^n}{(1+\alpha)^{2n}}.$$

Since $\alpha > 0$ therefore $\kappa > 0$. By considering the possible cases of $|1-2\kappa|$ we can easily obtain the upper bounds of $|A_3|$.

1. when $\kappa \geq 1$ which implies $(1-2\kappa) \leq -1$,

$$|A_3| \leq \frac{2\alpha^{n-1}}{(2+\alpha)^n}(2\kappa - 1)$$

2. when $0 < \kappa \leq 1$, then $|1-2\kappa| \leq 1$ and therefore we have

$$|A_3| \leq \frac{2\alpha^{n-1}}{(2+\alpha)^n}$$

which completes the upper bound in Theorem 2.2.1(ii).

Next, writing $A_2 = -a_2$ and $A_3 = 2a_2^2 - a_3$, we get $A_4 = 5a_2(a_3 - a_2^2) - a_4$ and using equation (2.13) we have,

$$A_4 = 5\frac{\alpha^{n-1}c_1}{(1+\alpha)^n}\left[\frac{\alpha^{n-1}c_2}{(2+\alpha)^n} + \frac{(1-\alpha)}{2}\frac{(\alpha^{n-1}c_1)^2}{(1+\alpha)^{2n}} - \frac{(\alpha^{n-1}c_1)^2}{(1+\alpha)^{2n}}\right]$$

$$- \frac{\alpha^{n-1}}{(3+\alpha)^n}\left[c_3 + \frac{(1-\alpha)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n}\left(\frac{c_1c_2}{(2+\alpha)^n} + \frac{(1-2\alpha)\alpha^{n-1}c_1^3}{6(1+\alpha)^{2n}}\right)\right].$$

Using estimates in Lemma 2.1.3 and Lemma 2.1.6, give

$$|A_4| \leq \alpha^{n-1}\left\{\frac{2}{(3+\alpha)^n} + 2(2)|\nu(\alpha,n)|\ max\left(1, \left|1 - 2\frac{\mu(\alpha,n)}{\nu(\alpha,n)}\right|\right)\right\}, \qquad (2.14)$$

with $\mu(\alpha,n)$ and $\nu(\alpha,n)$ given by equation (2.9) and equation (2.10). From equation (2.14), to obtain the upper bounds for $|A_4|$, we consider the following cases:

- $|1-2\mu| \geq 1$ which implies $\frac{\mu}{\nu} \leq 0$ or $\frac{\mu}{\nu} \geq 1$
- $|1-2\mu| \leq 1$ which implies $0 \leq \frac{\mu}{\nu} \leq 1$
- $\nu = 0$

which with some elementary manipulation gives the upper bound.

The bounds for $|A_2|$, $|A_3|$ and $|A_4|$ are all sharp. Result (i), (ii) for $\kappa \geq 1$ and (iii) for $0 \leq \nu \leq \mu$ or $\nu < 0, \mu > 0$ and $\mu \leq \nu \leq 0$ or $\nu > 0, \mu < 0$ are sharp for

$$f_0(z) = z\left(1 + 2\sum_{k=1}^{\infty}\frac{\alpha^n z^k}{(k+\alpha)^n}\right)^{\frac{1}{\alpha}}.$$

This function is derived from the following relationship

$$\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n \left(\frac{1+z}{1-z}\right).$$

The upperbound for $|A_3|$ in the case $0 < \kappa \leq 1$ is sharp for the function

$$f_1(z) = z\left(1 + 2\sum_{k=1}^{\infty} \frac{\alpha^n z^{2k}}{(2k+\alpha)^n}\right)^{\frac{1}{\alpha}},$$

and similarly for $|A_4|$ when $\nu = 0$, the bound is sharp for

$$f_2(z) = z\left(1 + 2\sum_{k=1}^{\infty} \frac{\alpha^n z^{3k}}{(3k+\alpha)^n}\right)^{\frac{1}{\alpha}}.$$

This completes the proof.

**Theorem 2.2.2.** *Let $f \in \mathcal{B}_n(\alpha)$ and $f^{-1}(w) = w + \sum_{k-2}^{\infty} A_k w^k$. Then*

$$|A_3 - tA_2^2| \leq \begin{cases} \frac{2\alpha^{n-1}}{(2+\alpha)^n} - \frac{2\alpha^{2n-2}}{(1+\alpha)^{2n}}(3-2t+\alpha) & \text{for } t \leq \frac{3+\alpha}{2} - \frac{(1+\alpha)^{2n}}{\alpha^{n-1}(2+\alpha)^n}, \\ \frac{2\alpha^{n-1}}{(2+\alpha)^n} & \text{for } \frac{3+\alpha}{2} - \frac{(1+\alpha)^{2n}}{\alpha^{n-1}(2+\alpha)^n} \leq t \leq \frac{3+\alpha}{2}, \\ \frac{2\alpha^{2n-2}}{(1+\alpha)^{2n}}(3-2t+\alpha) - \frac{2\alpha^{n-1}}{(2+\alpha)^n} & \text{for } t \geq \frac{3+\alpha}{2}, \end{cases}$$

*If $t \geq \frac{3+\alpha}{2}$ or $t \leq \frac{3+\alpha}{2} - \frac{(1+\alpha)^{2n}}{\alpha^{n-1}(2+\alpha)^n}$, equality holds if and only if $f$ is given by $\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n\left(\frac{1+z}{1-z}\right)$. If $\frac{3+\alpha}{2} - \frac{(1+\alpha)^{2n}}{\alpha^{n-1}(2+\alpha)^n} \leq t \leq \frac{3+\alpha}{2}$, equality holds if and only if $f$ is given by $\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n\left(\frac{1+z^2}{1-z^2}\right)$.*

*Proof* : Using equation (2.11) and equation (2.12), write

$$\begin{aligned} A_3 - tA_2^2 &= (2-t)a_2^2 - a_3 \\ &= (2-t)\left(\frac{\alpha^{n-1}c_1}{(1+\alpha)^n}\right)^2 - \left(\frac{\alpha^{n-1}c_2}{(2+\alpha)^n} + \frac{(1-\alpha)\alpha^{2n-2}c_1^2}{2(1+\alpha)^{2n}}\right) \\ &= -\frac{\alpha^{n-1}}{(2+\alpha)^n}\left[c_2 - \frac{\alpha^{n-1}(2+\alpha)^n}{(1+\alpha)^{2n}}\left(\frac{3-2t+\alpha}{2}\right)c_1^2\right]. \end{aligned}$$

Using Lemma 2.1.3(ii), gives

$$\begin{aligned} |A_3 - tA_2^2| &= \left|-\frac{\alpha^{n-1}}{(2+\alpha)^n}\left(c_2 - \frac{\alpha^{n-1}(2+\alpha)^n}{(1+\alpha)^{2n}}\left(\frac{3-2t+\alpha}{2}\right)c_1^2\right)\right| \\ &\leq \frac{2\alpha^{n-1}}{(2+\alpha)^n}\, max\{1, |1-2\mu|\} \\ where \quad \mu &= \frac{\alpha^{n-1}(2+\alpha)^n}{(1+\alpha)^{2n}}\left(\frac{3-2t+\alpha}{2}\right). \end{aligned}$$

By considering the possible cases of $|1-2\mu|$ we can easily obtain the upper bounds of $|A_3 - tA_2^2|$ for the different range of values of $t$.

1. when $\mu \leq 0$ which means $(1-2\mu) \geq 1$ and also that $t \geq \frac{3+\alpha}{2}$,

$$\begin{aligned} |A_3 - tA_2^2| &\leq \frac{2\alpha^{n-1}}{(2+\alpha)^n}(-1+2\mu) \\ &= \frac{2\alpha^{2n-2}}{(1+\alpha)^{2n}}(3-2t+\alpha) - \frac{2\alpha^{n-1}}{(2+\alpha)^n},\ for\ \mu \leq 0. \end{aligned}$$

2. when $\mu \geq 1$ which implies $(1-2\mu) \leq -1$ and also

$$t \leq \frac{3+\alpha}{2} - \frac{(1+\alpha)^{2n}}{\alpha^{n-1}(2+\alpha)^n}.$$

Thus,

$$\begin{aligned} |A_3 - tA_2^2| &\leq \frac{2\alpha^{n-1}}{(2+\alpha)^n}(1-2\mu) \\ &= \frac{2\alpha^{n-1}}{(2+\alpha)^n} - \frac{2\alpha^{2n-2}}{(1+\alpha)^{2n}}(3-2t+\alpha),\ for\ \mu \geq 1. \end{aligned}$$

3. Finally, when $0 \leq \mu < 1$, then $|1-2\mu| \leq 1$ and therefore we have

$$|A_3 - tA_2^2| \;\leq\; \frac{2\alpha^{n-1}}{(2+\alpha)^n} \qquad for \;\; 0 \leq \mu \leq 1,$$

which completes the proof of Theorem 2.2.2. Next we will consider to the more general differential operator.

### 2.3 Al-Oboudi differential operator

In this section we investigate the inclusion relation, coefficient estimates for class $B_n^\lambda(\alpha)$ as well as its inverse. The problem of maximising $|a_3 - \mu a_2^2|$ in $\mathcal{S}$ and its various subclasses has been extensively studied by many authors. The next result concerns Fekete-Szegö type, for the inverse function $F$ that involved Al-Oboudi differential operator.

In Abdul Halim (1990), $\mathcal{B}_n^1(\alpha)$ was considered by Halim, who proved the following theorem.

**Theorem 2.3.1.** *$\mathcal{B}_{n+1}^1(\alpha) \subset \mathcal{B}_n^1(\alpha)$ for $n \geq 1$.*

**Corollary 2.3.2.** *$\mathcal{B}_n^1(\alpha) \subset \mathcal{S}$.*

We now extend these results for class $\mathcal{B}_n^\lambda(\alpha)$.

**Theorem 2.3.3.** *For $n \geq 1$, $\lambda > 0$ and $\alpha \geq 1$,*

$$\mathcal{B}_{n+1}^\lambda(\alpha) \subset \mathcal{B}_n^\lambda(\alpha),$$

*Proof* : Let $f$ be given by (2.1) belong to $\mathcal{B}_{n+1}^{\lambda}(\alpha)$. Then from (2.6), we have

$$Re\left\{1+\frac{1}{2}\sum_{k=2}^{\infty}\left[\frac{1-\lambda+\lambda(\alpha+k-1)}{1-\lambda+\lambda\alpha}\right]^{n+1}H_k(\alpha)z^{k-1}\right\}>\frac{1}{2}. \tag{2.15}$$

For fixed $n, \lambda$ and $\alpha \geq 1$,

$$\begin{aligned}\frac{D_{\lambda}^{n}f(z)^{\alpha}}{[1-\lambda+\lambda\alpha]^{n}z^{\alpha}} &= 1+\sum_{k=2}^{\infty}\left[\frac{1-\lambda+\lambda(\alpha+k-1)}{1-\lambda+\lambda\alpha}\right]^{n}H_k(\alpha)z^{k-1}\\ &= \left(1+\frac{1}{2}\sum_{k=2}^{\infty}\left[\frac{1-\lambda+\lambda(\alpha+k-1)}{1-\lambda+\lambda\alpha}\right]^{n+1}H_k(\alpha)z^{k-1}\right)\\ &\quad *\left(1+2\sum_{k=2}^{\infty}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+k-1)}\right]z^{k-1}\right).\end{aligned}$$

Suppose $c_0 = 1$ and $c_k = \frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+k-1)}$ for $k = 1, 2, \ldots$. Then it is obvious that for $\alpha \geq 1$, $\{c_k\}$ is a convex null sequence since

$$\begin{aligned}\frac{c_0-c_1}{c_1-c_2} &= \left(\frac{\lambda}{1-\lambda+\lambda(\alpha+1)}\right)\left(\frac{[1-\lambda+\lambda(\alpha+1)][1-\lambda+\lambda(\alpha+2)]}{\lambda[1-\lambda+\lambda\alpha]}\right)\\ &= 1+\frac{2\lambda}{1-\lambda+\lambda\alpha}\\ &\geq 1\end{aligned}$$

and furthermore for $\alpha \geq 1$, $c_k - c_{k+1} \geq 0$. Applying Lemma 2.1.2, gives

$$Re\left\{1+2\sum_{k=2}^{\infty}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+k-1)}\right]z^{k-1}\right\}>0.$$

Then, by taking $q(z) = 1+\frac{1}{2}\sum_{k=2}^{\infty}\left[\frac{1-\lambda+\lambda(\alpha+k-1)}{1-\lambda+\lambda\alpha}\right]^{n+1}H_k(\alpha)z^{k-1}$, $(H_1(\alpha) = 1)$ and using (2.15) in Lemma 2.1.1, we obtain $Re\left\{\frac{D_{\lambda}^{n}f^{\alpha}(z)}{[1-\lambda+\lambda\alpha]^{n}z^{\alpha}}\right\} > 0$, which proves the result.

**Remark 2.3.1.** *When $\lambda = 1$, we obtain Theorem 2.3.1.*

For $\mathcal{B}_n^1(\alpha)$, Abdul Halim (2003) gave estimates for the modulus of the coefficients $a_2, a_3$ and $a_4$ of $f$. The author proved the following:

**Theorem 2.3.4.** *If $\alpha > 0$, $n = 0, 1, 2, \ldots,$ and $f \in \mathcal{B}_n^1(\alpha)$ ($n$ is fixed) with $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then the following inequalities hold:*

$$|a_2| \leq \frac{2\alpha^{n-1}}{(1+\alpha)^n}$$

$$|a_3| \leq \begin{cases} \frac{2\alpha^{n-1}}{(2+\alpha)^n}\left(1 - \left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\alpha^2+2\alpha}{\alpha^2+2\alpha+1}\right)^n\right), & \text{for } 0 < \alpha < 1, \\ \\ \frac{2\alpha^{n-1}}{(2+\alpha)^n}, & \text{for } \alpha \geq 1, \end{cases}$$

$$|a_4| \leq \begin{cases} \frac{2\alpha^{n-1}}{(3+\alpha)^n} + \frac{4(1-\alpha)\alpha^{2n-2}}{(1+\alpha)^n(2+\alpha)^n}\left(1 + \frac{(1-2\alpha)(2+\alpha)^n\alpha^{n-1}}{3(1+\alpha)^{2n}}\right), & \text{for } 0 < \alpha < 1, \\ \\ \frac{2\alpha^{n-1}}{(3+\alpha)^n}, & \text{for } \alpha \geq 1. \end{cases}$$

**Remark 2.3.2.** *When $n = 1$, the above results are reduced to those obtained by Singh (1973).*

Using the similar approach, we extend Halim's results to the class $\mathcal{B}_n^\lambda(\alpha)$.

**Theorem 2.3.5.** *If $\alpha > 0$, $n = 0, 1, 2, \ldots$ and $f \in \mathcal{B}_n^\lambda(\alpha)$ ($n$ is fixed) with $f$ of the form (2.1), then the inequalities hold:*

$$(i)\quad |a_2| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n$$

$$(ii)\quad |a_3| \leq \begin{cases} \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n & \textit{for } \alpha \geq 1, \\ \\ \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n + \frac{2}{\alpha}\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n} & \textit{for } 0<\alpha<1. \end{cases}$$

$$(iii)\quad |a_4| \leq \begin{cases} \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n & \textit{for } \alpha \geq 1, \\ \\ \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + \frac{4}{\alpha}\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-2\alpha}{3\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{3n} & \\ \\ \quad + \frac{4}{\alpha}\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n & \textit{for } 0<\alpha<1. \end{cases}$$

**Remark 2.3.3.** *When $\lambda = 1$, the above results reduce to those obtained by Abdul Halim (2003) and for $n = 1$ the results are obtained by Singh (1973).*

*Proof* : Since $f \in \mathcal{B}_n^\lambda(\alpha)$, $\exists p \in P$ such that for $z \in \mathcal{U}$,

$$\frac{D_\lambda^n f(z)^\alpha}{z^\alpha} = [1-\lambda+\lambda\alpha]^n p(z).$$

Using (2.6) the above can be written as,

$$1 + \sum_{k=2}^{\infty}\left[\frac{1-\lambda+\lambda(\alpha+k-1)}{1-\lambda+\lambda\alpha}\right]^n H_k(\alpha) z^{k-1} = 1 + \sum_{i=1}^{\infty} c_i z^i. \tag{2.16}$$

On comparing coefficients in (2.16), the following relationships hold

$$a_2 = \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n c_1, \tag{2.17}$$

$$a_3 = \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_2 + \frac{(1-\alpha)}{2}a_2^2, \tag{2.18}$$

$$a_4 = \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n c_3 + (1-\alpha)a_2a_3 + \frac{(1-\alpha)(\alpha-2)}{6}a_2^3. \tag{2.19}$$

Inequality Theorem 2.3.5(i) follows easily from (2.17) since by Lemma 2.1.3(i), $|c_1| \leq 2$ for all $\alpha > 0$. Eliminating $a_2$ in (2.18), gives

$$\begin{aligned}
|a_3| &= \left|\frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_2 + \left(\frac{1-\alpha}{2\alpha^2}\right)\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n} c_1^2\right| \\
&= \left|\frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n\right. \\
&\qquad \times\left.\left(c_2 - \left[\frac{\alpha-1}{2\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n c_1^2\right)\right| \\
&= \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n |c_2 - \mu c_1^2| \\
&\leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n max\ \{1, |1-2\mu|\},
\end{aligned}$$

where Lemma 2.1.3(ii) is used with

$$2\mu = \left[\frac{\alpha-1}{\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n.$$

Since $\mu \geq 0$ for $\alpha \geq 1$, both inequalities in Theorem 2.3.5(ii) are easily obtained.

We now prove inequality Theorem 2.3.5(iii). From (2.17), (2.18) and (2.19), we have

$$a_4 = \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n \\ \times\left(c_3 + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_1c_2 \right. \\ \left. + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-2\alpha}{6\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n} c_1^3\right) \tag{2.20}$$

First, consider the case $0 < \alpha < 1/2$. The triangle inequality with Lemma 2.1.3(i) results in the inequality

$$|a_4| \le \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n \\ \left(1 + 2\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \right. \\ \left. + 2\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-2\alpha}{3\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n}\right)$$

which is the first inequality in Theorem 2.3.5(iii). For the case $1/2 \le \alpha < 1$, we use Carathéodory-Toeplitz result which state that for some $\varepsilon$ with $|\varepsilon| < 1$,

$$c_2 = \frac{c_1^2}{2} + \varepsilon\left(2 - \frac{|c_1|^2}{2}\right). \tag{2.21}$$

Subtituting (2.21) into (2.20), gives

$$a_4 = \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n \\ \left(c_3 + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_1\left(\frac{c_1^2}{2}\right) \right. \\ + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_1\left(2\varepsilon - \frac{|c_1|^2}{2}\varepsilon\right) \\ \left. + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-2\alpha}{6\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n} c_1^3\right)$$

$$= \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n$$
$$\left(c_3 + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_1\right.$$
$$\left.\times\left[\frac{c_1^2}{2}+2\varepsilon-\frac{|c_1|^2}{2}\varepsilon+\left[\frac{1-2\alpha}{6\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n c_1^2\right]\right).$$

Thus

$$|a_4| \le \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n$$
$$\left(|c_3| + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n |c_1|\right.$$
$$\left.\times\left|\frac{|c_1|^2}{2}\,w+2\varepsilon-\frac{|c_1|^2}{2}\varepsilon\right|\right).$$

where

$$w = 1+\left[\frac{1-2\alpha}{3\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n.$$

Since $0 < w \le 1$ and $|\varepsilon| < 1$, it is easily shown that

$$|a_4| \le \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n$$
$$\left(|c_3| + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n |c_1|\right.$$
$$\left.\times\left[\frac{|c_1|^2}{2}\,(w-1)+2\right]\right).$$

and the result follows upon using $|c_1| \le 2$ and $|c_3| \le 2$.

Finally, consider the case $\alpha \geq 1$. Here, we use a method introduced by Nehari and Netanyahu (1957) which was also used by Singh (1973) and the author in Abdul Halim (1989).

First, let $h$ and $g$ be defined as in Lemma 2.1.5, and since $p \in P$, Lemma 2.1.4 indicates that

$$1 + G(z) = 1 + \frac{1}{2}\sum_{k=1}^{\infty} g_k c_k z^k$$

also belongs to $P$. Next, it follows from (2.8) that,

$$|D_3| = \left|\frac{1}{2}g_3 c_3 - \frac{1}{2}\gamma_1 g_1 g_2 c_1 c_2 + \frac{1}{8}\gamma_2 g_1^3 c_1^3\right|. \tag{2.22}$$

Rewrite (2.20) as

$$\begin{aligned}
&\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda\alpha}\right]^n \alpha a_4 \\
&\qquad = c_3 + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_1 c_2 \\
&\qquad + \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-2\alpha}{6\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n} c_1^3
\end{aligned} \tag{2.23}$$

and compare it with (2.22). The required result easily follows since, by Lemma 2.1.5,

$$|A_3| = \alpha\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda\alpha}\right]^n |a_4| \leq 2.$$

This however is only true if we can show the existence of functions $h$ and $\psi$ in $P$ where $\psi(z) = 1 + g(z)$. To be simple, we choose $\psi(z) = (1+z)/(1-z)$. Thus, now it remains to construct and show that $h \in P$. Since $g_1 = g_2 = g_3 = 2$, it follows from (2.22) and (2.23) that

$$2\gamma_1 = \left[\frac{\alpha-1}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n, \tag{2.24}$$

$$\gamma_2 = \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{1-2\alpha}{6\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n}. \tag{2.25}$$

However, from (2.7), we have

$$\gamma_1 = \frac{1}{2}\left(1+\frac{1}{2}h_1\right), \tag{2.26}$$

$$\gamma_2 = \frac{1}{4}\left(1+h_1+\frac{1}{2}h_2\right). \tag{2.27}$$

Solving for $h_1$ by eliminating $\gamma_1$ from (2.24) and (2.26), we obtain

$$|h_1| = 2\left|\left[\frac{\alpha-1}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n-1\right|. \tag{2.28}$$

Quite trivially, it can be seen that $|h_1| \leq 2$ for $\alpha \geq 1$. In a similar manner, eliminating $\gamma_2$ from (2.25) and (2.27) and using $h_1$ given by (2.28), we have

$$\begin{aligned} h_2 = 2\Bigg[1-\frac{2}{3}\left[\frac{\alpha-1}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \\ \times\left(\left[\frac{1-2\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n+3\right)\Bigg]. \end{aligned} \tag{2.29}$$

Now we let,

$$\begin{aligned}
\mu_1 &= 1 - \left[\frac{\alpha-1}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \\
\mu_2 &= \left[\frac{\alpha-1}{\alpha}\right]\left[\frac{1-\lambda+\lambda(\alpha+3)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \\
\pi &= 1 - \frac{2}{3}\left(\left[\frac{1-2\alpha}{\alpha}\right]\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n + 3\right).
\end{aligned}$$

Form equation (2.30), upon simplification we can construct $h_2 = 2[1-\mu_2(1-\pi)]$. Further, with a little bit of manipulation, it can be shown that $|\pi| \leq 1$, then for $\alpha \geq 1$, it is obivious that $|h_2| \leq 2$.

Next, we construct $h$ by first setting it to be of the form

$$h(z) = \frac{\mu_1(1-z)}{1+z} + \frac{\mu_2(1+\pi z^2)}{1-\pi z^2}.$$

It is readily seen that for $\alpha \geq 1$, both $\mu_1$ and $\mu_2$ are nonnegative and $\mu_1 + \mu_2 = 1$. By some elementary manipulation it can be shown that the coefficients of $z$ and $z^2$ in the expansion of $h$ are respectively those given by (2.28) and (2.29). Hence $h \in P$ and thus $|a_4| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n$, the second inequality in Theorem 2.3.5(iii). This completes the proof.

**Theorem 2.3.6.** *If $\alpha > 0, n \geq 1$ and $f \in \mathcal{B}_n^\lambda(\alpha)$ and $F(w) = f^{-1}(w) = w + \sum_{k=2}^{\infty} A_k w^k$, then the following bounds are true:*

$$(i) \quad |A_2| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n$$

$$(ii) \quad |A_3| \leq \begin{cases} \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n (2\kappa-1) & \textit{for } \kappa \geq 1, \\ \\ \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n & \textit{for } 0 < \kappa \leq 1. \end{cases}$$

$$(iii) \quad |A_4| \leq \begin{cases} \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n & \textit{for } \nu = 0, \\ \\ -\frac{4(4+\alpha)}{\alpha^2}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n \left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n & \\ \qquad + \frac{10}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n & \textit{for } 0 \leq \mu \leq \nu, \\ \\ \frac{4(4+\alpha)}{\alpha^2}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n \left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n & \\ \qquad - \frac{6}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n & \textit{for } \nu \leq \mu \leq 0, \\ \\ \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n - \phi & \textit{for } 0 < \nu \leq \mu \ \textit{ or } \nu < 0, \mu \geq 0, \\ \\ \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + \phi & \textit{for } \mu \leq \nu < 0 \ \textit{ or } \nu > 0, \mu \leq 0, \end{cases}$$

*where*

$$2\kappa = \left(\frac{3+\alpha}{\alpha}\right)\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n$$

$$\begin{aligned}\phi = &\frac{8(4+\alpha)(2+\alpha)}{3\alpha^3}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{3n}\\ &-\frac{4(4+\alpha)}{\alpha^2}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n\end{aligned}$$

$$\mu(\alpha,n) = \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n - \frac{(4+\alpha)(2+\alpha)}{3\alpha^3}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{3n}$$

$$\nu(\alpha,n) = \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n - \frac{4+\alpha}{\alpha^2}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n$$

*Proof*: Let $F(w) = f^{-1}(w) = w + \sum_{k=2}^{\infty} A_k w^k$. Since $f(w + \sum_{k=2}^{\infty} A_k w^k) = w$, we then have,

$$\begin{aligned}&(w + A_2w^2 + A_3w^3 + ...) + a_2(w + A_2w^2 + A_3w^3 + ...)^2\\ &\quad + a_3(w + A_2w^2 + A_3w^3 + ...)^3 + a_4(w + A_2w^2 + A_3w^3 + ...)^4 + ... = w.\end{aligned}$$

Equating coefficients, we obtain

$$A_2 + a_2 = 0 \tag{2.30}$$

$$A_3 + 2a_2A_2 + a_3 = 0 \tag{2.31}$$

$$A_4 + a_2(2A_3 + A_2^2) + 3a_3A_2 + a_4 = 0. \tag{2.32}$$

(i) From (2.17) and (2.30), the result follows trivially from Lemma 2.1.3(i). In similar manner, substituting (2.17) and (2.18) in (2.31), gives

$$
\begin{aligned}
|A_3| &= \left| \left(\frac{3+\alpha}{2}\right)\left(\frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n c_1\right)^2 - \left(\frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_2\right)\right| \\
&= \left| -\frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \right. \\
&\qquad \left. \times \left(c_2 - \left[\frac{(3+\alpha)[1-\lambda+\lambda(\alpha+2)]^n[1-\lambda+\lambda\alpha]^n}{2\alpha[1-\lambda+\lambda(\alpha+1)]^{2n}}\right] c_1^2\right)\right| \\
&= \left| -\frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n (c_2 - \kappa c_1^2)\right| \\
&\leq \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n max\{1, |1-2\kappa|\}.
\end{aligned}
$$

$$
where \quad 2\kappa = \frac{(3+\alpha)[1-\lambda+\lambda(\alpha+2)]^n[1-\lambda+\lambda\alpha]^n}{\alpha[1-\lambda+\lambda(\alpha+1)]^{2n}} > 0.
$$

Application of Lemma 2.1.3(ii) gives the upper bound in (ii).

(iii) Next using (2.17)-(2.19) in (2.32) and after some simplification

$$
\begin{aligned}
-A_4 &= \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n c_3 + \left(\frac{(4+\alpha)(2+\alpha)}{3\alpha^3}\right)\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{3n} c_1^3 \\
&\quad - \left(\frac{4+\alpha}{\alpha^2}\right)\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n \left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n c_1c_2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n (c_3 - 2c_1c_2 + c_1^3) \\
&\quad + \left(\frac{(4+\alpha)(2+\alpha)}{3\alpha^3}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{3n} - \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n\right) c_1^3 \\
&\quad + \left(\frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n \right. \\
&\qquad \left. - \frac{4+\alpha}{\alpha^2}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n \left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n\right) c_1c_2.
\end{aligned}
$$

Estimates in Lemma 2.1.3 and Lemma 2.1.6, give

$$|A_4| \leq \left\{ \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + 2(2)|\nu(\alpha,n)|\ max\left(1, \left|1-2\frac{\mu(\alpha,n)}{\nu(\alpha,n)}\right|\right) \right\},$$

where

$$\mu(\alpha,n) = \frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n - \frac{(4+\alpha)(2+\alpha)}{3\alpha^3}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{3n},$$

$$\nu(\alpha,n) = \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n - \frac{4+\alpha}{\alpha^2}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n.$$

Next we consider the following cases:

- $|1-2\frac{\mu}{\nu}| \geq 1$ which implies $\frac{\mu}{\nu} \leq 0$ or $\frac{\mu}{\nu} \geq 1$.

  For $\frac{\mu}{\nu} \leq 0$,

$$|A_4| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + 4(\nu)\left(1-\frac{2\mu}{\nu}\right), \qquad \text{for } \nu > 0, \mu \leq 0$$

$$|A_4| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + 4(\nu)\left(-1+\frac{2\mu}{\nu}\right), \qquad \text{for } \nu < 0, \mu \geq 0.$$

  For $\frac{\mu}{\nu} \geq 1$,

$$|A_4| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + 4(-\nu)\left(1-\frac{2\mu}{\nu}\right), \qquad \text{for } \mu \geq \nu > 0$$

$$|A_4| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + 4(-\nu)\left(-1+\frac{2\mu}{\nu}\right), \qquad \text{for } \mu \leq \nu < 0$$

- $|1-2\frac{\mu}{\nu}| \leq 1$ which implies $0 \leq \frac{\mu}{\nu} \leq 1$.

  For $0 \leq \frac{\mu}{\nu} \leq 1$,

$$|A_4| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + 4(\nu)(-1), \qquad \text{for } \nu \leq \mu \leq 0$$

$$|A_4| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+3)}\right]^n + 4(\nu)(1), \qquad \text{for } 0 \leq \mu \leq \nu$$

- $\nu = 0$.

By using some elementary manipulations, we obtain the upper bounds for $|A_4|$.

**Theorem 2.3.7.** *Let* $\mathcal{B}_n^{\lambda}(\alpha)$ *and* $f^{-1} = w + \sum_{n=2}^{\infty} A_n w^n$. *Then*

$$|A_3 - tA_2^2| \leq \begin{cases} \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n & \text{for } \frac{3+\alpha}{2} - \delta \leq t \leq \frac{3+\alpha}{2}, \\ \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n - \frac{2}{\alpha}\left(\frac{3-2t+\alpha}{\alpha}\right)\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n} & \text{for } t \leq \frac{3+\alpha}{2} - \delta, \\ \frac{2}{\alpha}\left(\frac{3-2t+\alpha}{\alpha}\right)\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n} - \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n & \text{for } t \geq \frac{3+\alpha}{2}, \end{cases}$$

$$\textit{where} \qquad \delta = \alpha\left[\frac{1-\lambda+\lambda(\alpha+1)}{1-\lambda+\lambda(\alpha+2)}\right]^n\left[\frac{1-\lambda+\lambda(\alpha+1)}{1-\lambda+\lambda\alpha}\right]^n.$$

*Proof* : Using (2.17) and (2.18), we write

$$
\begin{aligned}
A_3 - tA_2^2 &= (2-t)a_2^2 - a_3 \\
&= -\frac{1}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \\
&\quad \times \left(c_2 - \left(\frac{3-2t+\alpha}{2\alpha}\right)\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n \left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n c_1^2\right).
\end{aligned}
$$

Using Lemma 2.1.3(ii), gives

$$
|A_3 - tA_2^2| \leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \; max\{1, |1-2\mu|\}
$$

where

$$
\mu = \left(\frac{3-2t+\alpha}{2\alpha}\right)\left[\frac{1-\lambda+\lambda(\alpha+2)}{1-\lambda+\lambda(\alpha+1)}\right]^n \left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^n.
$$

By considering the possible cases of $|1-2\mu|$ we can easily obtain the upper bounds of $|A_3 - tA_2^2|$ for the different range of values of $t$,

(i) $|1-2\mu| \geq 1$ if and only if $\mu \leq 0$ or $\mu \geq 1$.

For $\mu \leq 0$ which means $t \geq \frac{3+\alpha}{2}$,

$$
\begin{aligned}
|A_3 - tA_2^2| &\leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n (-1+2\mu) \quad for\ \mu \leq 0, \\
&= -\frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n + \frac{2}{\alpha}\left(\frac{3-2t+\alpha}{\alpha}\right)\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n}.
\end{aligned}
$$

For $\mu \geq 1$ which implies $t \leq \frac{3+\alpha}{2} - \alpha\left[\frac{1-\lambda+\lambda(\alpha+1)}{1-\lambda+\lambda(\alpha+2)}\right]^n \left[\frac{1-\lambda+\lambda(\alpha+1)}{1-\lambda+\lambda\alpha}\right]^n$,

$$
\begin{aligned}
|A_3 - tA_2^2| &\leq \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n (1-2\mu) \quad for\ \mu \geq 1 \\
&= \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n - \frac{2}{\alpha}\left(\frac{3-2t+\alpha}{\alpha}\right)\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+1)}\right]^{2n}.
\end{aligned}
$$

(ii) $|1-2\mu| \leq 1$ if and only if $0 \leq \mu \leq 1$,

$$|A_3 - tA_2^2| \;\; \leq \;\; \frac{2}{\alpha}\left[\frac{1-\lambda+\lambda\alpha}{1-\lambda+\lambda(\alpha+2)}\right]^n \qquad for \;\; 0 \leq \mu \leq 1.$$

This complete the proof.

# CHAPTER 3

## PRESERVING INTEGRAL OPERATORS

### 3.1 Introduction

Let $\mathcal{A}_p$ be given by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad (0 \leq \alpha < p, p \in \mathbb{N}). \tag{3.1}$$

For a function $f(z) \in \mathcal{A}_p$ which are analytic in $\mathcal{U}$, Shams *et al.* (2006) defined the integral operator $I_p^\sigma$ by

$$\begin{aligned} I_p^\sigma f(z) &= \frac{(p+1)^\sigma}{z^p \Gamma(\sigma)} \int_0^z t^{p-1} \left( log \ \frac{z}{t} \right)^{\sigma - 1} f(t) \ dt \\ &= z^p + \sum_{n=1}^{\infty} \left( \frac{p+1}{p+1+n} \right)^\sigma a_{p+n} z^{p+n}, \quad (\sigma > 0). \end{aligned}$$

Motivated essentially by the Jung-Kim-Srivastava integral operator in Jung *et. al.* (1993), Liu and Owa (2003) introduced and investigated the following integral operator:

$$\begin{aligned} J_\beta^\sigma f(z) &= \binom{p+\sigma+\beta-1}{p+\beta-1} \frac{\sigma}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\sigma-1} t^{\beta-1} f(t) \ dt \\ &= z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} a_{p+n} z^{p+n}, \quad (\sigma > 0, \ \beta > -1), \end{aligned}$$

where $\Gamma$ denotes the Gamma function, $f \in \mathcal{A}_p$ is assumed to be given by (3.1), and (in general)

$$\binom{p+\beta-1}{\sigma} := \frac{\Gamma(p+\beta)}{\Gamma(p+\beta-\sigma)\Gamma(\sigma+1)} =: \binom{p+\beta-1}{p+\beta-1-\sigma}.$$

If $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ and $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$ are analytic in $\mathcal{U}$, then their Hadamard product (or convolution), denoted by $f * g$, is the function defined by the power series

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}, \quad (z \in \mathcal{U}).$$

In order to prove our results, we will use the following lemmas.

An infinite sequence $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ of nonnegative numbers is said to be a convex null sequence if

1. $\lambda_n \to 0$ as $n \to \infty$ and

2. $\lambda_0 - \lambda_1 \geq \lambda_1 - \lambda_2 \geq \cdots \geq \lambda_k - \lambda_{n+1} \geq \cdots \geq 0.$

**Lemma 3.1.1.** *Jahangiri (2003). Suppose $\{\lambda_n\}_{n=0}^{\infty}$ is a convex null sequence. Then the function $p(z) = \frac{\lambda_0}{2} + \lambda_1 z + \lambda_2 z^2 + \ldots\ z \in \mathcal{U}$, is analytic in $\mathcal{U}$ with Re $p(z) > 0$.*

**Remark 3.1.1.** *If $\{\lambda_n\}_{n=0}^{\infty}$ is a convex null sequence, then by above Lemma, Re $p(z) >$ $\frac{1}{2}$.*

**Lemma 3.1.2.** *Jahangiri (2003). Let $p$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and Re$\{p(z)\} > \frac{1}{2}$ in $\mathcal{U}$. For any function $q$ analytic in $\mathcal{U}$, the function $p * q$ takes value in the convex hull image on $\mathcal{U}$ under $q$.*

**Lemma 3.1.3.** *Babalola (2007). Let $\theta$ be a real number and $l$ be a positive integer. If $-1 < \gamma \leq A$, then*

$$\frac{1}{1+\gamma} + \sum_{k=1}^{l} \frac{cos\ k\theta}{k+\gamma} \geq 0$$

*where the constant $A = 4.5678018...$ is the best possible.*

**Lemma 3.1.4.** *Babalola (2007). For $z \in \mathcal{U}$ and $-1 < \gamma \leq A = 4.5678018...$,*

$$Re \sum_{k=1}^{l} \frac{z^k}{k+\gamma} \geq -\frac{1}{1+\gamma}.$$

In this Chapter, we give results on preservation of integral operators for functions belonging to the class $\mathcal{ST}_{p,g}(\beta)$, $\mathcal{CV}_{p,g}(\beta)$ and $\mathcal{CCV}_{p,g}(\beta)$. These classes are defined using the above characterization and convolution.

## 3.2 Starlike and Convex function

For a fixed function $g \in A_p$, the class $\mathcal{ST}_{p,g}(\beta)$ which consists of functions $f \in A_p$ satisfying

$$\frac{(g*f)(z)}{z^p} \neq 0$$

in $\mathcal{U}$ and

$$Re\ \frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} > \beta$$

where $0 \leq \beta < 1$. In a similar manner $\mathcal{CV}_{p,g}(\beta)$ is defined using the $1+\frac{zf''(z)}{f'(z)}$ characterization. Our results show the preservation of the integral operators $I_p^\sigma$ and $J_\beta^\sigma$ for the classes $\mathcal{ST}_{p,g}(\beta)$ and $\mathcal{CV}_{p,g}(\beta)$.

**Theorem 3.2.1.** *If $f \in \mathcal{ST}_{p,g}(\beta)$ then both $I_p^\sigma f$ and $J_\beta^\sigma f$ also belong to $\mathcal{ST}_{p,g}(\beta)$. ($\mathcal{ST}_{p,g}(\beta)$ is preserved by the integral operators $I_p^\sigma f$ and $J_\beta^\sigma f$).*

*Proof* : Given a fixed $g \in \mathcal{A}_p$. If $f \in \mathcal{ST}_{p,g}(\beta)$ then $\exists h$ such that Re $h(z) > \beta$ and the following is true

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} = h(z) = 1 + \frac{1}{p}\sum_{n=1}^{\infty} c_{p+n}z^{p+n}. \tag{3.2}$$

Suppose $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n}$ and $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}z^{p+n}$. Then

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} = 1 + \frac{1}{p}\left(\frac{\sum_{n=1}^{\infty} na_{p+n}b_{p+n}z^n}{1+\sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^n}\right).$$

From (3.2), we have

$$1 + \frac{1}{p}\left(\frac{\sum_{n=1}^{\infty} na_{p+n}b_{p+n}z^n}{1+\sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^n}\right) = 1 + \frac{1}{p}\sum_{n=1}^{\infty} c_{p+n}z^n$$

and after simplifying, gives

$$\begin{aligned}\sum_{n=1}^{\infty} na_{p+n}b_{p+n}z^n &= \sum_{n=1}^{\infty} c_{p+n}z^n + \left(\sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^n\right)\left(\sum_{n=1}^{\infty} c_{p+n}z^n\right)\\ &= \sum_{n=1}^{\infty} c_{p+n}z^n + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} a_{p+k}b_{p+k}c_{p+n-k}\right)z^n.\end{aligned}$$

Equating coefficients, provides the following relation

$$na_{p+n}b_{p+n} = \sum_{k=1}^{n-1} a_{p+k}b_{p+k}c_{p+n-k} \tag{3.3}$$

where $a_p = b_p = 1$.

Since

$$I_p^\sigma f(z) \;=\; z^p + \sum_{n=1}^{\infty} \left(\frac{p+1}{p+1+n}\right)^\sigma a_{p+n}z^{p+n},$$

by using similar method we establish the following,

$$\begin{aligned} \frac{1}{p}\frac{z(g * I_p^\sigma)'(z)}{(g * I_p^\sigma)(z)} &= 1 + \frac{1}{p}\left[\frac{\sum_{n=1}^{\infty}(\frac{p+1}{p+1+n})^\sigma na_{p+n}b_{p+n}z^n}{1+\sum_{n=1}^{\infty}(\frac{p+1}{p+1+n})^\sigma a_{p+n}b_{p+n}z^n}\right] \\ &= 1 + \frac{1}{p}\sum_{n=1}^{\infty} d_{p+n}z^n, \;\; say. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(\frac{p+1}{p+1+n}\right)^\sigma na_{p+n}b_{p+n}z^n \\ &\qquad = \sum_{n=1}^{\infty} d_{p+n}z^n + \left(\sum_{n=1}^{\infty} d_{p+n}z^n\right)\left(\sum_{n=1}^{\infty}\left(\frac{p+1}{p+1+n}\right)^\sigma a_{p+n}b_{p+n}z^n\right) \\ &\qquad = \sum_{n=1}^{\infty} d_{p+n}z^n + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} a_{p+k}b_{p+k}\left(\frac{p+1}{p+1+k}\right)^\sigma d_{p+n-k}\right)z^n, \end{aligned}$$

which, again upon equating coefficients gives

$$\left(\frac{p+1}{p+1+n}\right)^\sigma na_{p+n}b_{p+n} = \sum_{k=0}^{n-1}\left(\frac{p+1}{p+1+k}\right)^\sigma a_{p+k}b_{p+k}d_{p+n-k}. \tag{3.4}$$

Substituting (3.3) into (3.4) results in

$$\left(\frac{p+1}{p+1+n}\right)^\sigma\left(\sum_{k=0}^{n-1} a_{p+k}b_{p+k}c_{p+n-k}\right) = \sum_{k=0}^{n-1}\left(\frac{p+1}{p+1+k}\right)^\sigma a_{p+k}b_{p+k}d_{p+n-k}$$

which upon simplification gives the following relation

$$\begin{aligned}\left(\frac{p+1}{p+1+n}\right)^{\sigma} a_{p+k}b_{p+k}c_{p+n-k} &= \left(\frac{p+1}{p+1+k}\right)^{\sigma} a_{p+k}b_{p+k}d_{p+n-k}\\ d_{p+k} &= \left(\frac{p+n+1-k}{p+1+n}\right)^{\sigma} c_{p+k}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{p}\frac{z(g*I_p^{\alpha})'(z)}{(g*I_p^{\alpha})(z)} &= 1+\frac{1}{p}\sum_{n=1}^{\infty} d_{p+n}z^n\\ &= 1+\frac{1}{p}\sum_{n=1}^{\infty}\left(\frac{p+1}{p+1+n}\right)^{\sigma} c_{p+n}z^n\\ &= \left(1+\sum_{n=1}^{\infty}\left(\frac{p+1}{p+1+n}\right)^{\sigma} z^n\right)*\left(1+\frac{1}{p}\sum_{n=1}^{\infty} c_{p+n}z^n\right)\\ &= q(z)*h(z)\end{aligned}$$

where

$$q(z)=1+\sum_{n=1}^{\infty}\left(\frac{p+1}{p+1+n}\right)^{\sigma} z^n = 1+\sum_{n=1}^{\infty}\lambda_n z^n.$$

It is obvious that the infinite sequence $\{\lambda_n\}_{n=0}^{\infty}$ where $\lambda_0 = 1$ is a convex null sequence. Therefore by Lemma 3.1.1 the $Re\ q(z) > \frac{1}{2}$. We know that $h(z) > \beta$ and by using Lemma 3.1.2 this imply,

$$Re\ \frac{1}{p}\frac{z(g*I_p^{\sigma})'(z)}{(g*I_p^{\sigma})(z)} > \beta,$$

which indicate preservation of $I_p^{\sigma} f$.

Next, for

$$J_{\beta}^{\sigma} f(z) = z^p + \sum_{n=1}^{\infty}\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} a_{p+n}z^{p+n}$$

we have

$$
\begin{aligned}
\frac{1}{p}\frac{z(g*J_\beta^\sigma)'(z)}{(g*J_\beta^\sigma)(z)} &= 1+\frac{1}{p}\left[\frac{\sum_{n=1}^{\infty} na_{p+n}b_{p+n}\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}z^n}{1+\sum_{n=1}^{\infty} a_{p+n}b_{p+n}\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}z^n}\right] \\
&= h(z) \\
&= 1+\frac{1}{p}\sum_{n=1}^{\infty} s_{p+n}z^n, (say)
\end{aligned}
$$

which in comparison, gives

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\right] na_{p+n}b_{p+n}z^n \\
&\qquad = \sum_{n=1}^{\infty} s_{p+n}z^n + \left[\sum_{n=1}^{\infty} a_{p+n}b_{p+n}\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\ z^n\right]\left[\sum_{n=1}^{\infty} s_{p+n}z^n\right] \\
&\qquad = \sum_{n=1}^{\infty} s_{p+n}z^n + \sum_{n=2}^{\infty}\left[\sum_{k=1}^{n-1} a_{p+k}b_{p+k}\frac{\Gamma(p+k+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+k+\sigma+\beta)\Gamma(p+\beta)}s_{p+n-1}\right] z^n.
\end{aligned}
$$

Equating coefficient results in,

$$
\begin{aligned}
&na_{p+n}b_{p+n}\left[\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\right] \\
&\qquad = \sum_{k=0}^{n-1}\left[\frac{\Gamma(p+k+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+k+\sigma+\beta)\Gamma(p+\beta)}\right] a_{p+k}b_{p+k}s_{p+n-k}
\end{aligned}
\tag{3.5}
$$

and substitution of (3.3) into (3.5), gives

$$
s_{p+n-k} = \left[\frac{\Gamma(p+n+\beta)}{\Gamma(p+n+\sigma+\beta)}\right]\left[\frac{\Gamma(p+k+\sigma+\beta)}{\Gamma(p+k+\beta)}\right] c_{p+n-k}.
$$

Hence,

$$
\begin{aligned}
\frac{1}{p}\frac{z(g*J_\beta^\sigma)'(z)}{(g*J_\beta^\sigma)(z)} &= 1+\frac{1}{p}\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\right] c_{p+n-k}z^n \\
&= \left(1+\sum_{n=1}^{\infty} q_n z^n\right) * \left(1+\frac{1}{p}\sum_{n=1}^{\infty} c_{p+n}z^n\right) \\
&= q(z) * h(z)
\end{aligned}
$$

where

$$\{q_n\} = \left\{ \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} \right\},$$

is a convex null sequence. This is obvious since for $\sigma > 0$, $\beta > 0$, $n > 1$ and $p > 1$,

$$\begin{aligned}
\frac{q_n - q_{n+1}}{q_{n+1} - q_{n+2}} &= \left[\frac{p+n+\sigma+\beta}{p+n+\beta}\right]\left[\frac{p+n+\sigma+\beta+1}{p+n+\sigma+\beta}\right] \\
&= 1 + \frac{\sigma+1}{p+n+\beta} \\
&\geq 1
\end{aligned}$$

and thus,

$$q_n - q_{n+1} \geq q_{n+1} - q_{n+2} > 0.$$

Lemma 3.1.1 implies

$$p(z) = \frac{q_0}{2} + q_1(z) + q_2 z^2 + \dots$$

is analytic with Re $p(z) > 0$. Furthermore, as $q_0 = 1$

$$\begin{aligned}
Re\left(1 + \sum_{n=1}^{\infty} q_n z^n\right) &= Re\left(1 + p(z) - \frac{q_0}{2}\right) \\
&= Re\left(p(z) + \frac{1}{2}\right) \\
&> \frac{1}{2}.
\end{aligned}$$

Using Lemma 3.1.1 and Lemma 3.1.2, with $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$,we obtain $q$ is analytic in $\mathcal{U}$ with $q(0) = 1$ and Re $q(z) > \frac{1}{2}$ where $q * h$ takes its values in the convex hull of $h(\mathcal{U})$, thus completing the proof.

The next result is an analogous result for the class $\mathcal{CV}_{p,g}(\beta)$.

**Theorem 3.2.2.** *If $f \in \mathcal{CV}_{p,g}(\beta)$ then $I_p^\sigma$ and $J_\beta^\sigma$ also belong to $\mathcal{CV}_{p,g}(\beta)$.*

In the next section, we consider preservation of integral operators for $f \in \mathcal{CCV}_{p,g}(\beta)$ by using same method but some extra condition is needed.

## 3.3 Close-to-convex function

For a fixed function $g \in \mathcal{A}_p$, the class $\mathcal{CCV}_{p,g}(\beta)$, which consists of functions $f \in \mathcal{A}_p$ satisfying

$$\frac{(g * \psi)(z)}{z^p} \neq 0$$

in $\mathcal{U}$ and

$$Re\ \frac{1}{p}\left\{\frac{z(g*f)'(z)}{(g*\psi)(z)}\right\} > \beta, \quad (0 \leq \beta < 1)$$

for some $\psi \in \mathcal{ST}_{p,g}(\beta)$ and we let $g \in \mathcal{ST}$. Our results show the preservation of the integral operators $I_p^\sigma$ and $J_\beta^\sigma$ for the classes $\mathcal{CCV}_{p,g}(\beta)$.

**Theorem 3.3.1.** *If $f \in \mathcal{CCV}_{p,g}(\beta)$ then $I_p^\sigma f$ and $J_\beta^\sigma f$ belongs to $\mathcal{CCV}_{p,g}(\beta)$ for a fixed $g \in \mathcal{ST}$. ($\mathcal{CCV}_{p,g}(\beta)$ is preserved by the integral operators $I_p^\sigma f$ and $J_\beta^\sigma f$.)*

*Proof* : First we let $\psi_1(z) = z^p + \sum_{n=1}^{\infty} z^{p+n}$ and since $g(z) \in \mathcal{ST}$ this implies that $\psi_1(z) \in \mathcal{ST}_{p,g}(\beta)$. We note that $f \in \mathcal{CCV}_{p,g}(\beta)$, then

$$h(z) = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\psi_1)(z)} \;=\; \frac{1+\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)a_{p+n}b_{p+n}z^n}{1+\sum_{n=1}^{\infty}b_{p+n}z^n}$$
$$= 1+\frac{\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)a_{p+n}b_{p+n}z^n-\sum_{n=1}^{\infty}b_{p+n}z^n}{1+\sum_{n=1}^{\infty}b_{p+n}z^n}.$$

If we let $h(z)=1+\frac{1}{p}\sum_{n=1}^{\infty}r_{p+n}z^n$, then the following is established

$$\frac{\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)a_{p+n}b_{p+n}z^n-\sum_{n=1}^{\infty}b_{p+n}z^n}{1+\sum_{n=1}^{\infty}b_{p+n}z^n} = \frac{1}{p}\sum_{n=1}^{\infty}r_{p+n}z^n,$$

which upon simplification gives the following relation

$$\sum_{n=1}^{\infty}(p+n)a_{p+n}b_{p+n}z^n-\sum_{n=1}^{\infty}p\; b_{p+n}z^n=\sum_{n=1}^{\infty}r_{p+n}z^n+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}b_{p+k}r_{p+n-k}\right)z^n.$$

Comparing coefficients, we obtain the following relation

$$(p+n)a_{p+n}b_{p+n}-p\; b_{p+n} \;=\; \sum_{k=0}^{n-1}b_{p+k}r_{p+n-k} \tag{3.6}$$

with $b_p=1$ and $n=1,2,3,\ldots$.

Next we let $\psi_2(z)=z^p+\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}z^{p+n}$ and note that $\psi_2(z)\in S_{p,g}(\beta)$. Now we consider integral operator $I_p^{\sigma}$,

$$\frac{1}{p}\frac{z(g*I_p^{\sigma}f)'(z)}{(g*\psi_2)(z)}$$
$$=\frac{1+\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}\left(\frac{p+n}{p}\right)a_{p+n}b_{p+n}z^n}{1+\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}b_{p+n}z^n}$$
$$=1+\frac{\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}\left(\frac{p+n}{p}\right)a_{p+n}b_{p+n}z^n-\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}b_{p+n}z^n}{1+\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}b_{p+n}z^n}$$
$$:=1+\frac{1}{p}\sum_{n=1}^{\infty}s_{p+n}z^n,$$

then

$$\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}(p+n)a_{p+n}b_{p+n}z^n-\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}pb_{p+n}z^n$$
$$=\sum_{n=1}^{\infty}s_{p+n}z^n+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\left(\frac{p+1}{p+k+1}\right)^{\sigma}b_{p+k}s_{p+n-k}\right)z^n.$$

Hence by equating coefficient gives

$$\begin{aligned}\left(\frac{p+1}{p+n+1}\right)^{\sigma}(p+n)a_{p+n}b_{p+n}-\left(\frac{p+1}{p+n+1}\right)^{\sigma}pb_{p+n}\\ =s_{p+n}+\sum_{k=1}^{n-1}\left(\frac{p+1}{p+k+1}\right)^{\sigma}b_{p+k}s_{p+n-k}\\ =\sum_{k=0}^{n-1}\left(\frac{p+1}{p+k+1}\right)^{\sigma}b_{p+k}s_{p+n-k}\end{aligned}\tag{3.7}$$

where $b_p=1$ and $n=1,2,3,\ldots$. From (3.6) and (3.7), we have

$$\left(\frac{p+1}{p+n+1}\right)^{\sigma}\sum_{k=0}^{n-1}b_{p+k}r_{p+n-k}=\sum_{k=0}^{n-1}\left(\frac{p+1}{p+k+1}\right)^{\sigma}b_{p+k}s_{p+n-k},$$

and thus the relation,

$$s_{p+k}=\left(\frac{p+1+n-k}{p+n+1}\right)^{\sigma}r_{p+k}.\tag{3.8}$$

Hence using (3.8) we obtain

$$\begin{aligned}\frac{1}{p}\frac{z(g*I_p^{\sigma}f)'(z)}{(g*\psi_2)(z)} &= 1+\frac{1}{p}\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}r_{p+n}z^n\\ &:= q(z)*h(z),\end{aligned}$$

where

$$q(z) = 1+\sum_{n=1}^{\infty}\left(\frac{p+1}{p+n+1}\right)^{\sigma}z^n=1+\sum_{n=1}^{\infty}\lambda_n z^n,$$

$$h(z) = 1+\frac{1}{p}\sum_{n=1}^{\infty}r_{p+n}z^n=\frac{1}{p}\frac{z(g*f)'(z)}{(g*\psi_1)(z)}.$$

It is elementary to show that the infinite sequence $\{\lambda_n\}_{n=0}^{\infty}$ where $\lambda_0 = 1$ is a convex null sequence. This is obvious since for $\sigma > 0$, $n > 1$ and $p > 1$

$$\begin{aligned} \lambda_n - \lambda_{n+1} &\geq \lambda_{n+1} - \lambda_{n+2} \\ \left(\frac{p+1}{p+n+1}\right)^{\sigma} - \left(\frac{p+1}{p+n+2}\right)^{\sigma} &\geq \left(\frac{p+1}{p+n+2}\right)^{\sigma} - \left(\frac{p+1}{p+n+3}\right)^{\sigma} \\ &> 0. \end{aligned}$$

Using Lemma 3.1.1 and Lemma 3.1.2, we obtain $q$ is analytic in $\mathcal{U}$ with $q(0) = 1$ and Re $q(z) > \frac{1}{2}$ where $q * h$ takes its values in the convex hull of $h(\mathcal{U})$, thus imply,

$$Re\ \frac{1}{p}\frac{z(g * I_p^{\sigma} f)'(z)}{(g * \psi_2)(z)} > \beta \Rightarrow I_p^{\sigma} f \in \mathcal{CCV}_{p,g}(\beta).$$

This complete the proof.

Let $\psi_3(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} z^{p+n}$ and note that $\psi_3(z) \in S_{p,g}(\beta)$. Next using the previous method we now consider operator, $J_{\beta}^{\sigma}$,

$$\begin{aligned} &\frac{1}{p}\frac{z(g * J_{\beta}^{\sigma})'(z)}{(g * \psi_3)(z)} \\ &\quad = \left[\frac{1 + \sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right) \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} a_{p+n} b_{p+n} z^n}{1 + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} b_{p+n} z^n}\right] \\ &\quad = 1 + \left[\frac{\sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right) \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} a_{p+n} b_{p+n} z^n - \sum_{n=1}^{\infty} \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} b_{p+n} z^n}{1 + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)} b_{p+n} z^n}\right] \\ &\quad := 1 + \frac{1}{p}\sum_{n=1}^{\infty} t_{p+n} z^n \end{aligned}$$

which in comparison, gives

$$\sum_{n=1}^{\infty}(p+n)\,\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}a_{p+n}b_{p+n}z^n$$
$$-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}pb_{p+n}z^n$$
$$=\sum_{n=1}^{\infty}t_{p+n}z^n+\left[\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}b_{p+n}z^n\right]\left[\sum_{n=1}^{\infty}t_{p+n}z^n\right]$$
$$=\sum_{n=1}^{\infty}t_{p+n}z^n+\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n-1}\frac{\Gamma(p+k+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+k+\sigma+\beta)\Gamma(p+\beta)}b_{p+k}t_{p+n-k}\right]z^n.$$

Equating coefficient results in,

$$(p+n)\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}a_{p+n}b_{p+n}-\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}pb_{p+n}$$
$$=t_{p+n}+\sum_{k=1}^{n-1}\frac{\Gamma(p+k+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+k+\sigma+\beta)\Gamma(p+\beta)}b_{p+k}t_{p+n-k}$$
$$=\sum_{k=0}^{n-1}\frac{\Gamma(p+k+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+k+\sigma+\beta)\Gamma(p+\beta)}b_{p+k}t_{p+n-k} \tag{3.9}$$

with $b_p=1$ and $n=1,2,3,\ldots$. Then by substituting (3.6) into (3.9), gives

$$\left[\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\right]\sum_{k=0}^{n-1}b_{p+k}r_{p+n-k}$$
$$=\sum_{k=0}^{n-1}\frac{\Gamma(p+k+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+k+\sigma+\beta)\Gamma(p+\beta)}b_{p+k}t_{p+n-k}$$

upon simplification

$$t_{p+k}=\left[\frac{\Gamma(p+n+\beta)}{\Gamma(p+n+\sigma+\beta)}\right]\left[\frac{\Gamma(p+n-k+\sigma+\beta)}{\Gamma(p+n-k+\beta)}\right]r_{p+k}. \tag{3.10}$$

Hence using (3.10) we obtain

$$\begin{aligned}\frac{1}{p}\frac{z(g*J^{\sigma}_{\beta})'(z)}{(g*\psi_3)(z)} &= 1+\frac{1}{p}\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\right] r_{p+n}z^n \\ &= w(z)*h(z)\end{aligned}$$

where

$$w(z) = 1+\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\right] z^n = 1+\sum_{n=1}^{\infty}\lambda_n z^n,$$

$$h(z) = 1+\frac{1}{p}\sum_{n=1}^{\infty} r_{p+n}z^n = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\psi_1)(z)}.$$

It is elementary to show that,

$$\{\lambda_n\} = \left\{\frac{\Gamma(p+n+\beta)\Gamma(p+\sigma+\beta)}{\Gamma(p+n+\sigma+\beta)\Gamma(p+\beta)}\right\},$$

is a convex null sequence where $\lambda_0 = 1$. This is obvious since for $\sigma > 0$, $\beta > 0$, $n > 1$ and $p > 1$

$$\begin{aligned}\frac{\lambda_n-\lambda_{n+1}}{\lambda_{n+1}-\lambda_{n+2}} &= \left[\frac{p+n+\sigma+\beta}{p+n+\beta}\right]\left[\frac{p+n+\sigma+\beta+1}{p+n+\sigma+\beta}\right] \\ &= 1+\frac{\sigma+1}{p+n+\beta} \\ &\geq 1\end{aligned}$$

and thus,

$$\lambda_n-\lambda_{n+1} \geq \lambda_{n+1}-\lambda_{n+2} > 0.$$

Using Lemma 3.1.1 and Lemma 3.1.2, with $w(z) = 1+\sum_{n=1}^{\infty}\lambda_n z^n$, we obtain $w$ is analytic in $\mathcal{U}$ with $w(0) = 1$ and Re $w(z) > \frac{1}{2}$ where $w*h$ takes its values in the convex hull of $h(\mathcal{U})$, thus completing the proof.

## 3.4 Bounded turning

Let $f \in \mathcal{A}_p$ be given by (3.1) and define the integral operator $L_{p,c}$ as follows

$$\begin{aligned} L_{p,c}(f) &= \frac{c+p}{z^c}\int_0^z f(t)t^{c-1}\,dt, \\ &= z^p + \sum_{n=1}^{\infty}\frac{c+p}{c+p+n}a_{p+n}z^{p+n}. \quad (c+p>0, z\in\mathcal{U}) \end{aligned}$$

with its sequence of partial sums denoted by

$$L_k = z^p + \sum_{n=1}^{k}\frac{c+p}{c+p+n}a_{p+n}z^{p+n}. \quad (z\in\mathcal{U}) \tag{3.11}$$

The above integral operator was introduced by Reddy and Padmanabhan (1982). In particular, the operator $F_{1,c}$ was studied earlier by Bernardi (1969).

A function $f \in \mathcal{A}_p$ is denoted by $\mathcal{P}(p,\alpha)$ if it satisfies

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha, \qquad (0\le\alpha<p,\ z\in\mathcal{U}).$$

The classes $\mathcal{P}(1,0)$ and $\mathcal{P}(p,0)$ were investigated by MacGregor (1962) and Umezawa (1957), respectively. In fact, the class $\mathcal{P}(p,\alpha)$ is a subclass of the class $\mathcal{A}$. In 2010, Darus *et. al.* proved the following theorem:

**Theorem 3.4.1.** *Darus (2010). Let $f\in\mathcal{A}$. If $\frac{1}{2}<\alpha<1$ and $f(z)\in\mathcal{P}(1,\alpha)$ then $L_k(z)\in\mathcal{P}(1,\frac{3-(c+1)(1-\alpha)}{3})$.*

We now extend the result for $p$-valent functions.

**Theorem 3.4.2.** *If* $\frac{1}{2} < \frac{\alpha}{p} < 1$ *and* $f \in \mathcal{P}(p, \alpha)$*, then* $L_k(z) \in \mathcal{P}\left(p, \frac{p+\alpha(c+p)}{p(c+p+1)}\right)$.

*Proof* : Since

$$Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \frac{\alpha}{p}, \ \left(\frac{1}{2} < \frac{\alpha}{p} < 1, \ z \in \mathcal{U}\right)$$

this implies

$$Re\left\{1 + \sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right) a_{p+n}z^n\right\} > \frac{\alpha}{p} > \frac{1}{2}.$$

Now for $\frac{1}{2} < \frac{\alpha}{p} < 1$ we have

$$Re\left\{1 + \sum_{n=1}^{\infty}\left(\frac{p+n}{p-\alpha}\right) a_{p+n}z^n\right\} > Re\left\{1 + \sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right) a_{p+n}z^n\right\}$$

then

$$Re\left\{1 + \sum_{n=1}^{\infty}\left(\frac{p+n}{p-\alpha}\right) a_{p+n}z^n\right\} > \frac{1}{2}.$$

Next, we write

$$\begin{aligned}
\frac{1}{p}\frac{L_k'(z)}{z^{p-1}} &= 1 + \sum_{n=1}^{k}\frac{(p+n)(c+p)}{p(c+p+n)}a_{p+n}z^n \\
&= \left(1 + \sum_{n=1}^{\infty}\left(\frac{p+n}{p-\alpha}\right) a_{p+n}z^n\right) * \left(1 + \sum_{n=1}^{k}\frac{(p-\alpha)(c+p)}{p(c+p+n)}z^n\right) \\
&:= P * Q
\end{aligned}$$

Making use of Lemma 3.1.4, we obtain

$$\begin{aligned}
Re\ Q(z) = Re\left\{1 + \left(\frac{(p-\alpha)(c+p)}{p}\right)\sum_{n=1}^{k}\frac{z^n}{c+p+n}\right\} &\geq 1 - \frac{(p-\alpha)(c+p)}{p(c+p+1)} \\
&= \frac{p+\alpha(c+p)}{p(c+p+1)}.
\end{aligned}$$

Therefore, using Lemma 3.1.2, gives the result.

# CHAPTER 4

# PARTIAL SUMS

## 4.1 Introduction

For functions $f \in \mathcal{A}$, there have been interest by authors in seeking the properties of its partial sums. For $f$ given by (1.1), we denote the partial sums $f_k$ as

$$f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \qquad (z \in \mathcal{U}).$$

See Ibrahim *et. al.* (2010), Latha (2006) and Goyal (2008) for some of these properties. In particular, for $f \in \mathcal{CV}(0)$, Sheil-Small (1970) showed that

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > \frac{1}{2}, \quad (n \geq 1).$$

Properties on the real part of the radius of $f$ to its partial sums were investigated for a variety of other classes as well. For example, in Brickman (1973), if $f \in \mathcal{CV}(\alpha)$ and $0 \leq \alpha < 1,\ \alpha \neq \frac{1}{2}$, then the sharp lower bound was obtained as

$$Re\left\{\frac{f(z)}{f_1(z)}\right\} \geq \frac{1}{1-2\alpha}\frac{1}{|z|}[1-(1+|z|)^{2\alpha-1}].$$

Let $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subfamilies of $\mathcal{ST}$ and $\mathcal{CV}$, respectively, whose functions are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0).$$

Furthermore, for $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, Silvia (1985) established

$$\begin{aligned} Re\left\{\frac{f(z)}{f_n(z)}\right\} &\geq \frac{1}{2-\alpha}, \\ Re\left\{\frac{f(z)}{f_n(z)}\right\} &\geq \frac{3-\alpha}{4-2\alpha}. \end{aligned}$$

In 1975, Silverman showed that if $f$ of the form (1.1) satisfies

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha, \tag{4.1}$$

then $f \in \mathcal{ST}(\alpha)$ and if $f$ of the form (1.1) satisfies

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha, \tag{4.2}$$

then $f \in \mathcal{CV}(\alpha)$. After that, Silverman (1997) initiated interest into seeking sharp lower bounds on the real part of the quotients between the normalized starlike and convex functions and their sequence of partial sums. Silverman determined the following results:

**Theorem 4.1.1.** *If $f$ of the form (1.1) satisfies condition (4.1), then*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{k}{k+1-\alpha},$$

*where $f_k$ denotes the $k$-th partial sums of $f$. The result is sharp for every $k$, with extremal function*

$$f(z) = z + \frac{1-\alpha}{k+1-\alpha} z^{k+1}. \tag{4.3}$$

**Theorem 4.1.2.** *If $f$ of the form (1.1) satisfies condition (4.2), then*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{k(k+2-\alpha)}{(k+1)(k+1-\alpha)}.$$

*The result is sharp for every k, with extremal function*

$$f(z) = z + \frac{1-\alpha}{(k+1)(k+1-\alpha)} z^{k+1}. \tag{4.4}$$

**Theorem 4.1.3.** *If $f$ is of the form (1.1), then*

$$\begin{aligned} &(i) \quad Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{k+1-\alpha}{k+2-2\alpha}, \\ &(ii) \quad Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(k+1)(k+1-\alpha)}{(k+1)(k+1-\alpha)+(1-\alpha)}, \end{aligned}$$

*where $f$ in (i) satisfies condition (4.1) and $f$ in (ii) satisfies (4.2). Equalities hold in (i) and (ii) for the functions given by (4.3) and (4.4), respectively.*

**Theorem 4.1.4.** *If $f$ of the form (1.1), satisfies condition (4.1), then*

$$\begin{aligned} &(i) \quad Re\left\{\frac{f'(z)}{f'_k(z)}\right\} \geq \frac{\alpha k}{k+1-\alpha}, \\ &(ii) \quad Re\left\{\frac{f'_k(z)}{f'(z)}\right\} \geq \frac{k+1-\alpha}{(k+1-\alpha)+(k+1)(1-\alpha)}. \end{aligned}$$

*In both cases, the extremal function is given by (4.3).*

**Theorem 4.1.5.** *If $f$ of the form (1.1) satiesfies condition (4.2), then*

$$\begin{aligned} &(i) \quad Re\left\{\frac{f'(z)}{f'_k(z)}\right\} \geq \frac{k}{k+1-\alpha}, \\ &(ii) \quad Re\left\{\frac{f'_k(z)}{f'(z)}\right\} \geq \frac{k+1-\alpha}{k+2-2\alpha}. \end{aligned}$$

*In both cases, the extremal function is given by (4.4).*

In this chapter, we generalized the idea of Silverman for $p$-valent function which belongs to $ST(\alpha)$ and $CV(\alpha)$. By using the same method, we also obtained the lower bounds for *uniformly convex* and *parabolic starlike functions* which satisfy certain conditions.

## 4.2 $p$-valent starlike and convex function

Let $\mathcal{A}_p$ be given by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (0 \leq \alpha < p,\ p \in \mathbb{N}). \tag{4.5}$$

and its sequence of partial sums is denoted by $f_k(z) = z^p + \sum_{n=1}^{k} a_{p+n} z^{p+n}$. As defined in Section 1.2, a $p$-valent function $f \in \mathcal{A}_p$ is *starlike* and *convex* if it satisfies the conditions $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$ and $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$, respectively. A sufficient condition for a function $f$ of the form (4.5) to be in $\mathcal{ST}_p(\alpha)$ is that

$$\sum_{n=1}^{\infty}(p+n-\alpha)|a_{p+n}| \leq p-\alpha, \tag{4.6}$$

and to be in $\mathcal{CV}_p(\alpha)$ is that

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)|a_{p+n}| \leq p(p-\alpha). \tag{4.7}$$

Further, we note that these sufficient conditions are also necessary for functions of the form (4.5) with positive and negative coefficients (see Owa (1985)). The known result that $\text{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0,\ (z \in \mathcal{U})$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|w(z)| \leq |z|$ is used to obtain bounds for the above ratios. Further works by several other authors which provide interesting developments concerning partial sums of analytic functions can be found in Aouf (2006), Aouf (2009) and Cho (2004).

**Theorem 4.2.1.** *Suppose $f$ of the form (4.5) satisfies condition (4.6), then*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{k+1}{k+p+1-\alpha}, \quad (z \in \mathcal{U})$$

*where $f_k$ denotes the $k$-th partial sums of $f$. The result is sharp for every $k$, with extremal function*

$$f(z) = z^p - \frac{p-\alpha}{p+n-\alpha} z^{p+n}. \tag{4.8}$$

*Proof* : First, write

$$\begin{aligned}\frac{k+p+1-\alpha}{p-\alpha}&\left[\frac{f(z)}{f_k(z)} - \frac{k+1}{k+p+1-\alpha}\right] \\ &= \frac{1+\sum_{n=1}^{k} a_{p+n}z^n + \left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}{1+\sum_{n=1}^{\infty} a_{p+n}z^n} \\ &= \frac{1+w(z)}{1-w(z)}.\end{aligned}$$

Thus

$$w(z) = \frac{\left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}{2+2\sum_{n=1}^{k} a_{p+n}z^n + \left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} |a_{p+n}|}{2-2\sum_{n=1}^{k} |a_{p+n}| - \left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} |a_{p+n}|}.$$

Since the numerator is positive, $|w(z)| \leq 1$ if and only if

$$2\left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} |a_{p+n}| \leq 2 - 2\sum_{n=1}^{k} |a_{p+n}|.$$

This is equivalent to

$$\sum_{n=1}^{k} |a_{p+n}| + \left(\frac{k+p+1-\alpha}{p-\alpha}\right) \sum_{n=k+1}^{\infty} |a_{p+n}| \leq 1. \tag{4.9}$$

Thus, it is sufficient to show that the expression on the left of (4.9) is bounded above by $\sum_{n=1}^{\infty} \left(\frac{p+n-\alpha}{p-\alpha}\right) |a_{p+n}|$ which is equivalent to

$$\sum_{n=1}^{k} \left(\frac{n}{p-\alpha}\right) |a_{p+n}| + \sum_{n=k+1}^{\infty} \left(\frac{n-k-1}{p-\alpha}\right) |a_{p+n}| \geq 0.$$

This is evident true using the hypothesis. The proof is complete.

To see that function $f(z)$ given by (4.8) gives the sharp result, let $z = re^{\frac{i\pi}{n}}$,

$$\frac{f(z)}{f_k(z)} = 1 - \frac{p-\alpha}{p+n-\alpha} z^n \rightarrow 1 - \frac{p-\alpha}{p+n-\alpha} = \frac{n}{p+n-\alpha} = \frac{k+1}{p+k+1-\alpha}$$

when r $\rightarrow 1^+$ and $n = k+1$.

**Theorem 4.2.2.** *If $f$ of the form (4.5) satisfies condition (4.7), then*

$$Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{(k+1)(2p+k+1-\alpha)}{(p+k+1)(p+k+1-\alpha)}, \quad (z \in \mathcal{U})$$

*The result is sharp for every k, with extremal function*

$$f(z) = z^p - \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} z^{p+n}. \tag{4.10}$$

*Proof* : In a similar manner, consider

$$\frac{(p+k+1)(p+k+1-\alpha)}{p(p-\alpha)}\left[\frac{f(z)}{f_k(z)}-\frac{(k+1)(2p+k+1-\alpha)}{(p+k+1)(p+k+1-\alpha)}\right]$$

$$= \frac{1+\sum_{n=1}^{k} a_{p+n}z^n + \left(\frac{(p+k+1)(p+k+1-\alpha)}{p(p-\alpha)}\right)\sum_{k+1}^{\infty} a_{p+n}z^n}{1+\sum_{n=1}^{k} a_{p+n}z^n}$$

$$= \frac{1+w(z)}{1-w(z)}$$

where

$$w(z)=\frac{\left(\frac{(p+k+1)(p+k+1-\alpha)}{p(p-\alpha)}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}{2+2\sum_{n=1}^{k} a_{p+n}z^n+\left(\frac{(p+k+1)(p+k+1-\alpha)}{p(p-\alpha)}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}.$$

Triangle inequality gives

$$|w(z)|\leq\frac{\left(\frac{(p+k+1)(p+k+1-\alpha)}{p(p-\alpha)}\right)\sum_{n=k+1}^{\infty}|a_{p+n}|}{2-2\sum_{n=1}^{k}|a_{p+n}|-\left(\frac{(p+k+1)(p+k+1-\alpha)}{p(p-\alpha)}\right)\sum_{n=k+1}^{\infty}|a_{p+n}|}.$$

Furthermore, $|w(z)|\leq 1$ if and only if

$$\sum_{n=1}^{k}|a_{p+n}|+\left(\frac{(p+k+1)(p+k+1-\alpha)}{p(p-\alpha)}\right)\sum_{n=k+1}^{\infty}|a_{p+n}|\leq 1. \qquad (4.11)$$

Equivalently, we verify that the expression on the left of (4.11) is bounded above by, $\sum_{n=1}^{\infty}\frac{(p+n)(p+n-\alpha)}{p(p-\alpha)}|a_{p+n}|$. Using (4.7) the following

$$\frac{1}{p(p-\alpha)}\left[\sum_{n=1}^{k} n(2p+n-\alpha)|a_{p+n}|\right] + \frac{1}{p(p-\alpha)}\left[\sum_{n=k+1}^{\infty}[(p+n)(p+n-\alpha)-(p+k+1)(p+k+1-\alpha)]|a_{p+n}|\right]\geq 0$$

is easily established and the proof is complete.

Next, results for the lower bounds of $Re\left\{\frac{f_k(z)}{f(z)}\right\}$ for $f \in \mathcal{ST}_p(\alpha)$ and $f \in \mathcal{CV}_p(\alpha)$ are obtained.

**Theorem 4.2.3.** *Given $f$ is of the form (4.5), then*

$$\begin{aligned}
&(i)\quad Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{k+p+1-\alpha}{k+2p+1-2\alpha},\\
&(ii)\quad Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(p+k+1)(p+k+1-\alpha)}{(p+k+1)(p+k+1-\alpha)+p(p-\alpha)},
\end{aligned}$$

*where $f$ in (i) satisfies condition (4.6) and $f$ in (ii) satisfies (4.7). Equalities hold in (i) and (ii) for the functions given by (4.8) and (4.10), respectively.*

*Proof* : We only prove (i) since the proof of (ii) is similar. First, write

$$\begin{aligned}
&\frac{k+2p+1-2\alpha}{p-\alpha}\left[\frac{f_k(z)}{f(z)} - \frac{k+p+1-\alpha}{k+2p+1-2\alpha}\right]\\
&= \frac{1+\sum_{n=1}^{k} a_{p+n}z^n - \left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}{1+\sum_{n=1}^{\infty} a_{p+n}z^n}\\
&= \frac{1+w(z)}{1-w(z)},
\end{aligned}$$

where

$$w(z) = \frac{-\left(\frac{k+2p+1-2\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}{2+2\sum_{n=1}^{k} a_{p+n}z^n - \left(\frac{k+1}{p-\alpha}\right)\sum_{n=k+1}^{\infty} a_{p+n}z^n}.$$

Easily,

$$|w(z)| \leq \frac{\left(\frac{k+2p+1-2\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} |a_{p+n}|}{2-2\sum_{n=1}^{k} |a_{p+n}| - \left(\frac{k+1}{p-\alpha}\right)\sum_{n=k+1}^{\infty} |a_{p+n}|},$$

and $|w(z)| < 1$ implies

$$\sum_{n=1}^{\infty} |a_{p+n}| + \left(\frac{k+p+1-\alpha}{p-\alpha}\right)\sum_{n=k+1}^{\infty} |a_{p+n}| \leq 1. \tag{4.12}$$

Since the left hand side expression of (4.12) is bounded above by,

$\sum_{n=1}^{\infty}\left(\frac{p+n-\alpha}{p-\alpha}\right)|a_{p+n}| \leq 1$, the proof is complete.

**Theorem 4.2.4.** *If $f$ of the form (4.5), satisfies condition (4.6), then*

$$\begin{array}{lll} (i) & Re\left\{\dfrac{f'(z)}{f_k'(z)}\right\} \geq \dfrac{\alpha(k+1)}{p(k+p+1-\alpha)}, & (z \in \mathcal{U}) \\ (ii) & Re\left\{\dfrac{f_k'(z)}{f'(z)}\right\} \geq \dfrac{p(k+p+1-\alpha)}{2p(k+p+1-\alpha)-\alpha(k+1)}, & (z \in \mathcal{U}). \end{array}$$

*In both cases, the extremal function is given by (4.8).*

*Proof* : We prove only (i). Write

$$\begin{aligned} &\frac{p(k+1+p-\alpha)}{(k+p+1)(p-\alpha)}\left[\frac{f'(z)}{f_k'(z)} - \frac{\alpha(k+1)}{p(k+1+p-\alpha)}\right] \\ &= \frac{1+\sum_{n=1}^{k}\left(\frac{p+n}{p}\right)a_{p+n}z^n + \left(\frac{p(k+1+p-\alpha)}{(k+p+1)(p-\alpha)}\right)\sum_{n=k+1}^{\infty}\left(\frac{p+n}{p}\right)a_{p+n}z^n}{1+\sum_{n=1}^{k}\left(\frac{p+n}{p}\right)a_{p+n}z^n} \\ &= \frac{1+w(z)}{1-w(z)}, \end{aligned}$$

which on estimations the modulus of $w$ gives

$$|w(z)| \leq \frac{\left(\frac{p(k+1+p-\alpha)}{(k+p+1)(p-\alpha)}\right)\sum_{n=k+1}^{\infty}\left(\frac{p+n}{p}\right)|a_{p+n}|}{2-2\sum_{n=1}^{k}\left(\frac{p+n}{p}\right)|a_{p+n}| - \left(\frac{p(k+1+p-\alpha)}{(k+p+1)(p-\alpha)}\right)\sum_{n=k+1}^{\infty}\left(\frac{p+n}{p}\right)|a_{p+n}|}.$$

Thus, $|w(z)| \leq 1$ if and only if the following is true,

$$\sum_{n=1}^{k}\left(\frac{p+n}{p}\right)|a_{p+n}| + \left(\frac{p(k+1+p-\alpha)}{(k+p+1)(p-\alpha)}\right)\sum_{n=k+1}^{\infty}\left(\frac{p+n}{p}\right)|a_{p+n}| \leq 1. \quad (4.13)$$

Since the left hand side of (4.13) is bounded above by,

$\sum_{n=1}^{\infty}\left(\frac{p+n-\alpha}{p-\alpha}\right)|a_{p+n}|$, thus using (4.6), the proof is complete.

**Theorem 4.2.5.** *If $f$ of the form (4.5) satisfies condition (4.7), then*

$$(i) \quad Re\left\{\frac{f'(z)}{f'_k(z)}\right\} \geq \frac{k+1}{k+p+1-\alpha}, \quad (z \in \mathcal{U})$$
$$(ii) \quad Re\left\{\frac{f'_k(z)}{f'(z)}\right\} \geq \frac{k+p+1-\alpha}{k+2p+1-2\alpha}, \quad (z \in \mathcal{U}).$$

*In both cases, the extremal function is given by (4.10).*

*Proof* : It is well known that $f \in \mathcal{CV}_p(\alpha) \Leftrightarrow \frac{zf'}{p} \in \mathcal{ST}_p(\alpha)$. In particular, $f$ satisfies condition (4.7) if and only if $\frac{zf'}{p}$ satisfies condition (4.6). Thus, (i) is an immediate consequence of Theorem 4.2.1 and (ii) follows directly from Theorem 4.2.3(i).

## 4.3 Uniformly Convex & Parabolic Starlike Functions

Let $\mathcal{A}$ be the class consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}). \tag{4.14}$$

with its sequence of partial sums denoted by $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$.

Goodman (1991a/1991b) introduced the concepts of *uniformly convex functions*, $\mathcal{UCV}$ and *uniformly starlike functions*, $\mathcal{UST}$. The corresponding "uniform classes" are defined in the following way, by their geometrical mapping properties.

$$f \in \mathcal{UCV} \quad \Leftrightarrow \quad Re\left\{ 1 + (z-\xi)\frac{f''(z)}{f'(z)} \right\} \geq 0, \quad (z, \xi) \in \mathcal{U}.$$

$$f \in \mathcal{UST} \quad \Leftrightarrow \quad Re\left\{ \frac{f(z)-f(\xi)}{(z-\xi)f'(z)} \right\} \geq 0, \quad (z, \xi) \in \mathcal{U}.$$

Rönning (1993a) further found a more applicable one variable analytic characterization and further proved that a function $f$ of the form (4.14) is in $\mathcal{UCV}$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathcal{U}).$$

In the same paper, Rönning also introduced the class of parabolic starlike functions $\mathcal{PST}$ such that $f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{PST}$. It is established that

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathcal{U}).$$

Rönning (1993a/1993b) further generalized the classes $\mathcal{UCV}$ and $\mathcal{PST}$ by introducing a parameter $\alpha$ in the following way. A function $f \in \mathcal{A}$ is in $\mathcal{PST}(\alpha)$ if it satisfies the analytic characterization

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} - \alpha \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (-1 \leq \alpha \leq 1,\ z \in \mathcal{U}),$$

and $f \in \mathcal{UCV}(\alpha)$, the class of *uniformly convex functions of order* $\alpha$, if it satisfies

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} - \alpha \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad (-1 \leq \alpha \leq 1,\ z \in \mathcal{U}).$$

Determining bounds for the coefficients have always fascinated researchers. Bharati *et. al.* (1997) obtained coefficient properties for the various generalized related classes. A sufficient condition for $f$ of the form (4.14) to be in $\mathcal{PST}(\alpha)$, $(-1 \leq \alpha < 1)$ is given by

$$\sum_{n=2}^{\infty}(2n-1-\alpha)|a_n| \leq 1-\alpha, \tag{4.15}$$

and for $f \in \mathcal{UCV}(\alpha)$, $(-1 \leq \alpha < 1)$ is that the following condition is true

$$\sum_{n=2}^{\infty} n(2n-1-\alpha)|a_n| \leq 1-\alpha. \tag{4.16}$$

Further works by several other authors which provide interesting developments concerning partial sums of analytic functions can be found in Aouf (2006), Aouf (2009) and Cho (2004). Motivated by Silverman's work in Silverman (1997), we establish the lower bounds for Re $(f(z)/f_k(z))$, Re $(f_k(z)/f(z))$, Re $(f'(z)/f_k'(z))$ and Re $(f_k'(z)/f'(z))$. The known result that Re $\left(\frac{1+w(z)}{1-w(z)}\right) > 0$, $(z \in \mathcal{U})$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|w(z)| \leq |z|$ is used to obtain bounds for the above ratios.

**Theorem 4.3.1.** *If $f$ is of the from (4.14) and satisfies condition (4.15), then*

$$Re\left(\frac{f(z)}{f_k(z)}\right) \geq \frac{2k}{2k+1-\alpha}.$$

*Proof* : Consider,

$$\begin{aligned}\frac{2k+1-\alpha}{1-\alpha}&\left[\frac{f(z)}{f_k(z)} - \frac{2k}{2k+1-\alpha}\right] \\ &= \frac{1+\sum_{n=2}^{k} a_n z^{n-1} + \sum_{n=k+1}^{\infty}\left(\frac{2k+1-\alpha}{1-\alpha}\right) a_n z^{n-1}}{1+\sum_{n=2}^{k} a_n z^{n-1}} \\ &= \frac{1+A(z)}{1+B(z)}.\end{aligned}$$

Writing $\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)}$ we obtain,

$$w(z) = \frac{\sum_{n=k+1}^{\infty}\left(\frac{2k+1-\alpha}{1-\alpha}\right) a_n z^{n-1}}{2+2\sum_{n=2}^{k} a_n z^{n-1} + \sum_{n=k+1}^{\infty}\left(\frac{2k+1-\alpha}{1-\alpha}\right) a_n z^{n-1}}.$$

For $|z| = r < 1$, we have

$$|w(z)| \leq \frac{\sum_{n=k+1}^{\infty} \left(\frac{2k+1-\alpha}{1-\alpha}\right) |a_n|}{2 - 2\sum_{n=2}^{k} |a_n| - \sum_{n=k+1}^{\infty} \left(\frac{2k+1-\alpha}{1-\alpha}\right) |a_n|}.$$

To establish $|w(z)| \leq 1$, it is required to show that the following

$$\sum_{n=2}^{k} |a_n| + \sum_{n=k+1}^{\infty} \left(\frac{2k+1-\alpha}{1-\alpha}\right) |a_n|$$

is bounded by $\sum_{n=2}^{\infty} \left(\frac{2n-1-\alpha}{1-\alpha}\right) |a_n|$. This is evident since

$$\sum_{n=2}^{k} 2\left(\frac{n-1}{1-\alpha}\right) |a_n| + \sum_{n=k+1}^{\infty} 2\left(\frac{n-k-1}{1-\alpha}\right) |a_n| \geq 0.$$

This completes the proof.

**Theorem 4.3.2.** *If $f$ of the form (4.14) satisfies condition (4.16), then*

$$Re\left(\frac{f(z)}{f_k(z)}\right) \geq \frac{k(2k+3-\alpha)}{(k+1)(2k+1-\alpha)}.$$

*Proof* : We write

$$\frac{(k+1)(2k+1-\alpha)}{1-\alpha}\left[\frac{f(z)}{f_k(z)} - \frac{k(2k+3-\alpha)}{(k+1)(2k+1-\alpha)}\right]$$

$$= \frac{1 + \sum_{n=2}^{k} a_n z^{n-1} + \sum_{n=k+1}^{\infty} \left(\frac{(k+1)(2k+1-\alpha)}{1-\alpha}\right) a_n z^{n-1}}{1 + \sum_{n=2}^{k} a_n z^{n-1}}$$

$$= \frac{1+w(z)}{1-w(z)}$$

where

$$w(z) = \frac{\sum_{n=k+1}^{\infty} \left(\frac{(k+1)(2k+1-\alpha)}{1-\alpha}\right) a_n z^{n-1}}{2 + 2\sum_{n=2}^{k} a_n z^{n-1} + \sum_{n=k+1}^{\infty} \left(\frac{(k+1)(2k+1-\alpha)}{1-\alpha}\right) a_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\sum_{n=k+1}^{\infty}\left(\frac{(k+1)(2k+1-\alpha)}{1-\alpha}\right)|a_n|}{2-2\sum_{n=2}^{k}|a_n| - \sum_{n=k+1}^{\infty}\left(\frac{(k+1)(2k+1-\alpha)}{1-\alpha}\right)|a_n|}.$$

Next, we show that

$$\sum_{n=2}^{k}|a_n| + \sum_{n=k+1}^{\infty}\left(\frac{(k+1)(2k+1-\alpha)}{1-\alpha}\right)|a_n|$$

is bounded above by $\sum_{n=2}^{\infty}\frac{n(2n-1-\alpha)}{1-\alpha}|a_n|$. Since, it is evident that

$$\begin{aligned}\sum_{n=2}^{k}\frac{n(2n-1-\alpha)}{1-\alpha}|a_n| &= \sum_{n=2}^{k}|a_n| + \sum_{n=2}^{k}\frac{(n-1)(2n+1-\alpha)}{1-\alpha}|a_n| \\ &> \sum_{n=2}^{k}|a_n|\end{aligned}$$

and

$$\begin{aligned}\sum_{n=k+1}^{\infty}\frac{n(2n-1-\alpha)}{1-\alpha}|a_n| &= \sum_{n=k+1}^{\infty}\frac{(n-1)(2n+1-\alpha)}{1-\alpha}|a_n| + \sum_{n=k+1}^{\infty}|a_n| \\ &> \sum_{n=k+1}^{\infty}\frac{(k+1)(2k+1-\alpha)}{1-\alpha}|a_n|\end{aligned}$$

hence, using the fact that (4.16) is true, thus we establish $|w(z)| \leq 1$.

**Theorem 4.3.3.** *Given $f$ is of the form (4.14), then*

$$\begin{aligned}&(i) \quad Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{2k+1-\alpha}{2(k+1-\alpha)}, \quad (z \in \mathcal{U}) \\ &(ii) \quad Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{(k+1)(2k+1-\alpha)}{(k+1)(2k+1-\alpha)+(1-\alpha)}\end{aligned}$$

*where $f$ in (i) satisfies condition (4.15) and $f$ in (ii) satisfies (4.16).*

*Proof* : We prove (i) since the proof of (ii) is similar. First, write

$$\frac{2(k+1-\alpha)}{1-\alpha}\left[\frac{f_k(z)}{f(z)}-\frac{2k+1-\alpha}{2(k+1-\alpha)}\right]$$
$$=\frac{1+\sum_{n=2}^{k}a_nz^{n-1}-\sum_{n=k+1}^{\infty}\left(\frac{2k+1-\alpha}{1-\alpha}\right)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}}$$
$$=\frac{1+w(z)}{1-w(z)},$$

where

$$w(z)=\frac{-\sum_{n=k+1}^{\infty}\left(\frac{2(k+1-\alpha)}{1-\alpha}\right)a_nz^{n-1}}{2+2\sum_{n=2}^{k}a_nz^{n-1}-\sum_{n=k+1}^{\infty}\left(\frac{2k}{1-\alpha}\right)a_nz^{n-1}}$$

and

$$|w(z)|\le\frac{\sum_{n=k+1}^{\infty}\left(\frac{2(k+1-\alpha)}{1-\alpha}\right)|a_n|}{2-2\sum_{n=2}^{k}|a_n|-\sum_{n=k+1}^{\infty}\left(\frac{2k}{1-\alpha}\right)|a_n|}.$$

Finally, $|w(z)|\le 1$ because

$$\sum_{n=2}^{k}|a_n|+\sum_{n=k+1}^{\infty}\left(\frac{2k+1-\alpha}{1-\alpha}\right)|a_n|$$

is bounded by $\sum_{n=2}^{\infty}\left(\frac{2n-1-\alpha}{1-\alpha}\right)|a_n|$ and using the hypothesis the proof is complete.

**Theorem 4.3.4.** *If $f$ of the form (4.14) satisfies condition (4.15), then*

$$(i)\quad Re\left(\frac{f'(z)}{f_k'(z)}\right)\ge\frac{k(1+\alpha)}{2k+1-\alpha},\quad (z\in\mathcal{U})$$
$$(ii)\quad Re\left(\frac{f_k'(z)}{f'(z)}\right)\ge\frac{2k+1-\alpha}{(2k+1-\alpha)+(1-\alpha)(k+1)},\quad (z\in\mathcal{U}).$$

*Proof* : We prove only (i). Let

$$\frac{2k+1-\alpha}{(k+1)(1-\alpha)}\left(\frac{f'(z)}{f_k'(z)}-\frac{k(1+\alpha)}{2k+1-\alpha}\right)$$
$$=\frac{1+\sum_{n=2}^{k}na_nz^{n-1}+\left(\frac{2k+1-\alpha}{(k+1)(1-\alpha)}\right)\sum_{n=k+1}^{\infty}na_nz^{n-1}}{1+\sum_{n=2}^{k}na_nz^{n-1}}$$
$$=\frac{1+w(z)}{1-w(z)}$$

where

$$w(z) = \frac{\left(\frac{2k+1-\alpha}{(k+1)(1-\alpha)}\right)\sum_{n=k+1}^{\infty} na_n z^{n-1}}{2 + 2\sum_{n=2}^{k} na_n z^{n-1} + \left(\frac{2k+1-\alpha}{(k+1)(1-\alpha)}\right)\sum_{n=k+1}^{\infty} na_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\left(\frac{2k+1-\alpha}{(k+1)(1-\alpha)}\right)\sum_{n=k+1}^{\infty} n|a_n|}{2 - 2\sum_{n=2}^{k} n|a_n| - \left(\frac{2k+1-\alpha}{(k+1)(1-\alpha)}\right)\sum_{n=k+1}^{\infty} n|a_n|}.$$

To establish $|w(z)| \leq 1$, it is similar to showing that

$$\sum_{n=2}^{k} n|a_n| + \left(\frac{2k+1-\alpha}{(k+1)(1-\alpha)}\right)\sum_{n=k+1}^{\infty} n|a_n| \leq 1.$$

Obviously, since

$$\begin{aligned}\sum_{n=2}^{k} \frac{2n-1-\alpha}{1-\alpha}|a_n| &= \sum_{n=2}^{k} n|a_n| + \sum_{n=2}^{k} \frac{(n-1)(1+\alpha)}{1-\alpha}|a_n| \\ &> \sum_{n=2}^{k} n|a_n|,\end{aligned}$$

and

$$\begin{aligned}\sum_{n=k+1}^{\infty} \frac{2n-1-\alpha}{1-\alpha}|a_n| &= \sum_{n=k+1}^{\infty} n|a_n| + \sum_{n=k+1}^{\infty} \frac{(n-1)(1+\alpha)}{1-\alpha}|a_n| \\ &> \sum_{n=k+1}^{\infty} \frac{2k+1-\alpha}{(k+1)(1-\alpha)} n|a_n|.\end{aligned}$$

Hence,

$$\sum_{n=2}^{\infty} \frac{2n-1-\alpha}{1-\alpha}|a_n| > \sum_{n=2}^{k} n|a_n| + \sum_{n=k+1}^{\infty} \frac{2k+1-\alpha}{(k+1)(1-\alpha)n|a_n|},$$

which by hypothesis proves the result.

**Theorem 4.3.5.** *If $f$ of the form (4.14) and satisfies condition (4.16), then*

$$(i) \quad Re\left(\frac{f'(z)}{f_k'(z)}\right) \geq \frac{2k}{2k+1-\alpha}, \quad (z \in \mathcal{U}),$$
$$(ii) \quad Re\left(\frac{f_k'(z)}{f'(z)}\right) \geq \frac{2k+1-\alpha}{2(k+1-\alpha)}, \quad (z \in \mathcal{U}).$$

$Proof$ : Since is known that $f \in \mathcal{UCV}(\alpha) \Leftrightarrow zf' \in \mathcal{PST}(\alpha)$, therefore, $f$ satisfies condition (4.15) if and only if $zf'$ satisfies condition (4.16). Thus, (i) is an immediate consequence of Theorem 4.3.1 and (ii) follows directly from Theorem 4.3.3(i).

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