CHAPTER ONE

INTRODUCTION

1.1 Introduction

The differential equations (DE) appear naturally in many areas of science and humanities. Ordinary differential equations (ODE) have found a wide range of application in biological, physical, social and engineering systems which are dynamic in character. They can be used effectively to analyze the evolutionary trend of such systems, they also aid in the formulation of these systems and the qualitative examination of their stability and adaptability to external stimuli. Many phenomena in different branches of sciences are interpreted in terms of second order DE and their solutions. For example, the so–called Emden-Fowler differential equation arises in the study of gas dynamics and fluid mechanics. This equation appears also in the study of chemically reacting systems.

Since the classic work of Atkinson (1955), there has much interest in the problem of determining oscillation criteria for second order non-linear DE. The study of the oscillation of second–order nonlinear ODE's with alternating coefficients is of special interest because of the fact that many physical systems are modeled by second order nonlinear ODE. Some of the most important and useful tests have involved the average behavior of the integral of the alternating coefficient. These tests have been motivated by the averaging criterion of Kameneve (1978) and its generalizations. The use of averaging functions in the study of oscillation dates back to the work of Wintner (1949) and Hartman (1952).

Although differential equations of second-order have been studied extensively, the study of qualitative behavior of third-order differential equations has received considerably less attention in the literature, however certain results for third-order differential equations have been known for a long time and their applications in mathematical modeling in biology and physics. In 1961 Hanan (1961) studied the oscillation and non-oscillation of two different types of third order differential equations and gave definitions of two types of the solutions. The paper was the starting point for many investigations to the asymptotic behavior of third-order equations.

The purpose of this thesis is to study the problem of oscillation of second order non-linear ordinary differential equations of the form

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)\Phi\left(g(x(t)), r(t)\dot{x}(t)\right) = H(t, x(t)) \tag{1.1}$$

and oscillation of second non-linear equations with damping term of the form

$$\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + h(t)\dot{x}(t) + q(t)\Phi\left(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)\right) = H\left(t, x(t), \dot{x}(t)\right), \tag{1.2}$$

where r, h and q are continuous functions on the interval $[t_0, \infty)$, $t_0 \geq 0$, $r(t)$ is a positive function and $\Psi \in C(R, R^+)$, g is continuously differentiable function on the real line R except possibly at 0 with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$, Φ is a continuous function on RxR with $u\Phi(u,v) > 0$ for all $u \neq 0$ and $\Phi(\lambda u, \lambda v) = \lambda\Phi(u,v)$ for any $\lambda \in (0,\infty)$ and H is a continuous function on $[t_0,\infty)\times\mathrm{R}\times\mathrm{R}$ with $H(t, x(t), \dot{x}(t))/g(x(t)) \leq p(t)$ for all $x \neq 0$ and $t \geq t_0$.

The thesis also deals study of problem of oscillation of third order non-linear equations of the form

$$\left(r(t)\,f(\ddot{x}(t))\right)^{\bullet}+q(t)g_1(\dot{x}(t)) = H\left(t,x(t),\dot{x}(t),\ddot{x}(t)\right), \qquad (1.3)$$

where q and r are defined as above, g_1 is continuously differentiable function on the real line R except possibly at 0 with $yg_1(y)>0$ and $g_1'(y)\geq k>0$ for all $y\neq 0$, f is a continuous function on R and $H_1:[t_0,\infty)\times R\times R\times R\to R$ is a continuous function such that $H_1(t,x,y,z)/g_1(y)\leq p(t)$ for all $y\neq 0$ and $t\geq t_0$.

We list some basic definitions and Elementary results which will be needed in the next chapters.

1.2 The Basic Definitions

Definition 1.2.1

A point $t=\tau\geq 0$ is called a zero of the solution $x(t)$ of the differential equation if $x(\tau)=0$.

Definition 1.2.2

A solution $x(t)$ of the differential equation is said to be oscillatory if it has arbitrary large zeros. Otherwise it is said to be non-oscillatory.

Definition 1.2.3

Differential equation is called oscillatory if all its solutions are oscillatory. Otherwise it is called non oscillatory.

Definition 1.2.4

The differential equations (1.1) and (1.2) are called

(1) Sub-linear if the function g satisfies that

$$0<\int_{0}^{\varepsilon}\frac{du}{g(u)}<\infty \quad and \quad 0<\int_{0}^{-\varepsilon}\frac{du}{g(u)}<\infty \quad for\ all\ \varepsilon>0$$

(2) Super-linear if the function g satisfies that

$$0<\int_{\varepsilon}^{\infty}\frac{du}{g(u)}<\infty \quad and \quad 0<\int_{-\varepsilon}^{-\infty}\frac{du}{g(u)}<\infty \quad for\ all\ \varepsilon>0$$

(3) A mixed type if the function g satisfies that

$$0<\int_{0}^{\infty}\frac{du}{g(u)}<\infty \quad and \quad 0<\int_{0}^{-\infty}\frac{du}{g(u)}<\infty.$$

1.3 Elementary Results

The following theorems play an important role in the theory of oscillation of the solutions of the linear differential equations:

1.3.1 Sturm's Comparison Theorem (Bartle (1970))

Let $q_1(t), q_2(t)$ and $r(t)$ be continuous functions on (a,b) and $r(t)>0$ on (a,b). Assume that $x_1(t)$ and $x_2(t)$ are real solutions of

$$\left(r(t)\dot{x}(t)\right)^{\bullet}+q_1(t)\ x(t)=0 \tag{1.4}$$

and

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q_2(t)\, x(t) = 0 \qquad (1.5)$$

respectively on (a,b). Further, let $q_2(t) \geq q_1(t)$ for $t \in (a,b)$. Then, between any two consecutive zeros t_1, t_2 of $x_1(t)$ in (a,b), there exists at least one of $x_2(t)$ unless $q_1(t) = q_2(t)$ on $[t_1, t_2]$. Moreover, in this case the conclusion is still true if the solution $x_2(t)$ is linearly independent of $x_1(t)$.

1.3.2 Sturm's Separation Theorem (Bartle (1970))

If $x_1(t)$ and $x_2(t)$ are linearly independent solutions of the equation

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)\, x(t) = 0. \qquad (1.6)$$

Then, between any two consecutive zeros of $x_1(t)$, there is precisely one zero of $x_2(t)$. Therefore the solutions of the second order linear differential equations are either all oscillatory or all non-oscillatory. The story of non-linear equations is not the same. The nonlinear differential equations may have both oscillatory solutions.

The importance of classification of the second order differential equations into oscillatory categories is due to the following well-known fact: A non – trivial solution of the second order ordinary differential equation must change its sign whenever it vanishes, since $x(t)$ and $\dot{x}(t)$ cannot vanish simultaneously (in this case the zeros of $x(t)$ are said to be isolated).

The following theorem is quite useful element of our study in the following chapters:

1.3.3 The Bonnet's Theorem (Ross (1984))

Suppose that h is a continuous function on [a,b], ρ is a non- negative function and an increasing function on the interval [a,b]. Then there exists a point c in [a,b] such that

$$\int_a^b \rho(s)\,h(s)\;ds = \rho(b)\int_c^b h(s)\,ds.$$

If ρ is a decreasing function on [a,b], then there exists a point c in [a,b] such that

$$\int_a^b \rho(s)\,h(s)\;ds = \rho(a)\int_a^c h(s)\,ds$$

This theorem is a part of the second mean value theorem of integrals (Ross (1984)).

1.4 Riccati Technique

In the study of oscillation theory of differential equations, there are two techniques which are used to reduce the higher-order equations to the first-order Riccati equation or inequality. The first one is the Riccati transformation technique. The second one is called the generalized Riccati technique. This technique can introduce some new oscillation criteria and can be applied to different equations which cannot be covered by the results established by the Riccati technique.

Riccati Transformation Technique:

(1) If $x(t)$ is a non-vanishing solution of equation (1.6) on the interval (a,b), then $\omega(t) = r(t)\dot{x}(t)/x(t)$ is a solution of

$$\dot{\omega}(t) + q(t) + r^{-1}(t)\omega^2(t) = 0 \quad for\, t \in (a,b) \tag{1.7}$$

(2) If $\omega(t)$ is a solution of equation (1.7) on (a,b), then

$$x(t) = e^{\left[\int_a^b \omega(s) r^{-1}(s) ds\right]}$$

is a non-vanishing solution of equation (1.6) on (a,b).

1.5 Applications Of Oscillatory Differential Equations

Ordinary differential equations have a variety of applications in science, mechanical engineering, aerospace engineering and physical systems, we explore three of them: Undamped simple pendulum, damped simple pendulum and a half cylinder rolling on a horizontal plane.

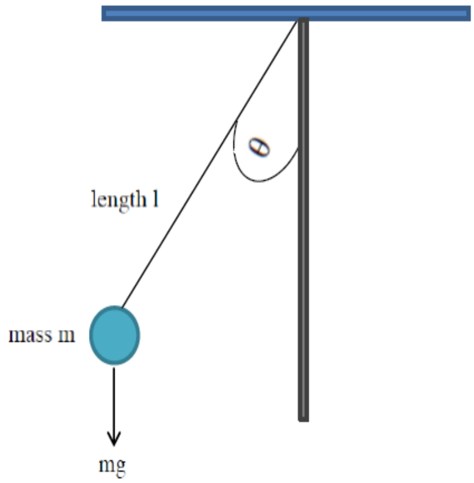
Example 1.1

Consider a pendulum with mass *m* at the end of a rigid rod of length *L* attached to say a fixed frictionless pivot which allows the pendulum to move freely under gravity in the vertical plane as illustrated in Figure 1.1. The angular equation of motion of the pendulum is given as a nonlinear differential equation

$$\frac{d^2\theta}{dt^2} + K \sin\theta = 0, \tag{1.8}$$

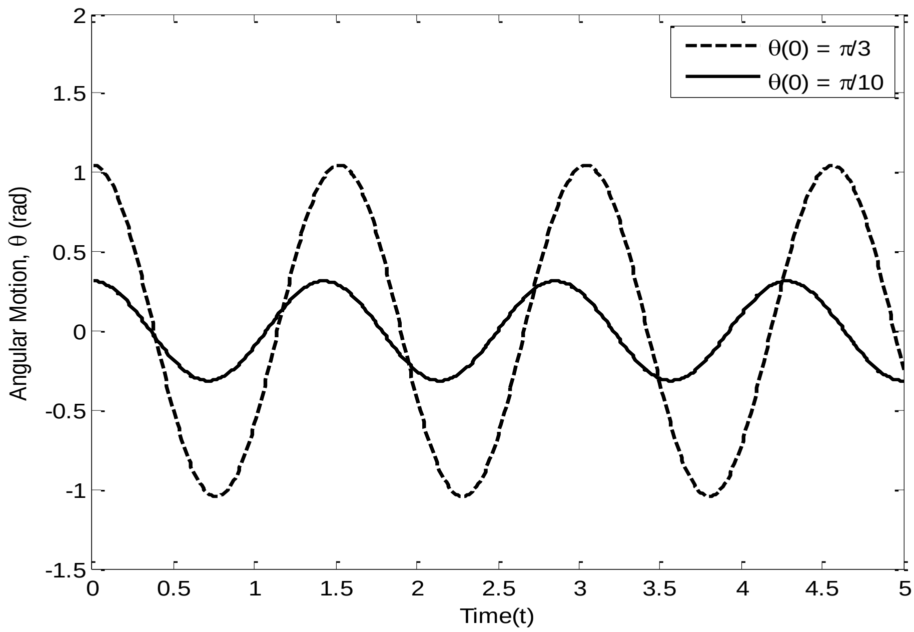
where K = g/L.

Figure 1.1: Simple pendulum



The numerical solutions curves of the equation (1.8) when m=100, g =9.81 and L=1 and for different initial angles π/3 and π/10 with zero initial velocity are

Figure 1.2: Numerical solutions of $d^2\theta/dt^2 + K\sin\theta = 0$



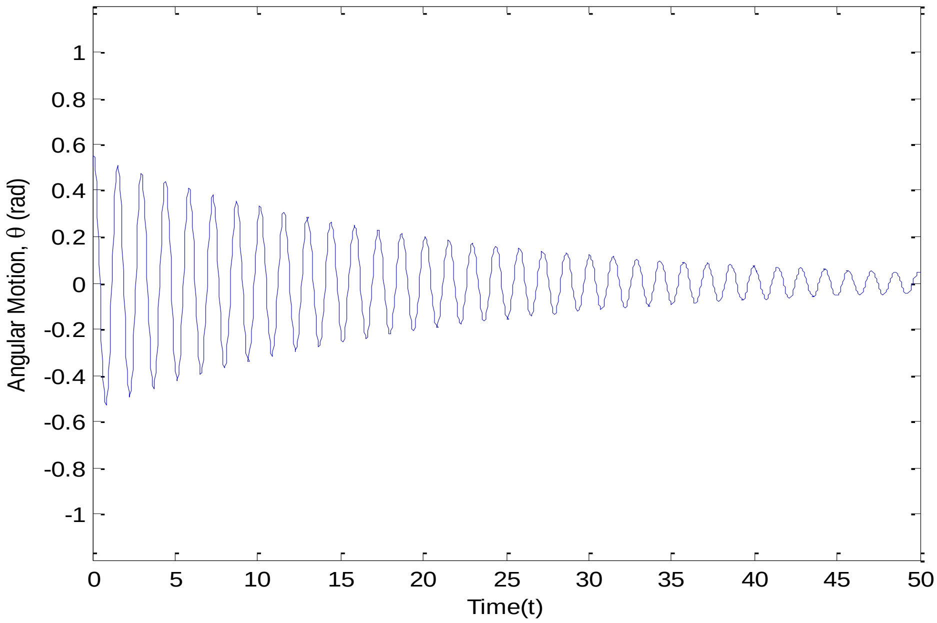
Example 1.2

As in example 1.1, we consider the motion of a simple pendulum that subject to a frictional force or damping force. We assume that the damping coefficient α, so the nonlinear differential equation represents this motion as follows:

$$\frac{d^2\theta}{dt^2} + \alpha \frac{d\theta}{dt} + K \sin\theta = 0, \qquad (1.9)$$

The numerical solution curve of equation (1.9) when $\alpha = 0.1$ is

Figure 1.3: Numerical solution of $d^2\theta / dt^2 + \alpha\, d\theta / dt + K \sin\theta = 0$



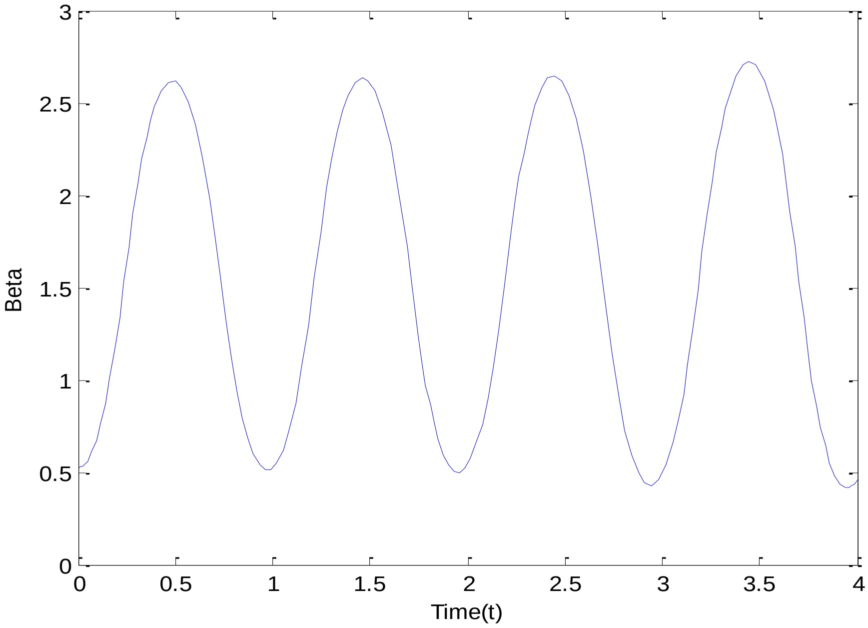
Example 1.3

The general governing differential equation of motion of the half cylinder rolling on a horizontal plane is

$$\left(\frac{3mr^2}{2} - 2mEr\sin(x(t))\right)\ddot{x}(t) - mEr\cos(x(t))\,\dot{x}^2(t) = \beta E\cos(x(t)).$$

The numerical solution curve of this equation when m =4, $E = {}^{4r}\!/_{3\pi}$ and r = 0.1 is

Figure 1.4: Solution curve of half cylinder rolling on a horizontal plane.



1.6 Thesis Organization

Besides the introductory chapter (chapter 1) about the oscillation of second and third order non-linear ordinary differential equations, the thesis is organized as follows:

Chapter 2: This chapter will contain the literature review of the main results of the oscillation of second and third order ordinary differential equations which are given in the literature.

Chapter 3: This chapter is devoted to study of the oscillation of the second order equation (1.1) and contains some oscillation criteria for oscillation equation (1.1). The oscillation results obtained will be illustrated by some examples and their numerical solutions that are found by using Runge Kutta method of fourth order.

Chapter 4: This chapter is devoted to study of the oscillation of the second order equation with damping term (1.2). Some sufficient conditions for oscillation equation (1.2) will be given in this chapter. The oscillation results obtained will be illustrated by some examples and their numerical solutions that are found by using Runge Kutta method of fourth order.

Chapter 5: this chapter is concerned with oscillation of third order ordinary differential equation (1.3) and includes oscillation results for oscillation of equation (1.3) and an illustrative example with its numerical solution obtained by using Runge Kutta method of fourth order for these results presented.

Chapter 6: This chapter contains the conclusion with suggestions for future work and references.

CHAPTER TWO

LITERATURE REVIEW

In this Chapter, we will review the literature within the context of our study of oscillations of ODEs of the 2nd Order and the 3rd Order. We will see that most of the previous oscillation results depend on the behavior of the integral of the coefficients and a reduction of order of the ODEs and using the Riccati technique to establish some sufficient conditions. Our results improve and extend almost of these existing results in the literature.

2.1 Oscillations Of Second Order Differential Equations

Oscillatory and non-oscillatory behavior of solutions for various classes of second order has been studied extensively in literature as Atkinson (1955), Bihari (1963), Bhatia (1966), Grace (1992), Ayanlar & Tiryaki (2000), Elabbasy et al. (2005), Lee & Yeh (2007), Berkani (2008), Remili (2010). Various researchers have studied particular cases of the equations (1.1) and (1.2). These particular cases can be classified as follows:

The homogeneous linear equations

$$\ddot{x}(t)+q(t)x(t)=0. \tag{2.1}$$

$$\left(r(t)\dot{x}(t)\right)^{\bullet}+q(t)\,x(t)=0. \tag{2.2}$$

The homogeneous non-linear equations

$$\ddot{x}(t) + q(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = 0, \ \gamma > 0. \tag{2.3}$$

$$\ddot{x}(t) + q(t)g(x(t)) = 0. \tag{2.4}$$

$$\ddot{x}(t) + q(t)\ \Phi(x(t), \dot{x}(t)) = 0. \tag{2.5}$$

$$\left(r(t)\dot{x}(t) \right)^{\bullet} + q(t)\ g(x(t)) = 0. \tag{2.6}$$

$$\left(r(t)\Psi(x(t))\dot{x}(t) \right)^{\bullet} + q(t)\ g(x(t)) = 0. \tag{2.7}$$

The homogeneous non-linear equations with damping term

$$\ddot{x}(t) + h(t)\dot{x}(t) + q(t)g(x(t)) = 0. \tag{2.8}$$

$$\left(r(t)\dot{x}(t) \right)^{\bullet} + h(t)\dot{x}(t) + q(t)g(x(t)) = 0. \tag{2.9}$$

$$\left(r(t)\Psi(x(t))\dot{x}(t) \right)^{\bullet} + h(t)\dot{x}(t) + q(t)\ g(x(t)) = 0. \tag{2.10}$$

The non-homogeneous non-linear equations

$$\left(r(t)\dot{x}(t) \right)^{\bullet} + \Phi(t, x(t)) = H(t, x(t), \dot{x}(t)). \tag{2.11}$$

$$\left(r(t)\Psi(x(t))\ \dot{x}(t) \right)^{\bullet} + q(t)\ g(x(t)) = H(t). \tag{2.12}$$

The non-homogeneous non-linear equations with damping term

$$\left(r(t)\Psi(x(t))\dot{x}(t) \right)^{\bullet} + h(t)\dot{x}(t) + q(t)g(x(t)) = H(t), \tag{2.13}$$

where, the functions $r, \Psi, h, g, q,\ \Phi$ and H are defined as in the equation (1.2).

The investigation of the oscillation of (1.1) and (1.2) may be done by following many criteria. Many of these criteria depend on determining integral tests involving the function q to obtain oscillation criteria.

For convenience of writing, we adopt the following notations: $\int^{\infty}$ is written, it is to be assumed that $\int^{\infty} = \lim_{t\to\infty} \int^{t}$ and that this limit exists in the extended real numbers.

2.1.1 Oscillation Of Homogenous Linear Equations

2.1.1.1 Oscillation Of Equations Of Type (2.1)

This section is devoted to the oscillation criteria for the second order linear differential equation of the form (2.1) . The oscillation of equation (2.1) has brought the attention of many authors as Wintner (1949), Kamenev (1978), Philos (1983) and Yan (1986), since the early paper by Fite (1918). Among the numerous papers dealing with this subject we refer in particular to the following :

Theorem 2.1.1 Fite (1918)

If $q(t) > 0$ for all $t \ge t_0$ and

$$\int_{t_0}^{\infty} q(s)\, ds = \infty,$$

then, every solution of the equation (2.1) is oscillatory. The following theorem extended the result of Fite (1918) to an equation in which q is of arbitrary sign.

Theorem 2.1.2 Wintner (1949)

Suppose that

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\ du\, ds = \infty.$$

Then, every solution of the equation (2.1) is oscillatory.

Example 2.1.1

Consider the following differential equation

$$\ddot{x}(t) + (2 - 3\cos t)x(t) = 0,\ t \geq 0.$$

Theorem 2.1.2 ensures that the given equation is oscillatory, however Theorem 2.1.1 fails.

Hartman (1952) also studied the equation (2.1) and improved Wintner's result (1949) by proving the condition given in Theorem 2.1.3.

Theorem 2.1.3 Hartman (1952)

Suppose that

$$-\infty < \liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\ du\, ds < \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\ du\, ds \leq \infty,$$

then, equation (2.1) is oscillatory. In the following, Kamenev (1978) has proved a new integral criterion for the oscillation of the differential equation (2.1) based on the use of the n^{th} primitive of the coefficient $q(t)$ which has Wintner's result (1949) as a particular case.

Theorem 2.1.4 Kamenev (1978)

The equation (2.1) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds = \infty,$$

for some integer $n \geq 3$. Philos (1983) improved the above Kamenev's result (1978).

Theorem 2.1.5 Philos (1983)

Let ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that

(*i*) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{(t-s)^{n-3}}{\rho(s)} \left[(n-1)\rho(s) - (t-s)\, \dot{\rho}(s) \right]^2 ds < \infty$ for some integer $n \geq 3$,

(*ii*) $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s)\, q(s)\, ds = \infty.$

Then, every solution of the equation (2.1) is oscillatory.

Remark 2.1.1: By setting $\rho(t) = 1$ in the above Theorem 2.1.5, Theorem 2.1.5 leads to Kamenev's Result (1978) (Theorem 2.1.4).

Yan (1986) presented another new oscillation theorem for equation (2.1).

Theorem 2.1.6 Yan (1986)

Suppose that there exists an integer $n \geq 3$ with

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds < \infty.$$

Let $\Omega(t)$ be a continuous function on $[t_0, \infty)$ with

$$\liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \; q(s) \; ds \geq \Omega(T) \quad \textit{for every } T \geq t_0 .$$

Then equation (2.1) is oscillatory if

$$\int_{t_0}^{\infty} \Omega_+(s) \; ds = \infty ,$$

where $\Omega_+(t) = \max \{\Omega(t), 0\}, \; t \geq t_0$.

Also, Philos (1989) extended the Kamenev's result (1978) as follows

Theorem 2.1.7 Philos (1989)

Let H and h be two continuous functions $h, H : D = \{(t,s) : t \geq s \geq t_0\} \to \mathrm{R}$ and H has a continuous and non-positive partial derivative on D with respect to the second variable such that $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ *for* $t > s \geq t_0$ and

$$-\frac{\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D.$$

Then, equation (2.1) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s) q(s) - \frac{1}{4} h^2(t,s) \right) ds = \infty .$$

Also, Philos (1989) extended and improved Yan's result (1986) in the following theorem:

Theorem 2.1.8 Philos (1989)

Let H and h be as in Theorem 2.1.7, moreover, suppose that

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\leq\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}h^2(t,s)\;ds<\infty.$$

Assume that $\Omega(t)$ as in Theorem 2.1.6 with

$$\int_{t_0}^{\infty}\Omega_+^2(s)ds=\infty.$$

Then, the equation (2.1) is oscillatory if

$$\limsup_{t\to\infty}\;\frac{1}{H(t,T)}\int_{T}^{t}\left(H(t,s)\,q(s)-\frac{1}{4}h^2(t,s)\right)ds\geq\Omega(T)\;\; for\;\; every\; T\geq t_0.$$

2.1.1.2 Oscillation Of Equation Of Type (2.2)

This section is devoted to the study of the oscillation of the equation of type (2.2). It is interesting to discuss conditions for the alternating coefficient $q(t)$ which are sufficient for all solutions of equation (2.2) to be oscillated. An interesting case is that of finding oscillations criteria of equation (2.2) which involve the average behavior of the integral of q. The problem has received the attention of many authors in recent years as Moore

(1955) and Popa (1981). Among numerous papers dealing with such averaging techniques of the oscillation of equations of type (2.2), we mention the following: Moore (1955) gave the following oscillation criteria for equations of type (2.2).

Theorem 2.1.9 Moore (1955)

Suppose that the function ρ satisfies $\rho \in C^2[t_0, \infty)$, $\rho(t) > 0$

$$\int_{t_0}^{\infty} \frac{ds}{r(s)\rho^2(s)} = \infty$$

and

$$\int_{t_0}^{\infty} \rho(s)\left(\left(r(s)\dot{\rho}(s)\right)^{\bullet} + \rho(s)q(s)\right)ds = \infty.$$

Then, equation (2.2) is oscillatory.

In fact, Popa (1981) extended Kamenev's oscillation criterion (1978) to apply on equation of the form (2.2). He proved the following two theorems:

Theorem 2.1.10 Popa (1981)

If *r*(t) is bounded above and

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds = \infty,$$

where *n* is an integer and $n > 2$, then, the equation (2.2) is oscillatory.

Theorem 2.1.11 Popa (1981)

If $\frac{\dot{r}(t)}{r(t)}$ is bounded and

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \frac{q(s)}{r(s)} ds = \infty,$$

where *n* is an integer and $n > 2$, then the equation (2.2) is oscillatory.

2.1.2 Oscillation Of Homogenous Nonlinear Equations

2.1.2.1 Oscillation Of Equations Of Types (2.3) and (2.4)

This section is devoted to the oscillation criteria for the second order nonlinear differential equations of the form (2.3) and (2.4). The oscillation of equation (2.4) has brought the attention of many authors since the earliest work by Atkinson (1955). The equation (2.3) is also known Emden-Fowler equation (EF). Clearly equation (EF) is sub-linear if $\gamma < 1$ and super linear if $\gamma > 1$.

The oscillation problem for second order nonlinear differential equation is of particular interest. Many physical systems are modeled by nonlinear ordinary differential equations. For example, equation (EF) arises in the study of gas dynamics and fluid mechanics, nuclear physics and chemically reacting systems. The study of Emden–Fowler equation originates from earlier theorems concerning gaseous dynamics in astrophysics around the turn of the century. For more details for the equation we refer to the paper by Wong (1973) for a detailed account of second order nonlinear oscillation and its physical motivation. There has recently been an increase in studying the oscillation for equations (2.4) and (EF). We list some of more important oscillation criteria as follows.

The following theorem gives the necessary and sufficient conditions for oscillation of equation (2.4) with $g(x) = x^{2n+1}$, $n = 1,2,3,...$

Theorem 2.1.12 Atkinson (1955)

Suppose that $q(t) > 0$ on $[t_0, \infty)$ and

$$g(x) = x^{2n+1} \ , \ n = 1,2,...,$$

The equation (2.4) is oscillatory if

$$\int_{t_0}^{\infty} sq(s)ds = \infty.$$

Waltman (1965) extended Wintner's result (1949) for the equation (2.4) without any restriction on the sign of $q(t)$.

Theorem 2.1.13 Waltman (1965)

Suppose that

$$g(x) = x^{2n+1}, \ n = 1,2,...$$

and

$$\int_{t_0}^{\infty} q(s)ds = \infty.$$

Then every solution of equation (2.4) is oscillatory. Kiguradze (1967) established the following theorem for the Emden–Fowler equation (2.3).

Theorem 2.1.14 Kiguradze (1967)

The equation (2.3) is oscillatory for $\gamma > 1$ if

$$\int^{\infty} \rho(t) q(t)\, dt = \infty$$

for a continuous, positive and concave function $\rho(t)$. Wong (1973) extended Wintner's oscillation criteria (1949) to apply on the equation (2.3).

Theorem 2.1.15 Wong (1973)

Let $\gamma > 1$. Equation (2.3) is oscillatory if

$$\liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds = -\lambda > -\infty, \ \ \lambda > 0$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s) q(s) ds = \infty.$$

Onose (1975) proved the theorem of Wong (1973) (Theorem 2.1.15) for the sublinear Emden–Fowler differential equation and also study the extension of Wong's result (1973) to the more general super-linear differential equation of the form equation (2.4) as in the following three theorems:

Theorem 2.1.16 Onose (1975)

Suppose that

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} q(s)\,ds > -\lambda > -\infty,\ \lambda > 0,$

(2) $\limsup_{t\to\infty} \int_{t_0}^{t} q(s)\,ds = \infty,$

(3) $\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\,du\,ds = \infty.$

Then, the equation (2.3) is oscillatory for $0 < \gamma < 1$.

Theorem 2.1.17 Onose (1975)

Assume that

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} q(s)\,ds \ge 0,$

(2) $\limsup_{t\to\infty} \int_{t_0}^{t} q(s)ds = \infty.$

Then, the equation (2.4) is oscillatory.

Theorem 2.1.18 Onose (1975)

Suppose that

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} q(s)\,ds > -\lambda > -\infty,\ \lambda > 0,$

(2) $\limsup_{t\to\infty} \int_{t_0}^{t}\int_{t_0}^{s} q(u)\, duds = \infty.$

Then, the equation (2.4) is oscillatory.

Yeh (1982) established new integral criteria for the equation (2.4) which has Wintner's result (1949) as a particular case.

Theorem 2.1.29 Yeh (1982)

Suppose that

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s) ds = \infty,$$

for some integer *n > 2.* Then, the equation (2.4) is oscillatory.

Philos (1984) gave a new oscillation criteria for the differential equation (2-3) if $0 < \gamma < 1$.

Theorem 2.1.20 Philos (1984)

Let ρ be a positive continuous differentiable function on the interval $[t_0, \infty)$ such that

$$\gamma\rho(t)\ddot{\rho}(t) + (t-\gamma)\dot{\rho}^2(t) \le 0 \quad \textit{for all } t \ge t_0.$$

Then, the equation (2.3) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \rho(s) q(s)\, ds = \infty \text{ for some integer } n \ge 2.$$

Philos (1985) improved Onose's result (1975) for equation (2.4).

Theorem 2.1.21 Philos (1985)

Suppose that ρ be a positive twice continuously differentiable function on $[t_0,\infty)$ such that

$$\dot{\rho}(t) \geq 0 \quad and \quad \ddot{\rho}(t) \leq 0 \quad on \quad [t_0,\infty),$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds > -\infty,$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)\rho(s)q(s)ds = \infty.$$

Then, the equation (2.4) is oscillatory.

Wong and Yeh (1992) improved Wong's result (1973) for equation (2.3) to the more general equation (2.4).

Theorem 2.1.22 Wong and Yeh (1992)

Suppose that

$$\liminf_{t\to\infty}\int_{T}^{t}q(s)\,ds \geq 0$$

for large $T \geq t_0$ and there exists a positive concave function ρ on $[t_0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^{t} (t-s)^{\beta} \rho(s)\, q(s)\, ds = \infty,$$

for some $\beta \geq 0$. Then the super-linear differential equation (2.4) is oscillatory.

Theorem 2.1.23 Philos and Purnaras (1992)

Suppose that

(1) $\liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\; ds > -\infty$ for some integer $n \geq 1$,

(2) $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left(\int_{t_0}^{s} q(u)\, du \right)^2 ds = \infty.$

Then, the super-linear differential equation (2.4) is oscillatory.

2.1.2.2 Oscillation Of Equations Of Type (2.5)

This section is devoted to the oscillation criteria for half-linear second order differential equations of the form (2.5). The oscillation of the equation has brought the attention of some authors since the early paper by Bihari (1963).

Theorem 2.1.24 Bihari (1963)

If $q(t) > 0$ *for all* $t \geq t_0$ and

$$\int_{t_0}^{\infty} q(s) ds = \infty,$$

then, every solution of the equation (2.5) is oscillatory. The following theorem extended the result of Bihari (1963) to an equation in which q is of arbitrary sign.

Theorem 2.1.25 Kartsatos (1968)

Suppose that

(i) There exists a constant $B^* > 0$ such that

$$G(m) = \int_0^m \frac{ds}{\Phi(1,s)} \geq -B^* \text{ for all } m \in \mathrm{R},$$

(ii) $\int_{t_0}^{\infty} q(s)ds = \infty.$

Then, every solution of equation (2.5) is oscillatory.

2.1.2.3 Oscillation Of Equations Of Type (2.6)

This section is devoted to the oscillation criteria for the second order nonlinear differential equation of the form (2.6). Bhatia (1966) presented the following oscillation criteria for the general equation (2.6) which contains as a special case of Waltman's result (1965) for the nonlinear case.

Theorem 2.1.26 Bhatia (1966)

Suppose that

(1) $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty,$

(2) $\int_{t_0}^{\infty} q(s)\, ds = \infty.$

Then, the equation (2.6) is oscillatory.

E. M. Elabbasy (1996) improved and extended the results of Philos (1983) to the equation (2.6).

Theorem 2.1.27 E. M. Elabbasy (1996)

Suppose that

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)q(s)ds > -\infty,$

(2) $\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\left(\int_{t_0}^{s} \rho(u)q(u)du\right)^2 ds = \infty,$

where $\rho:[t_0,\infty)\to(0,\infty)$ is continuously differentiable function such that

$$\dot{\rho}(t) \geq 0, \left(r(t)\rho(t)\right)^{\bullet} \geq 0,\ \left(r(t)\rho(t)\right)^{\bullet\bullet} \leq 0 \ \mathit{and} \ \left(r(t)\dot{\rho}(t)\right)^{\bullet} \leq 0.$$

Then, the equation (2.6) is oscillatory.

2.1.2.4 Oscillation Of Equations Of Type (2.7)

Oscillation of the equations of type (2.7) has been considered by many authors who presented some oscillation criteria for solutions of the equation (2.7). Grace (1992) studied the equation (2.7) and gave some sufficient conditions for oscillation of equation (2.7) in some theorems for example the following two theorems:

Theorem 2.1.28 Grace (1992)

Suppose that

(1) $\frac{g'(x)}{\Psi(x)} \geq k > 0$ for $x \neq 0$,

Moreover, there exists a differentiable function $\rho:[t_0,\infty) \to (0,\infty)$ and the functions h, H are defined as in Philos's result (1989) (in Theorem 2.1.7). Moreover, suppose that

(2) $$0 < \inf_{s \geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty,$$

(3) $$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s) \left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(t,s)} \right) ds < \infty.$$

If there exists a continuous function $\Omega(t)$ on $[t_0,\infty)$ such that

(4) $$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4k} \left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(t,s)} \right)^2 \right] ds \geq \Omega(T),$$

for every large $T \geq t_0$, and

(5) $$\lim_{t\to\infty} \int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{r(s)\rho(s)} ds = \infty,$$

where $\Omega_+(t) = \max\{\Omega(t), 0\}$, then, the equation (2.7) is oscillatory.

Theorem 2.1.29 Grace (1992)

Suppose that the condition (1) from Theorem 2.1.28 holds and functions h, H, ρ are defined as in Theorem 2.1.28 and $\dot{\rho}(t) \geq 0$ *and* $(r(t)\dot{\rho}(t))^{\bullet} \leq 0$ *for* $t \geq t_0$.

Moreover, suppose that

$$(1) \quad \liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)q(s)ds > -\infty,$$

and

$$(2) \quad \lim_{t\to\infty} \int_{t_0}^{\infty} \frac{ds}{r(s)\rho(s)} = \infty.$$

Then, the equation (2.7) is oscillatory if the conditions (4) and (5) hold.

2.1.3 Oscillation Of Homogenous Nonlinear Equations With Damping Term

2.1.3.1 Oscillation Of Equations Of Type (2.8)

In last three decades, oscillation of nonlinear differential equations with damping term has become an important area of research due to the fact that such equations appear in many real life problems. Oscillation of non-linear equation (2.8) has been considered by many authors, for example Yeh (1982) considered the equation (2.8) and presented some oscillation criteria for equation (2.8).

Theorem 2.1.30 Yeh (1982)

Suppose that

$$\limsup_{t\to\infty}\int_{t_0}^{t}(t-s)^{n-1}sq(s)ds=\infty,$$

$$\lim_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}s\left[(t-s)(h(s)-\frac{1}{n})+n-1\right]^2(t-s)^{n-3}ds<\infty$$

for some integer $n\geq 3,$ are sufficient conditions for the oscillation of equation (2.8).

2.1.3.2 Oscillation Of Equations Of Type (2.9)

This section is devoted to the oscillation criteria for the second order nonlinear differential equation with damping term of the form (2.9). Nagabuchi and Yamamoto (1988) have extended and improved the result of Yeh (1982) for equation (2.8) to the equation (2.9).

Theorem 2.1.31 Nagabuchi and Yamamoto (1988)

The equation (2.9) is oscillatory if there exists a continuously differentiable function $\rho(t)$ on $[t_0,\infty)$ and a constant $\alpha\in(1,\infty)$ such that

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}\rho(s)q(s)-\frac{1}{4k}\left[(t-s)\frac{\rho(s)h(s)}{r(s)}+\alpha\rho(s)-(t-s)\dot{\rho}(s)\right]^2$$
$$\times(t-s)^{\alpha-2}\frac{r(s)}{\rho(s)}ds=\infty.$$

Grace (1992) established some oscillation conditions for equation (2.9) in some three theorems for example, the following theorem:

Theorem 2.1.32 Grace (1992)

Let $g'(x) \geq k > 0 \;\; for\, x \neq 0$ and the functions h, H, ρ are defined as in Theorem 2.1.28 such that the conditions (1) and (2) from Theorem 2.1.29 and $\gamma_1(t) = r(t)\dot{\rho}(t) - h(t)\rho(t) \geq 0$ and $\dot{\gamma}_1(t) \leq 0 \;\; for\, t \geq t_0$. Then, the superlinear equation (2.9) is oscillatory if there exists a continuous function $\Omega(t)$ on $[t_0, \infty)$ such that the condition (5) from Theorem 2.1.28 and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4k}\left(h(t,s) - \gamma(s)\sqrt{H(t,s)}\;\right)^2 \right] ds \geq \Omega(T),$$

for every large $T \geq t_0$ and $\gamma_1(t) = \left(r(t)\dot{\rho}(t) - h(t)\rho(t) \right) \big/ r(t)\rho(t)$.

Kirane and Rogovchenko (2001) were concerned with the problem of oscillation of nonlinear second order equation with damping (2.9) and presented some oscillation theorems for solutions of (2.9). One among the theorems is the next one.

Theorem 2.1.33 Kirane and Rogovchenko (2001)

Assume that

$$\frac{g(x)}{x} \geq k > 0 \;\; for\; x \neq 0.$$

Suppose further that the functions h, H, ρ are defined as in Theorem 2.1.28 and there exists a function $f \in C^1([t_0, \infty); (0, \infty))$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_T^t\left[H(t,s)\psi(s)-\frac{r(s)\rho(s)}{4k}\left(h(t,s)-Q^2(t,s)\gamma(s)\sqrt{H(t,s)}\ \right)^2\right]ds=\infty,$$

where

$$\psi(s)=\rho(s)\left(kq(s)-h(s)f(s)-\left[r(s)f(s)\right]^{\bullet}+r(s)f^2(s)-h(s)\rho(s)\right),\rho(s)=\exp(-2\int^s f(u)du)$$

and

$$Q(t,s)=h(t,s)+h(s)(r(s))^{-1}(H(t,s))^{1/2}.$$

Then, the equation (2.9) is oscillatory.

Elabbasy, et al. (2005) have studied the oscillatory behavior of equation (2.9) and improved a number of existing oscillation criteria.

Theorem 2.1.34 Elabbasy, et al. (2005)

Assume that the condition (1) from Theorem 2.1.29 holds and there exists $\rho:[t_0,\infty)\to(0,\infty)$ such that

$$\dot{\rho}(t)\geq 0,\ (r(t)\rho(t))^{\bullet}\geq 0, (r(t)\rho(t))^{\bullet\bullet}\leq 0,\ \left[r(t)\rho(t)-\rho(t)h(t)\right]^{\bullet}\leq 0,$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^t\left[\int_{t_0}^s\rho(u)q(u)du\right]ds=\infty.$$

Then, the equation (2.9) is oscillatory.

Lu and Meng (2007) have considered the equation (2.9) and given several oscillation conditions. They improved and extended result of Philos (1989) and result of Yu (1993). They needed the following lemma to simplify proofs of their results. First they recalled a class functions defined on $D = \{(t,s): t \geq s \geq t_0\}$. A function $H \in C(D,R)$ is said to belong the class W if

(1) $H(t,t) = 0$, for $t \geq t_0$ and $H(t,s) > 0$ when $t \neq s$;

(2) $H(t,s)$ has partial derivatives on D such that

$$\frac{\partial}{\partial t} H(t,s) = h_1(t,s)\sqrt{H(t,s)}, \quad -\frac{\partial}{\partial s} H(t,s) = -h_2(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D.$$

for some $h_1, h_2 \in L^1_{loc}(D,R)$.

Lemma 2.1.1 Lu and Meng (2007)

Let $A_0, A_1, A_2 \in C([t_0,\infty),R)$ with $A_2 > 0$, and $w \in C^1([t_0,\infty),R)$. If there exists $(a,b) \subset [t_0,\infty)$ and $c \in (a,b)$ such that

$$\dot{w} \leq -A_0(s) + A_1(s)w - A_2(s)w^2, \; s \in (a,b),$$

then

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)A_0(s) - \frac{1}{4A_2(s)}\Phi_1^2(s,a) \right] ds + \frac{1}{H(b,c)}\int_c^b \left[H(b,s)A_0(s) - \frac{1}{4A_2(s)}\Phi_2^2(b,s) \right] ds \leq 0,$$

for every $H \in W$,

where

$$\Phi_1(s,a) = h_1(s,a) + A_1(s)\sqrt{H(s,a)} \ \ and \ \ \Phi_2(b,s) = h_2(b,s) - A_1(s)\sqrt{H(b,s)}$$

Theorem 2.1.35 Lu and Meng (2007)

Suppose that there exists a function $\rho \in C^1[[t_0,\infty),(0,\infty)]$ such that the condition (1) from Theorem 2.1.29 is satisfied, and

$$\xi(t) \geq 0,\ \dot{\xi}(t) \leq 0,\ t \geq t_0,\ \int^{\infty} \eta(s)ds = \infty,$$

where $\xi(t) = r(t)\dot{\rho}(t) - h(t)\rho(t),\ \eta(t) = \dfrac{1}{r(t)\rho(t)},\ v[t,T] = \eta(t)\left(\int_T^t \eta(s)ds\right)^{-1}.$

If for every $T \geq t_0$, there exists an interval $(a,b) \subset [T,\infty)$, and that there exists $c \in (a,b)$, $H \in W$ and for any constant $D > 0$ such that

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)\rho(s)q(s) - \frac{1}{4Dv[s,t_0]}\Phi_1^2(s,a)\right]ds + \frac{1}{H(b,c)}\int_c^b \left[H(b,s)\rho(s)q(s) - \frac{1}{4Dv[s,t_0]}\Phi_2^2(b,s)\right]ds > 0,$$

where

$$\Phi_1(s,a) = h_1(s,a) + \xi(s)\eta(s)\sqrt{H(s,a)} \ \ and \ \ \Phi_2(b,s) = h_2(b,s) - \xi(s)\eta(s)\sqrt{H(b,s)}.$$

Then, the equation (2.9) is oscillatory.

Rogovchenko and Tuncay (2008) have considered the nonlinear equation (2.9) and established some sufficient conditions for oscillation of solution of equation (2.9) by giving many theorems for example, the following theorem:

Theorem 2.1.36 Rogovchenko and Tuncay (2008)

Suppose that $g'(x)$ exists and $g'(x) \geq k > 0$ *for* $x \neq 0$.

Suppose, further, that there exists a function $\sigma \in C^1([t_0, \infty), R)$ such that, for some $\beta \geq 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_T^t \left[H(t,s)\psi(s) - \frac{\beta}{4k} v(s) r(s) h^2(t,s) \right] ds = \infty,$$

where $v(s) = \exp\left(-2k \int^t \left(\sigma(s) - \frac{h(s)}{2kr(s)} \right) ds \right)$

and $\psi(s) = v(t)\left(q(t) + kr(t)\sigma^2(t) - h(t)\sigma(t) - [r(s)\sigma(t)]^\bullet \right)$. Then, the equation (2.9) is oscillatory.

2.1.3.3 Oscillation Of Equations Of Type (2.10)

Oscillatory behavior of nonlinear second order differential equation with damping (2.10) has been studied by many authors. Grace (1992) considered the equation (2.10), presented some oscillation results for equation (2.10) and extended and improved a number of previously known oscillation results.

Theorem 2.1.37 Grace (1992)

Suppose that the condition (1) from Theorem 2.1.28 holds. Moreover, assume that the functions ρ, h and H are defined as in Theorem 2.1.28 and

(1) $h(t) \leq 0$, $(h(t)\rho(t))^\bullet \geq 0$ for $t \geq t_0$.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4k}\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2\right]ds=\infty,$$

then, superlinear equation (2.10) is oscillatory.

Theorem 2.1.38 Grace (1992)

Suppose that the condition (1) from Theorem 1.5.28 holds and

$$\int^{\infty}\frac{\Psi(u)}{g(u)}du<\infty \quad and \quad \int^{-\infty}\frac{\Psi(u)}{g(u)}du<\infty.$$

Let the functions ρ, h and H are defined as in Theorem 2.1.28 such that the condition (1) from Theorem 2.1.37 holds and

$$\dot{\rho}(t)\geq 0 \ and \ \left(r(t)\dot{\rho}(t)\right)^{\bullet}\leq 0 \text{ for } t\geq t_0.$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4k}h^2(t,s)\right]ds=\infty,$$

then, superlinear equation (2.10) is oscillatory.

Rogovchenko and Tuncay (2007) were concerned with equation (2.10) and obtained some oscillation criteria for oscillation of equation (2.10). They gave some oscillation theorems and proved these theorems by using their following lemma which is a particular case of Tiryaki and Zafer's lemma 1.1 (2005).

Let $D = \{(t,s) : -\infty < s \le t < \infty\}$. A function $H(t,s)$ belongs to the class W if

(1) $H \in C(D,[0,\infty))$;

(2) $H(t,t) = 0$, for $t \ge t_0$ and $H(t,s) > 0$ for $-\infty < s \le t < \infty$;

(3) $H(t,s)$ has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ satisfying

$$\frac{\partial}{\partial t} H(t,s) = h_1(t,s)\sqrt{H(t,s)}, \quad -\frac{\partial}{\partial s} H(t,s) = -h_2(t,s)\sqrt{H(t,s)},$$

where $h_1, h_2 \in L^1_{loc}(D,R)$.

Lemma 2.1.2 Rogovchenko and Tuncay (2007)

Suppose that a function $U \in C^1(D,R)$ satisfies the inequality

$$\dot{U}(t) \le -\alpha(t) - \beta(t)U^2(t),$$

for all $t \in (a,b) \subset [t_0,\infty)$, where the functions $\alpha \in C([t_0,\infty),R)$, $\beta \in C([t_0,\infty),(0,\infty))$.

Then, for any $c \in (a,b)$ and for any $H \in W$,

$$\frac{1}{H(c,a)}\int_a^c \left[\alpha(s)H(s,a) - \frac{1}{4\beta(S)}h_1^2(s,a)\right]ds + \frac{1}{H(b,c)}\int_c^b \left[\alpha(s)H(b,s) - \frac{1}{4\beta(s)}h_2^2(b,s)\right]ds \le 0.$$

Theorem 2.1.39 Rogovchenko and Tuncay (2007)

Let g be continuously differentiable and satisfy for all $x \in R$,

$$g'(x) \ge k > 0.$$

Assume that, for all $x \in R$,

$$0 < C \le \Psi(x) \le C_1.$$

Suppose also that there exists a function $\sigma \in C^1([t_0, \infty), R)$ such that, for some $H \in W$ and $c \in (a,b)$,

$$\frac{1}{H(c,a)} \int_a^c \left[\varphi(s) H(s,a) - \frac{C_1}{4k} \nu(s) r(s) h_1^2(s,a) \right] ds + \frac{1}{H(b,c)} \int_c^b \left[\varphi(s) H(b,s) - \frac{C_1}{4k} \nu(s) r(s) h_2^2(b,s) \right] ds > 0,$$

where

$$\varphi(s) = \nu(t) \left[q(t) - \sigma'(t) + \frac{\sigma(t)(k\sigma(t) - h(t))}{C_1 r(t)} + \left(\frac{1}{C_1} - \frac{1}{C} \right) \frac{h^2(t)}{4kr(t)} \right]$$

and

$$\nu(t) = \exp\left[\int^t \frac{h(s) - 2k\sigma(s)}{C_1 r(s)} ds \right].$$

Then, every solution of the equation (2.10) has at least one zero in (a,b).

2.1.4 Oscillation Of Nonhomogeneous Nonlinear Equations

2.1.4.1 Oscillation Of Equations Of Type (2.11)

This section is devoted to study the oscillation of equation (2.11). Many authors are concerned with the oscillation criteria of solutions of the homogeneous second order nonlinear differential equations. However, few authors studied the non-homogeneous equations. Greaf, et al. (1978) considered the non-homogeneous equation (2.11) and gave some oscillation sufficient conditions for this for the non-homogeneous equation, for instance, the next three theorems.

Theorem 2.1.40 Greaf, et al. (1978)

Suppose that

(1) $\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty$,

(2) $\int_{t_0}^{\infty} \left(q(s) - p(s)\right) ds = \infty$,

Then, all solutions of equation (2.11) are oscillatory.

Example 2.1.2

Consider the differential equation:

$$\left(\dot{x}(t) \big/ t^2 \right)^{\bullet} + \left(\frac{1}{3} + 2\cos t \right) x^3(t) = \frac{\sin t \; x^7(t)}{\left(x^4(t) + 1\right) t^2}, \; t > 0.$$

Theorem 2.1.40 ensures that the given equation is oscillatory.

Theorem 2.1.41 Greaf, et al. (1978)

Suppose that the condition (1) from Theorem 2.1.40 holds and

(1) $\int_{t_0}^{\infty} \left(q(s) - p(s)\right) ds < \infty$,

(2) $\liminf_{t\to\infty} \int_{t_0}^{t} \left(q(s) - p(s) \right) ds \ge 0$ for all large $T \ge t_0$,

(3) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} \left(q(u) - p(u) \right) du ds = \infty.$

Then, the superlinear differential equation (2.11) is oscillatory.

Theorem 2.1.42 Greaf, et al. (1978)

Suppose that

$$\int_{t_0}^{\infty} \frac{M}{r(s)} ds - \int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} \left(q(u) - p(u) \right) du ds = -\infty,$$

for every constant M, then the sub-linear differential equation (2.11) is oscillatory.

Example 2.1.3

Consider the differential equation

$$\left(t^2 \dot{x}(t) \right)^{\bullet} + \left(\frac{1}{2} + \cos t \right) x^{\frac{1}{3}}(t) = x^{\frac{4}{3}}(t) \Big/ 4\left(|x(t)| + 1 \right), \ t > 0 ,$$

Theorem 2.1.42 ensures that the given equation is oscillatory.

Remili (2010) was concerned with non-homogeneous nonlinear equation (2.11) and presented oscillation criteria for equation (2.11) which contain results of Greaf, et al. (1978) as particular case. Two theorems are presented here.

Theorem 2.1.43 Remili (2010)

Let $\rho(t)$ be a positive continuously differentiable function on $[T,\infty)$ such that $\dot{\rho} \geq 0$ on $[T,\infty)$. Equation (2-11) is oscillatory if

(1) $$\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{\rho(s)r(s)} = \infty,$$

(2) $$\int_{t_0}^{\infty} R(s)ds = \infty,$$

where $R(t) = \rho(t)[q(t) - p(t)] - \dfrac{1}{4k}\dfrac{\dot{\rho}^2(t)r(t)}{\rho(t)}$.

Theorem 2.1.44 Remili (2010)

If the condition (1) from Theorem 2.1.43 holds and $\rho(t)$ is defined as Theorem 2.1.43 such that

(1) $$\int_{t_0}^{\infty}\rho(s)\big(q(s) - p(s)\big)ds < \infty,$$

(2) $$\liminf_{t\to\infty}\left[\int_{T}^{t} R(s)ds\right] \geq 0 \text{ for all large } T \geq t_0,$$

(3) $$\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty} R(u)duds = \infty.$$

Then, the super linear differential equation (2.11) is oscillatory.

2.1.4.2 Oscillation Of Equation Of Type (2.12)

Oscillation of solutions of the nonhomogeneous non-linear equation (2.12) has been studied by many authors. Manojlovic (1991) has considered the nonhomogeneous non-linear equation (2.12) and established some oscillation theorems for solutions of this equation. For example the following theorem:

Define the sets $D_0 = \{(t,s) : t > s \geq t_0\}$, $D = \{(t,s) : t \geq s \geq t_0\}$ and introduce the function $H \in C(D)$ which satisfies the following conditions:

(1) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $(t,s) \in D_0$,

(2) H has a continuous and non-positive partial derivative on D_0 with respect to the second variable, as well as a continuous function $h : D_0 \to R$ such that

$$-\frac{\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \text{ for } (t,s) \in D_0.$$

Theorem 2.1.45 Manojlovic (1991)

Suppose that for any $\lambda_1 > 0$ there exists $\lambda_2 > 0$ such that

$$\frac{g'(x)}{\Psi(x)} \geq \lambda_2 \text{ for } |x| \geq \lambda_1.$$

Suppose, furthermore, that

$$\int_{t_0}^{t} r(s)h^2(t,s)ds < \infty \text{ for } t \geq t_0,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\big(q(s)-K|H(s)|\big)-Lr(s)h^2(t,s)\right]ds=\infty$$

for every $T \geq t_0$ and any positive constants *K* and *L*. Then, any solution of equation (2.12) is either oscillatory or satisfies $\liminf_{t\to\infty}|x(t)| = 0.$

2.1.5 Oscillation Of Non-homogeneous Nonlinear Equations With Damping Term

2.1.5.1 Oscillation Of Equations Of Type (2.13)

This section is devoted to study the oscillation of equation (2.13). Many authors are concerned with the oscillation criteria of solutions of the non-homogeneous non-linear second order equations with damping term. Berkani (2008) considered the equation (2.13) and presented some sufficient condition for all solutions of equation (2.13) to be oscillatory.

Theorem 2.1.46 Berkani (2008)

Assume that for some constants $\mathrm{K}, \mathrm{C}, \mathrm{C}_1$ and for all $x \neq 0$,

$$\frac{g(x)}{x|x|} \geq K,$$

$$0 < C \leq \Psi(x) \leq C_1.$$

Suppose further there exists a continuous function $u(t)$ such that $u(a) = u(b) = u(c) = 0,\ u(t)$ is differentiable on the open set $(a,c)\cup(c,b)$ and satisfies the inequalities

$$\int_a^c \left[\left(\sqrt{Kq(s)H|(s)|} - \frac{h^2(s)}{2Cr(s)}\right)u^2(s) - 2C_1 r(s)\left(u'(s)\right)^2\right]ds \geq 0$$

and

$$\int_c^b \left[\left(\sqrt{Kq(s)H|(s)|} - \frac{h^2(s)}{2Cr(s)}\right)u^2(s) - 2C_1 r(s)\left(u'(s)\right)^2\right]ds \geq 0.$$

Then, every solution of equation (2.13) has a zero in $[a,b]$.

2.2 oscillation Of Third Order Differential Equations

In the relevant literature, until now, oscillatory and non-oscillatory behavior of solutions for various classes of linear and non-linear third order differential equations has been the subject of intensive investigations for many authors. There are many papers dealing with particular cases of the equations (1.3). These particular cases can be classified as follows:

The homogeneous linear equations are given below

$$\dddot{x}(t) + q(t)x(t) = 0. \tag{2.14}$$

$$\dddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t) = 0. \tag{2.15}$$

$$\dddot{x}(t) + a(t)\ddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t) = 0. \tag{2.16}$$

The non-homogeneous linear equations is given below

$$\dddot{x}(t)+ a(t)\ddot{x}(t)+b(t)\dot{x}(t)+c(t)x(t) = f(t). \qquad (2.17)$$

The homogeneous non-linear equations are given below

$$\dddot{x}(t)+b(t)\dot{x}(t)+c(t)x^{\alpha}(t) = 0. \qquad (2.18)$$

$$\dddot{x}(t)+b(t)\dot{x}(t)+c(t)f(x(t)) = 0. \qquad (2.19)$$

$$\dddot{x}(t)+q(t)f(x(t)) = 0. \qquad (2.20)$$

The paper which was presented by Hanan (1961) was the starting point for many investigations to asymptotic behavior of third order equations.

2.2.1 Oscillation Of Homogenous Linear Equations

2.2.1.1 Oscillation Of Equations Of Type (2.14)

This section is devoted to the oscillation criteria for the third order linear differential equation of the form (2.14). Hanan (1961) considered the equation (2.14) presented the following oscillation criteria for equation (2.14):

Theorem 2.2.1 Hanan (1961)

The equation (2.14) is non-oscillatory if

$$\int_{t_0}^{\infty} s^2 q(s)\ ds < \infty,$$

or

$$\limsup_{t\to\infty} t^3 q(t) < \frac{2}{3\sqrt{3}},$$

and the equation (2.14) is oscillatory if

$$\liminf_{t\to\infty} t^3 q(t) > \frac{2}{3\sqrt{3}}.$$

Later, in 2001, Adamets and Lomtatidze (2001) also studied oscillatory properties os solutions of the equation (2.14) where q is eventually of one sign $[0,\infty)$.

Mehri (1976) considered the third equation (2.14) and presented the following oscillation result.

Theorem 2.2.2 Mehri (1976)

The equation (2.14) is oscillatory if

$$\int_{t_0}^{\infty} q(s)\ ds = \infty.$$

2.2.1.2 Oscillation Of Equations Of Type (2.15)

This section is devoted to the oscillation criteria for the equation of the form (2.15). Hanan (1961) also derived some oscillation criteria for equation (2.15) and proved the following theorem:

Theorem 2.2.3 Hanan (1961)

If $2c(t) - b'(t) > 0,$ and there exists a number $0 < k < 1$ such that

$$\liminf_{t\to\infty} \ t^2 b(t) > k$$

and

$$\liminf_{t\to\infty} \ t^3 c(t) > -k,$$

Then, the equation (2.15) is oscillatory. Lazer (1966) studied oscillation of equation (2.15) and proved the following:

Theorem 2.2.4 Lazer (1966)

Assumed that if $b(t) < 0$ and

$$\int_{t_0}^{\infty} \left[c(s) - \frac{2}{3\sqrt{3}} \left(-b(t)\right)^{\frac{2}{3}} \right] ds = \infty,$$

Then, the equation (2.15) is oscillatory.

2.2.1.3 Oscillation of Equations of type (2.16)

This section is devoted to the oscillation criteria for the equation of the form (2.16). Parhi and Das (1993) considered the linear equation (2.16) and presented the following theorem:

Theorem 2.2.5 Parhi and Das (1993)

Supposed that $a(t) \ge 0,\ b(t) \le 0, c(t) > 0, a'(t) \ge b(t),\ c(t) \ge 0$ and proved that if

$$\int_{t_0}^{\infty}\left[\frac{2a^3(s)}{27}+c(s)-\frac{a(s)b(s)}{3}-\frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3}-b(s)\right)^{\frac{3}{2}}\right]ds=\infty,$$

then, the equation (2.16) is oscillatory.

Theorem 2.2.6 Parhi and Das (1993)

Supposed that $a(t)\le 0,\ b(t)\le 0, c(t)>0, a'(t)\ge b(t)$ and proved that if

$$\int_{t_0}^{\infty}\left[\frac{2a^3(s)}{27}+c(s)-\frac{a(s)b(s)}{3}-\frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3}-b(s)+a'(s)\right)^{\frac{3}{2}}\right]ds=\infty,$$

then, the equation (2.16) is oscillatory.

2.2.2 Oscillation Of Non-homogenous Linear Equations

2.2.2.1 Oscillation Of Equations Of Type (2.17)

This section is devoted to the oscillation criteria for the equations of the form (2.17). Das (1995) studied the equation (2.17) and established some new oscillation criteria for the equation (2.17).

Theorem 2.2.7 Das (1995)

Supposed that $a(t)>0,\ b(t)\le 0, c(t)>0, c'(t)>0,\ a'(t)\ge b(t),\ f(t)>0,\ f'(t)<0$

and proved that if

$$\int_{t_0}^{\infty}\left[\frac{2a^3(s)}{27}+c(s)-\frac{a(s)(b(s)-a'(s))}{3}-\frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3}-(b(s)-a'(s))\right)^{\frac{3}{2}}\right]ds=\infty,$$

then, every solution of equation (2.17) oscillates.

2.2.3 Oscillation Of Homogenous Non-linear Equations

2.2.3.1 Oscillation Of Equations Of Types (2.18) and (2.19)

This section is devoted to the oscillation criteria for third order non-linear equations of the form (2.18) and (2.19). Waltman (1966) considered the equations (2.18) and (2.19) and established two theorems for oscillation.

Theorem 2.2.8 Waltman (1966)

Supposed that the equation (2.18) is oscillatory if $b(t)$ and $c(t)$ are continuous functions and $b'(t) < 0,$ α is a ratio of two odd positive integers and

$$A + Bt - \int_{t_0}^{t} Q(s)ds < 0,$$

for large *t*, where $A,\ B$ and $Q(t) = \int_{0}^{t} c(s)ds.$

Theorem 2.2.9 Waltman (1966)

Supposed that the equation (2.19) is oscillatory if $b(t)$ and $c(t)$ are positive continuous functions such that $kc(t) - b'(t) > 0,\ f(u)/u \geq k > 0$ and

$$\int_{t_0}^{\infty} s[kc(s) - b'(s)]ds = \infty.$$

Heidel (1968) also considered the non-linear third order equation (2.18) and investigated the behavior of non-oscillatory and oscillatory of solutions of equation (2.18). He proved the following oscillation theorem:

Theorem 2.2.10 Heidel (1968)

Proved that if $b(t) \leq 0,\ c(t) \leq 0$ and $-\dfrac{2}{t^2} \leq b(t) \leq 0,$

$$\int_{t_0}^{\infty} s^{2-\alpha} c(s)\, ds = -\infty, 0 < \alpha < 1,$$

$$\int_{t_0}^{\infty} s^4 [b'(s) - 2c(s)]ds = \infty,$$

then, the equation (2.18) is oscillatory.

2.2.3.2 Oscillation Of Equations Of Type (2.20)

This section is devoted to the oscillation criteria for third order non-linear equations of the form (2.20). Ramili (2007) studied non-oscillatory for the third order non-linear equation (2.20) and presented the following:

Theorem 2.2.11 Ramili (2007)

Supposed that $f(u)f''(u) - 2f'^2(u) \leq 0$ for every $u \geq 0,$

$$0 < \int^{\infty} \frac{du}{f(u)} < \infty \quad \text{and} \quad 0 < \int^{-\infty} \frac{du}{f(u)} < \infty,$$

$$\liminf_{t\to\infty} \int_{T_0}^{\infty} R(s)q(s)\,ds > -\infty,$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_{T_0}^{t} \int_{T_0}^{s} R(s)q(s)\,ds = \infty,$$

then, every solution of equation (2.20) is non-oscillatory.

CHAPTER THREE

OSCILLATION OF SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH ALTERNATING COFFICIENTS

3.1 Introduction

In this chapter, we are concerned with the problem of oscillation of second order non-linear ordinary differential equation of the form

$$\left(r(t)\dot{x}(t) \right)^{\bullet} + q(t)\Phi\left(g(x(t)), r(t)\dot{x}(t) \right) = H\left(t, x(t)\right), \tag{1.1}$$

where q and r are continuous functions on the interval $[t_0, \infty)$, $t_0 \geq 0$, $r(t)$ is a positive function, g is continuously differentiable function on the real line R except possibly at 0 with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$, Φ is a continuous function on RxR with $u\Phi(u,v) > 0$ for all $u \neq 0$ and $\Phi(\lambda u, \lambda v) = \lambda\Phi(u,v)$ for any $\lambda \in (0,\infty)$ and H is a continuous function on $[t_0, \infty) \times$R with $H(t, x(t))/g(x(t)) \leq p(t)$ for all $x \neq 0$ and $t \geq t_0$.

3.2 Second Order Nonlinear ODE Of Type (1.1)

In this chapter, we present the results of our study of finding the sufficient conditions for oscillation of solutions of ordinary differential equations of second order of type (1.1). The present oscillation results have among other finding extended and improved many previous oscillation results, for examples, such as the works of Bihari (1963), Kartsatos (1968), Philos and Purnaras (1992), Philos (1989), El-abbasy (1996), and El-Abbasy, et al. (2005). We have established some new sufficient conditions which guarantee that our differential equations are oscillatory. A number of theorems and illustrative examples for oscillation differential equation of type (1.1) are shown. Further, a number of numerical examples are given to illustrate the theorems. These numerical examples are computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of proposed research study.

3.3 Oscillation Theorems

Theorem 3.3.1: Suppose that

(1) $\frac{1}{\Phi(1,v)} < \frac{1}{C_0}, C_0 \in (0,\infty),$

(2) $G(m) = \int_0^m \frac{1}{\Phi(1,s)} ds > -B^*,\ B^* \in (0,\infty)$ for every m $\in$ R,

(3) $\int_T^\infty [C_0 q(s) - p(s)] ds = \infty,\ T \geq t_0,$

where $p : [t_0, \infty) \to (0,\infty)$, then every solution of equation (1.1) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution $x(t) > 0$ of equation (1.1) such that $x(t) > 0$ on $[T,\infty)$ for some $T \geq t_0 \geq 0.$ Define

$$\omega(t) = \frac{r(t)\dot{x}(t)}{g(x(t))},\ t \geq T.$$

This and the equation (1.1) imply

$$\dot{\omega}(t) \le p(t) - q(t)\Phi(1, \omega(t)),\ t \ge T.$$

$$\frac{\dot{\omega}(t)}{\Phi(1, \omega(t))} \le \frac{p(t)}{\Phi(1, \omega(t))} - q(t),\ t \ge T.$$

By the condition (1), we have

$$\frac{\dot{\omega}(t)}{\Phi(1, \omega(t))} \le \frac{1}{C_0} p(t) - q(t),\ t \ge T.$$

Integrate from T to t and from condition (2), we obtain

$$\int_T^t [C_0 q(s) - p(s)]ds \le -C_0 \int_T^t \frac{\dot{\omega}(s)}{\Phi(1, \omega(s))} ds = -C_0 G(\omega(s)) + C_0 G(\omega(T)) \le C_0 B^* + C_0 G(\omega(T)).$$

Thus, we have

$$\int_T^\infty [C_0 q(s) - p(s)]ds < \infty,$$

which contradicts to the condition (3). Hence, the proof is completed.

Example 3.3.1

Consider the differential equation

$$\left(t\,\dot{x}(t)\right)^{\bullet} + (1 + 3\cos t)x(t) = \frac{x(t)\cos x(t)}{t^3},\ t > 0$$

Here $r(t) = t,\ q(t) = (1 + 3\cos t),\ g(x) = x,\ \Phi(u, v) = u$ and

$$\frac{H(t, x(t))}{g(x(t))} = \frac{\cos x(t)}{t^3} \le \frac{1}{t^3} = p(t).$$

All conditions of Theorem 3.3.1 are satisfied and hence every solution of the given equation is oscillatory. To demonstrate that our result in Theorem 3.3.1 is true, we also find the numerical solution of the given differential equation in Example 3.3.1 using the Runge Kutta method of fourth order (RK4) for different step sizes *h*.

We have

$$\ddot{x} = f(t, x, \dot{x}) = x\cos x - 3.99x$$

with initial conditions $x(1) = 1, \dot{x}(1) = -0.5$ on the chosen interval [1,50] and finding the values of the functions *r, q* and f where we consider $H(t, x(t)) = f(t)l(x)$ at *t=1* *n =750, n =1500, n=2250* and *n*=3000 and the step sizes *h* =0.065, *h* =0.032, *h* =0.021 and *h* = 0.016.

Table 3.1: Comparison of the numerical solutions of ODE 3.1 with different steps sizes

t_k	h=0.065	h=0.032	h=0.021	h=0.016	$\lvert x_4 - x_1 \rvert$	Error %	$\lvert x_4 - x_2 \rvert$	Error %	$\lvert x_4 - x_3 \rvert$	Error %
	$x_1(t_k)$	$x_2(t_k)$	$x_3(t_k)$	$x_4(t_k)$						
1	1	1	1	1	0	0	0	0	0	0
5.9	-1.01759033	-1.01759222	-1.01759231	-1.01759233	0.000002	0.00019654	0.00000011	0.00001080	0.00000002	0.00000196
10.8	0.83533792	0.83534724	0.83534771-	0.83534779	0.00000987	0.00118154	0.00000055	0.00006584	0.00000008	0.00000957
12.76	-1.01965909	-1.01966331	1.01966351	1.01966354	0.00000445	0.00043641	0.00000023	0.00002255	0.00000003	0.00000294
17.66	0.84203045	0.84204651	0.84204731	0.84204744	0.00001699	0.00201770	0.00000093	0.00011044	0.00000013	0.00001543
20.404	-0.29548472	-0.29545661	-0.29545516	-0.29545492	0.0000298	0.01008614	0.00000169	0.00057199	0.00000024	0.00008123
24.52	0.84862257	0.84864561	0.84864674	0.84864693	0.00002436	0.00287045	0.00000132	0.00015554	0.00000019	0.00002238
27.068	-0.61356734	-0.61353503	-0.61353338	-0.61353310	0.00003424	0.00558079	0.00000193	0.00031457	0.00000028	0.00004563
30.4	-0.74862337	-0.74859198	-0.74859037	-0.74859011	0.00003326	0.00444301	0.00000187	0.00024980	0.00000026	0.00003473
32.36	0.36385799	0.36381122	0.36380887	0.36380848	0.00004951	0.01360880	0.00000274	0.00075314	0.00000039	0.00010719
37.26	-0.74073290	-0.74069241	-0.74069037	-0.74069003	0.00004287	0.00578784	0.00000208	0.00028081	0.00000034	0.00004590
40.2	-1.02666771	-1.02667995	-1.02668048	-1.02668057	0.00001286	0.00125258	0.00000062	0.00006038	0.00000009	0.00008766
42.16	0.97986943	0.97984966	0.97984863	0.97984858	0.00002085	0.00212787	0.00000108	0.00011022	0.00000005	0.00000105
47.06	-1.02810204	-1.02811583	-1.02811641	-1.02811650	0.00001446	0.00140645	0.00000067	0.00006516	0.00000009	0.00000875
50	-0.54365482	-0.54373039	-0.54373397	-0.54373455	0.00007973	0.01466340	0.00000416	0.00076507	0.00000042	0.00007724

Figure 3.1(a): Solution curves of ODE 3.1.

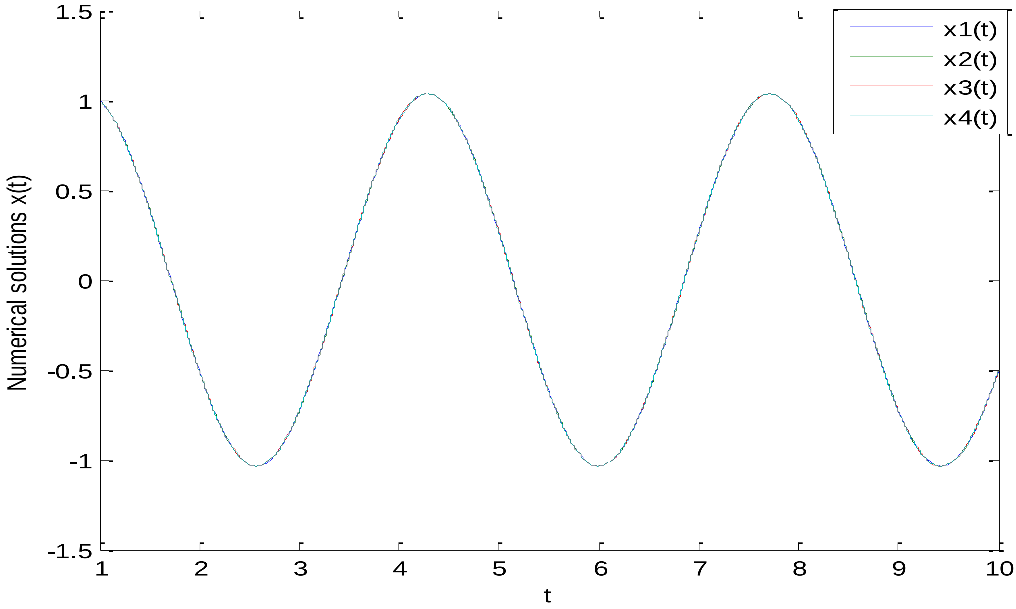
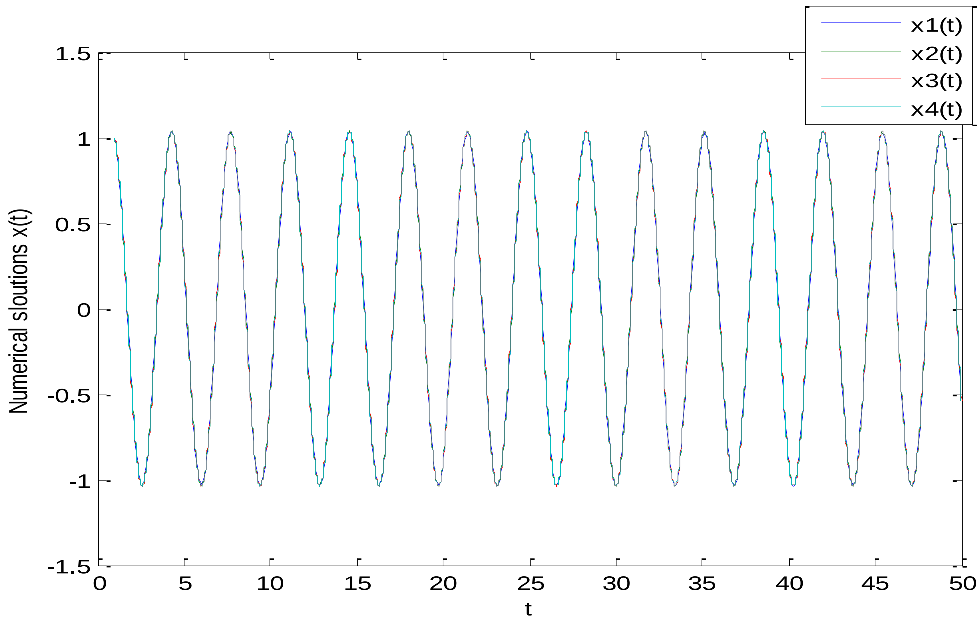


Figure 3.1 (b): Solution curves of ODE 3.1.



Remark 3.3.1: Theorem 3.3.1 is the extension of the results of Bihari (1963) and the results of Kartsatos (1968) who have studied the equation (2.5) as mentioned in chapter two. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 3.3.1 because their equation is aparticular

case of our equation when $r(t) \equiv 1$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t)) \equiv 0$.

Theorem 3.3.2

If the conditions (1) and (2) hold and assume that ρ be a positive continuous differentiable function on the interval $[t_0,\infty)$ with $\rho(t)$ is a decreasing function on the interval $[t_0,\infty)$ and such that

(4) $$\lim_{t\to\infty}\int_T^t \rho(s)\left[C_0 q(s) - p(s)\right]ds = \infty,$$

where p: $[t_0,\infty)\to(0,\infty)$, then, every solution of equation (1.1) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution $x(t) > 0$ of equation (1.1) such that $x(t) > 0$ on $[T,\infty)$ for some $T \geq t_0 \geq 0$. Define

$$\omega(t) = \frac{\rho(t) r(t)\dot{x}(t)}{g\left(x(t)\right)},\ t \geq T.$$

Thus and (1.1) imply

$$\left(\rho(t)\frac{\omega(t)}{\rho(t)}\right)^{\bullet} \leq \rho(t)p(t) - \rho(t)q(t)\Phi\left(1, \omega(t)/\rho(t)\right) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t),\ t \geq T.$$

Thus, we obtain

$$\rho(t)\left(\frac{\omega(t)}{\rho(t)}\right)^{\bullet} \leq \rho(t)p(t) - \rho(t)q(t)\Phi\left(1, \omega(t)/\rho(t)\right),\ t \geq T.$$

After dividing the last inequality by $\Phi\left(1, \omega(t)/\rho(t)\right) > 0$, integrating from T to t and using condition (1), we obtain

$$\int_T^t \rho(s)\left[C_0 q(s) - p(s)\right] ds \le -C_0 \int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1, \omega(s)/\rho(s)\right)} ds \ , \ t \ge T. \quad (3.3.1)$$

By the Bonnet's theorem, we see that for each t ≥ T, there exists $a_t \in [T,t]$ such that

$$-\int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1, \omega(s)/\rho(s)\right)} ds = -\rho(T) \int_T^{a_t} \frac{\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1, \omega(s)/\rho(s)\right)} ds \qquad (3.3.2)$$

From the inequality (3.3.2) in inequality (3.3.1), we have

$$\int_T^t \rho(s)\left[C_0 q(s) - p(s)\right] ds \le -C_0 \rho(T) \int_{\omega(T)/\rho(T)}^{\omega(a_t)/\rho(a_t)} \frac{du}{\Phi\left(1, u\right)} = C_0 \rho(T) \left[-G\left(\frac{\omega(a_t)}{\rho(a_t)}\right) + G\left(\frac{\omega(T)}{\rho(T)}\right) \right]$$

$$\le C_0 \rho(T) B^* + C_0 \rho(T) G\left(\frac{\omega(T)}{\rho(T)}\right) < \infty,$$

which contradicts to the condition (4). Hence the proof is completed.

Example 3.3.2

Consider the following differential equation

$$\left(t^3 \dot{x}(t)\right)^{\bullet} + \left(\frac{t^5 + 4t^5 \cos t}{t^5 + 1}\right)\left(x^9(t) + \frac{x^{27}(t)}{x^{18}(t) + (t^3 \dot{x}(t))^2}\right) = \frac{2t^5 x^9(t) \sin((x(t))}{\left(t^{10} + t^5\right)}, \ t > 0.$$

Here $r(t) = t^3, q(t) = \dfrac{t^5 + 4t^5 \cos t}{t^5 + 1}$, $g(x) = x^9, \Phi(u,v) = u + \dfrac{u^3}{u^2 + v^2}$ and

$$\frac{H(t, x(t)}{g(x(t)} = \frac{2t^5 \sin((x(t))}{\left(t^{10} + t^5\right)} \le \frac{2t^5}{\left(t^{10} + t^5\right)} = p(t).$$

Taking $\rho(t) = \frac{t^5 + 1}{t^5}$ such that

$$\lim_{t \to \infty} \int_T^t \rho(s)\left[C_0 q(s) - p(s)\right] ds = \infty.$$

We get all conditions of Theorem 3.3.2 are satisfied and hence every solution of the given equation is oscillatory. The numerical solutions of the given differential equation are found out using the Runge Kutta method of fourth order (RK4) for different step sizes *h*.

We have

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^9(t)\sin(x(t)) + 2.49\left(x^9(t) + \frac{x^{27}(t)}{x^{18}(t) + \dot{x}^2(t)} \right)$$

with initial conditions $x(1) = -0.5$, $\dot{x}(1) = 0$ on the chosen interval [1,50] and finding the values of the functions *r, q* and *f* where we consider $H(t, x(t)) = f(t)l(x)$ at *t=1*, *n =2250*, *n =2500*, *n=2750* and *n*=3000 and the step sizes *h* =0.0217, *h* =0.0196, *h* =0.0178 and *h* = 0.0163.

Table 3.2: Comparison of the numerical solutions of ODE 3.2 with different step sizes

t_k	h=0.0217	h=0.0196	h=0.0178	h=0.0163	$\lvert x_4 - x_1 \rvert$	Error %	$\lvert x_4 - x_2 \rvert$	Error %	$\lvert x_4 - x_3 \rvert$	Error %
	$x_1(t_k)$	$x_2(t_k)$	$x_3(t_k)$	$x_4(t_k)$						
1	-0.5	-0.5	-0.5	-0.5	0	0	0	0	0	0
5.9	-0.15690175	-0.15690048	-0.15689975	-0.15689930	0.00000245	0.00156151	0.00000118	0.00075207	0.00000045	0.00028680
10.8	0.18620527	0.18620947	0.18621184	0.18621326	0.00000799	0.00429077	0.00000379	0.00203530	0.00000142	0.00076256
12.76	0.16971561	0.16970981	0.16970654	0.16970459	0.00001102	0.00649363	0.00000522	0.00307593	0.00000195	0.00114905
17.66	-0.17338580	-0.17339683	-0.17340299	-0.17340663	0.00002083	0.01201222	0.0000098	0.00565145	0.00000364	0.00209911
20.404	0.67611109	0.67612577	0.67613395	0.67613877	0.00002768	0.00409383	0.000013	0.00192268	0.00000482	0.00071287
24.52	0.23522632	0.23524740	0.23525912	0.23526601	0.00003969	0.01687026	0.00001861	0.00791019	0.00000689	0.00292859
27.068	-0.46746030	-0.46748592	-0.46750016	-0.46750851	0.00004821	0.01031211	0.00002259	0.00483199	0.00000835	0.00178606
30.4	0.75706545	0.75703400	0.75701654	0.75700630	0.00005915	0.00781367	0.0000277	0.00365915	0.00001024	0.00135269
32.36	-0.99324347	-0.99322602	-0.99321634	-0.99321066	0.00003281	0.00330342	0.00001536	0.00154649	0.00000568	0.00057188
37.26	-0.69554475	-0.69549747	-0.69547130	-0.69545598	0.00008877	0.01276428	0.00004149	0.00596587	0.00001532	0.00220287
40.2	0.07171299	0.07165729	0.07162647	0.07160847	0.00010452	0.14596038	0.00004882	0.06817629	0.000018	0.02513669
42.16	-0.35297498	-0.35291389	-0.35288011	-0.35286038	0.0001146	0.03247743	0.00005351	0.01516463	0.00001973	0.00559144
47.472	-0.59796627	-0.59789246	-0.59785169	-0.59782789	0.00013838	0.02314713	0.00006457	0.01080076	0.0000238	0.00398107
50	-0.61431326	-0.61439928	-0.61444676	-0.61447445	0.00016119	0.02623217	0.00007517	0.01223321	0.00002769	0.00450628

Figure 3.2 (a): Solutions curves of ODE 3.2

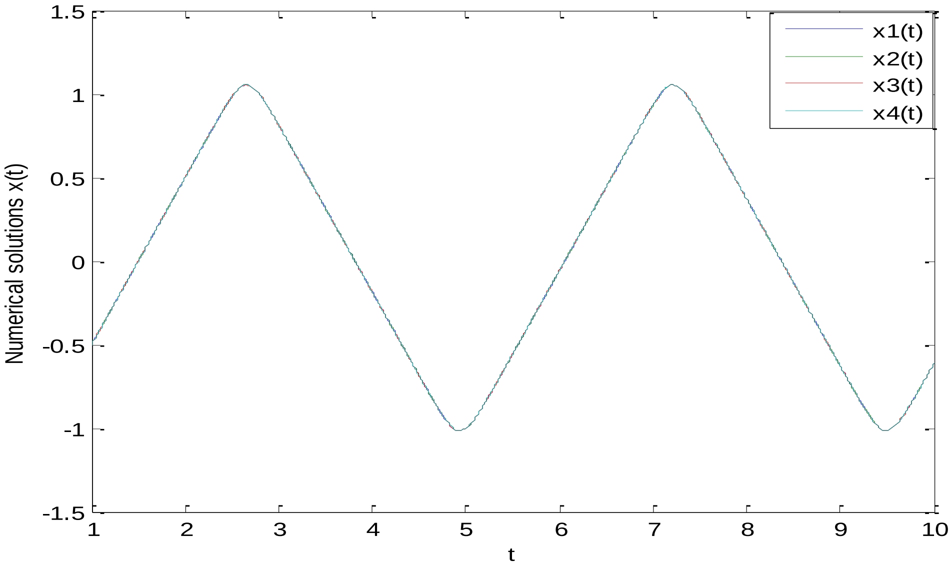
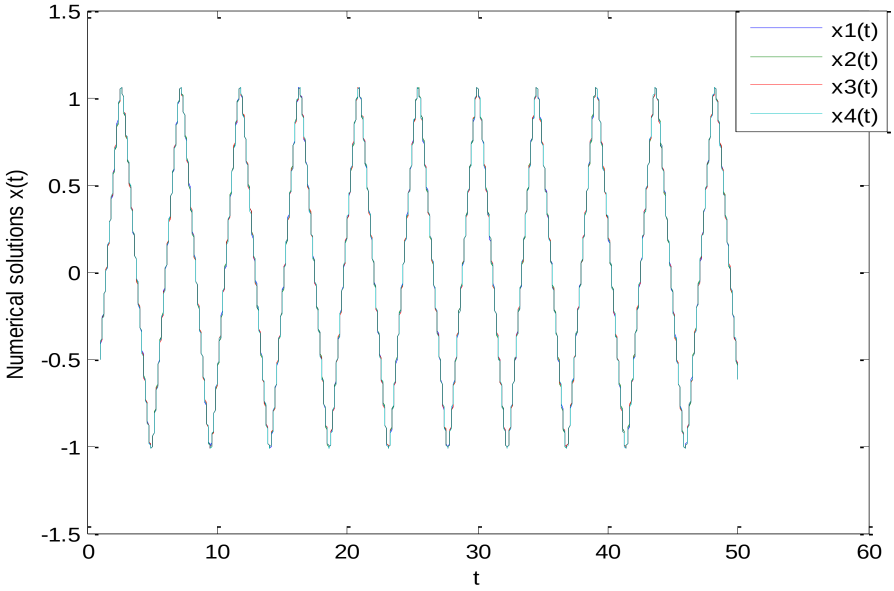


Figure 3.2 (b): Solutions curves of ODE 3.2



Remark 3.3.2

Theorem 3.3.2 extends results of Bihari (1963) and Kartsatos (1968), who have studied the equation (1.1) when $r(t) \equiv 1$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t)) \equiv 0$. Also, Theorem 3.3.2 is the extension of El-Abbasy (2005) who studied the equation (2.6) which is a special case of the equation (1.1) as mentioned in chapter two. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 3.3.2 because their equations are particular cases of the equation (1.1). Theorem 3.3.2 is the extension of Theorem 3.3.1 as well.

Theorem 3.3.3

Suppose that condition (1) holds and

(5) $q(t) > 0, \quad \textit{for all } t > 0.$

Furthermore, suppose that there exists a positive continuous differentiable function ρ on the interval $[t_0, \infty)$ with $\dot{\rho}(t) \geq 0$ and $(\dot{\rho}(t) r(t))^{\bullet} \leq 0$ such that condition (4) holds and

(6) $$\int_{t_0}^{\infty} \frac{ds}{r(s)\rho(s)} = \infty \quad \textit{for every } t \geq t_0.$$

Then every solution of superlinear equation (1.1) is oscillatory.

Proof

If $x(t)$ is oscillatory on $[T, \infty)$, $T \geq t_0 \geq 0$, then $\dot{x}(t)$ is oscillatory on $[T, \infty)$ and if $\dot{x}(t)$ is oscillatory on $[T, \infty)$, then, $\ddot{x}(t)$ is oscillatory on $[T, \infty)$. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0 \textit{ on } [T, \infty)$ for some $T \geq t_0 \geq 0$. We have three cases of $\dot{x}(t)$:

(i) $\dot{x}(t) > 0 \quad \textit{for every } t \geq T.$

(ii) $\dot{x}(t) < 0 \quad \textit{for every } t \geq T.$

(iii) $\dot{x}(t)$ is oscillatory.

If $\dot{x}(t) > 0$ for $t \geq T,\ T \geq t_0$ and we define

$$\omega(t) = \frac{\rho(t) r(t) \dot{x}(t)}{g(x(t))},\ t \geq T.$$

This and (1.1) imply

$$\dot{\omega}(t) \leq \rho(t) p(t) - \rho(t) q(t) \Phi(1, \omega(t)/\rho(t)) + \frac{\dot{\rho}(t) r(t) \dot{x}(t)}{g(x(t))},\ \ t \geq T.$$

From condition (1), we have

$$\dot{\omega}(t) \leq -\rho(t)\left[C_0 q(t) - p(t)\right] + \frac{\dot{\rho}(t) r(t) \dot{x}(t)}{g(x(t))},\ t \geq T.$$

Integrate the last inequality from *T* to *t*, we obtain

$$\omega(t) \leq \omega(T) - \int_T^t \rho(s)\left[C_0 q(s) - p(s)\right] ds + \int_T^t \frac{\dot{\rho}(s) r(s) \dot{x}(s)}{g(x(s))} ds. \qquad (3.3.3)$$

By Bonnet's theorem since $\dot{\rho}(t) r(t)$ is non-increasing, for a fixed $t \geq T$, there exists $\beta_t \in [T, t]$ such that

$$\int_T^t \frac{\dot{\rho}(s) r(s) \dot{x}(s)}{g(x(s))} ds = \dot{\rho}(T) r(T) \int_T^{\beta_t} \frac{\dot{x}(s)}{g(x(s))} ds = \dot{\rho}(T) r(T) \int_{x(T)}^{x(\beta_t)} \frac{du}{g(u)}.$$

Since $\dot{\rho}(t) r(t) \geq 0$ and the equation (1.1) is superlinear, we have

$$\int_{x(T)}^{x(\beta_t)} \frac{du}{g(u)} < \begin{cases} 0 & ,\ if\ x(\beta_t) < x(T) \\ \displaystyle\int_{x(T)}^{\infty} \frac{du}{g(u)} & ,\ if\ x(\beta_t) \geq x(T). \end{cases}$$

We have

$$-\infty < \int_T^t \dot{\rho}(s) r(s) \frac{\dot{x}(s)}{g(x(s))} ds \le A, \tag{3.3.4}$$

where $A = \dot{\rho}(T) r(T) \int_{x(T)}^{\infty} \frac{du}{g(u)}$.

Thus, from (3.3.4) in (3.3.3), we obtain

$$\omega(t) \le \omega(T) + A - \int_T^t \rho(s)\left[C_0 q(s) - p(s)\right] ds .$$

By the condition (4), we get $\omega(t) < 0$, then, $\dot{x}(t) < 0$ for $t \ge T_1$, $T_1 \ge T$. This is a contradiction.

If $\dot{x}(t) < 0$ for every $t \ge T_2 \ge T$. The condition (4) implies that there exists $T_3 \ge T_2$ such that

$$\int_{T_3}^t \rho(s)\left(C_0 q(s) - p(s)\right) ds \ge 0 \quad \text{for all } t \ge T_3.$$

Thus, from equation (1.1) multiplied by $\rho(t)$, we obtain

$$\rho(t)\left(r(t)\dot{x}(t)\right)^{\bullet} + \rho(t) g\left(x(t)\right) q(t) \Phi\left(1, \omega(t)/\rho(t)\right) \le \rho(t) g\left(x(t)\right) p(t),\ t \ge T_3.$$

By condition (1), we have

$$\rho(t)\left(r(t)\dot{x}(t)\right)^{\bullet} \le -\rho(t) g(x(t))\left(C_0 q(t) - p(t)\right).$$

Integrate the last inequality from T_3 to *t*, we obtain

$$\rho(t)r(t)\dot{x}(t) \le \rho(T_3)r(T_3)\dot{x}(T_3) + \int_{T_3}^{t} \dot{\rho}(s)r(s)\dot{x}(s)ds - g(x(t))\int_{T_3}^{t} \rho(s)\big(C_0 q(s) - p(s)\big)\,ds$$
$$+\int_{T_3}^{t} g'(x(s))\dot{x}(s)\int_{T_3}^{s} \rho(u)\big(Cq(u) - p(u)\big)duds$$
$$\le \rho(T_3)r(T_3)\dot{x}(T_3),\ t \ge T_3.$$

Integrate the last inequality divided by $r(t)\rho(t)$ from T_3 to *t* and by condition (6), we have

$$x(t) \le x(T_3) + \rho(T_3)r(T_3)\dot{x}(T_3)\int_{T_3}^{t} \frac{ds}{r(s)\rho(s)} \to -\infty,$$

as $t \to \infty$, contradicting the fact $x(t) > 0$ for all $t \ge T$. Thus, we have $\dot{x}(t)$ is oscillatory and this leads to (1.1) is oscillatory. Hence the proof is completed**.**

Example 3.3.3

Consider the following differential equation

$$\left(\frac{t+1}{t}\dot{x}(t)\right)^{\bullet} + 9t\left(x^5(t) + \frac{x^{15}(t)}{3x^{10}(t) + 6\left(\left(t+1\right)\dot{x}(t)/t\right)^2}\right) = \frac{x^5(t)\sin(x(t))}{t^3},\ t > 0.$$

We have $r(t) = \dfrac{(t+1)}{t},\ q(t) = 9t,\ g(x) = x^5,$

$$\Phi(u,v) = u + \frac{u^3}{3u^2 + 6v^2}\ \ and\ \ \frac{H(t,x(t))}{g(x(t))} = \frac{\sin(x(t))}{t^3} \le \frac{1}{t^3} = p(t) \text{ for all } x \ne 0\ and\ t > 0.$$

Taking $\rho(t) = t$ such that

$$\lim_{t\to\infty}\int_{t_0}^{t} \rho(s)\big(C_0 q(s) - p(s)\big)ds = \lim_{t\to\infty}\int_{t_0}^{t} s\left(9C_0 s - \frac{1}{s^3}\right) ds = \infty.$$

All conditions of Theorem 3.3.3 are satisfied, then, the given equation is oscillatory. Also the numerical solutions of the given differential equation are computed using the Runge Kutta method of fourth order (RK4) for different steps sizes *h*.

We have

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = 0.5x^5(t)\sin x - 4.5(x^5(t) + \frac{x^{15}(t)}{3x^{10}(t) + 24\dot{x}^2(t)})$$

with initial conditions $x(1) = 0.5$, $\dot{x}(1) = 0$ on the chosen interval [1,50] and finding the values of the functions *r*, *q* and *f* where we consider $H(t,x) = f(t)\,l(x)$ at *t=1, n =2250, n =2500, n=2750* and *n*=3000 and the step sizes *h* =0.0217, *h* =0.0196, *h* =0.0178 and *h* = 0.01

Table 3.3: Comparison of the numerical solutions of ODE 3.3 with different step sizes

t_k	h=0.0217	h=0.0196	h=0.0178	h=0.0163	$\lvert x_4 - x_1 \rvert$	Error %	$\lvert x_4 - x_2 \rvert$	Error %	$\lvert x_4 - x_3 \rvert$	Error %
	$x_1(t_k)$	$x_2(t_k)$	$x_3(t_k)$	$x_4(t_k)$						
1	0.5	0.5	0.5	0.5	0	0	0	0	0	0
5.9	-0.13172968	-0.13172968	-0.13172968	-0.13172967	0.00000001	0.00000759	0.00000001	0.00000759	0.00000001	0.00000759
10.8	-0.31893197	-0.31893196	-0.31893195	-0.31893195	0.00000002	0.00000627	0.00000001	0.00000313	0	0
12.76	-0.02652583	-0.02652582	-0.02652582	-0.02652581	0.00000002	0.00007539	0.00000001	0.00003769	0.00000001	0.00003769
17.66	0.46639589	0.46639588	0.46639587	0.46639586	0.00000003	0.00000643	0.00000002	0.00000428	0.00000001	0.00000214
20.404	0.08656452	0.08656450	0.08656449	0.08656449	0.00000003	0.00000346	0.00000001	0.00001155	0	0
24.52	-0.47953084	-0.47953084	-0.47953084	-0.47953084	0	0	0	0	0	0
27.068	-0.27366699	-0.27366697	-0.27366695	-0.27366694	0.00000005	0.00001827	0.00000003	0.00001096	0.00000001	0.00000365
30.4	0.22558962	0.22558964	0.22558965	0.22558966	0.00000004	0.00001773	0.00000002	0.00000886	0.00000001	0.00000443
32.36	0.47799336	0.47799337	0.47799338	0.47799338	0.00000002	0.00000418	0.00000001	0.00000209	0	0
37.26	-0.04805195	-0.04805199	-0.04805201	-0.04805202	0.00000007	0.00014567	0.00000003	0.00006243	0.00000001	0.00002081
40.2	-0.45877839	-0.45877841	-0.45877842	-0.45877843	0.00000004	0.00000871	0.00000002	0.00000435	0.00000001	0.00000217
42.16	-0.39586501	-0.39586497	-0.39586495	-0.39586493	0.00000008	0.00002020	0.00000004	0.00001010	0.00000002	0.00000505
47.06	0.32893993	0.32893999	0.32894000	0.32894002	0.00000009	0.00002736	0.00000003	0.00000912	0.00000002	0.00000608
50	0.42195375	0.42195370	0.42195367	0.42195367	0.00000008	0.00001895	0.00000003	0.00000710	0	0

Figure 3.3(a): Solutions curves of ODE 3.3.

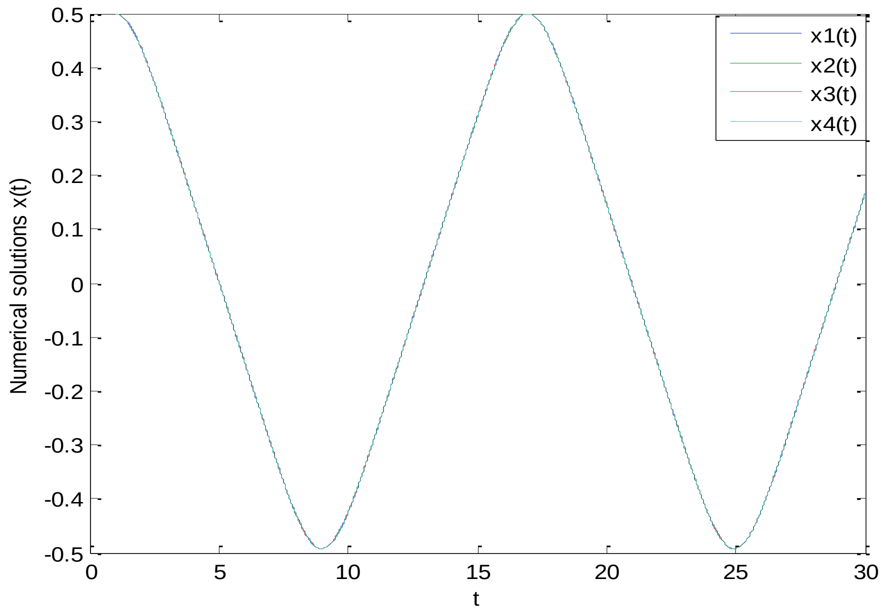
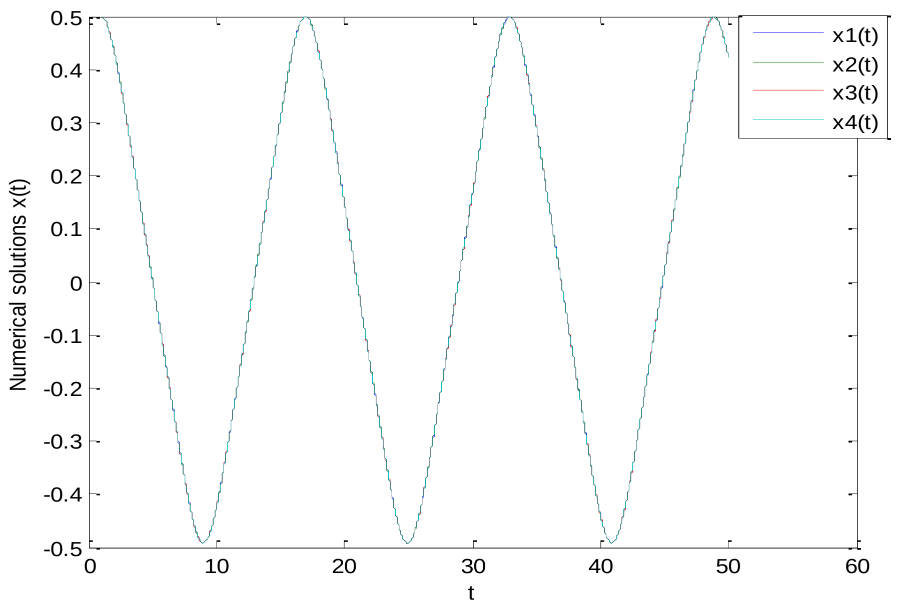


Figure 3.3(b): Solutions curves of ODE 3.3



Remark 3.3.3

Theorem 3.3.3 extends result of Philos (1983) who has studied the equation (1.1) as $r(t) \equiv 1$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$, result of

Bhatia (1966) who has considered the equation (1.1) as $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t,x(t)) \equiv 0$, and result of Philos and Punaras (1992) who have studied the equation (1.1) as $r(t) \equiv 1$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t,x(t)) \equiv 0$. Our result can be applied on their equations (2.1), (2.6) and (2.4) respectively, but their oscillation results cannot be applied on the given equation in Example 3.3.3 because their equations are particular cases of our equation (1.1).

Theorem 3.3.4

Suppose that the conditions (1), (5) and (6) hold and there exists a continuously differentiable function ρ which is defined as Theorem 3.3.3 such that

(7) $$\int_{t_0}^{\infty} \Psi(s)\,ds = \infty,$$

where $\Psi(t) = \rho(t)\big(C_0 q(t) - p(t)\big) - \dfrac{\dot{\rho}^2(t) r(t)}{4k\rho(t)}$.

Then, every solution of equation (1.1) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t)>0$ *on* $[T,\infty)$ for some $T \geq t_0 \geq 0$. Define

$$\omega(t) = \frac{r(t)\dot{x}(t)}{g(x(t))},\ t \geq T.$$

This, conditions (1), (5) and the equation (1.1) imply

$$\left(\frac{r(t)\dot{x}(t)}{g(x(t))}\right)^{\bullet} \leq -\left(C_0 q(t) - p(t)\right) - \frac{k\,r(t)\dot{x}^2(t)}{g^2(x(t))}\ , \ t \geq T. \qquad (3.3.5)$$

We multiply the last inequality (3.3.5) by $\rho(t)$ and integrate form T to t, we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \leq C_1 - \int_T^t \rho(s)\left(C_0 q(s) - p(s)\right) ds + \int_T^t \left[\dot{\rho}(s)\omega(s) - k\frac{\rho(s)}{r(s)}\omega^2(s)\right] ds,$$

where

$$C_1 = \frac{\rho(T)r(T)\dot{x}(T)}{g(x(T))}.$$

$$\begin{aligned}\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} &\leq C_1 - \int_T^t \rho(s)\left(C_0 q(s) - p(s)\right) ds - \int_T^t k\frac{\rho(s)}{r(s)}\left(\eta^2(s) - \left(\frac{\dot{\rho}(s)r(s)}{2k\rho(s)}\right)^2\right) ds \\ &\leq C_1 - \int_T^t \left[\rho(s)\left(C_0 q(s) - p(s)\right) - \frac{\dot{\rho}^2(s)r(s)}{4k\rho(s)}\right] ds \\ &\leq C_1 - \int_T^t \Psi(s) ds,\end{aligned}$$

where $\eta(t) = \omega(t) - \dfrac{\dot{\rho}(t)r(t)}{2k\rho(t)}$.

By the condition (7), we get

$$\lim_{t\to\infty} \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} = -\infty,$$

and there exists $T_1 \geq T$ such that $\dot{x}(t) < 0$ for $t \geq T_1$. The condition (7) also implies that there exists $T_2 \geq T_1$ such that

$$\int_{T_1}^{T_2} \rho(s)\big(C_0 q(s) - p(s)\big)\, ds = 0 \quad and \quad \int_{T_2}^{t} \big(C_0 q(s) - p(s)\big)\, ds \geq 0 \qquad for \;\; t \geq T_2.$$

Multiplying equation (1.1) by $\rho(t)$ and by conditions (1) and (5), we have

$$\rho(t)\left(r(t)\dot{x}(t)\right)^{\bullet} + C_0 \rho(t) g(x(t)) q(t) \leq \rho(t) g(x(t)) p(t),\; t \geq T_2.$$

Integrate the last inequality from T_2 to t, we obtain

$$\rho(t) r(t)\dot{x}(t) \leq \rho(T_2) r(T_2)\dot{x}(T_2) + \int_{T_2}^{t} \dot{\rho}(s) r(s)\dot{x}(s) ds - g(x(t))\int_{T_2}^{t} \rho(s)\big(C_0 q(s) - p(s)\big) ds$$
$$+ \int_{T_2}^{t} g'(x(s))\dot{x}(s)\int_{T_2}^{s} \rho(u)\big(Cq(u) - p(u)\big)\, du\, ds,\; t \geq T_2.$$

By the Bonnet's theorem, for $t \geq T_2$ there exists $\gamma_t \in [T_2, t]$ such that

$$\rho(t) r(t)\dot{x}(t) \leq \rho(T_2) r(T_2)\dot{x}(T_2) + \dot{\rho}(T_2) r(T_2)\big[x(\gamma_t) - x(T_2)\big] - g(x(t))\int_{T_2}^{t} \rho(s)\big(Cq(s) - p(s)\big)\, ds$$
$$+ \int_{T_2}^{t} g'(x(s))\dot{x}(s)\int_{T_2}^{s} \rho(u)\big(Cq(u) - p(u)\big) du\, ds$$
$$\leq \rho(T_2) r(T_2)\dot{x}(T_2),\; t \geq T_2.$$

Dividing the last inequality by $\rho(t) r(t)$, integrate from T_2 to t and the condition (6), we obtain

$$x(t) \leq x(T_2) + \rho(T_2) r(T_2)\dot{x}(T_2)\int_{T_2}^{t} \frac{ds}{\rho(s) r(s)} \to -\infty,\; as\;\; t \to \infty,$$

which is a contradiction to the fact that $x(t) > 0$ for $t \ge T$. Hence the proof is completed.

Example 3.3.4

Consider the following differential equation

$$\left(\frac{1}{t^3}\dot{x}(t)\right)^{\bullet} + t^3\left(x^3(t) + \frac{x^{15}(t)}{5x^{12}(t) + 8\left(\dot{x}(t)/t^3\right)^4}\right) = \frac{x^3(t)\sin(x(t))}{t^5},\ t > 0.$$

We note here

$$r(t) = \frac{1}{t^3},\ q(t) = t^3,\ g(x) = x^3,\ \Phi(u,v) = u + \frac{u^5}{5u^4 + 8v^4},$$

$$\frac{H(t,x(t))}{g(x(t))} = \frac{\sin(x(t))}{t^5} \le \frac{1}{t^5} = p(t) \quad \textit{for all } x \ne 0 \textit{ and } t > 0.$$

Taking $\rho(t) = t$ *and*

$$\int_{t_0}^{\infty} \Psi(s)ds = \int_{t_0}^{\infty}\left[\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{\dot{\rho}^2(s)r(s)}{4k\rho(s)}\right] ds = \infty.$$

All conditions of Theorem 3.3.4 are satisfied. Thus, the given equation is oscillatory. We also compute the numerical solutions of the given differential equation using the Runge Kutta method of fourth order (RK4) for different step sizes *h*.

We have

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^3(t)\sin(x(t)) - \left(x^3(t) + \frac{x^{15}(t)}{5x^{12}(t) + 8\dot{x}^4(t)} \right)$$

with initial conditions $x(1) = -0.5,\ \dot{x}(1) = 0$ on the chosen interval [1,50] and finding the values of the functions *r*, *q* and *f* where we consider $H(t,x) = f(t)\,l(x)$ at *t=1*, *n =2250*, *n =2500*, *n=2750* and *n*=3000 and the step sizes *h* =0.0217, *h* =0.0196, *h* =0.0178 and *h* = 0.016.

Table 3.4: Comparison of the numerical solutions of ODE .3.4 for different steps sizes

t_k	h=0.0217	h=0.0196	h=0.0178	h=0.0163	$\lvert x_4 - x_1 \rvert$	Error %	$\lvert x_4 - x_2 \rvert$	Error %	$\lvert x_4 - x_3 \rvert$	Error %
	$x_1(t_k)$	$x_2(t_k)$	$x_3(t_k)$	$x_4(t_k)$						
1	-0.5	-0.5	-0.5	-0.5	0	0	0	0	0	0
5.9	0.18716061	0.18716060	0.18716060	0.18716060	0.00000001	0.00000534	0	0	0	0
10.8	1.72347511	1.72347511	1.72347511	1.72347512	0.00000001	0.00000058	0.00000001	0.00000058	0.00000001	0.0000058
12.76	0.51782045	0.51782043	0.51782041	0.51782041	0.00000004	0.00002534	0.00000002	0.00001267	0	0
17.66	1.60782741	1.60782743	1.60782744	1.60782745	0.00000004	0.00000248	0.00000002	0.00000124	0.00000001	0.00000062
20.6	-0.15016993	-0.15016997	-0.15017000	-0.15017001	0.00000008	0.00005327	0.00000004	0.00002663	0.00000001	0.00000665
24.52	1.40402218	1.40402223	1.40402226	1.40402227	0.00000009	0.00000644	0.00000004	0.00000284	0.00000001	0.00000071
27.068	0.58008968	0.58008962	0.58008958	0.58008955	0.00000013	0.00002241	0.00000007	0.00001206	0.00000003	0.00000517
30.4	0.17266125	0.17266133	0.17266139	0.17266142	0.00000017	0.00009845	0.00000009	0.00005212	0.00000003	0.00001737
32.36	1.72449224	1.72449226	1.72449228	1.72449228	0.00000004	0.00000231	0.00000002	0.00000115	0	0
37.26	-0.15917045	-0.15917033	-0.15917026	-0.15917022	0.00000023	0.00014449	0.00000011	0.00006910	0.00000004	0.00002513
40.2	1.61225739	1.61225733	1.61225729	1.61225727	0.00000012	0.00000744	0.00000006	0.00000372	0.00000002	0.00000124
42.16	-0.15566964	-0.15566978	-0.15566986	-0.15566991	0.00000027	0.00017344	0.00000013	0.00008351	0.00000005	0.00003211
47.06	1.72512330	1.72512328	1.72512326	1.72512325	0.00000005	0.00000289	0.00000003	0.00000173	0.00000001	0.00000057
50	-0.79791323	-0.79791338	-0.79791347	-0.79791352	0.00000029	0.00003634	0.00000014	0.00001754	0.00000005	0.00000626

Figure 3.4(a): Solutions curves of ODE 3.4

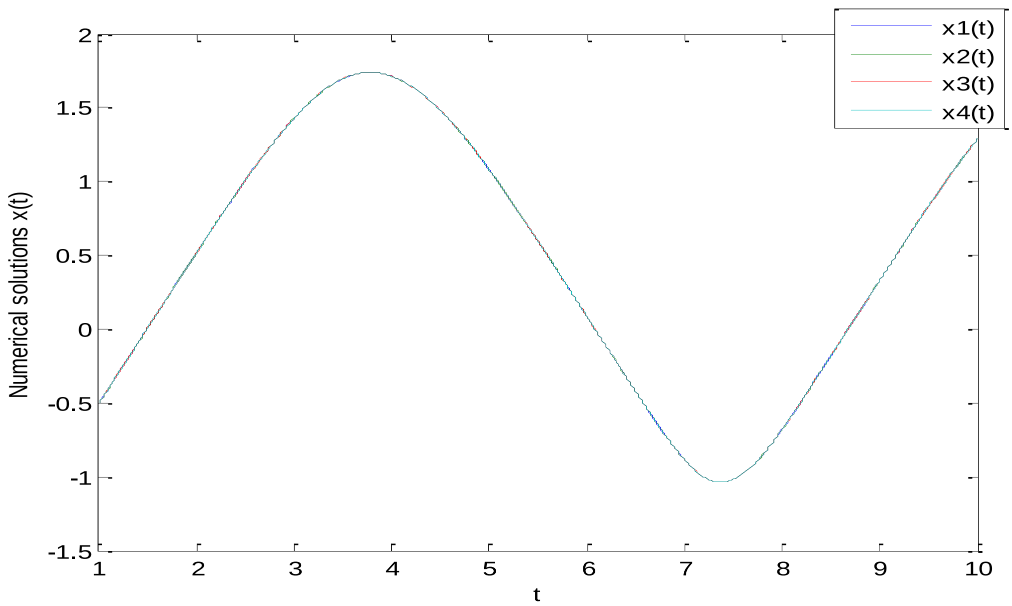
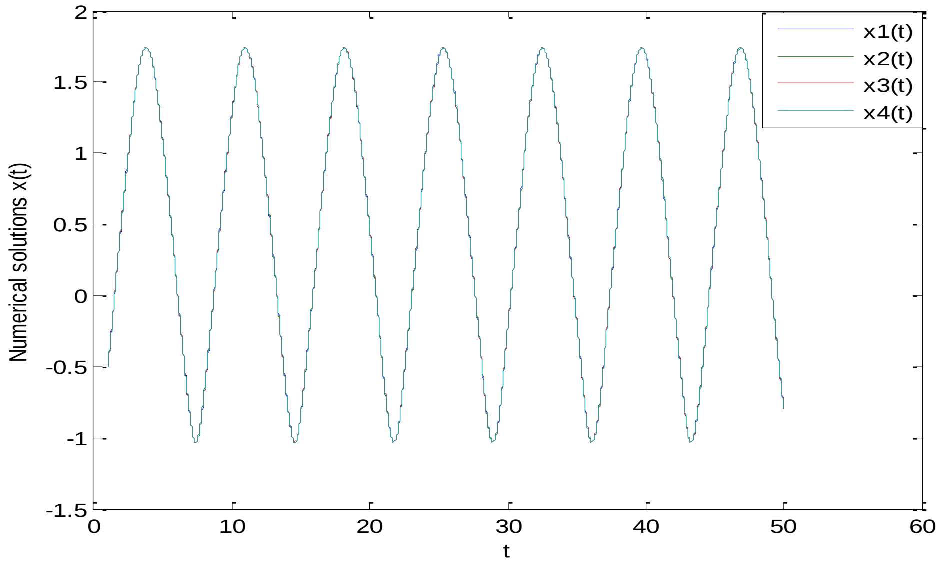


Figure 3.4(b): Solutions curves of ODE 3.4.



Remark 3.3.4

Theorem 3.3.4 is the extension of the results of Bihari (1963) and the results of Kartsatos (1968) who have studied the equation (1.1) as $r(t) \equiv 1,\ \ g(x(t)) \equiv x(t),\ \ \Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t)) \equiv 0$ and results of El-abbasy (2000) who has studied the equation (1.1) as $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0.$ Our result can be applied on their equations (2.5) and (2.6) respectively, but their oscillation results cannot be applied on the given equation in Example 3.3.4 because their equations are particular cases of our equation (1.1).

Theorem 3.3.5: Suppose that the conditions (1) and (5) hold and there exists continuously differentiable function ρ is defined as in Theorem 3.3.3 such that the condition (6) holds and

(8) $$\int_{t_0}^{\infty} \rho(s)[C_0 q(s) - p(s)]ds < \infty,$$

(9) $$\liminf_{t\to\infty} \left[\int_{t_0}^{\infty} \Psi(s)\, ds \right] \geq 0 \quad \text{for all large } t,$$

(10) $$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{\rho(s) r(s)} \int_{s}^{\infty} \Psi(u)\, du ds = \infty.$$

Then, every solution of superlinear equation (1.1) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t)>0$ $on\ [T, \infty)$ for some $T \geq t_0 \geq 0.$ Define

$$\omega(t) = \frac{r(t)\dot{x}(t)}{g(x(t))},\ t \geq T.$$

This and (1.1) imply

$$\dot{\omega}(t) = \left(\frac{r(t)\dot{x}(t)}{g(x(t))}\right)^{\bullet} \leq p(t) - q(t)\Phi(1, \omega(t)) - \frac{k\, r(t)\dot{x}^2(t)}{g^2(x(t))},\ t \geq T. \tag{3.3.6}$$

From condition (1), for $t \geq T$, we have

$$\left(\frac{r(t)\dot{x}(t)}{g(x(t))}\right)^{\bullet} \leq -[C_0 q(t) - p(t)] - \frac{k\, r(t)\dot{x}^2(t)}{g^2(x(t))},\ t \geq T.$$

We multiply the last inequality by $\rho(t)$ and integrate form *T* to *t*, we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \leq \frac{\rho(T)r(T)\dot{x}(T)}{g(x(T))} - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)\, ds + \int_T^t \left[\dot{\rho}(s)\omega(s) - k\frac{\rho(s)}{r(s)}\omega^2(s)\right] ds$$

$$Let \quad C_1 = \frac{\rho(T)r(T)\dot{x}(T)}{g(x(T))} \ and \ \eta(t) = \omega(t) - \frac{\dot{\rho}(t)r(t)}{2k\rho(t)}.$$

Thus, we obtain

$$\begin{aligned}
\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} &\leq C_1 - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)\, ds - \int_T^t k\frac{\rho(s)}{r(s)}\left(\eta^2(s) - \left(\frac{\dot{\rho}(s)r(s)}{2k\rho(s)}\right)^2\right) ds \\
&\leq C_1 - \int_T^t \left[\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{\dot{\rho}^2(s)r(s)}{4k\rho(s)}\right] ds \\
&\leq C_1 - \int_T^t \Psi(s) ds.
\end{aligned} \tag{3.3.7}$$

From inequality (3.3.7), we have

$$\int_T^t \Psi(s)ds \le \frac{\rho(T)r(T)\dot{x}(T)}{g(x(T))} - \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))}$$

Now, we consider three cases for $\dot{x}(t)$

Case 1: If $\dot{x}(t) > 0$ *for* $t \ge T_1 \ge T$, then we get

$$\int_{T_1}^t \Psi(s)ds \le \frac{\rho(T_1)r(T_1)\dot{x}(T_1)}{g(x(T_1))} - \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))}.$$

Thus, for all $t \ge T_1$, we obtain

$$\int_t^\infty \Psi(s)ds \le \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))}.$$

We divide the last inequality $\rho(t)\,r(t)$ and integrate from T_1 to t, we obtain

$$\int_{T_1}^t \frac{1}{\rho(s)r(s)}\int_s^\infty \Psi(u)duds \le \int_{T_1}^t \frac{\dot{x}(s)}{g(x(s))}ds.$$

Since the equation (1.1) is superlinear, we get

$$\int_{T_1}^t \frac{1}{\rho(s)r(s)}\int_s^\infty \Psi(u)duds \le \int_{T_1}^t \frac{\dot{x}(s)}{g(x(s))}ds = \int_{x(T_1)}^{x(t)} \frac{du}{g(u)} < \infty,$$

This contradicts condition (10).

Case 2: If $\dot{x}(t)$ is oscillatory, then there exists a sequence τ_n in $[T,\infty)$ such that $\dot{x}(\tau_n)=0$. Choose N large enough so that (9) holds. Then from inequality (3.3.7), we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le C_{\tau_n} - \int_{\tau_n}^{t} \Psi(s)ds.$$

So

$$\limsup_{t\to\infty} \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le C_{\tau_n} + \limsup_{t\to\infty}\left\{-\int_{\tau_n}^{t}\Psi(s)ds\right\} = C_{\tau_n} - \liminf_{t\to\infty}\left\{\int_{\tau_n}^{t}\Psi(s)ds\right\} < 0,$$

which contradicts the fact that $\dot{x}(t)$ oscillates.

Case 3: If $\dot{x}(t) < 0$ for $t \ge T_2 \ge T$, the condition (9) implies that there exists $T_3 \ge T_2$ such that

$$\int_{T_3}^{t} \rho(s)\big(C_0 q(s) - p(s)\big)\, ds \ge 0 \qquad for\ t \ge T_3.$$

Multiplying the equation (1.1) by $\rho(t)$ and from the condition (1), for $t \ge T_3$, we have

$$\rho(t)\left(r(t)\dot{x}(t)\right)^{\bullet} + C_0\rho(t)g(x(t))q(t) \le \rho(t)g(x(t))p(t),\ t \ge T_3$$

Integrate the last inequality from T_3 to t, we obtain

$$\rho(t)r(t)\dot{x}(t) \le \rho(T_3)r(T_3)\dot{x}(T_3) + \int_{T_3}^{t} \dot{\rho}(s)r(s)\dot{x}(s)ds - g(x(t))\int_{T_3}^{t} \rho(s)\big(C_0 q(s) - p(s)\big)ds$$

$$+ \int_{T_3}^{t} g'(x(s))\dot{x}(s)\int_{T_3}^{s} \rho(u)\big(C_0 q(u) - p(u)\big)du\, ds,\ t \ge T_3$$

$$\le \rho(T_3)r(T_3)\dot{x}(T_3) - g(x(t))\int_{T_3}^{t} \rho(s)\big(C_0 q(s) - p(s)\big)ds$$

$$+ \int_{T_3}^{t} g'(x(s))\dot{x}(s)\int_{T_3}^{s} \rho(u)\big(C_0 q(u) - p(u)\big)du\, ds,\ t \ge T_3$$

$$\le \rho(T_3)r(T_3)\dot{x}(T_3),\ t \ge T_3.$$

Dividing the last inequality by $\rho(t)r(t)$ and integrate from T_3 to t we obtain

$$x(t) \le x(T_3) + \rho(T_3)r(T_3)\dot{x}(T_3)\int_{T_3}^{t} \frac{ds}{\rho(s)r(s)} \to -\infty,\ as\ t \to \infty$$

which is a contradiction to the fact that $x(t) > 0$ for $t \ge T$. Hence, the proof is completed.

Example 3.3.5

Consider the following differential equation

$$\left(\dot{x}(t)\big/t\right)^{\bullet} + \frac{1}{t^3}\frac{x^{35}(t)}{2x^{28}(t) + \left(\dot{x}(t)\big/t\right)^4} = -\frac{x^{11}(t)}{x^4(t)+1},\ t > 0.$$

Here $r(t) = 1/t$, $q(t) = 1/t^3$, $g(x) = x^7$, $\Phi(u,v) = u^5\big/\left(2u^4 + v^4\right)$ and

$$\frac{H(t,x(t))}{g(x(t))} = -\frac{x^4(t)}{x^4(t)+1} \le 0 = p(t) \quad \textit{for all } x \ne 0\, \textit{and } t > 0.$$

Taking $\rho(t) = 4 > 0$ such that

$$(1)\ \int_{t_0}^{\infty} \rho(s)\big(C_0 q(s) - p(s)\big)\, ds = \frac{2C_0}{t_0^2} < \infty,$$

$$(2)\ \liminf_{t\to\infty} \left\{ \int_{t_0}^{t} \Psi(s) ds \right\} = \liminf_{t\to\infty} \left\{ \int_{t_0}^{t} \left[\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{\rho^{\bullet 2}(s) r(s)}{4k\rho(s)} \right] ds \right\} = \frac{2C}{t_0{}^2} > 0,$$

$$(3)\ \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{\rho(s) r(s)} \int_{s}^{\infty} \Psi(u)\, duds = \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{\rho(s) r(s)} \int_{s}^{\infty} \left[\rho(u)\big(C_0 q(u) - p(u)\big) - \frac{\rho^{\bullet 2}(u) r(u)}{4k\rho(u)} \right] duds = \infty.$$

All conditions of Theorem 3.3.5 are satisfied and hence every solution of the given equation is oscillatory. To demonstrate that our result in Theorem 3.3.5 is true we also find the numerical solution of the given differential equation in Example 3.3.5 using the Runge Kutta method of fourth order.

We have

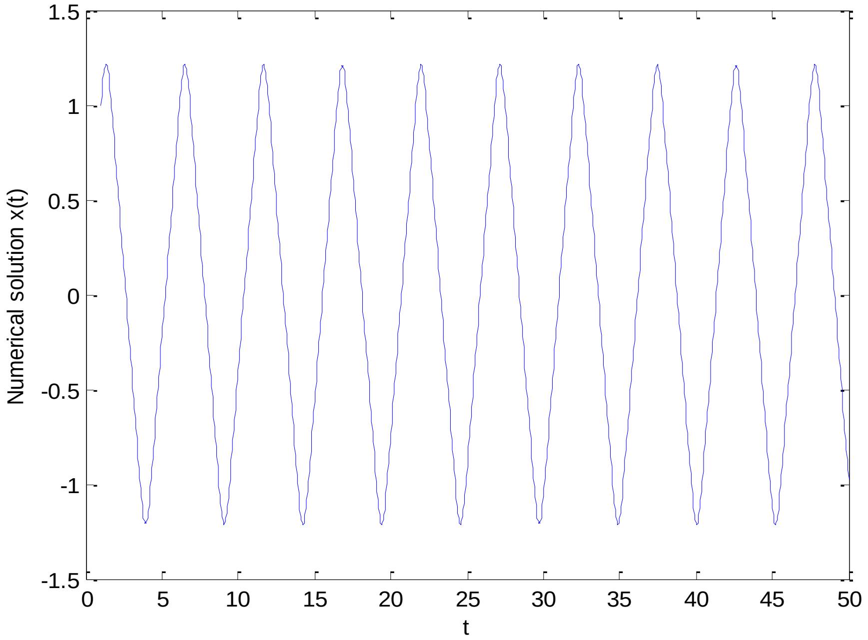
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = -\frac{x^{11}}{x^4 + 1} - \frac{x^{35}}{2x^{28} + \dot{x}^4(t)}$$

with initial conditions $x(1) = 1,\ \dot{x}(1) = 1$ on the chosen interval $[1,50]$ and finding the values the functions r, q and f where we consider $H(t,x) = f(t)\, l(x)$ at t=1, n=980 and h=0.05.

Table 3.5: Numerical solution of ODE 3.5

k	t_k	$x(t_k)$
1	1	1
81	5	-0.22568469
182	10.5	0.13417542
261	14	-1.09294017
321	17	1.14607460
397	20.8	0.11187711
461	24	-0.76853441
521	27	1.16599637
581	30	-1.06554696
702	36.3	0.13116006
801	41	-0.35706690
921	47	0.53245699
981	50	-0.97421735

Figure 3.5: Solution curve of ODE 3.5



Remark 3.3.5

Theorem 3.3.5 extends result of Wong and Yeh (1992), result of Philos (1989), result of Onose (1975) and result of Philos and Purnaras (1992) who have studied the special case of the equation (1.1) as $r(t) \equiv 1, \Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$ and result of E. M. Elabbasy (2000) who has studied the special case of the equation (1.1) as $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0.$ Our result can be applied on their equations (2.4) and (2.9) ,as mentioned in Chapter Two, but their oscillation results cannot be applied on the given equation in Example 3.3.5 because their equations are particular cases of our equation (1.1).

Theorem 3.3.6

Suppose that the conditions (1) and (5) hold. Assume that there exists continuously differentiable function ρ is defined as in Theorem 3.3.3 such that condition (6) holds and

(11) $\dfrac{r(t)}{\rho(t)} \leq \beta t, \beta \in (0, \infty),$

(12) $\liminf_{t\to\infty} \int_T^t \Psi(s)\, ds > -\infty$ for all large t.

(13) $\limsup_{t\to\infty} \dfrac{1}{t}\int_T^t \dfrac{1}{\rho(s)} \int_T^s \Psi(u) du ds = \infty,$

where $\Psi(t) = \rho(t)(C_0 q(t) - p(t)) - \dfrac{\dot{\rho}^2(t) r(t)}{4k\rho(t)}$ and $p:[t_0, \infty) \to (0, \infty).$ Then, every solution of equation (1.1) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ *on* $[T, \infty)$ for some $T \geq t_0 \geq 0.$ We define the function $\omega(t)$ as

$$\omega(t) = \frac{r(t)\dot{x}(t)}{g(x(t))},\ t \ge T.$$

This, the equation (1.1) and condition (1), we obtain

$$\left(\frac{r(t)\dot{x}(t)}{g(x(t))}\right)^{\bullet} \le -\left(C_0 q(t) - p(t)\right) - \frac{r(t)g'(x(t))\dot{x}^2(t)}{g^2(x(t))}, t \ge T$$

We integrate the last inequality multiplied by $\rho(t)$ form T to t, we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le C_1 - \int_T^t \rho(s)\left(C_0 q(s) - p(s)\right) ds + \int_T^t \frac{\dot{\rho}(s)r(s)\dot{x}(s)}{g(x(s))}\, ds$$
$$-\int_T^t \frac{\rho(s)r(s)g'(x(t))\dot{x}^2(s)}{g^2(x(s))}\, ds, \tag{3.3.8}$$

where

$$C_1 = \frac{\rho(T)r(T)\dot{x}(T)}{g(x(T))}.$$

Thus, we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le C_1 - \int_T^t \rho(s)\left(C_0 q(s) - p(s)\right) ds - \int_T^t \left[k\frac{\rho(s)}{r(s)}\omega^2(s) - \dot{\rho}(s)\omega(s)\right] ds$$
$$\le C_1 - \int_T^t \rho(s)\left(C_0 q(s) - p(s)\right) ds - \int_T^t k\frac{\rho(s)}{r(s)}\left(\eta^2(s) - \left(\frac{\dot{\rho}(s)r(s)}{2k\rho(s)}\right)^2\right) ds,$$

where $\eta(t) = \omega(t) - \dfrac{\dot{\rho}(t)r(t)}{2k\rho(t)}$.

Thus, we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \leq C_1 - \int_T^t \Psi(s)ds - \int_T^t \frac{k\rho(s)}{r(s)}\eta^2(s)\ ds. \tag{3.3.9}$$

Also, from the inequality (3.3.9) divided by $\rho(t)$, we have

$$\frac{1}{\rho(t)}\int_T^t \Psi(s)ds \leq \frac{C_1}{\rho(t)} - \omega(t). \tag{3.3.10}$$

Now, we have three cases for $\dot{x}(t)$.

Case 1**:** If $\dot{x}(t)$ is oscillatory, then, there exists a sequence τ_n in $[T,\infty)$ with $\lim\limits_{t\to\infty} \tau_n = \infty$ and such that $\dot{x}(\tau_n) = 0$. Then, from the inequality (3.3.9), we have

$$\int_T^{\tau_n} \frac{k\rho(s)}{r(s)}\eta^2(s)\ ds \leq C_1 - \int_T^{\tau_n} \Psi(s)ds.$$

Hence, by the condition (12), we get

$$\int_T^{\infty} \frac{k\rho(s)}{r(s)}\eta^2(s)\ ds < \infty.$$

This gives, for a positive constant N

$$\int_T^{\infty} \frac{k\rho(s)}{r(s)}\eta^2(s)\ ds < N \quad \text{for every } t \geq T. \tag{3.3.11}$$

Further, by using the Schwarz's inequality, for $t \geq T,$ we obtain

$$\left| -\int_T^t \eta(s)\ ds \right|^2 = \left| -\int_T^t \sqrt{\frac{k\rho(s)}{r(s)}}\eta(s)\sqrt{\frac{r(s)}{k\rho(s)}}\ ds \right|^2 \leq \int_T^t \frac{k\rho(s)}{r(s)}\eta^2(s)\,ds\int_T^t \frac{r(s)}{k\rho(s)}ds \leq \frac{N}{k}\int_T^t \frac{r(s)}{\rho(s)}ds.$$

By condition (11), the last inequality becomes

$$\left| -\int_T^t \eta(s)\ ds \right|^2 \leq \frac{N}{k}\beta\int_T^t sds = \frac{N\beta}{2k}(t^2 - T^2) \leq \frac{N\beta}{2k}t^2.$$

Then,

$$-\int_T^t \eta(s)\ ds = -\int_T^t \omega(s) - \frac{\dot{\rho}(s)r(s)}{2k\rho(s)}\ ds \leq \sqrt{\frac{N\beta}{2k}}t.$$

Thus, for $t \geq T,$ we have

$$-\int_T^t \omega(s)\,ds \leq \sqrt{\frac{N\beta}{2k}}t \tag{3.3.12}$$

Integrate the inequality (3.3.10) and from (3.3.12), we obtain

$$\begin{aligned}\int_T^t \frac{1}{\rho(s)}\int_T^s \Psi(u)duds &\leq C_1\int_T^t \frac{ds}{\rho(s)} - \int_T^t \omega(s)ds \\ &\leq \frac{C_1}{\rho(T)}(t-T) + \sqrt{\frac{N\beta}{2k}}t \leq \frac{C_1}{\rho(T)}t + \sqrt{\frac{N\beta}{2k}}t.\end{aligned}$$

Dividing the last inequality by *t* and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_T^t \frac{1}{\rho(s)}\int_T^s \Psi(u)duds \leq \frac{C_1}{\rho(T)} + \sqrt{\frac{N\beta}{2k}} < \infty,$$

which contradicts condition (13).

Case 2: If $\dot{x}(t) > 0 \;\; for \;\; t \geq T_1 \geq T$, then, from (3.3.10), we get

$$\frac{1}{\rho(t)}\int_{T_1}^{t}\Psi(s)ds \leq \frac{C_1}{\rho(t)}.$$

Integrate the last inequality, dividing by *t* and taking the limit superior on both sides, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_{T_1}^{t}\frac{1}{\rho(s)}\int_{T_1}^{s}\Psi(u)duds \leq \frac{C_1}{\rho(T_1)} < \infty,$$

which also contradicts condition (13).

Case 3: if $\dot{x}(t) < 0$ for $t \geq T_2 \geq T$. Then from inequality (3.3.8), we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \leq C_1 - \int_{T_2}^{t}\rho(s)\big(C_0 q(s) - p(s)\big)\,ds - \int_{T_2}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds.$$

Now, we have two cases for $\int_{T_2}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds$: If this integral is finite, in this case, we can get a contradiction by the procedure of case (1). If this integral is infinite, from condition (12), we obtain

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \leq C_1 + \lambda - \int_{T_2}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds.$$

Also, from the last inequality, we obtain

$$-\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \geq -\left(C_1+\lambda\right)+\int_{T_2}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds$$

$$\geq N^* + \int_{T_2}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds, \qquad (3.3.13)$$

where $N^* = -(C_1+\lambda)$.

We consider a $T_3 \geq T_2$ such that

$$N_1 = N^* + k\int_{T_2}^{T_3}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds > 0$$

Hence, for all $t \geq T_3$, we get

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \leq -\left[N^* + \int_{T_3}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds\right].$$

From the last inequality, we get

$$\frac{\rho(t)r(t)g'(x(t))\dot{x}^2(t)}{g^2(x(t))}\Bigg/\left[N^* + \int_{T_3}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds\right] \leq -\frac{g'(x(t))\dot{x}(s)}{g(x(s))}.$$

Integrate the last inequality from T_3 to t, we have

$$\ln\left\{N^* + \int_{T_3}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds\right\}_{T_3}^{t} = \ln\left\{N^* + \int_{T_3}^{t}\frac{\rho(s)r(s)g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds \Big/ N_1\right\} \geq \ln\left\{\frac{g(x(T_3))}{g(x(t))}\right\}.$$

Thus,

$$N^* + \int_{T_3}^{t} \frac{\rho(s) r(s) g'(x(s)) \dot{x}^2(s)}{g^2(x(s))} ds \geq N_1 \frac{g(x(T_3))}{g(x(t))}, \; t \geq T_3.$$

From (3.3.13), we have

$$-\frac{\rho(t) r(t) \dot{x}(t)}{g(x(t))} \geq N_1 \frac{g(x(T_3))}{g(x(t))}, \; t \geq T_3.$$

Since $N_1 g(x(T_3)) > 0.$ Thus, from the last inequality, we obtain

$$x(t) \leq x(T_3) - N_1 g(x(T_3)) \int_{T_3}^{t} \frac{ds}{\rho(s) r(s)} \; , t \geq T_3.$$

which leads to $\lim_{t \to \infty} x(t) = -\infty$, which is a contradiction to the fact that $x(t) > 0$ for $t \geq T$.

Hence, the proof is completed.

Example 3.3.6

Consider the following differential equation

$$\left(t^{3/4} \dot{x}(t)\right)^{\bullet} + (t^3 + 2)\left(x^3(t) + \frac{x^{21}(t)}{2x^{18}(t) + \left(t^{3/4} \dot{x}(t)\right)^6} \right) = \frac{x^3(t) \sin t}{t^8}, \; t > 0.$$

We have $r(t) = t^{3/4}$, $q(t) = t^3 + 2$, $g(x) = x^3$, $\Phi(u,v) = u + \dfrac{u^7}{2u^6 + v^6}$ and

$\frac{H(t,x(t))}{g(x(t))} = \frac{\sin t}{t^8} \le \frac{1}{t^8} = p(t)$ for all $x \ne 0$ and $t > 0$. Taking $\rho(t) = 5$ such that

$$\int_{t_0}^{\infty} \frac{ds}{\rho(s)r(s)} = \infty \quad and \quad \frac{r(t)}{\rho(s)} = t^{3/4} \le t\,,\ \beta = 1,$$

$$\liminf_{t\to\infty} \left\{ \int_T^t \Psi(s)ds \right\} = \liminf_{t\to\infty} \left\{ \int_T^t \left[\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{\rho^{\bullet 2}(s)r(s)}{4k\rho(s)} \right] ds \right\}$$
$$= \liminf_{t\to\infty} \left\{ \int_T^t C_0 (s^3 + 2) - \frac{1}{s^8}\, ds \right\} = \infty > -\infty,$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_T^t \frac{1}{\rho(s)} \int_T^s \Psi(u)\, du ds = \limsup_{t\to\infty} \frac{1}{t} \int_T^t \frac{1}{\rho(s)} \int_T^s \left[\big(C_0 q(u) - p(u)\big) - \frac{\rho^{\bullet 2}(u)r(u)}{4k\rho(u)} \right] du ds$$
$$= \limsup_{t\to\infty} \frac{1}{t} \int_T^t \int_T^s C_0 (u^3 + 2) - \frac{1}{u^8}\, du\, ds = \infty.$$

All conditions of Theorem 3.3.6 are satisfied and hence every solution of the given equation is oscillatory. To demonstrate that our result in Theorem 3.3.6 is true, we also find the numerical solution of the given differential equation in Example 3.3.6 using the Runge Kutta method of fourth order.

We have

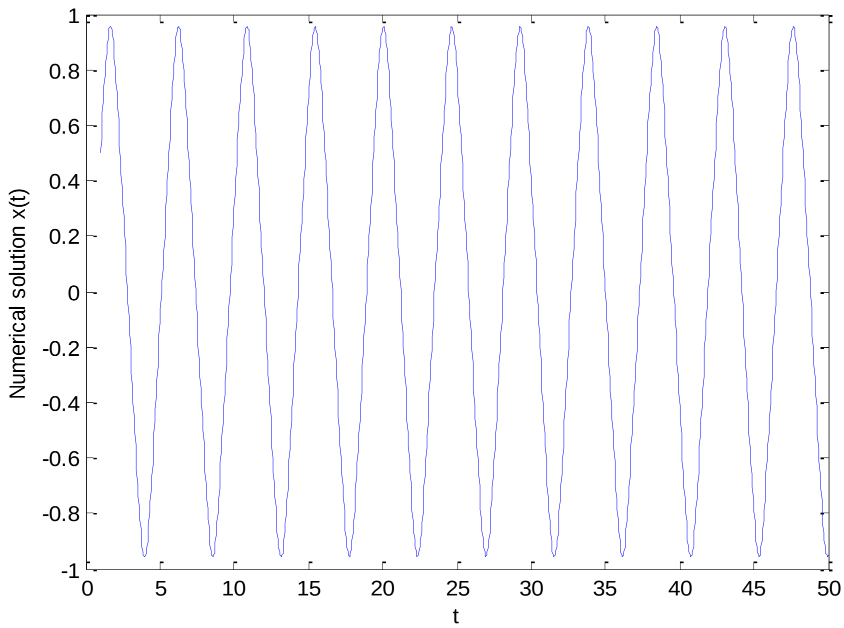
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^3 - 3\left(x^3 + \frac{x^{21}}{2x^{18} + \dot{x}^6(t)} \right)$$

with initial conditions $x(1) = 0.5, \dot{x}(1) = 1$ on the chosen interval $[1,50]$ and finding the values the functions *r*, *q* and f where we consider $H(t,x) = f(t)l(x)$ at *t=1*, *n*=980 and *h*=0.05.

Table 3.6: Numerical solution of ODE 3.6

k	**t_k**	**$x(t_k)$**
1	1	0.5
81	5	-0.11388331
181	10	0.29975744
222	12.05	-0.04307199
313	16.6	0.00691001
405	21.25	-0.04618202
461	24	0.45734920
251	27	-0.95229708
581	30	0.41341535
682	35.05	-0.05082571
821	42	0.10467651
916	48.8	-0.00391035
981	50	-0.95114725

Figure 3.6: Solution curve of ODE 3.6



Remark 3.3.6: Theorem 3.3.6 extends result of Popa (1981) for the equation (2.2), result of Wong (1973) for the equation (2.3), results of Onose (1975), Philos (1985) and Yeh (1982) for the equation (2.4) and result of E. Elabbasy (2000) for the equation (2.6). Our result can be applied on their equations (2.2), (2.3), (2.4) and (2.6) respectively, but their previous oscillation results cannot be applied on the given equation in Example 3.3.6 because their equations are particular cases of our equation (1.1).

Theorem 3.3.7: Suppose, in addition to the condition (1) holds that

(14) $\int_T^\infty \frac{ds}{r(s)} \le k_1 \,,\; k_1 > 0.$

(15) There exists a constant B^* such that

$$G(m) = \int_0^m \frac{ds}{\Phi(1,s)} > B^* m \,,\; B^* \in (-\infty, 0) \;\; and \;\; m \in \mathrm{R}.$$

Furthermore, suppose that there exists a positive continuous differentiable function ρ on the interval $[t_0, \infty)$ with $\rho(t)$ is a non-decreasing function on the interval $[t_0, \infty)$ such that

(16) $\limsup_{t \to \infty} \int_T^t \frac{1}{r(s)\rho(s)} \int_T^s \rho(u)[C_0 q(u) - p(u)] du\, ds = \infty,$

where, $p : [t_0, \infty) \to (0, \infty)$. Then, every solution of superlinear equation (1.1) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ *on* $[T, \infty)$ *for some* $T \geq t_0 \geq 0$. Define

$$\omega(t) = \frac{\rho(t) r(t) \dot{x}(t)}{g(x(t))},\ t \geq T.$$

Thus and (1.1) imply

$$\dot{\omega}(t) \leq \rho(t) p(t) - \rho(t) q(t) \Phi(1, \omega(t)/\rho(t)) + \frac{\dot{\rho}(t)}{\rho(t)} \omega(t),\ t \geq T.$$

Thus, we have

$$\rho(t) \left(\frac{\omega(t)}{\rho(t)} \right)^{\bullet} \leq \rho(t) p(t) - \rho(t) q(t) \Phi(1, \omega(t)/\rho(t)),\ \ t \geq T.$$

Dividing the last inequality by $\Phi(1, \omega(t)/\rho(t)) > 0$ and by condition (1), we obtain

$$\rho(t)[C_0 q(t) - p(t)] \leq -C_0 \frac{\rho(t)(\omega(t)/\rho(t))^{\bullet}}{\Phi(1, \omega(t)/\rho(t))},\ t \geq T$$

Integrate from T to t, we have

$$\int_T^t \rho(s)[C_0 q(s) - p(s)] ds \leq -C_0 \int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1, \omega(s)/\rho(s))} ds\ , t \geq T \qquad (3.3.14)$$

By Bonnet's theorem, since $\rho(t)$ is a non-decreasing function on the interval $[t_0, \infty)$, there exists $T_1 \in [T, t]$ such that

$$\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds = \rho(t)\int_{T_1}^t \frac{(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds. \qquad (3.3.15)$$

Using the inequality (3.3.15) in the inequality (3.3.14), we have

$$\begin{aligned}\int_T^t \rho(s)[C_0 q(s) - p(s)]\,ds &\le -C_0\rho(t)\int_{\omega(T_1)/\rho(T_1)}^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)} \\ &\le -C_0\rho(t)\left[-\int_0^{\omega(T_1)/\rho(T_1)} \frac{du}{\Phi(1,u)} + \int_0^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)}\right] \\ &\le C_0\rho(t)G\left(\frac{\omega(T_1)}{\rho(T_1)}\right) - C_0\rho(t)G\left(\frac{\omega(t)}{\rho(t)}\right).\end{aligned}$$

Thus, by condition (15), we obtain

$$\int_T^t \rho(s)[C_0 q(s) - p(s)]ds \le C_0\rho(t)G\left(\frac{\omega(T_1)}{\rho(T_1)}\right) - C_0 B^{*}\omega(t).$$

Integrating the last inequality divided by $\rho(t)\,r(t)$ from T to t, we obtain

$$\int_T^t \frac{1}{r(s)\rho(s)}\int_T^s \rho(u)[C_0 q(u) - p(u)]du\,ds \le C_0 G\left(\frac{\omega(T_1)}{\rho(T_1)}\right)\int_T^t \frac{ds}{r(s)} - C_0 B^{*}\int_T^t \frac{\dot{x}(s)}{g(x(s))}ds$$

Taking the limit superior on both sides, by condition (14) and since the equation (1.1) is superlinear, we have

$$\limsup_{t\to\infty}\int_T^t \frac{1}{r(s)\rho(s)}\int_T^s \rho(u)[C_0 q(u) - p(u)]du\,ds \le \limsup_{t\to\infty}\left\{k_1 C_0 G\left(\frac{\omega(T_1)}{\rho(T_1)}\right) - C_0 B^{*}\int_{x(T)}^{x(t)} \frac{du}{g(u)}\right\} < \infty,$$

as $t \to \infty$, which contradicts to the condition (16). Hence the proof is completed.

Example 3.3.7

Consider the following differential equation

$$\left(t^2 \dot{x}(t)\right)^{\bullet} + t^6\left(\frac{x^9(t)}{x^6(t)+\left(\dot{x}(t)/t^2\right)^2}\right) = \frac{x^3(t)\cos x}{t^7},\ t > 0.$$

We note that $r(t) = t^2$, $q(t) = t^6$, $g(x) = x^3$ and $\Phi(u,v) = \dfrac{u^3}{u^2+v^2}$ such that

$$G(m) = \int_0^m \frac{ds}{\Phi(1,s)} = \int_0^m (1+s^2)ds = \int_0^m -1\,ds = -m \geq -(1)m, thus,\ B = -1, B \in R^- and\ for\ all\ m \in R.$$

$$\frac{H(t,x(t))}{g(x(t))} = \frac{\cos x}{t^7} \leq \frac{1}{t^7} = p(t),\ for\ all\ t > 0\ and\ x \neq 0.$$ Let $\rho(t) = t^4$ such that

$$\limsup_{t\to\infty}\int_T^t \frac{1}{r(s)\rho(s)}\int_T^s \rho(u)\left[C_0 q(u) - p(u)\right]du\,ds = \limsup_{t\to\infty}\int_T^t \frac{1}{s^6}\int_T^s u^4\left[C_0 u^6 - \frac{1}{u^7}\right]du\,ds = \infty.$$

All conditions of Theorem 3.3.7 are satisfied and hence every solution of the given equation is oscillatory. The numerical solution of the given differential equation using Runge Kutta method of fourth order (RK4) is as follows:

We have

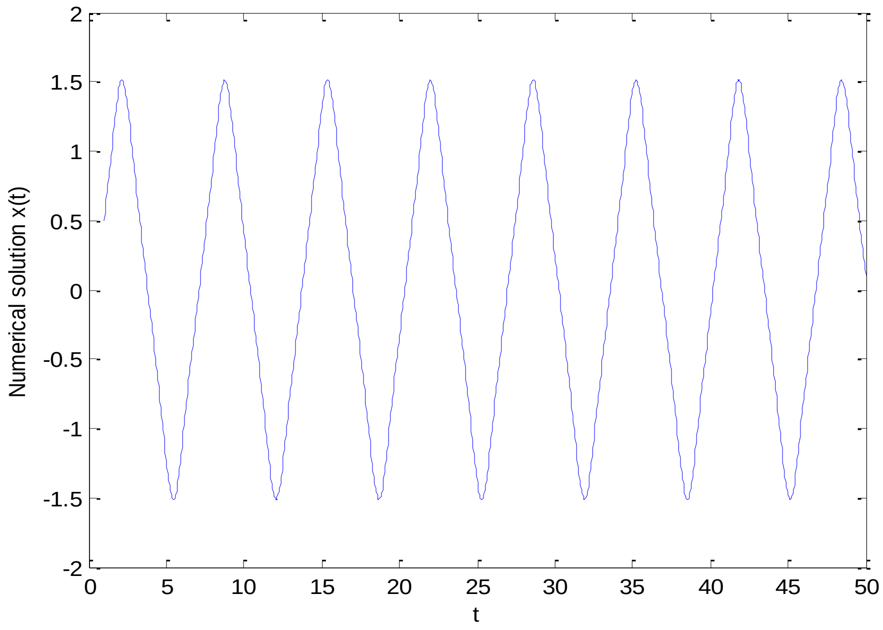
$$\ddot{x}(t) = f(t, x(t), p = \dot{x}(t)) = x^3(t)\cos(x(t)) - \frac{x^9(t)}{x^6(t) + \dot{x}^2(t)}$$

with initial conditions $x(1) = 0.5,\ \dot{x}(1) = 1$ on the chosen interval $[1,50]$ and finding the values of the functions *r*, *q* and f where we consider $H(t,x(t)) = f(t)\,l(x)$ at *t=1* *n*=980 and *h*=0.05.

Table 3.7: Numerical solution of ODE 3.7

k	t_k	$x(t_k)$
1	1	0.5
81	5	-1.23206499
181	10	0.40954827
221	12	-1.50525503
321	17	0.02833177
381	20	-0.33103118
521	27	0.04981132
586	30.3	-0.04300197
661	34	0.43117751
721	37	-0.12794647
821	42	1.46730560
917	46.6	-0.18820133
981	50	0.09644083

Figure 3.7: Solution curve of ODE 3.7



Remark 3.3.7: Theorem 3.3.7 extends results of Bihari (1963) and Kartsatos (1968) who have studied the equation (2.5) as mentioned in chapter two. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 3.3.7 because their equation is a particular case of our equation (1.1) when $r(t) \equiv 1,\ g(x(t)) \equiv x(t),\ \Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t)) \equiv 0$.

Theorem 3.3.8: Suppose, in addition to the condition (5) holds that

(17) $\dot{r}(t) \leq 0$ *for all* $t \geq t_0$ *and* $(r(t)q(t))^{\bullet} \geq 0$ *for all* $t \geq t_0$.

(18) $\Phi(1, v) \geq v$ *for all* $v \neq 0$.

(19) $\limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_2 r(s)q(s) - \int_{t_0}^s p(u)du \right] ds = \infty,$

where $p:[t_0,\infty) \to (0,\infty)$, then, every solution of superlinear equation (1.1) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ *on* $[T,\infty)$ *for some* $T \geq t_0 \geq 0$. Define

$$\omega(t) = \frac{r(t)\dot{x}(t)}{g(x(t))},\ t \geq T.$$

From $\omega(t)$, equation (1.1) and condition (18), we get

$$\left(\frac{r(t)\dot{x}(t)}{g(x(t))} \right)^{\bullet} \leq p(t) - q(t)\omega(t).$$

Integrate the last inequality from T to t, we obtain

$$\frac{r(t)\dot{x}(t)}{g(x(t))} \le A_1 + \int_T^t p(s)ds - \int_T^t r(s)q(s)\frac{\dot{x}(s)}{g(x(s))}ds, \tag{3.3.16}$$

where

$$A_1 = \frac{r(T)\dot{x}(T)}{g(x(T))}.$$

By condition (17) and the Bonnet's Theorem, we see that for each $t \ge T,$ there exists $T_2 \in [T,t]$ such that

$$\int_T^t r(s)q(s)\frac{\dot{x}(s)}{g(x(s))}ds = r(t)q(t)\int_{T_2}^t \frac{\dot{x}(s)}{g(x(s))}ds = r(t)q(t)\int_{x(T_2)}^{x(t)}\frac{du}{g(u)}.$$

Since $r(t)q(t) \ge 0$ and the equation (1.1) is superlinear, we have

$$\int_{x(T_2)}^{x(t)}\frac{du}{g(u)} < \begin{cases} 0 & , \text{ if } x(t) < x(T_2) \\ \int_{x(T_2)}^{\infty}\frac{du}{g(u)} & , \text{ if } x(t) \ge x(T_2). \end{cases}$$

Thus, it follows that

$$\int_T^t r(s)q(s)\frac{\dot{x}(s)}{g(x(s))}ds \ge A_2 r(t)q(t), \text{ where } A_2 = \inf \int_{x(T_2)}^{x(t)}\frac{du}{g(u)}.$$

Thus, the inequality (3.3.16) becomes

$$\frac{r(t)\dot{x}(t)}{g(x(t))} \le A_1 + \int_T^t p(s)ds - A_2 r(t)q(t).$$

Integrate the last inequality from T to t, we have

$$\int_T^t \frac{r(s)\dot{x}(s)}{g(x(s))}ds \le A_1(t-T) - \int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds.$$

Since $r(t)$ is positive and non-increasing for $t \ge T,$ the equation (1.1) is superlinear and by Bonnet's theorem, there exists $\beta_t \in [t,T]$ such that

$$\int_T^t \frac{r(s)\dot{x}(s)}{g(x(s))}ds = r(T)\int_{x(T)}^{x(\beta_t)} \frac{du}{g(u)} \ge A_3 r(T), \text{ where } A_3 = \inf \int_{x(T)}^{x(\beta_t)} \frac{du}{g(u)}.$$

Thus, for $t \ge T,$ we have

$$\int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds \le A_1(t-T) - A_3 r(T).$$

Dividing the last inequality by t and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds \le \limsup_{t\to\infty} A_1(1-\frac{T}{t}) - \limsup_{t\to\infty} \frac{A_3 r(T)}{t} < \infty,$$

as $t \to \infty$, which contradicts to the condition (19). Hence the proof is completed.

Example 3.3.8

Consider the following differential equation

$$\left(\dot{x}(t)/t\right)^{\bullet} + t^4 \left(x^5(t) + \frac{x^{15}(t)}{x^{10}(t) + \left(\dot{x}(t)/t\right)^2} \right) = \frac{x^7(t)\cos(x(t))}{t^4(x^2(t)+1)},\ t > 0.$$

Here

$r(t) = 1/t$, $q(t) = t^4$, $g(x) = x^5$, $\Phi(u,v) = u + \dfrac{u^3}{u^2 + v^2}$ and

$$\frac{H(t,x(t))}{g(x(t))} = \frac{x^2(t)\cos(x(t))}{t^4(x^2(t)+1)} \le \frac{1}{t^4} = p(t) \quad \textit{for all } t > 0 \textit{ and } x \neq 0.$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds = \limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_2 s^3 - 6\int_T^s \frac{du}{u^4} \right] ds = \infty.$$

All conditions of Theorem 3.3.8 are satisfied and hence every solution of the given equation is oscillatory. The numerical solution of the given equation using the Runge Kutta method of fourth order (RK4) is as follows:

We have

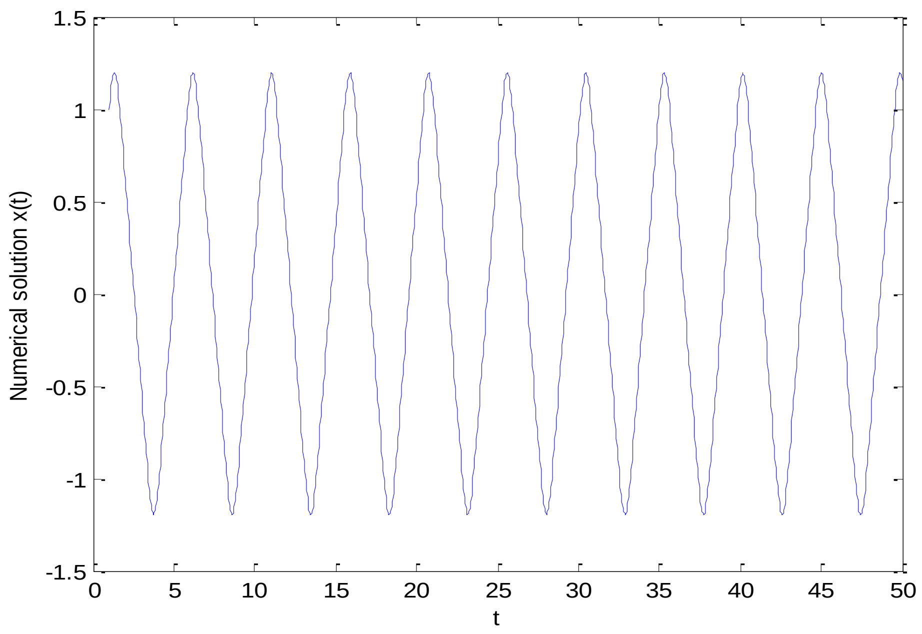
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = \frac{x^7(t)\cos(x(t))}{x^2(t)+1} - \left(x^5(t) + x^{15}(t) \Big/ \left(x^{10}(t) + \dot{x}^2(t) \right) \right)$$

with initial conditions $x(1) = 1,\ \dot{x}(1) = 1$ on the chosen interval $[1,50]$ and finding the values of the functions r, q and f where we consider $H(t, x(t)) = f(t)l(x)$ at t=*1*, n=980 and h=0.05.

Table 3.8: Numerical solution of ODE 3.8

k	t_k	$x(t_k)$
1	1	1
80	4.95	-0.02069572
181	10	0.20451305
226	12.25	-0.00138478
321	17	0.11785698
421	22	-0.04991059
475	24.7	0.36339219
521	27	-0.21766245
581	30	0.86993097
661	34	-0.10477736
761	39	0.06293869
821	42	-0.71934519
955	48.7	0.05390472
981	50	1.15722428

Figure 3.8: Solution curve of ODE 3.8



Remark 3.3.8:

Theorem 3.3.8 extends results of Bihari (1963), Kartsatos (1968) who have studied the equation (1.1) as $r(t) \equiv 1$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t)) \equiv 0$. Also, Theorem 3.3.8 extends results of Elabbasy (2000) who has considered the equation (1.1) as $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$. Our result can be applied on their equations (2.5) and (2.6), but their oscillation results cannot be applied on the given equation in Example 3.3.8 because their equations are particular cases of our equation (1.1).

Theorem 3.3.9: Suppose that the conditions (1) and (5) hold. Moreover, assume that there exist a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ and the continuous functions $h,\ H : D = \{(t,s) : t \ge s \ge t_0\} \to \mathrm{R}$, H has a continuous and non-positive partial derivative on D with respect to the second variable such that

$$H(t,t) = 0 \quad for\ t \ge t_o,\ H(t,s) > 0 \quad for\ t > s \ge t_0.$$

$$-\frac{\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \forall\,(t,s) \in D.$$

(20) If $\limsup_{t\to\infty} \dfrac{1}{H(t,t_0)} \displaystyle\int_{t_0}^{t} r(s)\rho(s)\sigma^2(t,s)\,ds < \infty$,

where $\sigma(t,s) = \left[h(t,s) - \dfrac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(t,s)} \right]$.

(21) $\limsup_{t\to\infty} \dfrac{1}{H(t,t_0)} \displaystyle\int_{t_0}^{t} H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds = \infty.$

Then, every solution of equation (1.1) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution x(t) of equation (1.1) such that $x(t) > 0$ *on* $[T, \infty)$ *for some* $T \geq t_0 \geq 0$. We define the function $\omega(t)$ as

$$\omega(t) = \frac{\rho(t) r(t) \dot{x}(t)}{g(x(t))},\ t \geq T.$$

This and the equation (1.1) imply

$$\dot{\omega}(t) \leq \rho(t) p(t) - \rho(t) q(t) \Phi(1, v(t)) + \frac{\dot{\rho}(t)}{\rho(t)} \omega(t) - \frac{k}{\rho(t) r(t)} \omega^2(t),\ t \geq T,$$

where $v(t) = \omega(t) / \rho(t)$.

Then, by condition (1), we have for all $t \geq T$

$$\dot{\omega}(t) \leq \rho(t) p(t) - C_0 \rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} \omega(t) - \frac{k}{\rho(t) r(t)} \omega^2(t),\ \ t \geq T.$$

Integrate the last inequality multiplied by $H(t,s)$ from T to t, we have

$$\begin{aligned}\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big) ds &\leq H(t,T)\omega(T) - \int_T^t \left[-\frac{\partial}{\partial s} H(t,s) \right] \omega(s) ds + \int_T^t \frac{\dot{\rho}(s)}{\rho(s)} H(t,s)\omega(s) ds \\ &\quad - \int_T^t \frac{kH(t,s)}{\rho(s) r(s)} \omega^2(s) ds \\ &\leq H(t,T)\omega(T) - \int_T^t \left[\frac{kH(t,s)}{\rho(s) r(s)} \omega^2(s) + \sigma(t,s)\sqrt{H(t,s)}\,\omega(s) \right] ds,\end{aligned}$$

where $\sigma(t,s) = h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}.$

Hence, we have

$$\int_T^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right) ds \le H(t,T)\omega(T) - \int_T^t \left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s) + \frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds$$
$$+ \int_T^t \frac{\rho(s)r(s)}{4k}\sigma^2(t,s)\,ds. \qquad (3.3.17)$$

Then, for $t \ge T,$ we have

$$\int_T^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right)ds \le H(t,T)\omega(T) + \frac{1}{4k}\int_T^t r(s)\rho(s)\sigma^2(t,s)\ ds, t \ge T.$$

Dividing the last inequality by $H(t,T)$, taking the limit superior as $t \to \infty$ and by condition (20), we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t H(t,s)\,\rho(s)\left(C_0 q(s) - p(s)\right)ds \le \omega(T)$$
$$+ \frac{1}{4k}\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\sigma^2(t,s)\,ds < \infty,$$

which contradicts to the condition (21). Hence, the proof is completed.

Example 3.3.9: Consider the differential equation

$$\left(\dot{x}(t)\big/t^2\right)^{\bullet} + t^5\left(x^3(t) + \frac{3x^9(t)}{4x^6(t) + \left(\dot{x}(t)\big/t^2\right)^2}\right) = \frac{x^3(t)\cos x(t)}{t^7},\ t > 0.$$

We have

$$r(t)=\frac{1}{t^2},\ q(t)=t^5,\ g(x)=x^3,\ \Phi(u,v)=u+\frac{3u^3}{4u^2+v^2}\ \text{ and}$$

$$\frac{H(t,x(t))}{g(x(t))}=\frac{\cos(x(t))}{t^7}\le\frac{1}{t^7}=p(t).$$

Let $H(t,s)=(t-s)^2\ge 0\ \ \forall\, t\ge s\ge t_0>0,$ then,

$$\frac{\partial}{\partial s}H(t,s)=-2(t-s)\ \ and\ \ thus\ \ h(t,s)=-2.$$

Taking $\rho(t)=t^2$ such that

$$(1)\quad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} r(s)\rho(s)\sigma^2(t,s)ds=\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t} r(s)\rho(s)\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_{T}^{t}\left(-2-\frac{2}{s}(t-s)\right)^2 ds=\frac{4}{T}<\infty,$$

$$(2)\quad \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t} H(t,s)\ \rho(s)\big(C_0q(s)-p(s)\big)ds\ =\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_{T}^{t}(t-s)^2\left(C_0s^5-\frac{1}{s^7}\right)ds$$

$$=\infty.$$

All conditions of Theorem 3.3.9 are satisfied, then, the given equation is oscillatory. Also the numerical solution of the given differential equation is computed using the Runge Kutta method of fourth order.

We have

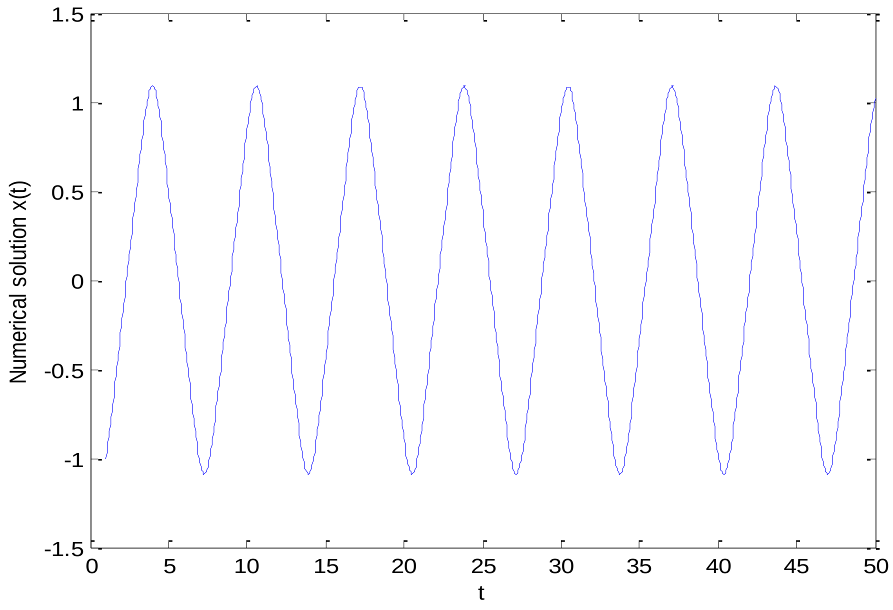
$$\ddot{x}(t)=\ f(t,x(t),\dot{x}(t))=x^3(t)\cos(x(t))-\left(x^3(t)+\frac{3x^9(t)}{4x^6(t)+\dot{x}^2(t)}\right)$$

with initial conditions $x(1) = -1,\ \dot{x}(1) = 0.5$ on the chosen interval $[1,50]$ and finding the values of the functions *r*, *q* and *f* where we consider $H(t,x) = f(t)\,l(x)$ at $t=1$, n=980 and h= 0.05.

Table 3.9: Numerical solution of ODE 3.9

k	t_k	$x(t_k)$
1	1	-1
81	5	0.50348046
226	12.25	-0.00392170
321	17	1.05359426
381	20	-0.89282510
461	24	1.06448349
521	27	-1.07378290
581	30	0.93949122
661	34	-1.04029283
721	37	1.08568566
781	40	-0.98145745
841	43	0.77568867
921	47	-1.08892372
981	50	1.01773294

Figure 3.9: Solution curve of ODE 3.9



Remark 3.3.9: Theorem 3.3.9 extends Kamenev's result (1978) and Philos's result (1989) who have studied a special case of the equation (1.1) as $r(t) \equiv 1,\ g(x(t)) \equiv x(t),\ \Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$. Our result can be applied on their equation however; their results cannot be applied to the given equation in Example 3.3.9.

Theorem 3.3.10

Suppose, in addition to the conditions (1), (5) and (20) hold that there exist continuous functions h and H are defined as in Theorem 3.3.9 and suppose that

$$(22) \qquad 0 < \inf_{s \geq t_0} \left[\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty.$$

If there exists a continuous function Ω *on* $[t_0, \infty)$ such that

$$(23)\qquad \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)\big(C_0q(s)-p(s)\big)-\frac{1}{4k}r(s)\rho(s)\sigma^2(t,s)\right]ds\ge\Omega(T)$$

for $T\ge t_0$, where $\sigma(t,s)=h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)},$ k is a positive constant and a differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ and

$$(24)\qquad \int_T^\infty\frac{\Omega_+^2(s)}{\rho(s)r(s)}ds=\infty,$$

where $\Omega_+(t)=\max\{\Omega(t),0\}$, then every solution of equation (1.1) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t)>0$ on $[T,\infty)$ for some $T\ge t_0\ge 0.$ Dividing inequality (3.3.17) by $H(t,T)$ and taking the limit superior as $t\to\infty,$ we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)\big(C_0q(s)-p(s)\big)-\frac{1}{4k}\rho(s)r(s)\sigma^2(t,s)\right]ds\le\omega(T)$$
$$-\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2ds$$
$$\le\omega(T)-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma\big(t,s)\big)\right]^2ds.$$

By condition (23), we get

$$\omega(T)\ge\Omega(T)+\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2ds.$$

This shows that

$$\omega(T) \geq \Omega(T) \quad \textit{for every} \;\; t \geq T, \tag{3.3.18}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s) + \frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\,\sigma(t,s) \right]^2 ds < \infty.$$

Hence,

$$\begin{aligned} \infty &> \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s) + \frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\,\sigma(t,s) \right]^2 ds \\ &\geq \liminf_{t\to\infty} \left[\frac{1}{H(t,t_0)} \int_{t_0}^t \frac{kH(t,s)}{\rho(s)r(s)}\omega^2(s)ds + \frac{1}{H(t,t_0)} \int_{t_0}^t \sigma(t,s)\sqrt{H(t,s)}\omega(s)ds \right]. \end{aligned} \tag{3.3.19}$$

Define

$$U(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{kH(t,s)}{\rho(s)r(s)}\omega^2(s)\,ds,\; t \geq t_0$$

and

$$V(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t \sigma(t,s)\sqrt{H(t,s)}\omega(s)ds\,,\; t \geq t_0.$$

Then, (3.3.19) becomes

$$\liminf_{t\to\infty} \left[U(t) + V(t)\right] < \infty. \tag{3.3.20}$$

Now, suppose that

$$\int_{t_0}^{\infty} \frac{\omega^2(s)}{\rho(s)r(s)} ds = \infty. \qquad (3.3.21)$$

Then, by condition (22) we can easily see that

$$\lim_{t\to\infty} U(t) = \infty. \qquad (3.3.22)$$

Let us consider a sequence $\{T_n\}_{n=1,2,3,...}$ in $[t_0, \infty)$ with $\lim_{n\to\infty} T_n = \infty$ and such that

$$\lim_{n\to\infty}[U(T_n) + V(T_n)] = \lim_{t\to\infty} \inf [U(t) + V(t)].$$

By inequality (3.3.20) there exists a constant N such that

$$U(T_n) + V(T_n) \le N, \; n = 1,2,3,... \qquad (3.3.23)$$

From inequality (3.3.22), we have

$$\lim_{n\to\infty} U(T_n) = \infty. \qquad (3.3.24)$$

And hence inequality (3.3.23) gives

$$\lim_{n\to\infty} V(T_n) = -\infty. \qquad (3.3.25)$$

By taking into account inequality (3.3.24), from inequality (3.3.23), we obtain

$$1 + \frac{V(T_n)}{U(T_n)} \le \frac{N}{U(T_n)} < \frac{1}{2}.$$

Provided that n is sufficiently large. Thus

$$\frac{V(T_n)}{U(T_n)} < -\frac{1}{2},$$

which by inequality (3.3.25) and inequality (3.3.23) we have

$$\lim_{n\to\infty} \frac{V^2(T_n)}{U(T_n)} = \infty. \tag{3.3.26}$$

On the other hand by Schwarz's inequality, we have

$$\begin{aligned} V^2(T_n) &= \frac{1}{H^2(T_n,t_0)}\left[\int_{t_0}^{T_n} \sigma(T_n,s)\sqrt{H(T_n,s)}\omega(s)ds\right]^2 \\ &\le \left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{\rho(s)r(s)}{k}\sigma^2(T_n,s)ds\right]\times\left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{kH(T_n,s)}{r(s)\rho(s)}\omega^2(s)ds\right] \\ &= \frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{\rho(s)r(s)}{k}\sigma^2(T_n,s)ds\times U(T_n). \end{aligned}$$

Thus, we have

$$\frac{V^2(T_n)}{U(T_n)} \le \frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{\rho(s)r(s)}{k}\sigma^2(T_n,s)ds \quad \text{for large } n.$$

By inequality (3.3.26), we have

$$\frac{1}{k}\lim_{n\to\infty} \frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} r(s)\rho(s)\sigma^2(T_n,s)ds = \infty.$$

Consequently,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} r(s)\rho(s)\sigma^2(t,s)ds = \infty,$$

which contradicts to the condition (20). Thus, inequality (3.3.21) fails and hence

$$\int_{t_0}^{\infty} \frac{\omega^2(s)}{r(s)\rho(s)}\, ds < \infty.$$

Hence from inequality (3.3.18), we have

$$\int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{r(s)\rho(s)}\, ds \le \int_{t_0}^{\infty} \frac{\omega^2(s)}{r(s)\rho(s)}\, ds < \infty,$$

which, contradicts to the condition (24). Hence the proof is completed.

Example 3.3.10

Consider the following differential equation

$$\left(\frac{\dot{x}(t)}{t^6}\right)^{\bullet} + \frac{1}{t^3}(x^7(t) + \left[\frac{x^{133}(t)}{9x^{126}(t) + 6\left(\dot{x}(t)\big/t^6\right)^{18}}\right] = -\frac{x^9(t)\sin^2 x(t)}{(x^2(t)+1)},\ t > 0.$$

We note that $r(t) = \frac{1}{t^6}$, $q(t) = \frac{1}{t^3}$, $g(x) = x^7$, $\Phi(u,v) = u + \frac{u^{19}}{9u^{18} + 6v^{18}}$ *and*

$$\frac{H(t,x(t))}{g(x(t))} = -\frac{x^2(t)\sin^2 x(t)}{(x^2(t)+1)} \le -\frac{x^2(t)}{(x^2(t)+1)} \le 0 = p(t) \quad \textit{for all } t \ge t_0.$$

We let $H(t,s) = (t-s)^2 > 0$ *for all* $t > s \ge t_0$, thus

$\frac{\partial}{\partial s} H(t,s) = -2(t-s) = h(t,s)\sqrt{H(t,s)}$ for all $t \ge t_0 > 0$. Taking $\rho(t) = 6$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\sigma^2(t,s)ds = \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds$$

$$= \limsup_{t\to\infty}\frac{24}{(t-T)^2}\int_T^t \frac{1}{s^6}ds = 0 < \infty,$$

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] = \inf_{s\geq t_0}\left(\liminf_{t\to\infty}\frac{(t-s)^2}{(t-t_0)^2}\right) = \inf_{s\geq t_0}(1) = 1,$$

$$\textit{thus}\ 0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] < \infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)\left(C_0q(s)-p(s)\right)-\frac{r(s)\rho(s)}{4k}\sigma^2(t,s)\right]ds$$

$$= \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[6C_0\frac{(t-s)^2}{s^3}-\frac{6}{ks^6}\right]ds = \frac{3C_0}{T^2} > \frac{3C_0}{4T^2}.$$

Set $\Omega(T)=\dfrac{3C_0}{4T^2}$, then $\Omega_+(T)=\dfrac{3C_0}{4T^2}$ and $\displaystyle\int_T^\infty\frac{\Omega^2{}_+(s)}{r(s)\rho(s)}ds=\frac{3C_0^2}{32}\int_T^\infty s^2ds=\infty.$

All conditions of Theorem 3.3.10 are satisfied. Thus, the given equation is oscillatory. We also compute the numerical solution of the given differential equation using the Runge Kutta method of fourth order (RK4). We have

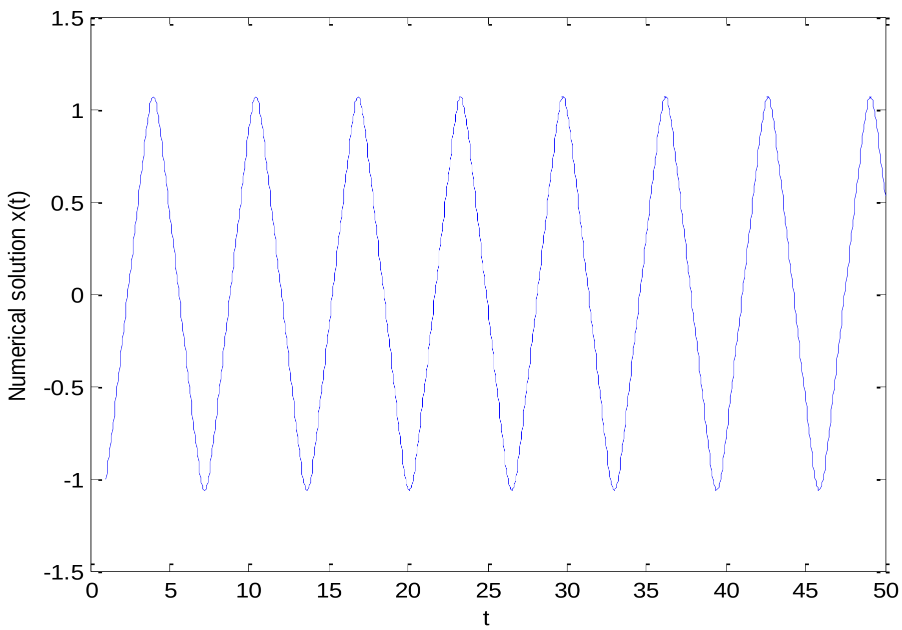
$$\ddot{x}(t) = f(t,x(t),\dot{x}(t)) = \frac{-x^9(t)\sin^2(x)}{x^2(t)+1} - \left(x^7(t)+\frac{x^{133}(t)}{9x^{126}(t)+6\left(\dot{x}(t)\right)^{18}}\right)$$

with initial conditions $x(1)=-1,\ \dot{x}(1)=0.5$ on the chosen interval $[1,50]$ and finding the values of the functions r, q and f where we consider $H(t,x)=f(t)\,l(x)$ at $t=1$, $n=980$ and $h=0.05$.

Table 3.10: Numerical solution of ODE 3.10

k	t_k	$x(t_k)$
1	1	-1
81	5	0.76515384
181	10	-0.14404191
221	12	1.02261405
321	17	-0.56631036
381	20	0.93295032
461	24	-0.96790375
561	27	0.94270152
581	30	-0.57991575
661	34	0.55142239
721	37	-0.15874611
798	40.9	0.06496234
921	47	-0.93061112
981	50	0.97197772

Figure 3.10: Solution curve of ODE 3.10.



Remark 3.3.10

Theorem 3.3.10 extends and improves the results of Philos (1989) and results of Yan (1986) who have studied the equation (1.1) $r(t) \equiv 1, \ g(x(t)) \equiv x(t), \ \Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$, as mentioned in Chapter Two. Our result can be applied on their equation (2.1), but their oscillation results cannot be applied on the given equation in Example 3.3.10.

Theorem 3.3.11

Suppose in addition to the condition (1) and (2) hold that assume that there exists ρ be a positive continuous differentiable function on the interval $[t_0, \infty)$ with $\rho(t)$ is increasing on the interval $[t_0, \infty)$ and such that

(25) $$\limsup_{t\to\infty} \frac{1}{\rho(t)} \int_T^t \rho(s)[C_0 q(s) - p(s)]ds = \infty,$$

where $p : [t_0, \infty) \to (0, \infty)$, then, every solution of equation (1.1) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ *on* $[T, \infty)$ *for some* $T \ge t_0 \ge 0.$ Define

$$\omega(t) = \frac{\rho(t) r(t) \dot{x}(t)}{g(x(t))}, \ t \ge T.$$

Thus and equation (1.1) imply

$$\dot{\omega}(t) \le \rho(t)p(t) - \rho(t)q(t)\Phi(1, \omega(t)/\rho(t)) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{\rho(t)r(t)}\omega^2(t),\ t \ge T.$$

Thus, we have

$$\rho(t)\left(\frac{\omega(t)}{\rho(t)}\right)^{\bullet} \le \rho(t)p(t) - \rho(t)q(t)\Phi(1, \omega(t)/\rho(t)) - \frac{k}{\rho(t)r(t)}\omega^2(t),\ t \ge T. \quad (3.3.27)$$

Dividing the last inequality by $\Phi(1, \omega(t)/\rho(t)) > 0$, we have

$$\frac{\rho(t)(\omega(t)/\rho(t))^{\bullet}}{\Phi(1, \omega(t)/\rho(t))} \le \frac{\rho(t)p(t)}{\Phi(1, \omega(t)/\rho(t))} - \rho(t)q(t),\ t \ge T$$

By condition (1), for $t \ge T$, we obtain

$$\rho(t)[C_0 q(t) - p(t)] \le -\frac{C_0\rho(t)(\omega(t)/\rho(t))^{\bullet}}{\Phi(1, \omega(t)/\rho(t))},\ t \ge T.$$

Integrate the last inequality from *T* to *t*, we obtain

$$\int_T^t \rho(s)[C_0 q(s) - p(s)]ds \le -C_0\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1, \omega(s)/\rho(s))}ds,\ t \ge T. \quad (3.3.28)$$

By Bonnet's theorem, we see that for each $t \ge T$, there exists $T_1 \in [T, t]$ such that

$$-\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1, \omega(s)/\rho(s))}ds = -\rho(t)\int_{T_1}^t \frac{(\omega(s)/\rho(s))^{\bullet}}{\Phi(1, \omega(s)/\rho(s))}ds. \quad (3.3.29)$$

Using the inequality (3.3.29) in the inequality (3.3.28), we have

$$\int_T^t \rho(s)[C_0 q(s) - p(s)]ds \le -C_0\rho(t)\int_{T_1}^t \frac{(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds = -C_0\rho(t)\int_{\omega(T_1)/\rho(T_1)}^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)}.$$

By condition (2), dividing the last inequality by $\rho(t)$ and taking the limit superior on both sides, we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{\rho(t)}\int_T^t \rho(s)[C_0 q(s) - p(s)]\, ds &\le -C_0 \limsup_{t\to\infty}\int_{\omega(T_1)/\rho(T_1)}^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)} \\ &\le -C_0 \limsup_{t\to\infty}\left[-\int_0^{\omega(T_1)/\rho(T_1)}\frac{du}{\Phi(1,u)} + \int_0^{\omega(t)/\rho(t)}\frac{du}{\Phi(1,u)}\right] \\ &\le C_0 \limsup_{t\to\infty}\left(G\left(\frac{\omega(T_1)}{\rho(T_1))}\right) + B^*\right) < \infty,\end{aligned}$$

as $t \to \infty$, which contradicts to the condition (25). Hence the proof is completed.

Example 3.3.11: Consider the differential equation

$$\left(t\,\dot{x}(t)\right)^{\bullet} + \left(\frac{t^3 + 3\cos t}{t^2}\right)x^3(t) = \frac{x^3(t)\sin x(t)}{t^4},\ t > 0.$$

Here $r(t) = t$, $q(t) = \dfrac{t^3 + 3\cos t}{t^2}$, $g(x) = x^3$, $\Phi(u,v) = u$ and

$\dfrac{H(t,x(t))}{g(x(t))} = \dfrac{\sin x(t)}{t^4} \le \dfrac{1}{t^4} = p(t)$ *for all* $t > 0$ *and* $x \ne 0$. Taking $\rho(t) = t^2$ *such that*

$$\limsup_{t\to\infty}\frac{1}{\rho(t)}\int_T^t \rho(s)(C_0 q(s) - p(s))\, ds = \limsup_{t\to\infty}\frac{1}{t^2}\int_T^t s^2\left(\frac{C_0 s^3 + 3C_0\cos s}{s^2} - \frac{1}{s^4}\right)ds = \infty.$$

All conditions of Theorem 3.3.11 are satisfied and hence, every solution of the given equation is oscillatory. To demonstrate that our result in Theorem 3.3.11 is true we also find the numerical solution of the given differential equation in Example 3.3.11 using the Runge Kutta method of fourth order (RK4).

We have

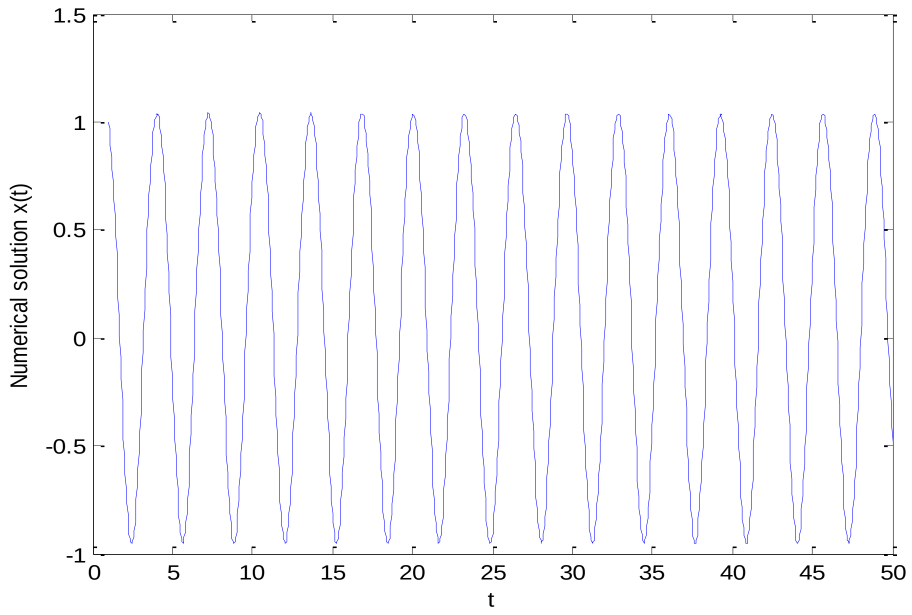
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^3 \sin(x) - 3.99x$$

with initial conditions $x(1) = 1, \dot{x}(1) = -0.5$ on the chosen interval $[1,50]$ and finding the values of the functions r, q and f where we consider $H(t, x(t)) = f(t)l(x)$ at $t=1$, n=980 and h=0.05.

Table 3.11: Numerical solution of ODE 3.11

k	t_k	$x(t_k)$
1	1	1
81	5	-0.18505453
181	10	0.72315333
221	12	-0.94485686
321	17	1.00637880
400	20.95	-0.15920533
461	24	0.23724881
529	27.4	-0.14883373
581	30	0.86387004
661	34	-0.50741992
719	36.9	0.06342616
785	40.2	-0.12804230
911	46.5	0.07913369
981	50	-0.48477630

Figure 3.11: Solution curve of ODE 3.11



Remark 3.3.11

Theorem 3.3.11 is the extension of the results of Bihari (1963), Kartsatos (1968) who have studied the equation (1.1) as $r(t) \equiv 1$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t)) \equiv 0$ and results of Wintiner (1949) and Kamenev (1978) who have studied the equation (1.1) as $r(t) \equiv 1$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$, as mentioned in Chapter Two. Our result can be applied on their equations, but their oscillation results cannot be applied on the given equation in Example 3.3.11 because their equations are particular cases of our equation (1.1).

Theorem 3.3.12: Suppose, in addition to the condition (2) holds that

(26) $$\frac{1}{C_1} < \frac{1}{\Phi(1,v)}, \; C_1 > 0,$$

(27) $$\frac{1}{\Phi(1,v)} \le v \ \ for\ all\ v \ne 0.$$

Assume that ρ be a positive continuous differentiable function on the interval $[t_0,\infty)$ with $\rho(t)$ is a decreasing function on the interval $[t_0,\infty)$ and such that

(28) $$\limsup_{t\to\infty} \int_T^t \rho(s)\left[q(s) - \frac{1}{4k^*}p^2(s)\right]ds = \infty,$$

where $p:[t_0,\infty) \to (0,\infty)$ and k^* is a positive constant, then, every solution of equation (1.1) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0\ on\ [T,\infty)\ for\ \ some\ T \ge t_0 \ge 0.$ By conditions (26) and (27) and from inequality (3.3.27) divided by $\Phi(1,\omega(t)/\rho(t)) > 0,$ we have

$$\frac{\rho(t)(\omega(t)/\rho(t))^{\bullet}}{\Phi(1,\omega(t)/\rho(t))} \le p(t)\omega(t) - \rho(t)q(t) - \frac{k^*}{r(t)\rho(t)}\omega^2(t),$$

where $k^* = k/C_1.$

Integrate the last inequality from T to t, we obtain

$$\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds \le -\int_T^t \rho(s)q(s)ds - \int_T^t \left[\frac{k^*}{r(s)\rho(s)}\omega^2(t) - p(s)\omega(s)\right] ds.$$

Thus, we have

$$\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds \leq -\int_T^t \rho(s)q(s)ds - \int_T^t\left(\sqrt{\frac{k^*}{r(s)\rho(s)}}\omega(t) - \frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{k^*}}p(s)\right)^2 ds$$
$$+\frac{1}{4k^*}\int_T^t r(s)\rho(s)p^2(s)ds.$$

Then, we get

$$\int_T^t \rho(s)\left[q(s) - \frac{1}{4k^*}r(s)p^2(s)\right]ds \leq -\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds. \tag{3.3.30}$$

By Bonnet's theorem, we see that for each $t \geq T$, there exists $a_t \in [T,t]$ such that

$$-\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds = -\rho(T)\int_T^{a_t}\frac{(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds. \tag{3.3.31}$$

From inequality (3.3.31) in inequality (3.3.30), the condition (2) and taking the limit superior on both sides, we obtain

$$\begin{aligned}
\limsup_{t\to\infty}\int_T^t \rho(s)\left[q(s) - \frac{1}{4k^*}r(s)p^2(s)\right]ds &\leq -\rho(T)\limsup_{t\to\infty}\int_{\omega(T)/\rho(T)}^{\omega(a_t)/\rho(a_t)}\frac{du}{\Phi(1,u)}\\
&\leq -\rho(T)\limsup_{t\to\infty}\left[-\int_0^{\omega(T)/\rho(T)}\frac{du}{\Phi(1,u)} + \int_0^{\omega(a_t)/\rho(a_t)}\frac{du}{\Phi(1,u)}\right]\\
&\leq \rho(T)\limsup_{t\to\infty}\left(G\left(\frac{\omega(T)}{\rho(T)}\right) - G\left(\frac{\omega(a_t)}{\rho(a_t)}\right)\right)\\
&\leq \rho(T)\limsup_{t\to\infty}\left(G\left(\frac{\omega(T)}{\rho(T)}\right) + B^*\right) < \infty,
\end{aligned}$$

as $t \to \infty$, which contradicts to the condition (28). Hence the proof is completed.

Example 3.3.12

Consider the following differential equation

$$\left(\frac{2\dot{x}(t)}{t^5+1}\right)^{\bullet} + \left(\frac{t^5+4t^5\cos t}{t^5+1}\right)\left(x^9(t) + \frac{x^{27}(t)}{x^{18}(t) + \left(2\dot{x}(t)\big/(t^5+1)\right)^2}\right) = \frac{x^9(t)\sin(x(t))}{t^2},\ t > 0.$$

Here $r(t) = \dfrac{2}{t^5+1}$, $q(t) = \dfrac{t^5+4t^5\cos t}{t^5+1}$, $g(x) = x^9$, $\Phi(u,v) = u + \dfrac{u^3}{u^2+v^2}$ and

$$\frac{H(t,x(t))}{g(x(t))} = \frac{\sin(x(t))}{t^2} \le \frac{1}{t^2} = p(t) \ \textit{for all } t > 0 \textit{ and } x \ne 0.\ \text{Let } \rho(t) = \frac{t^5+1}{t^5} > 0$$

such that

$$\limsup_{t\to\infty}\int_T^t \rho(s)\left[q(s) - \frac{1}{4k^*} r(s)p^2(s)\right]ds = \limsup_{t\to\infty}\int_T^t \frac{s^5+1}{s^5}\left[\frac{s^5+4s^5\cos s}{s^5+1} - \frac{1}{4k^*}\left(\frac{2}{s^5+1}\right)\frac{1}{s^2}\right]ds$$
$$= \infty.$$

We get all conditions of Theorem 3.3.12 are satisfied and hence, every solution of the given equation is oscillatory. The numerical solution of the given differential equation is found out using the Runge Kutta method of fourth order (RK4). We have

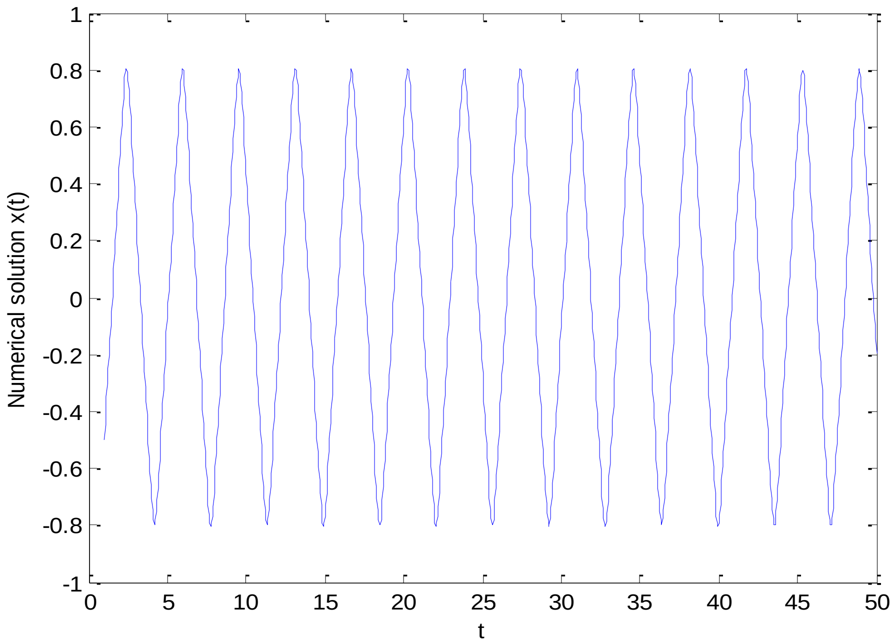
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^9(t)\sin(x(t)) - 42.49\left(x^9(t) + \frac{x^{27}(t)}{x^{18}(t) + \dot{x}^2(t)}\right)$$

with initial conditions $x(1) = -0.5,\ \dot{x}(1) = 1$ on the chosen interval $[1,50]$ and finding the values of the functions *r, q* and f where we consider $H(t,x(t)) = f(t)\,l(x)$ at *t=1*, *n*=980 and *h*=0.05.

Table 3.12: Numerical solution of ODE 3.12

k	**t_k**	**$x(t_k)$**
1	1	-0.5
83	5.1	0.02786237
191	10.5	-0.06638742
226	12.25	0.03073821
333	17.6	-0.01683783
381	20	0.62716312
479	24.9	-0.16418655
521	27	0.47451786
581	30	-0.10418323
661	34	0.31823882
721	37	-0.26359395
821	42	0.63188040
921	47	-0.75244510
981	50	-0.19568525

Figure 3.12: Solution curve of ODE 3.12



Remark 3.3.12

Theorem 3.3.12 extends and improves the results of Bihari (1963) and the results of Kartsatos (1966) who have studied the equation (2.5) as mentioned in chapter two. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 3.3.12 because their equation is a particular case of our equation (1.1) when $r(t) \equiv 1$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t)) \equiv 0$.

3.4 Conclusion

In this section, oscillation of second order nonlinear differential equation with alternating coefficients of type (1.1) has been investigated. Some oscillation results have been presented. These results contain the sufficient conditions for oscillation of solutions of the equation of type (1.1) which have been derived by using the generalized Riccati technique. Our results extend and improve many previous results that have been obtained before, for example, such as the works of Fite (1918), Wintner (1949), Atkinson (1955), Bihari (1963), Kartsatos (1968), Philos (1989), Philos and Purnaras (1992), El-abbasy (1996), and El-abbasy et al. (2005). All these previous results have been studied for particular cases of the equation (1.1) whereas our sufficient conditions have been derived for the generalized equation (1.1). A number of theorems and illustrative examples for oscillation differential equation of type (1.1) are given. Further, a number of numerical examples are given to illustrate the theorems which are computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of proposed research study.

CHAPTER FOUR

OSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

4.1 Introduction

In this chapter, we consider the second order nonlinear ordinary differential equation of the form

$$\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + h(t)\dot{x}(t) + q(t)\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) = H(t, x(t), \dot{x}(t)), \qquad (1.2)$$

where r, h and q are continuous functions on the interval $[t_0, \infty), \Psi \in C(R, R^+)$ and $r(t)$ is a positive function. g is a continuous function for $x \in (-\infty, \infty)$, continuously differentiable and satisfies $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$. The function Φ is a continuous function on RxR with $u\Phi(u,v) > 0$ for all $u \neq 0$ and $\Phi(\lambda u, \lambda v) = \lambda\Phi(u,v)$ for any $\lambda \in (0,\infty)$ and H is a continuous function on $[t_0, \infty)\times$R$\times$R with $H(t, x(t), \dot{x}(t))\big/g(x(t)) \leq p(t)$ for all $x \neq 0$ and $t \geq t_0$.

4.2 Second Order Nonlinear ODE With Damping Term of Type (1.2)

We consider a problem of finding the sufficient conditions for oscillation of solutions of ordinary differential equations of second order. The obtained oscillation results are motivated extended and improved many previous oscillation results, for examples, Bihari (1963), Kartsatos (1968), Greaf, et al. (1978), Grace (1992), Elabbasy et al. (2005), Lu & Meng (2007), Berkani (2008) and Remili (2010). Some new sufficient conditions are established which guarantee that our differential equations are oscillatory. A number of theorems and illustrative examples for oscillation differential equation (1.2) are given. Also, a number of numerical examples are given to illustrate the theorems. These numerical examples are computed by using Runge Kutta of fourth order in Matlab. The obtained results are compared with existing results to explain the motivation of proposed research study.

4.3 Oscillation Theorems

We state and prove here our oscillation theorems.

Theorem 4.3.1: Suppose that

(1) $a_1 \le \Psi(x) \le a_2$, $a_1,\ a_2 > 0$ and for $x \in R$,

(2) $G(m) = \int_0^m \frac{ds}{\Phi(1,s)} > -B^*,\ B^* > 0$ for every $m \in$ R.

Assume that there exists a positive continuous differentiable function ρ on the interval $[t_0, \infty)$, $\rho(t)$ is an increasing function on the interval $[t_0, \infty)$ and such that

(3) $$\limsup_{t\to\infty}\frac{1}{\rho(t)}\int_T^t \rho(s)\left[C_0 q(s)-p(s)-\frac{h^2(s)}{4a^* r(s)}\right]ds=\infty,$$

where $p:[t_0,\infty)\to(0,\infty)$, then, every solution of equation (1.2) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t)>0$ *on* $[T,\infty)$ for every $T\geq t_0\geq 0$. Define

$$\omega(t)=\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}\ ,\ t\geq T.$$

By equation (1.2) and condition (1), we obtain

$$\dot{\omega}(t)\leq \rho(t)p(t)-\frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))}-\rho(t)q(t)\Phi(1,\omega(t)/\rho(t))+\frac{\dot{\rho}(t)}{\rho(t)}\omega(t)$$
$$-\frac{k\rho(t)r(t)\Psi(x(t))\dot{x}^2(t)}{g^2(x(t))}. \tag{4.3.1}$$

Thus, we have

$$\rho(t)\left(\frac{\omega(t)}{\rho(t)}\right)^{\bullet}\leq \rho(t)p(t)-\rho(t)q(t)\Phi(1,\omega(t)/\rho(t))-\frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))}-\frac{k\rho(t)r(t)\Psi(x(t))\dot{x}^2(t)}{g^2(x(t))},\ t\geq T.$$

Dividing the last inequality by $\Phi(1,\omega(t)/\rho(t))>0$, we have

$$\frac{\rho(t)(\omega(t)/\rho(t))^{\bullet}}{\Phi(1,\omega(t)/\rho(t))}\leq\frac{\rho(t)p(t)}{\Phi(1,\omega(t)/\rho(t))}-\rho(t)q(t)-\frac{\rho(t)h(t)\dot{x}(t)}{\Phi(1,\omega(t)/\rho(t))g(x(t))}$$
$$-\frac{a_1 k\rho(t)r(t)\dot{x}^2(t)}{\Phi(1,\omega(t)/\rho(t))g^2(x(t))}. \tag{4.3.2}$$

By condition (1) and since $\Phi(1,\omega(t)/\rho(t))>0$ then, there exists a positive constant *C0* such that $\Phi(1,\omega(t)/\rho(t))>C_0$ thus, $0<\dfrac{1}{\Phi(1,\omega(t)/\rho(t))}<\dfrac{1}{C_0}$. Then, for $t\geq T$, we obtain

$$\rho(t)(C_0 q(t)-p(t))\leq -\frac{C_0\rho(t)(\omega(t)/\rho(t))^{\bullet}}{\Phi(1,\omega(t)/\rho(t))}-\frac{C_0\rho(t)h(t)\dot{x}(t)}{\Phi(1,\omega(t)/\rho(t))g(x(t))}-\frac{C_0 a_1 k\rho(t)r(t)\dot{x}^2(t)}{\Phi(1,\omega(t)/\rho(t))g^2(x(t))}.$$

Integrate the last inequality from *T* to *t*, we obtain

$$\int_T^t \rho(s)(C_0 q(s)-p(s))ds\leq -C_0\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds$$
$$-C_0\int_T^t\left[\frac{\rho(s)h(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}(s)}{g(x(s))}+\frac{a_1 k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}^2(s)}{g^2(x(s))}\right]ds. \quad (4.3.3)$$

Since $\rho(t)$ in the first integral in R. H. S. of the inequality (4.3.3) is an increasing function and by applying the Bonnet's theorem, we see that for each $t\geq T$, there exists $T_1\in[T,t]$ such that

$$-\int_T^t \frac{\rho(s)(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds=-\rho(t)\int_{T_1}^t \frac{(\omega(s)/\rho(s))^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds. \quad (4.3.4)$$

From the second integral in R. H. S. of (4.3.3), we have

$$-C_0\int_T^t\left[\frac{\rho(s)h(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}(s)}{g(x(s))}+\frac{a_1k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}^2(s)}{g^2(x(s))}\right]ds=$$

$$-C_0\int_T^t\left[\sqrt{\frac{a_1k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}}\frac{\dot{x}(s)}{g(x(s))}+\frac{1}{2}\sqrt{\frac{\rho(s)}{a_1kr(s)\Phi(1,\omega(s)/\rho(s))}}h(s)\right]^2ds$$

$$+\frac{C_0}{4a_1k}\int_T^t\frac{\rho(s)h^2(s)}{r(s)\Phi(1,\omega(s)/\rho(s))}ds$$

$$\leq\frac{1}{4a^*}\int_T^t\frac{\rho(s)h^2(s)}{r(s)}ds, \qquad (4.3.5)$$

where $a^*=a_1k$.

From inequalities (4.3.4) and (4.3.5) in inequality (4.3.3), we have

$$\int_T^t\rho(s)\left(C_0q(s)-p(s)-\frac{h^2(s)}{4a^*r(s)}\right)ds\leq -C_0\rho(t)\int_{T_1}^t\frac{(\omega(s)/\rho(s))^\bullet}{\Phi(1,\omega(s)/\rho(s))}ds$$

$$\leq -C_0\rho(t)\int_{\omega(T_1)/\rho(T_1)}^{\omega(t)/\rho(t)}\frac{du}{\Phi(1,u)}.$$

By condition (2), the last inequality divided by $\rho(t)$ and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty}\frac{1}{\rho(t)}\int_T^t\rho(s)\left(C_0q(s)-p(s)-\frac{h^2(s)}{4a^*r(s)}\right)ds\leq -C_0\limsup_{t\to\infty}\int_{\omega(T_1)/\rho(T_1)}^{\omega(t)/\rho(t)}\frac{du}{\Phi(1,u)}$$

$$\leq -C_0\limsup_{t\to\infty}\left[-\int_0^{\omega(T_1)/\rho(T_1)}\frac{du}{\Phi(1,u)}+\int_0^{\omega(t)/\rho(t)}\frac{du}{\Phi(1,u)}\right]$$

$$\leq C_0\limsup_{t\to\infty}\left[G\left(\frac{\omega(T_1)}{\rho(T_1)}-G\left(\frac{\omega(t)}{\rho(t)}\right)\right)\right]$$

$$\leq C_0\limsup_{t\to\infty}\left(G\left(\frac{\omega(T_1)}{\rho(T_1))}\right)+B^*\right)<\infty,$$

$as\ t\to\infty$, which contradicts to the condition (3). Hence the proof is completed.

Example 4.3.1

Consider the differential equation

$$\left(\frac{x^2(t)+2}{x^2(t)+1}\dot{x}(t)\right)^{\bullet}+\frac{\dot{x}(t)}{t^3}+\left(\frac{t^3+3\cos t}{t^2}\right)x^5(t)=\frac{2t^2x^5(t)\cos(\dot{x}(t))}{(t^6+t^4)},\ t>0.$$

We have $r(t)=1,\ h(t)=\frac{1}{t^3},\ q(t)=\frac{t^3+3\cos t}{t^2},\ g(x)=x^5,\ \Phi(u,v)=u$ and

(1) $\Psi(x)=\frac{x^2(t)+2}{x^2(t)+1}>0\, and\ \ 1\le\Psi(x)=1+\frac{1}{x^2(t)+1}\le 2\ \ for\ all\ x\in R.$

(2) $\frac{H(t,x(t),\dot{x}(t))}{g\big(x(t)\big)}=\frac{2t^2\cos(\dot{x}(t))}{(t^6+t^4)}\le\frac{2}{t^4}=p(t)$ for all $t>0$ and $x\neq 0.$

Taking $\rho(t)=t^2$ such that

(3) $$\limsup_{t\to\infty}\frac{1}{\rho(t)}\int_T^t\rho(s)\left(C_0q(s)-p(s)-\frac{h^2(s)}{4a^*r(s)}\right)ds$$
$$=\limsup_{t\to\infty}\frac{1}{t^2}\int_T^t s^2\left(\frac{C_0s^3+3C_0\cos s}{s^2}-\frac{2}{s^4}-\frac{1}{4a^*s^6}\right)ds=\infty.$$

All conditions of Theorem 4.3.1 are satisfied and hence every solution of the given equation is oscillatory. To demonstrate that our result in Theorem 4.3.1 is true, we also find the numerical solutions of the given differential equation in Example 4.3.1 using the Runge Kutta method (RK4) for different steps sizes.

We have

$$\ddot{x}(t)=f(t,x(t),\dot{x}(t))=x^5\cos(\dot{x})-3.9x^5$$

with initial conditions $x(1)=1,\ \dot{x}(1)=-1$ on the chosen interval $[1,50]$, the functions $\Psi(x)\equiv 1,\ h(t)\equiv 0$ and finding the values of the functions r, q and f, where we consider $H(t,x,\dot{x})=f(t)l(x,\dot{x})$ at t=1, n =*2250*, n =*2500*, n=*2750* and n=3000 and the steps sizes h =0.021, h =0.019, h =0.017 and h = 0.016.

Table 4.1: Comparison of the numerical solutions of ODE 4.1 with different steps sizes

	h=0.021	h=0.019	h=0.017	h=0.016	$\lvert x_4 - x_1 \rvert$	Error %	$\lvert x_4 - x_2 \rvert$	Error %	$\lvert x_4 - x_3 \rvert$	Error %
t_k	$x_1(t_k)$	$x_2(t_k)$	$x_3(t_k)$	$x_4(t_k)$						
1	1	1	1	1	0	0	0	0	0	0
5.9	-1.07394978	-1.07394981	-1.07394982	-1.07394984	0.00000006	0.00000558	0.00000003	0.00000279	0.00000002	0.00000186
10.8	0.05014663	0.05014678	0.05014687	0.05014694	0.00000031	0.00061818	0.00000016	0.00031906	0.00000007	0.00013958
12.76	-0.30438304	-0.30438320	-0.30438330	-0.30438337	0.00000033	0.00010841	0.00000017	0.00005585	0.00000007	0.00002299
18.64	0.91949065	0.91949076	0.91949084	0.91949089	0.00000024	0.00002610	0.00000013	0.00001413	0.00000005	0.00000543
20.404	-0.81159102	-0.81159115	-0.81159124	-0.81159130	0.00000028	0.00003450	0.00000015	0.00001848	0.00000006	0.00000739
24.52	-1.15967567	-1.15967570	-1.15967572	-1.15967573	0.00000006	0.00000517	0.00000003	0.00000258	0.00000001	0.00000086
27.068	0.60735840	0.60735832	0.60735826	0.60735820	0.00000020	0.00003292	0.00000012	0.00001975	0.00000006	0.00000987
30.4	1.07023911	1.07023912	1.07023912	1.07023910	0.00000001	0.00000093	0.00000002	0.00000186	0.00000002	0.00000186
32.36	-0.96141821	-0.96141824	-0.96141823	-0.96141822	0.00000001	0.00000104	0.00000002	0.00000208	0.00000001	0.00000104
37.26	1.09808651	1.09808651	1.09808652	1.09808654	0.00000003	0.00000273	0.00000003	0.00000273	0.00000002	0.00000182
40.2	-0.12779001	-0.12779022	-0.12779029	-0.12779029	0.00000028	0.00021910	0.00000007	0.00005477	0	0
42.16	-0.12977901	-0.12977874	-0.12977865	-0.12977863	0.00000038	0.00029280	0.00000011	0.00008475	0.00000002	0.00001541
47.06	-1.00415879	-1.00415904	-1.00415916	-1.00415921	0.00000042	0.00004182	0.00000017	0.00001692	0.00000005	0.00000497
50	-0.96243978	-0.96243952	-0.96243942	-0.96243937	0.00000041	0.00004260	0.00000015	0.00001558	0.00000005	0.00000519

Figure 4.1(a): Solution curves of ODE 4.1

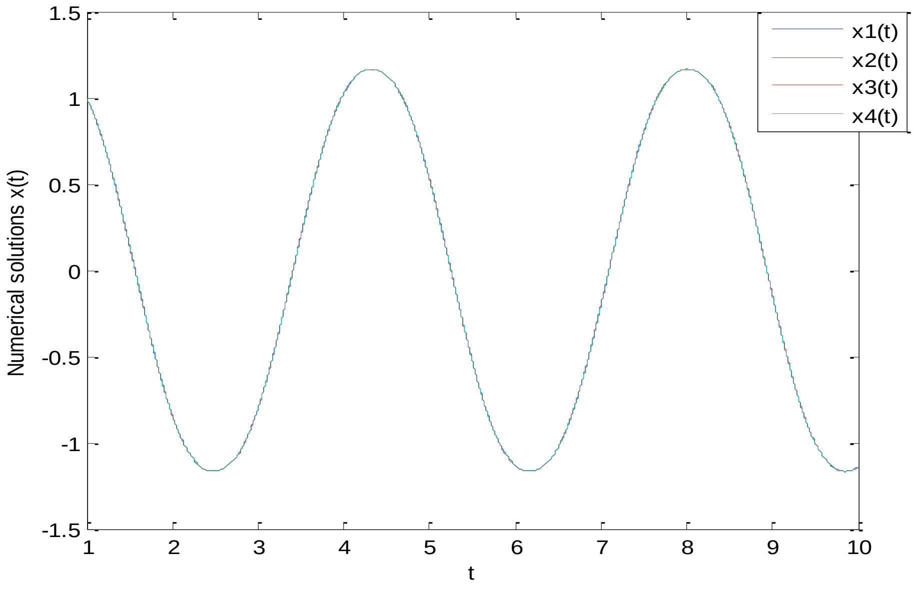
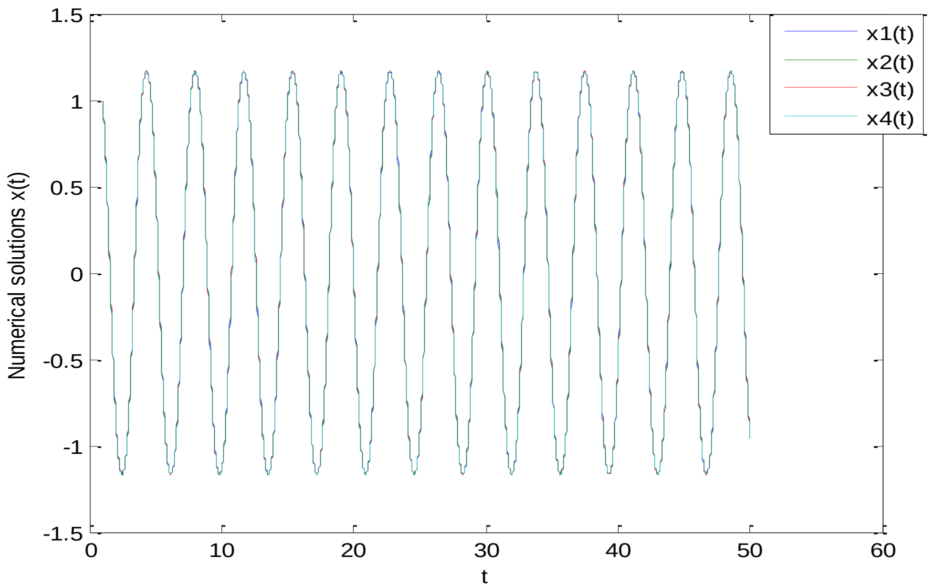


Figure 4.1(b): Solution curves of ODE 4.1



Remark 4.3.1: Theorem 4.3.1 is the extension of the results of Bihari (1963), Kartsatos (1968), who have studied the equation (1.2) when $r(t) \equiv 1$, $\Psi(x(t)) \equiv 1$, $h(t) \equiv 0$,

$g(x(t)) \equiv x(t),\ \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$ and results of Kamenev (1978) and Wintiner (1949) who have studied the equation (1.2) as $r(t) \equiv 1,\ \Psi(x(t)) \equiv 1,\ h(t) \equiv 0,\ g(x(t)) \equiv x(t),\ \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Our result can be applied on their equations, but their oscillation results cannot be applied on the given equation in Example 4.3.1 because their equations are particular cases of our equation (1.2).

Theorem 4.3.2: Suppose, in addition to the conditions (1) and (2) hold that

(4) $h(t) \geq 0,\ \textit{for all}\ t > t_0$.

(5) $\Phi(1, v) \leq C_1,\ C_1 > 0$ for $v \in R^+$,

(6) $\dfrac{1}{\Phi(1, v)} \leq v$ for all $v \in R^+$.

Assume that ρ be a positive continuous differentiable function on the interval $[t_0, \infty)$ with $\rho(t)$ is a decreasing function on the interval $[t_0, \infty)$ and such that

(7) $$\limsup_{t\to\infty} \int_T^t \rho(s)\left[q(s) - k_1 r(s) p^2(s) - k_2 \frac{h^2(s)}{r(s)}\right] ds = \infty,$$

where $k_1 = a_2 C_1 / 4k,\ k_2 = a_2 / 4k a_1^2 C_0$ and $p : [t_0, \infty) \to (0, \infty)$, then, every solution of the equation (1.2) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t) > 0\ on\ [T, \infty)$ for some $T \geq t_0 \geq 0$. From inequality (4.3.2) and by conditions (1) and (6), we have

$$\frac{\rho(t)\left(\omega(t)/\rho(t)\right)^{\bullet}}{\Phi\left(1,\omega(t)/\rho(t)\right)} \le \left(p(t) - \frac{h(t)}{r(t)\Psi(x(t))\Phi\left(1,\omega(t)/\rho(t)\right)}\right)\omega(t) - \rho(t)q(t)$$
$$- \frac{k}{a_2\rho(t)r(t)\Phi\left(1,\omega(t)/\rho(t)\right)}\omega^2(t).$$

Integrate the last inequality from T to t, we obtain

$$\int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}ds \le -\int_T^t \rho(s)q(s)ds$$
$$-\int_T^t \left(\sqrt{\frac{k}{a_2\rho(s)r(s)\Phi\left(1,\omega(s)/\rho(s)\right)}}\,\omega(s) - \frac{\left(p(s) - h(s)/r(s)\Psi(x(s))\Phi\left(1,\omega(s)/\rho(s)\right)\right)}{2\sqrt{k/a_2\rho(s)r(s)\Phi\left(1,\omega(s)/\rho(s)\right)}}\right)^2 ds$$
$$+\frac{a_2}{4k}\int_T^t \rho(s)r(s)\Phi\left(1,\omega(s)/\rho(s)\right)\left(p(s) - \frac{h(s)}{r(s)\Psi(x(s))\Phi\left(1,\omega(s)/\rho(s)\right)}\right)^2 ds.$$

Since $\Phi\left(1,\omega(s)/\rho(s)\right) > C_0$ and by conditions (4) and (5), we get

$$\int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}ds \le -\int_T^t \rho(s)q(s)ds + \frac{a_2}{4k}\int_T^t \rho(s)\left(C_1 r(s)p^2(s) + \frac{h^2(s)}{a_1^2 C_0 r(s)}\right)ds.$$

Thus,

$$\int_T^t \rho(s)\left[q(s) - k_1 r(s)p^2(s) - k_2\frac{h^2(s)}{r(s)}\right]ds \le -\int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}ds. \qquad (4.3.6)$$

Since $\rho(t)$ is a decreasing function and by the Bonnet's theorem, we see that for each $t \ge T,$ there exists $a_t \in [T,t]$ such that

$$-\int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}ds = -\rho(T)\int_T^{a_t} \frac{\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}ds. \qquad (4.3.7)$$

From inequality (4.3.7) in inequality (4.3.6), condition (2) and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty}\int_T^t \rho(s)\left[q(s)-k_1 r(s)p^2(s)-k_2\frac{h^2(s)}{r(s)}\right]ds \le -\rho(T)\limsup_{t\to\infty}\int_{\omega(T)/\rho(T)}^{\omega(a_t)/\rho(a_t)}\frac{du}{\Phi(1,u)}$$

$$\le -\rho(T)\limsup_{t\to\infty}\left[-\int_0^{\omega(T)/\rho(T)}\frac{du}{\Phi(1,u)}+\int_0^{\omega(a_t)/\rho(a_t)}\frac{du}{\Phi(1,u)}\right]$$

$$\le \rho(T)\limsup_{t\to\infty}\left(G\left(\frac{\omega(T)}{\rho(T)}\right)+B^*\right)<\infty,$$

as $t\to\infty$, which contradicts to the condition (7). Hence the proof is completed.

Example 4.3.2

Consider the following differential equation

$$\left(\frac{t^9(x^4(t)+4)}{(x^4(t)+3)}\dot{x}(t)\right)^{\bullet}+t^2\,\dot{x}(t)+\left(\frac{t^5+4t^5\cos t}{t^5+1}\right)\left(x^9(t)+\frac{x^{27}(t)}{x^{18}(t)+\left(\frac{t^9(x^4(t)+4)}{(x^4(t)+3)}\dot{x}(t)\right)^2}\right)=\frac{x^9(t)\sin(\dot{x}(t))}{t^9}.$$

Here $r(t)=t^9$, $h(t)=t^2$, $q(t)=\dfrac{t^5+4t^5\cos t}{t^5+1}$, $g(x)=x^9$, $\Phi(u,v)=u+\dfrac{u^3}{u^2+v^2}$,

$1\le\Psi(x)=\dfrac{x^4(t)+4}{x^4(t)+3}\le\dfrac{4}{3}$ *for all* $x\in R$ and $\dfrac{H(t,x(t),\dot{x}(t))}{g(x(t))}=\dfrac{\sin(\dot{x}(t))}{t^8}\le\dfrac{1}{t^8}=p(t)$ for all

$t>0$ *and* $x\ne 0$. Let $\rho(t)=\dfrac{t^5+1}{t^5}$, $\dot{\rho}(t)=-\dfrac{5}{t^6}$ *for all* $t>0$ and such that

$$\limsup_{t\to\infty}\int_T^t \rho(s)\left[q(s)-k_1 r(s)p^2(s)-k_2\frac{h^2(s)}{r(s)}\right]ds=\limsup_{t\to\infty}\int_T^t\frac{s^5+1}{s^5}\left[\frac{s^5+4s^5\cos s}{s^5+1}-\frac{k_1}{s^7}-\frac{k_2}{s^5}\right]ds$$

$$=\infty.$$

All conditions of Theorem 4.3.2 are satisfied and hence every solution of the given equation is oscillatory. The numerical solutions of the given differential equation are found out using the Runge Kutta method of fourth order (RK4) for different steps sizes. We have

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^9(t)\sin(\dot{x}(t)) - 2.49\left[x^9(t) + \frac{x^{27}(t)}{x^{18}(t) + \dot{x}^2(t)}\right].$$

with initial conditions $x(1) = 0.5,\ \dot{x}(1) = 1$ on the chosen interval $[1,50]$, the functions $\Psi(x) \equiv 1$ *and* $h(t) \equiv 0$ and finding the values of the functions *r*, *q* and f where we consider $H(t, x, \dot{x}) = f(t)\,l(x, \dot{x})$ at t=1, *n =2250*, *n =2500*, *n=2750* and *n*=3000 and the steps sizes *h* =0.021, *h* =0.019, *h* =0.017 and *h* = 0.016.

Table 4.2: Comparison of the numerical solutions of ODE 4.2 with different steps sizes

t_k	h=0.021	h=0.019	h=0.017	h=0.016	$\lvert x_4 - x_1\rvert$	Error %	$\lvert x_4 - x_2\rvert$	Error %	$\lvert x_4 - x_3\rvert$	Error %
	$x_1(t_k)$	$x_2(t_k)$	$x_3(t_k)$	$x_4(t_k)$						
1	0.5	0.5	0.5	0.5	0	0	0	0	0	0
5.9	1.04974322	1.04974334	1.04974341	1.04974344	0.00000022	0.00002095	0.00000001	0.00000952	0.00000003	0.00000285
10.8	0.39503481	0.39502606	0.39502114	0.39501821	0.0000166	0.00420233	0.00000785	0.00198725	0.00000293	0.00074173
12.76	-0.49793337	-0.49792335	-0.49791774	-0.49791441	0.00001896	0.00380788	0.00000894	0.00179548	0.00000333	0.00066878
18.64	1.04883740	1.04884026	1.04884184	1.04884277	0.00000537	0.00051199	0.00000251	0.00023931	0.00000093	0.00008866
20.404	-0.90604208	-0.90606849	-0.90608316	-0.90609177	0.00004969	0.00548399	0.00002328	0.00256927	0.00000861	0.00095023
24.52	-0.75039793	-0.75043872	-0.75046135	-0.75047460	0.00007667	0.01021620	0.00003588	0.00478097	0.00001325	0.00176554
27.068	1.01984960	1.01987480	1.01988876	1.01989693	0.00004733	0.00464066	0.00002213	0.00216982	0.00000817	0.00080106
30.4	0.13170811	0.13176014	0.13178894	0.13180579	0.00009768	0.07410903	0.00004565	0.03463429	0.00001685	0.01278396
32.36	0.04125134	0.04118152	0.04114289	0.04112031	0.00013103	0.31865032	0.00006121	0.14885588	0.00002258	0.05491203
37.26	-0.72936424	-0.72945572	-0.72950625	-0.72953575	0.00017151	0.02350947	0.00008003	0.01096999	0.0000295	0.00404366
40.2	0.83091185	0.83081053	0.83075457	0.83072191	0.00018994	0.02286445	0.00008862	0.01066782	0.00003266	0.00393152
42.16	-0.86320470	-0.86311129	-0.86305972	-0.86302964	0.00017506	0.02028435	0.00008162	0.00946085	0.00003008	0.00348539
47.06	-0.21886752	-0.21874634	-0.21867950	-0.21864055	0.00022697	0.10380965	0.00010579	0.04838535	0.00003895	0.01781462
50	-0.70821761	-0.70837999	-0.70846949	-0.70852162	0.00030401	0.04290765	0.00014163	0.01998950	0.00005213	0.00735757

Figure 4.2(a): Solution curves of ODE 4.2

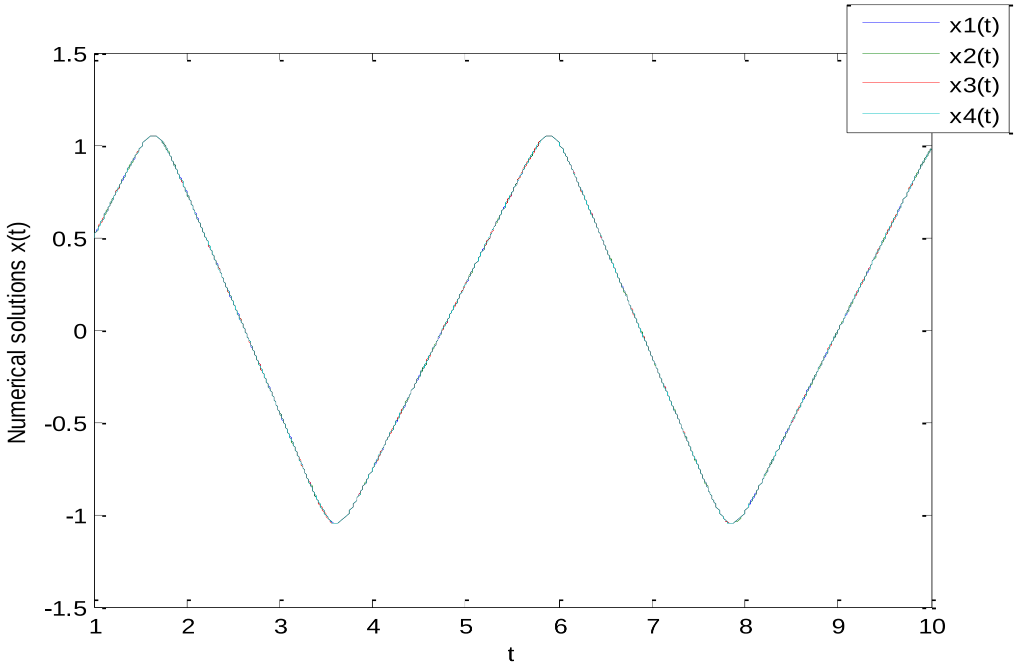
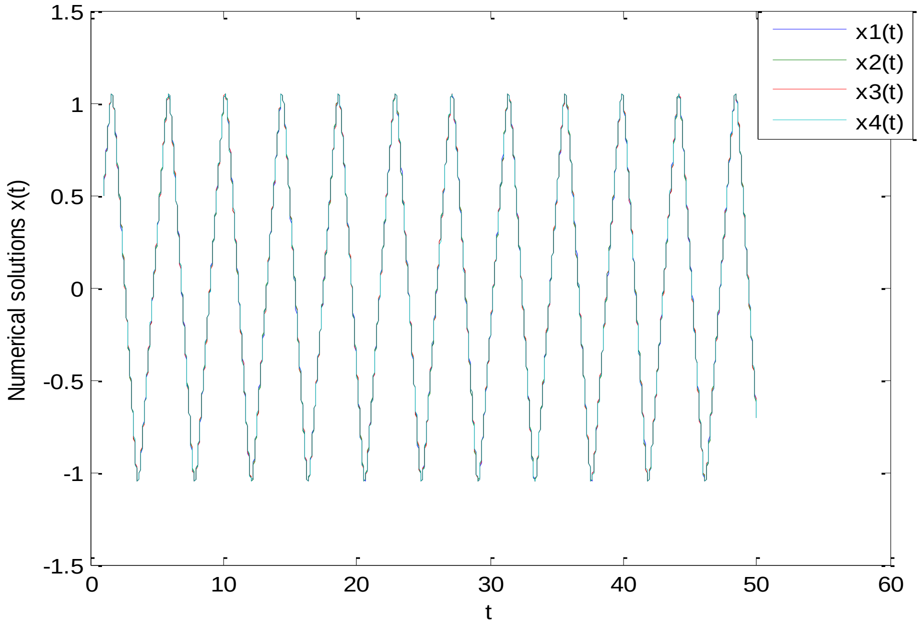


Figure 4.2(b): Solution curves of ODE 4.2



Remark 4.3.2:

Theorem 4.3.2 is the extension of the results of Bihari (1963) and Kartsatos (1968) who have studied the equation (1.2) when $r(t)\equiv 1,\ \ \Psi(x(t))\equiv 1,\ h(t)\equiv 0,\ \ g(x(t))\equiv x(t),$ $\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t))\equiv\Phi(x(t),\dot{x}(t))$ and $H(t,x(t),\dot{x}(t))\equiv 0.$ Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 4.3.2 because their equation are particular cases of our equation (1.2).

Theorem 4.3.3

Suppose, in addition to the conditions (1) and (4) hold that

(8) $q(t)>0$ for $t\geq t_0$.

(9) $$\int_{\pm\varepsilon}^{\pm\infty}\frac{\Psi(u)du}{g(u)}<\infty \quad for\ all\ \ \varepsilon>0.$$

Assume that there exist a differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ such that $(\rho r)^{\bullet}\leq 0$ and

(10) $$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left(\frac{\dot{\rho}^{2}(u)r(u)}{\rho(u)}+\frac{\rho(u)h(u)}{a_1^2 r(u)}\right)duds<\infty,$$

(11) $$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\rho(u)\left[C_0 q(u)-p(u)\right]duds=\infty.$$

Then, every solution of equation (1.2) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution *x*(t) of equation (1.2) such that $x(t) > 0 \ on \ [T, \infty)$ for some $T \geq t_0 \geq 0$. From the inequality (4.3.1) and condition (1), we have

$$\dot{\omega}(t) \leq \rho(t)p(t) - \frac{h(t)}{r(t)\Psi(x(t))}\omega(t) - \rho(t)q(t)\Phi(1, \omega(t)/\rho(t)) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{a_2\rho(t)r(t)}\omega^2(t).$$

Since $\Phi(1, \omega(t)/\rho(t)) > C_0$ and integrating the last inequality from *T* to *t* we have

$$\omega(t) \leq \omega(T) - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)ds - \int_T^t \left[\frac{k}{a_2\rho(s)r(s)}\omega^2(s) - \left(\frac{\dot{\rho}(s)}{\rho(s)} - \frac{h(s)}{r(s)\Psi(x(s))}\right)\omega(s)\right]ds.$$

Then, for $t \geq T$ and by the condition (4), we have

$$\begin{aligned}\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} &\leq \omega(T) - \int_T^t \rho(s)[C_0 q(s) - p(s)]ds + \frac{a_2}{4k}\int_T^t \rho(s)r(s)\left(\frac{\dot{\rho}(s)}{\rho(s)} - \frac{h(s)}{r(s)\Psi(x(s))}\right)^2 ds \\ &\leq \omega(T) - \int_T^t \rho(s)[C_0 q(s) - p(s)]ds + \frac{a_2}{4k}\int_T^t \left(\frac{\dot{\rho}^2(s)r(s)}{\rho(s)} + \frac{\rho(s)h^2(s)}{a_1^2 r(s)}\right)ds.\end{aligned}$$

Integrate the last inequality from *T* to *t*, we have

$$\begin{aligned}\int_T^t \frac{\rho(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds &\leq \omega(T)(t-T) - \int_T^t\int_T^s \rho(u)[C_0 q(u) - p(u)]duds \\ &+ \frac{a_2}{4k}\int_T^t\int_T^s \left(\frac{\dot{\rho}^2(u)r(u)}{\rho(u)} + \frac{\rho(u)h^2(u)}{a_1^2 r(u)}\right)duds. \qquad (4.3.8)\end{aligned}$$

Since $(\rho(t)r(t))$ is a non-increasing function and by the Bonnet's theorem, we see that for each $t \geq T$, there exists $\beta_t \in [T, t]$ such that

$$\int_T^t \frac{\rho(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds = \rho(T)r(T)\int_T^{\beta_t} \frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}ds = \rho(T)r(T)\int_{x(T)}^{x(\beta_t)} \frac{\Psi(u)du}{g(u)}.$$

Since $\rho(t)r(t) > 0$ and the condition (9), we have

$$\int_T^t \rho(s)r(s)\frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}ds \geq A_3, \text{ where } A_3 = \inf \rho(T)r(T)\int_{x(T_2)}^{x(\beta_t)} \frac{\Psi(u)du}{g(u)}.$$

Thus, the inequality (4.3.8) becomes

$$\int_T^t\int_T^s \rho(u)\left[C_0 q(u) - p(u)\right]duds \leq \omega(T)(t-T) - A_3 + \frac{a_2}{4k}\int_T^t\int_T^s \left(\frac{\dot{\rho}^2(u)r(u)}{\rho(u)} + \frac{\rho(u)h^2(u)}{a_1^2 r(u)}\right)duds.$$

Dividing the last inequality by t, taking the limit superior as $t \to \infty$, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_T^t\int_T^s \rho(u)\left(C_0 q(u) - p(u)\right)duds \leq \limsup_{t\to\infty}\omega(T)\left(1 - \frac{(T + A_3)}{t}\right)$$
$$+\frac{a_2}{4k}\limsup_{t\to\infty}\frac{1}{t}\int_T^t\int_T^s \left(\frac{\dot{\rho}^2(u)r(u)}{\rho(u)} + \frac{\rho(u)h^2(u)}{a_1^2 r(u)}\right)duds < \infty,$$

which contradicts to the condition (11). Hence the proof is completed.

Example 4.3.3

Consider the differential equation

$$\left(\frac{\left(x^2(t)+2\right)}{t^6\left(x^2(t)+1\right)}\dot{x}(t)\right)^{\bullet}+\frac{\dot{x}(t)}{t^9}+t^3\left[x^5(t)+\frac{x^{15}(t)}{x^{10}(t)+\left((x^2(t)+2)\,\dot{x}(t)\Big/ t^6(x^2(t)+2)\right)^2}\right]=\frac{x^5(t)\sin(\dot{x}(t))}{t^{10}}.$$

Here $r(t)=\dfrac{1}{t^6}>0,\ h(t)=\dfrac{1}{t^9},\ q(t)=t^3,\ g(x)=x^5,\Phi(u,v)=u+\dfrac{u^3}{u^2+v^2},$

$\Psi(x)=\dfrac{x^2(t)+2}{x^2(t)+1}>0\, and\ \ 1\le\Psi(x)=\dfrac{x^2(t)+2}{x^2(t)+1}=1+\dfrac{1}{x^2(t)+1}\le 2\ \ for\ all\ x\in R$ and

$$0<\int_{\pm\varepsilon}^{\pm\infty}\frac{\Psi(x)dx}{g(x)}\le\int_{\pm\varepsilon}^{\pm\infty}\frac{2dx}{x^5}=\frac{1}{2\varepsilon^4}<\infty\ \ for\ all\ \ \varepsilon>0.$$

$$\frac{H(t,x(t),\dot{x}(t))}{g\left(x(t)\right)}=\frac{\sin(\dot{x}(t))}{t^{10}}\le\frac{1}{t^{10}}=p(t)\ \ for\ all\ t>0\ \ and\ x\neq 0.$$

Taking $\rho(t)=t^4\ge 0\ \ \ for\ \ t>0,\ \dot{\rho}(t)=4t^3>0\ and\left(\rho(t)r(t)\right)^{\bullet}=-\dfrac{2}{t^3}\le 0\ for\ all\ t>0.$

(1) $$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left(\frac{\dot{\rho}^2(u)r(u)}{\rho(u)}+\frac{\rho(u)h(u)}{a_1^2r(u)}\right)duds=\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left(\frac{16}{u^4}+\frac{1}{u^8}\right)duds$$

$$=\frac{16}{3t_0^{\ 3}}+\frac{1}{7t_0^{\ 7}}<\infty.$$

(2) $$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\rho(u)\left[C_0q(u)-p(u)\right]du\,ds=\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}u^4\left(C_0u^3-\frac{1}{u^{10}}\right)duds=\infty.$$

It follows from Theorem 4.3.3 that the given equation is oscillatory. The numerical solutions of the given differential equation are found out using the Runge Kutta method of fourth order (RK4) for different steps sizes.

We have

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^5(t)\sin(\dot{x}(t)) - \left[x^5(t) + \frac{x^{15}(t)}{x^{10}(t) + \dot{x}^2(t)} \right]$$

with initial conditions $x(1) = 0.5, \dot{x}(1) = 1$ on the chosen interval $[1,50]$, the functions $\Psi(x) \equiv 1$ *and* $h(t) \equiv 0$ and finding the values of the functions *r*, *q* and *f* where we consider $H(t, x, \dot{x}) = f(t)l(x, \dot{x})$ at *t*=1, *n =2250, n =2500, n=2750* and *n*=3000 and the steps sizes *h* =0.021, *h* =0.019, *h* =0.017 and *h* = 0.016.

Table 4.3: Comparison of the numerical solutions of ODE 4.3 with different steps sizes

	h=0.021	**h=0.019**	**h=0.017**	**h=0.016**	$\lvert x_4 - x_1 \rvert$	**Error %**	$\lvert x_4 - x_2 \rvert$	**Error %**	$\lvert x_4 - x_3 \rvert$	**Error %**
t_k	**$x_1(t_k)$**	**$x_2(t_k)$**	**$x_3(t_k)$**	**$x_4(t_k)$**						
1	0.5	0.5	0.5	0.5	0	0	0	0	0	0
5.9	0.81878559	0.81878590	0.81878609	0.81878620	0.00000061	0.00007450	0.00000003	0.00003663	0.00000011	0.00001343
10.8	1.11327238	1.11327299	1.11327335	1.11327356	0.00000118	0.00010599	0.00000057	0.00005120	0.00000021	0.00001886
12.7	-1.16374641	-1.16374724	-1.16374772	-1.16374801	0.0000016	0.00013748	0.00000077	0.00006616	0.00000029	0.00002491
6	-0.18150657	-0.18150462	-0.18150350	-0.18150282	0.00000375	0.00206608	0.0000018	0.00099172	0.00000068	0.00037464
18.6	1.06756053	1.06755798	1.06755653	1.06755565	0.00000488	0.00045711	0.00000233	0.00021825	0.00000088	0.00008243
4	1.09703326	1.09703572	1.09703711	1.09703794	0.00000468	0.00042660	0.00000222	0.00020236	0.00000083	0.00007565
20.4	-0.91169694	-0.91169344	-0.91169146	-0.91169027	0.00000667	0.00073160	0.00000317	0.00034770	0.00000119	0.00013052
04	-0.20003829	-0.20004521	-0.20004913	-0.20005147	0.00001255	0.00627338	0.00000626	0.00312919	0.00000051	0.00025493
24.5	-0.20245965	-0.20245475	-0.20245198	-0.20245032	0.00000933	0.00460853	0.00000443	0.00218819	0.00000166	0.00081995
2	0.11735667	0.11736295	0.11736649	0.11736859	0.00001192	0.01015603	0.00000564	0.00480537	0.00000021	0.00178923
27.0	-1.12981135	-1.12981812	-1.12982191	-1.12982417	0.00001282	0.00113468	0.00000605	0.00053548	0.00000226	0.00020003
68	0.43716115	0.43716897	0.43717337	0.43717597	0.00001482	0.00338993	0.000007	0.00160118	0.0000026	0.00059472
30.4	0.75641819	0.75642768	0.75643299	0.75643613	0.00001794	0.00237164	0.00000845	0.00111708	0.00000314	0.00041510
32.3	-0.88199073	-0.88198042	-0.88197465	-0.88197124	0.00001949	0.00220982	0.00000918	0.00104085	0.00000341	0.00038663

Figure 4.3(a): Solution curve of ODE 4.3

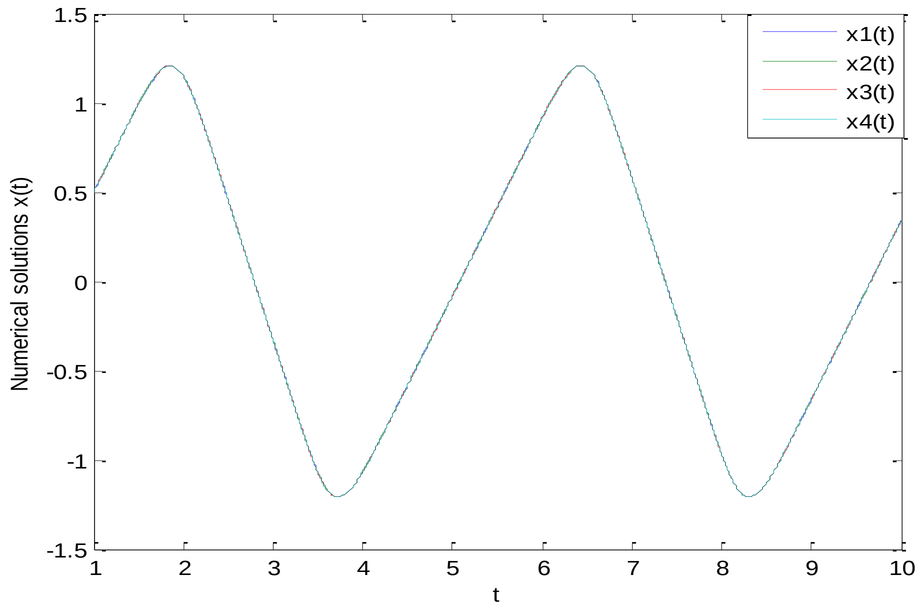
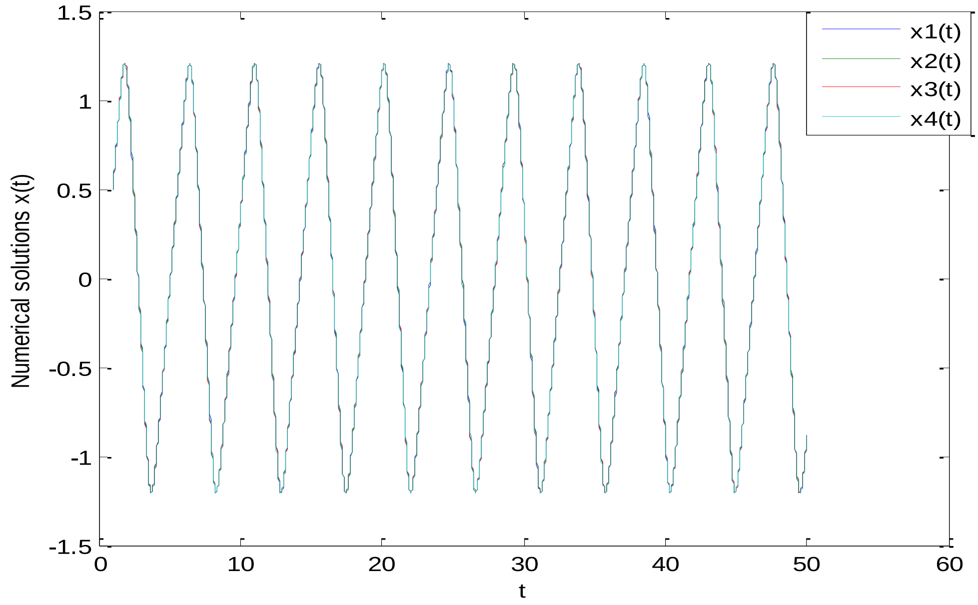


Figure 4.3(b): Solution curve of ODE 4.3



Theorem 4.3.4

Suppose that conditions (1) and (8) hold and

(12) $h(t) \le 0 \;\; for\, t \ge t_0$.

Furthermore, suppose that there exists a positive continuous differentiable function ρ on the interval $[t_0, \infty)$ with $\dot{\rho}(t) \geq 0$, $(\dot{\rho}(t)r(t))^{\bullet} \leq 0$, $(a_2\,\dot{\rho}(t)r(t) - \rho(t)h(t)) \geq 0$ and

$(a_2\,\dot{\rho}(t)r(t) - \rho(t)h(t))^{\bullet} \leq 0$ such that

(13) $$\int_{t_0}^{\infty} \frac{ds}{r(s)\rho(s)} = \infty, \quad forevery \;\; t \geq t_0 .$$

(14) $$\lim_{t\to\infty} \int_T^t \rho(s)[C_0 q(s) - p(s)]ds = \infty.$$

Then, every solution of super-linear equation (1.2) is oscillatory.

Proof

If $x(t)$ is oscillatory on $[T, \infty)$, $T \geq t_0 \geq 0$, then, $\dot{x}(t)$ is oscillatory on $[T, \infty)$ and if $\dot{x}(t)$ is oscillatory on $[T, \infty)$, then, $\ddot{x}(t)$ is oscillatory on $[T, \infty)$. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t) > 0 \;\; on \; [T, \infty)$.for some $T \geq t_0 \geq 0$ We have three cases of $\dot{x}(t)$:

(i) $\dot{x}(t) > 0 \quad for\, every\, t \geq T.$

(ii) $\dot{x}(t) < 0 \quad for\, every\, t \geq T.$

(iii) $\dot{x}(t)$ is oscillatory.

If $\dot{x}(t) > 0$ for $t \geq T$, $T \geq t_0$ and we define

$$\omega(t) = \frac{\rho(t)r(t)\Psi(x(t))\,\dot{x}(t)}{g(x(t))}, \;\; t \geq T.$$

Then, by equation (1.2) and condition (1), we get

$$\dot{\omega}(t) \le \rho(t)p(t) - \frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))} - \rho(t)q(t)\Phi(1,\omega(t)/\rho(t)) + \frac{a_2\,\dot{\rho}(t)r(t)\dot{x}(t)}{g(x(t))},\ t \ge T.$$

Since $\Phi(1,\omega(t)/\rho(t)) > 0$, then, there exists a positive constant C_0 such that $\Phi(1,\omega(t)/\rho(t)) > C_0 > 0$ for $t \ge T$, we have

$$\dot{\omega}(t) \le -\rho(t)\left[C_0 q(t) - p(t)\right] + \left(a_2\,\dot{\rho}(t)r(t) - \rho(t)h(t)\right)\frac{\dot{x}(t)}{g(x(t))},\ t \ge T.$$

Integrate the last inequality from *T* to *t*, we obtain

$$\omega(t) \le \omega(T) - \int_T^t \rho(s)\left[C_0 q(s) - p(s)\right]ds + \int_T^t \left(a_2\,\dot{\rho}(s)r(s) - \rho(s)h(s)\right)\frac{\dot{x}(s)}{g(x(s))}ds. \quad (4.3.9)$$

Since $(a_2\,\dot{\rho}(t)r(t) - \rho(t)h(t))$ is a non-increasing and by Bonnet's Theorem for a fixed $t \ge T$, there exists $\beta_t \in [T,t]$ such that

$$\begin{aligned}\int_T^t \left(a_2\,\dot{\rho}(s)r(s) - \rho(s)h(s)\right)\frac{\dot{x}(s)}{g(x(s))}ds &= \left(a_2\,\dot{\rho}(T)r(T) - \rho(T)h(T)\right)\int_T^{\beta_t}\frac{\dot{x}(s)}{g(x(s))}ds \\ &= \left(a_2\,\dot{\rho}(T)r(T) - \rho(T)h(T)\right)\int_{x(T)}^{x(\beta_t)}\frac{du}{g(u)}\end{aligned}$$

Since $(a_2\,\dot{\rho}(t)r(t) - \rho(t)h(t)) \ge 0$ and the equation (1.2) is superlinear, we have

$$\int_{x(T)}^{x(\beta_t)}\frac{du}{g(u)} < \begin{cases} 0 & ,\ if\ x(\beta_t) < x(T) \\ \displaystyle\int_{x(T)}^{\infty}\frac{du}{g(u)} & ,\ if\ x(\beta_t) \ge x(T). \end{cases}$$

We have

$$-\infty < \int_T^t \left(a_2 \dot{\rho}(s) r(s) - \rho(s) h(s) \right) \frac{\dot{x}(s)}{g(x(s))} ds \le A_1, \qquad (4.3.10)$$

where $A_1 = \left(a_2 \dot{\rho}(T) r(T) - \rho(T) h(T) \right) \int_{x(T)}^{\infty} \frac{du}{g(u)}.$

Thus, from (4.3.10) in (4.3.9), we obtain

$$\omega(t) \le \omega(T) + A_1 - \int_T^t \rho(s)\left[C_0 q(s) - p(s)\right] ds.$$

By the condition (14), we get $\omega(t) < 0$, then, $\dot{x}(t) < 0$ for $t \ge T_1, T_1 \ge T$. This is a contradiction.

If $\dot{x}(t) < 0$ for every $t \ge T_2 \ge T,$ the condition (14) implies that there exists $T_3 \ge T_2$ such that

$$\int_{T_3}^t \rho(t)\left[C_0 q(s) - p(s)\right] ds \ge 0 \quad \text{for all } t \ge T_3.$$

Thus, from equation (1.2) multiplied by $\rho(t)$, we obtain

$$\rho(t)\left(r(t)\Psi(x(t))\dot{x}(t) \right)^{\bullet} + \rho(t) h(t) \dot{x}(t) + \rho(t) g\left(x(t)\right) q(t) \Phi\left(1, \omega(t)/\rho(t)\right) \le \rho(t) g\left(x(t)\right) p(t),\ t \ge T_3.$$

By condition (12), we have

$$\rho(t)\left(r(t)\Psi(x(t))\dot{x}(t) \right)^{\bullet} \le -\rho(t) g(x(t))\left[C_0 q(t) - p(t)\right].$$

Integrate the last inequality from T_3 to t and by condition (1), we obtain

$$a_1\rho(t)r(t)\dot{x}(t) \le \rho(T_3)r(T_3)\Psi(x(T_3))\dot{x}(T_3) + a_1\int_{T_3}^{t}\dot{\rho}(s)r(s)\dot{x}(s)ds - \int_{T_3}^{t} g(x(s))\rho(s)\big(C_0q(s) - p(s)\big)ds$$

$$\le \rho(T_3)r(T_3)\Psi(x(T_3))\dot{x}(T_3) + a_1\int_{T_3}^{t}\dot{\rho}(s)r(s)\dot{x}(s)ds - g(x(t))\int_{T_3}^{t}\rho(s)\big(C_0q(s) - p(s)\big)ds$$

$$+\int_{T_3}^{t} g'(x(s))\dot{x}(s)\int_{T_3}^{s}\rho(u)\big(C_0q(u) - p(u)\big)duds,$$

$$\le \rho(T_3)r(T_3)\Psi(x(T_3))\dot{x}(T_3),\ t \ge T_3.$$

Integrate the last inequality divided by $\rho(t)r(t)$ from T_3 to t and by condition (13), we have

$$a_2x(t) \le a_2x(T_3) + \rho(T_3)r(T_3)\Psi(x(T_3))\dot{x}(T_3)\int_{T_3}^{t}\frac{ds}{r(s)\rho(s)} \to -\infty,$$

as $t \to \infty$, contradicting the fact $x(t) > 0$ for all $t \ge T$. Thus, we have $\dot{x}(t)$ is oscillatory and this leads to (1.2) is oscillatory. Hence the proof is completed.

Example 4.3.4

Consider the following differential equation

$$\left(\frac{(x^2(t)+5)}{t(x^2(t)+4)}\dot{x}(t)\right)^{\bullet} - \frac{\dot{x}(t)}{t^2} + 9t\left[x^5(t) + \frac{x^{15}(t)}{3x^{10}(t) + 6\left(\frac{(x^2(t)+5)\dot{x}(t)}{t(x^2(t)+4)}\right)^2}\right] = \frac{x^5(t)\sin(\dot{x}(t))}{t^3},\ t > 0.$$

We have $r(t)=\frac{1}{t}$, $h(t)=-\frac{1}{t^2}$, $q(t)=9t$, $g(x)=x^5$, $\Phi(u,v)=u+\frac{u^3}{3u^2+6v^2}$ and

(1) $H(t,x(t),\dot{x}(t))=\frac{x^5(t)\sin(\dot{x}(t))}{t^3}$ *and* $\frac{H(t,x(t),\dot{x}(t))}{g(x(t))}=\frac{\sin(\dot{x}(t))}{t^3}\le\frac{1}{t^3}=p(t)$ *for all* $x\neq 0$ *and* $t>0$.

(2) $\Psi(x)=\frac{x^2(t)+5}{x^2(t)+4}>0$ *and* $1\le\Psi(x)=\frac{x^2(t)+5}{x^2(t)+4}=1+\frac{1}{x^2(t)+4}\le\frac{5}{4}$ *for all* $x\in R$.

Taking $\rho(t)=2t>0$, $\dot{\rho}(t)=2>0$, $(\dot{\rho}(t)r(t))=\frac{2}{t}>0$, $(\dot{\rho}(t)r(t))^{\bullet}=-\frac{2}{t^2}<0$,

$(a_2\,\dot{\rho}(t)r(t)-\rho(t)h(t))=\frac{9}{2t}>0$, $(a_2\,\dot{\rho}(t)r(t)-\rho(t)h(t))^{\bullet}=-\frac{9}{2t^2}>0$ for all $t>0$ such that

(3) $\int_{t_0}^{\infty}\frac{ds}{\rho(s)r(s)}=\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{\rho(s)r(s)}=\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{2}=\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{2}=\lim_{t\to\infty}\frac{1}{2}(t-t_0)=\infty,$

(4) $\lim_{t\to\infty}\int_{t_0}^{t}\rho(s)(C_0q(s)-p(s))\,ds=\lim_{t\to\infty}\int_{t_0}^{t}2s\left(9C_0s-\frac{1}{s^3}\right)ds=\lim_{t\to\infty}\left[6C_0s^3+\frac{2}{s}\right]_{t_0}^{t}=\infty.$

All conditions of Theorem 4.3.4 are satisfied. Then the given equation is oscillatory. Also the numerical solutions of the given differential equation are computed using the Runge Kutta method of fourth order (RK4) for different steps sizes.

We have

$$\ddot{x}(t)=f(t,x(t),\dot{x}(t))=x^5(t)\sin(\dot{x}(t))-9\left[x^5(t)+\frac{x^{15}(t)}{3x^{10}(t)+6\dot{x}^2(t)}\right]$$

with initial conditions $x(1) = 0.5, \dot{x}(1) = 1$ on the chosen interval $[1,50]$, the functions $\Psi(x) \equiv 1$ and $h(t) \equiv 0$ and the finding the values of the functions *r, q* and *f* where we consider $H(t, x, \dot{x}) = f(t)l(x, \dot{x})$ at t=1, *n =2250, n =2500, n=2750* and *n*=3000 and the steps sizes *h* =0.021, *h* =0.019, *h* =0.017 and *h* = 0.016.

Table 4.4: Comparison of the numerical solutions of ODE 4.4 with different steps sizes

t_k	h=0.021	h=0.019	h=0.017	h=0.016	$\lvert x_4 - x_1 \rvert$	Error %	$\lvert x_4 - x_2 \rvert$	Error %	$\lvert x_4 - x_3 \rvert$	Error %
	$x_1(t_k)$	$x_2(t_k)$	$x_3(t_k)$	$x_4(t_k)$						
1	0.5	0.5	0.5	0.5	0	0	0	0	0	0
5.9	0.36223316	0.36223218	0.36223160	0.36223124	0.00000192	0.00053004	0.00000094	0.00025950	0.00000036	0.00009938
10.8	-0.77609392	-0.77609547	-0.77609635	-0.77609688	0.00000296	0.00038139	0.00000141	0.00018167	0.00000053	0.00006829
12.76	0.77780997	0.77781189	0.77781297	0.77781363	0.00000366	0.00047054	0.00000174	0.00022370	0.00000066	0.00008485
18.64	-0.83920231	-0.83920153	-0.83920109	-0.83920082	0.00000149	0.00017754	0.00000071	0.00008460	0.00000027	0.00003217
20.404	0.78635090	0.78635484	0.78635706	0.78635840	0.0000075	0.00095376	0.00000356	0.00045271	0.00000134	0.00017040
24.52	0.79864949	0.79864469	0.79864198	0.79864034	0.00000915	0.00114569	0.00000435	0.00054467	0.00000164	0.00020534
27.068	-0.15103522	-0.15102493	-0.15101915	-0.15101568	0.00001954	0.01293905	0.00000925	0.00612519	0.00000347	0.00229777
30.4	-0.63603046	-0.63601908	-0.63601269	-0.63600886	0.0000216	0.00339617	0.00001022	0.00160689	0.00000383	0.00060219
32.36	0.62445218	0.62443849	0.62443082	0.62442624	0.00002594	0.00415421	0.00001225	0.00196180	0.00000458	0.00073347
37.26	-0.54940701	-0.54942585	-0.54943638	-0.54944265	0.00003564	0.00648657	0.0000168	0.00305764	0.00000627	0.00114115
40.2	0.40322689	0.40320485	0.40319254	0.40318520	0.00004169	0.01034016	0.00001965	0.00487369	0.00000734	0.00182050
42.16	-0.32372898	-0.32370641	-0.32379379	-0.32368629	0.00004269	0.01318869	0.00002012	0.00621589	0.0001075	0.03321116
47.06	0.75149628	0.75151660	0.75152793	0.75153466	0.00003838	0.00510688	0.00001806	0.00240308	0.00000673	0.00089550
50	-0.11000764	-0.10997667	-0.10995940	-0.10994916	0.00005848	0.05318824	0.00002751	0.02502065	0.00001024	0.00931339

Figure 4.4(a): Solution curve of ODE 4.4

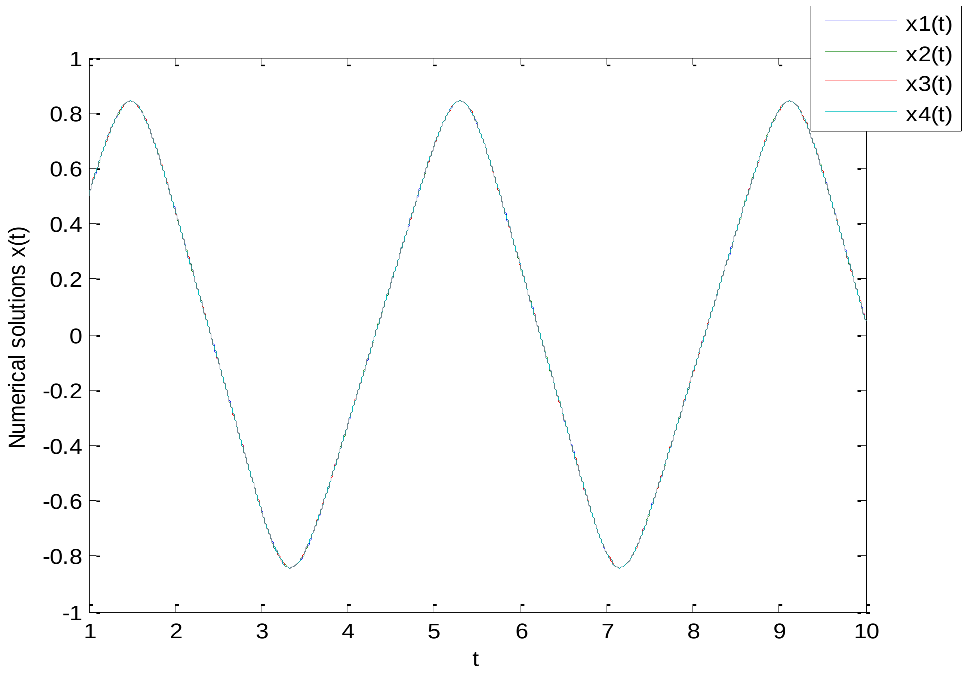
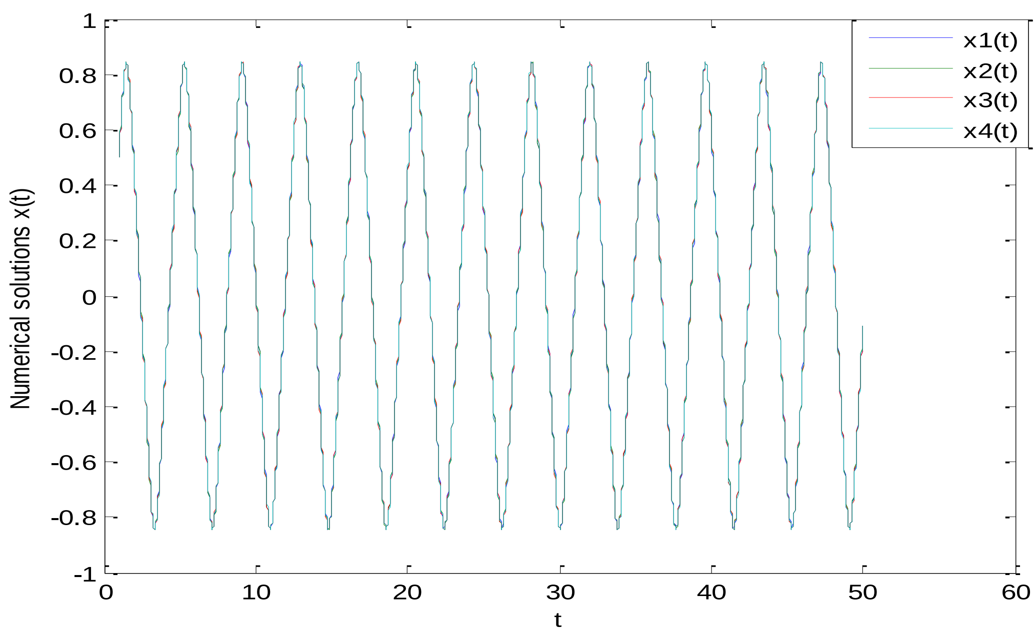


Figure 4.4(b): Solution curve of ODE 4.4



Remark 4.3.3

Theorem 4.3.4 extends results of Grace (1992) and El-abbasy et al. (2005), who have studied the equation (1.2) as $\Psi(x(t)) \equiv 1,\quad \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 4.3.4 because their equation is a particular case of our equation (1.2).

Theorem 4.3.5: Suppose that the conditions (1), (8), (12) and (13) hold and there exists a continuously differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ such that $(\rho(t)h(t))^{\bullet} \le 0,\ \dot{\rho}(t)r(t) \ge 0,\ \left(\dot{\rho}(t)r(t)\right)^{\bullet} \le 0\ \ on\ [t_0, \infty)$ and

$$(15)\qquad \int_{t_0}^{\infty} \Omega(s)\, ds = \infty,$$

where $\Omega(t) = \rho(t)\left(C_0 q(t) - p(t)\right) - \dfrac{\dot{\rho}^2(t) r(t)}{4k\rho(t)}$. Then, every solution of superlinear equation (1.2) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t)>0\ \ on\ [T, \infty)$ for some $T \ge t_0 \ge 0$. Define

$$\omega(t) = \frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))},\ t \ge T.$$

This, condition (8) and (1.2) imply

$$\left(\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}\right)^{\bullet} \leq -[C_0 q(t) - p(t)] - \frac{h(t)\dot{x}(t)}{g(x(t))} - \frac{kr(t)\Psi(x(t))\dot{x}^2(t)}{g^2(x(t))}, \; t \geq T. \quad (4.3.11)$$

We multiply the last inequality (4.3.11) by $\rho(t)$ and integrate form T to t, we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq C_1 - \int_T^t \rho(s)(C_0 q(s) - p(s))\,ds - \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds + \int_T^t \left[\dot{\rho}(s)\omega(s) - \frac{k}{a_2}\frac{\rho(s)}{r(s)}\omega^2(s)\right]ds,$$

where $C_1 = \dfrac{\rho(T)r(T)\Psi(x(T))\dot{x}(T)}{g(x(T))}$.

Thus

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq C_1 - \int_T^t \rho(s)(C_0 q(s) - p(s))\,ds - \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds - \int_T^t \frac{k}{a_2}\frac{\rho(s)}{r(s)}\left(\eta^2(s) - \left(\frac{a_2\dot{\rho}(s)r(s)}{2k\rho(s)}\right)^2\right)ds,$$

where $\eta(t) = \omega(t) - \dfrac{a_2\dot{\rho}(t)r(t)}{2k\rho(t)}$.

Thus, for $t \geq T$, we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq C_1 - \int_T^t \left[\rho(s)(C_0 q(s) - p(s)) - \frac{a_2\dot{\rho}^2(s)r(s)}{4k\rho(s)}\right]ds - \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds. \quad (4.3.12)$$

The second integral in R. H. S. of the inequality (4.3.12) is bounded from above. This can be by using the Bonnet's Theorem, for all $t \geq T$, there exists $a_t \in [T, t]$ such that

$$-\int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds = -\rho(T)h(T)\int_T^{a_t} \frac{\dot{x}(s)}{g(x(s))}ds = -\rho(T)h(T)\int_{x(T)}^{x(a_t)} \frac{du}{g(u)}.$$

Since $\left(-\rho(t)h(t)\right) \geq 0$ and the equation (1.2) is superlinear, we have

$$-\infty < \int_T^t -\rho(s)h(s)\frac{\dot{x}(s)}{g(x(s))}ds \leq B_1, \tag{4.3.13}$$

where $B_1 = -\rho(T)h(T)\int_{x(T)}^{\infty} \frac{du}{g(u)}$.

By inequality (4.3.13) and the condition (15), the inequality (4.3.12) becomes

$$\frac{\rho(t)r(t)\Psi(x(s))\dot{x}(t)}{g(x(t))} \leq C_1 + B_1 - \int_T^t \Omega(s)\,ds.$$

By the condition (15), we have

$$\lim_{t\to\infty} \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} = -\infty.$$

Thus, there exists $T_1 \geq T$ such that $\dot{x}(t) < 0$ for $t \geq T_1$. The condition (15) also implies that there exists $T_2 \geq T_1$ such that

$$\int_{T_1}^{T_2} \rho(s)\left[C_0 q(s) - p(s)\right]ds = 0 \quad and \quad \int_{T_2}^{t} \rho(s)\left[C_0 q(s) - p(s)\right]ds \geq 0 \quad for\ t \geq T_2.$$

Multiplying equation (1.2) by $\rho(t)$, from the conditions (12) and (8), we have

$$\rho(t)\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + C_0\rho(t)g(x(t))q(t) \leq \rho(t)g(x(t))p(t),\ t \geq T_2,$$

where $0 < C_0 = \min_{\omega(t)\in R} \Phi(1, \omega(t))$.

Integrate the last inequality from T_2 to t, we obtain

$$\rho(t)r(t)\Psi(x(t))\dot{x}(t) \le \rho(T_2)r(T_2)\Psi(x(T_2))\dot{x}(T_2) + \int_{T_2}^{t} \dot{\rho}(s)r(s)\Psi(x(s))\dot{x}(s)\,ds$$
$$- g(x(t))\int_{T_2}^{t} \rho(s)\big(C_0 q(s) - p(s)\big)ds + \int_{T_2}^{t} g'(x(s))\dot{x}(s)\int_{T_2}^{s} \rho(u)\big(C_0 q(u) - p(u)\big)du\,ds.$$

By the condition (1) and the Bonnet's Theorem, for $t \ge T_2$ there exists $\gamma_t \in [T_2, t]$ such that

$$a_2\rho(t)r(t)\dot{x}(t) \le \rho(T_2)r(T_2)\Psi(x(T_2))\dot{x}(T_2) + a_1\dot{\rho}(T_2)r(T_2)[x(\gamma_t) - x(T_2)] - g(x(t))\int_{T_2}^{t}\rho(s)\big(Cq(s) - p(s)\big)ds$$
$$+ \int_{T_2}^{t} g'(x(s))\dot{x}(s)\int_{T_2}^{s}\rho(u)\big(Cq(u) - p(u)\big)du\,ds,\ t \ge T_2.$$

Thus

$$a_2\rho(t)r(t)\dot{x}(t) \le \rho(T_2)r(T_2)\Psi(x(T_2))\dot{x}(T_2),\ t \ge T_2.$$

Dividing the last inequality by $\rho(t)r(t)$, integrate from T_2 to t and the condition (7), we obtain

$$a_2x(t) \le a_2x(T_2) + \rho(T_2)r(T_2)\Psi(x(T_2))\dot{x}(T_2)\int_{T_2}^{t}\frac{ds}{\rho(s)r(s)} \to -\infty,\ as\ t \to \infty,$$

which is a contradiction to the fact that $x(t) > 0$ for $t \ge T$. Hence the proof is completed.

Example 4.2.5

Consider the following differential equation

$$\left(\frac{x^6(t)+2}{t^3(x^6(t)+1)}\dot{x}(t)\right)^{\bullet} - t^2\dot{x}(t) + t^3\left[x^3(t) + \frac{x^{15}(t)}{5x^{12}(t)+8\left(\frac{x^6(t)+2}{t^3(x^6(t)+1)}\dot{x}(t)\right)^4}\right] = \frac{x^3(t)\sin(x(t)\dot{x}(t))}{t^5}.$$

We note that

$$r(t) = \frac{1}{t^3},\ h(t) = -t^2,\ q(t) = t^3,\ g(x) = x^3,\ \Phi(u,v) = u + \frac{u^5}{5u^4+8v^4}\ ,$$

$H(t,x(t),\dot{x}(t)) = \dfrac{x^3(t)\sin(x(t)\dot{x}(t))}{t^5}$, $\dfrac{H(t,x(t),\dot{x}(t))}{g(x(t))} = \dfrac{\sin(x(t)\dot{x}(t))}{t^5} \le \dfrac{1}{t^5} = p(t)$ for all $x \neq 0$ and $t > 0$. $\Psi(x) = \dfrac{x^6+2}{x^6+1}$ and $1 \le \Psi(x) \le 2$ for all $x \in \mathrm{R}$

Taking $\rho(t) = t$, $\dot{\rho}(t)r(t) = \dfrac{1}{t^3} > 0$, $(\rho(t)h(t))^{\bullet} = -3t^2 < 0$ *and* $\left(\dot{\rho}(t)r(t)\right)^{\bullet} = \left(\dfrac{1}{t^3}\right)^{\bullet} = \dfrac{-3}{t^4} < 0$ *for all* $t > 0$ *and* $\displaystyle\int_{t_0}^{\infty}\frac{ds}{\rho(s)r(s)} = \int_{t_0}^{\infty}s^2ds = \infty.$

(1) $$\int_{t_0}^{\infty}\Omega(s)ds = \int_{t_0}^{\infty}\left[\rho(s)\left(C_0q(s)-p(s)\right) - \frac{a_2\dot{\rho}^2(s)r(s)}{4k\rho(s)}\right]ds$$

$$= \int_{t_0}^{\infty}\left(s\left[C_0s^3 - \frac{1}{s^5}\right] - \frac{1}{2ks^4}\right)ds$$

$$= \left[\frac{C_0s^5}{5} + \frac{1}{3s^3} + \frac{1}{6ks^3}\right]_{t_0}^{\infty} = \infty.$$

All conditions of Theorem 4.3.5 are satisfied and hence every solution of the given equation is oscillatory. We also compute the numerical solution of the given differential equation using the Runge Kutta method of fourth order (RK4). We have

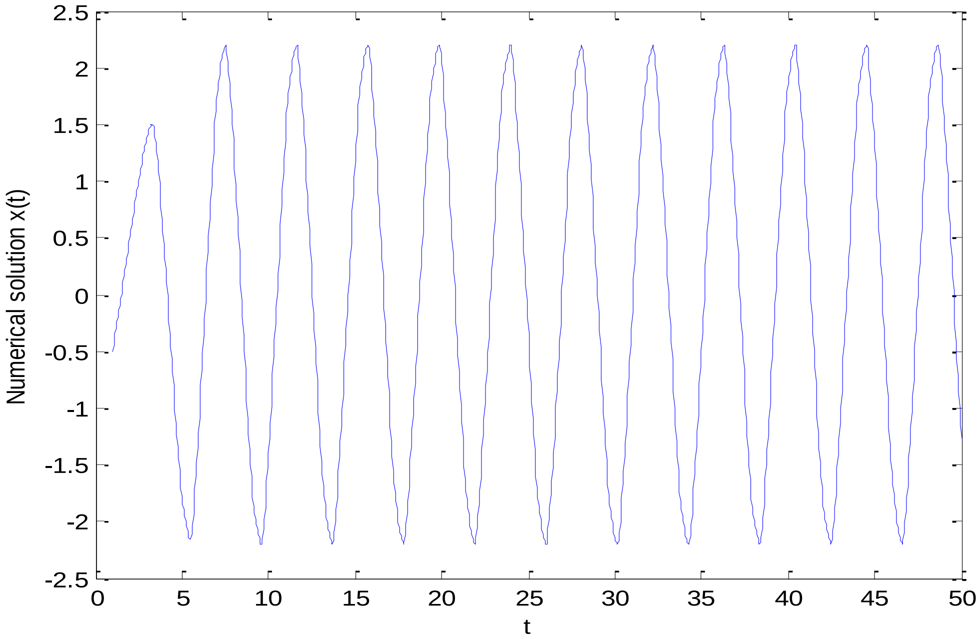
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^3(t)\sin(x(t)\dot{x}(t)) - \left(x^3(t) + \frac{x^{15}(t)}{5^{12}(t) + 8\dot{x}^4(t)} \right)$$

with initial conditions $x(1) = -0.5,\ \dot{x}(1) = 1$ on the chosen interval $[1,50]$, the functions $\Psi(x) \equiv 1$ and $h(t) \equiv 0$ and finding the values of the functions r, q and f where we consider $H(t, x, \dot{x}) = f(t)l(x, \dot{x})$ at t=1, n=980 and h=0.05.

Table 4.5: Numerical solution of ODE 4.5

k	**t_k**	**$x(t_k)$**
1	1	-0.5
62	4.05	0.22021420
181	10	-1.39354243
221	12	1.53595848
321	17	-1.01339800
381	20	2.04072897
461	24	2.19425748
561	29	-0.01291195
603	31.1	0.14141788
685	35.2	0.10102608
726	37.25	-0.08118400
781	40	1.86554418
821	42	-1.78419073
933	47.6	0.28301496
981	50	-1.28617425

Figure 4.5: Solution curve of ODE 4.5



Remark 4.3.4

Theorem 4.3.5 is the extension of the results of Greaf, et al. (1978) and Remili (2010) who have studied the equation (1.2) when $\Psi(x(t)) \equiv 1$, $h(t) \equiv 0$ and $\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv \Phi(t, x(t))$. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 4.3.5 because their equation is a particular case of our equation (1.2).

Theorem 4.3.6

Suppose, in addition to the conditions (1), (8) and (9) hold that

(16) $$\int_T^\infty \frac{ds}{r(s)} \le k_1\,,\ k_1 > 0\,.$$

(17) There exists a constant B^* such that

$$G(m)=\int_0^m \frac{ds}{\Phi(1,s)} > B^* m,\ B^*<0,\ for\ every\ m\in \mathrm{R}.$$

Furthermore, suppose that there exists a positive continuous differentiable function ρ on the interval $[t_0,\infty)$ with $\rho(t)$ is a non-decreasing function on the interval $[t_0,\infty)$ such that

$$(18)\qquad \limsup_{t\to\infty} \int_T^t \frac{1}{r(s)\rho(s)} \int_T^s \rho(u)\left[C_0 q(u)-p(u)-\frac{h^2(u)}{4a^* r(u)}\right] du\, ds=\infty,$$

where $p:[t_0,\infty)\to(0,\infty)$. Then, every solution of super-linear equation (1.2) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t)>0$ on $[T,\infty)$ for some $T\geq t_0\geq 0.$

Define

$$\omega(t)=\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))},\ t\geq T.$$

This and by condition (1) and equation (1.2), we have

$$\dot{\omega}(t)\leq \rho(t)p(t)-\frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))}-\rho(t)q(t)\Phi(1,\omega(t)/\rho(t))+\frac{\dot{\rho}(t)}{\rho(t)}\omega(t)-\frac{a_1 k\rho(t)r(t)\dot{x}^2(t)}{g^2(x(t))},\ t\geq T.$$

Thus for $t \geq T$, we have

$$\rho(t)\left(\frac{\omega(t)}{\rho(t)}\right)^{\bullet} \leq \rho(t)p(t) - \rho(t)q(t)\Phi(1,\omega(t)/\rho(t)) - \frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))} - \frac{a_1 k\rho(t)r(t)\dot{x}^2(t)}{g^2(x(t))},\ t \geq T.$$

Dividing the last inequality by $\Phi(1,\omega(t)/\rho(t)) > 0$, then, there exists a positive constant C_0 such that $\Phi(1,\omega(t)/\rho(t)) > C_0$ then, $0 < \dfrac{1}{\Phi(1,\omega(t)/\rho(t))} < \dfrac{1}{C_0}$. Thus, for $t \geq T$, we obtain

$$\rho(t)\left(C_0 q(t) - p(t)\right) \leq -\frac{C_0\rho(t)\left(\omega(t)/\rho(t)\right)^{\bullet}}{\Phi(1,\omega(t)/\rho(t))} - \frac{C_0\rho(t)h(t)\dot{x}(t)}{\Phi(1,\omega(t)/\rho(t))g(x(t))} - \frac{C_0 a_1 k\rho(t)r(t)\dot{x}^2(t)}{\Phi(1,\omega(t)/\rho(t))g^2(x(t))},\ t \geq T.$$

Integrate from T to t, we obtain

$$\int_T^t \rho(s)[C_0 q(s) - p(s)]ds \leq -C_0\int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi(1,\omega(s)/\rho(s))}ds$$
$$- C_0\int_T^t\left[\frac{\rho(s)h(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}(s)}{g(x(s))} + \frac{a_1 k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}^2(s)}{g^2(x(s))}\right]ds. \quad (4.3.14)$$

From the second integral in R. H. S. of (4.3.14), we have

$$-C_0\int_T^t\left[\frac{\rho(s)h(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}(s)}{g(x(s))} + \frac{a_1 k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}\frac{\dot{x}^2(s)}{g^2(x(s))}\right]ds$$
$$= -C_0\int_T^t\left[\sqrt{\frac{a_1 k\rho(s)r(s)}{\Phi(1,\omega(s)/\rho(s))}}\frac{\dot{x}(s)}{g(x(s))} + \frac{1}{2}\sqrt{\frac{\rho(s)}{a_1 k r(s)\Phi(1,\omega(s)/\rho(s))}}h(s)\right]^2 ds + \frac{C_0}{4a_1 k}\int_T^t \frac{\rho(s)h^2(s)}{\Phi(1,\omega(s)/\rho(s))r(s)}ds$$
$$\leq \frac{1}{4a^*}\int_T^t \frac{\rho(s)h^2(s)}{r(s)}ds, \quad (4.3.15)$$

where $a^* = a_1 k$.

By the Bonnet's Theorem, since $\rho(t)$ is a non-decreasing function on the interval $[t_0,\infty)$, there exists $T_1 \in [T,t]$ such that

$$\int_T^t \frac{\rho(s)\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}\,ds = \rho(t)\int_{T_1}^t \frac{\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}\,ds. \tag{4.3.16}$$

From inequalities (4.3.16) and (4.3.15) in inequality (4.3.14), we have

$$\begin{aligned}\int_T^t \rho(s)\left[C_0q(s)-p(s)-\frac{h^2(s)}{4a^*r(s)}\right]ds &\le -C_0\rho(t)\int_{T_1}^t \frac{\left(\omega(s)/\rho(s)\right)^{\bullet}}{\Phi\left(1,\omega(s)/\rho(s)\right)}\,ds = -C_0\rho(t)\int_{\omega(T_1)/\rho(T_1)}^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)}\\ &\le -C_0\rho(t)\left[-\int_0^{\omega(T_1)/\rho(T_1)} \frac{du}{\Phi(1,u)} + \int_0^{\omega(t)/\rho(t)} \frac{du}{\Phi(1,u)}\right]\\ &\le C_0\rho(t)G\left(\frac{\omega(T_1)}{\rho(T_1)}\right) - C_0\rho(t)G\left(\frac{\omega(t)}{\rho(t)}\right).\end{aligned}$$

By the condition (17), we obtain

$$\int_T^t \rho(s)\left[C_0q(s)-p(s)-\frac{h^2(s)}{4a^*r(s)}\right]ds \le C_0\rho(t)G\left(\frac{\omega(T_1)}{\rho(T_1)}\right) - C_0B^*\omega(t).$$

Integrating the last inequality divided by $\rho(t)\,r(t)$ from *T* to *t*, taking the limit superior on both sides and by conditions (16) and (17), we have

$$\begin{aligned}\limsup_{t\to\infty}\int_T^t \frac{1}{r(s)\rho(s)}\int_T^s \rho(u)\left[C_0q(u)-p(u)-\frac{h^2(s)}{4a^*r(s)}\right]du\,ds &\le \limsup_{t\to\infty}\left\{C_0G\left(\frac{\omega(T_1)}{\rho(T_1)}\right)\int_T^t \frac{ds}{r(s)}\right\}\\ &\quad -\left\{\limsup_{t\to\infty} C_0B^*\int_{x(T)}^{x(t)} \frac{\Psi(u)du}{g(u)}\right\} < \infty,\end{aligned}$$

as $t\to\infty$, which contradicts to the condition (18). Hence the proof is completed.

Example 4.3.6

Consider the following differential equation

$$\left(\frac{t^2(x^4(t)+2)}{(x^4(t)+1)}\dot{x}(t)\right)^{\bullet}+\frac{\dot{x}(t)}{t^3}+t^6\left(\frac{x^9(t)}{x^6(t)+\left((x^4(t)+2)\dot{x}(t)\big/t^2(x^4(t)+1)\right)^2}\right)=\frac{x^3(t)\cos(x(t))}{t^7},t>0.$$

We note that $r(t)=t^2,\ h(t)=\dfrac{1}{t^3},\ q(t)=t^6,\ g(x)=x^3,\ \Psi(x)=\dfrac{x^4(t)+2}{x^4(t)+1}>0$ and

$1\le\Psi(x)\le 2$ for all $x\in R$ and $\Phi(u,v)=\dfrac{u^3}{u^2+v^2}$ such that

(1) $G(m)=\int\limits_0^m\dfrac{ds}{\Phi(1,s)}=\int\limits_0^m(1+s^2)ds>\int\limits_0^m(-1)ds=-m\ge -m\ ,\ B=-1,\ B\in R^-\,and$ *for all* $m\in R.$

(2) $\dfrac{H(t,x(t),\dot{x}(t))}{g(x(t))}=\dfrac{\cos(x(t))}{t^7}\le\dfrac{1}{t^7}=p(t)$ *for all* $t>0$ *and* $x\neq 0.$

Let $\rho(t)=t^4$ such that

(3) $$\limsup_{t\to\infty}\int\limits_T^t\frac{1}{r(s)\rho(s)}\int\limits_T^s\rho(u)\left[C_0q(u)-p(u)-\frac{h^2(u)}{4a^*r(u)}\right]du\,ds$$

$$=\limsup_{t\to\infty}\int\limits_T^t\frac{1}{s^6}\int\limits_T^s u^4\left[C_0u^6-\frac{1}{u^7}-\frac{1}{4a^*u^6}\right]du\,ds$$

$$=\limsup_{t\to\infty}\left[C_0\frac{s^6}{66}-\frac{1}{14s^7}-\frac{1}{96a^*s^8}+(C_0\frac{T^{11}}{11}+\frac{1}{2T^2}+\frac{1}{4a^*T})\Big/5s^5\right]_T^t=\infty.$$

All conditions of Theorem 4.3.6 are satisfied and hence every solution of the given equation is oscillatory. The numerical solution of the given differential equation is found out using the Runge Kutta method of fourth order (RK4). We have

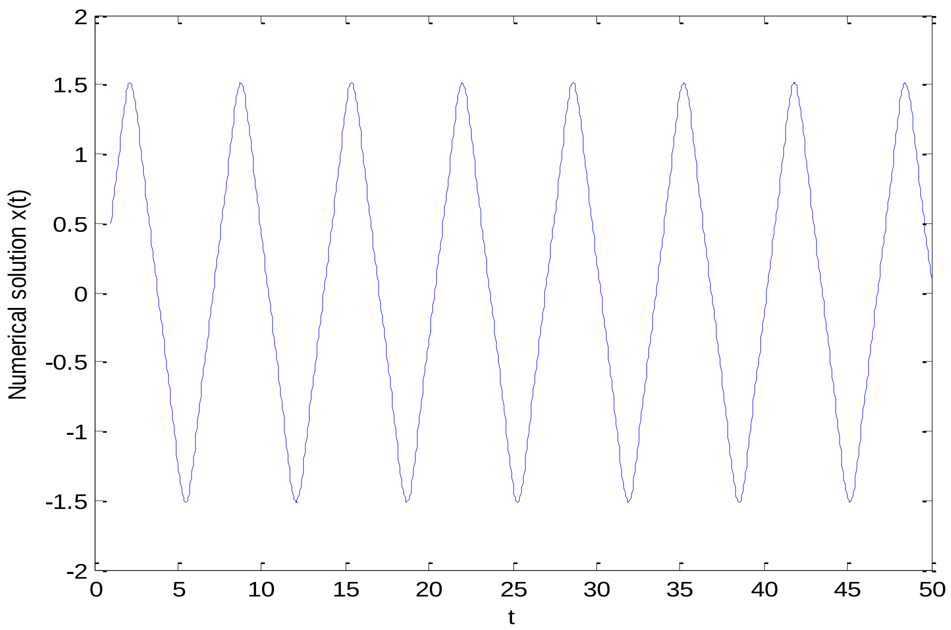
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^3(t)\cos(x(t)) - \frac{x^9(t)}{x^6(t) + \dot{x}^2(t)}$$

with initial conditions $x(1) = 0.5, \dot{x}(1) = 1$ on the chosen interval $[1,50]$, $\Psi(x) \equiv 1, h(t) \equiv 0$ and finding the values of the functions *r, q* and *f* where we consider $H(t, x, \dot{x}) = f(t)l(x, \dot{x})$ at *t*=1 n=980 and h=0.05.

Table 4.6: Numerical solution of ODE 4.6

k	**t_k**	**$x(t_k)$**
1	1	0.5
81	5	-1.23206499
181	10	0.40954827
221	12	-1.50525503
321	17	0.02833177
381	20	-0.33103118
461	24	-0.35258529
521	27	0.04981132
587	30.3	-0.43001970
653	33.6	0.03619157
721	37	-0.12794647
781	40	-0.17456968
821	42	1.46730560
921	47	0.20608737
981	50	0.09644083

Figure 4.6: Solution curve of ODE 4.6



Remark 4.3.5

Theorem 4.3.6 is the extension of the results of Bihari (1963) and Kartsatos (1968), who have studied the equation (1.2) when $r(t) \equiv 1$, $\Psi(x(t)) \equiv 1$, $h(t) \equiv 0$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 4.3.6 because their equation is a particular case of our equation (1.2).

Theorem 4.3.7

Suppose in addition to the conditions (1), (8) and (12) hold that there exists continuously differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ such that $\dot{\rho}(t) \geq 0$, condition (13) holds and

(19) $\liminf_{t\to\infty}\left[\int_{t_0}^{\infty}\Omega_1(s)\,ds\right]\geq 0$ for all large t,

(20) $\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty}\Omega_1(u)duds=\infty,$

where $\Omega_1(t)=\rho(t)\big(C_0q(t)-p(t)\big)-\frac{a_2}{4k}\frac{r(t)}{\rho(t)}\left(\dot{\rho}(t)-\frac{\rho(t)h(t)}{a_1r(t)}\right)^2.$

Then, every solution of superlinear equation (1.2) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t)>0$ *on* $[T,\infty)$ for some $T\geq t_0\geq 0.$ Define

$$\omega(t)=\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))},\ t\geq T.$$

This and (1.2) imply

$$\left(\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}\right)^{\bullet}\leq p(t)-\frac{h(t)\dot{x}(t)}{g(x(t))}-q(t)\Phi\big(1,\omega(t)\big)-\frac{r(t)\Psi(x(t))g'(x(t))\dot{x}^2(t)}{g^2(x(t))},\ t\geq T. \quad (4.3.17)$$

Since $\Phi(1,\omega(t)/\rho(t))>0,$ then there exists a positive constant C_0 such that $\Phi(1,\omega(t)/\rho(t))>C_0$, thus**,** we have

$$\left(\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}\right)^{\bullet}\leq -\big(C_0q(t)-p(t)\big)-\frac{h(t)\dot{x}(t)}{g(x(t))}-\frac{r(t)\Psi(x(t))g'(x(t))\dot{x}^2(t)}{g^2(x(t))},\ t\geq T.$$

We multiply the last inequality by $\rho(t)$ and integrate from T to t, we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \le C_1 - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)\, ds - \int_T^t \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}\, ds$$
$$+ \int_T^t \frac{\dot{\rho}(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}\, ds - \int_T^t \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\, ds, \quad (4.3.18)$$

where $C_1 = \dfrac{\rho(T)r(T)\Psi(x(T))\dot{x}(T)}{g(x(T))}$.

Thus, for we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \le C_1 - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)\, ds - \int_T^t \left[\frac{k\rho(s)}{r(s)\Psi(x(s))}\omega^2(s) - \left(\dot{\rho}(s) - \frac{\rho(s)h(s)}{r(s)\Psi(x(s))}\right)\omega(s)\right] ds$$
$$\le C_1 - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)\, ds$$
$$- \int_T^t \frac{k\rho(s)}{r(s)\Psi(x(s))}\left[\eta^2(s) - \frac{r^2(s)\Psi^2(x(s))}{4k^2\rho^2(s)}\left(\dot{\rho}(s) - \frac{\rho(s)h(s)}{r(s)\Psi(x(s))}\right)^2\right] ds$$
$$\le C_1 - \int_T^t \Omega_1(s)ds - \int_T^t \frac{k\rho(s)}{a_2 r(s)}\eta^2(s)ds$$
$$\le C_1 - \int_T^t \Omega_1(s)ds, \quad (4.3.19)$$

where $\eta(t) = \omega(t) - \dfrac{r(t)\Psi(x(t))}{2k\rho(t)}\left(\dot{\rho}(t) - \dfrac{\rho(t)h(t)}{r(t)\Psi(x(t))}\right).$

From inequality (4.3.19), we have

$$\int_T^t \Omega_1(s)ds \le \frac{\rho(T)r(T)\Psi(x(T))\dot{x}(T)}{g(x(T))} - \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}.$$

Now, we consider three cases for $\dot{x}(t)$.

Case 1: If $\dot{x}(t) > 0 \;\; for \;\; t \ge T_1 \ge T$, then, we get

$$\int_{T_1}^{t} \Omega_1(s)ds \le \frac{\rho(T_1)r(T_1)\Psi(x(T_1))\dot{x}(T_1)}{g(x(T_1))} - \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}.$$

Thus, for all $t \ge T_1$, we obtain

$$\int_{t}^{\infty} \Omega_1(s)ds \le \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}.$$

Integrate the last inequality divided by $\rho(t)\,r(t)$ from T_1 to t and by condition (1), we obtain

$$\int_{T_1}^{t} \frac{1}{\rho(s)r(s)} \int_{s}^{\infty} \Omega_1(u)duds \le a_2 \int_{T_1}^{t} \frac{\dot{x}(s)}{g(x(s))} ds.$$

Since the equation (1.2) is superlinear, we get

$$\int_{T_1}^{t} \frac{1}{\rho(s)r(s)} \int_{s}^{\infty} \Omega_1(u)duds \le a_2 \int_{x(T_1)}^{x(t)} \frac{du}{g(u)} < \infty.$$

This contradicts condition (20).

Case 2: If $\dot{x}(t)$ is oscillatory, then there exists a sequence τ_n in $[T,\infty)$ such that $\dot{x}(\tau_n) = 0$. Choose *M* large enough so that (19) holds. Then from inequality (4.3.19), we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \le C_{\tau_n} - \int_{\tau_n}^{t} \Omega_1(s)ds.$$

So

$$\limsup_{t\to\infty} \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \le C_{\tau_n} + \limsup_{t\to\infty}\left\{-\int_{\tau_n}^{t} \Omega_1(s)ds\right\} = C_{\tau_n} - \liminf_{t\to\infty}\left\{\int_{\tau_n}^{t} \Omega_1(s)ds\right\} < 0,$$

which contradicts the fact that $\dot{x}(t)$ oscillates.

Case 3: If $\dot{x}(t) < 0$ for $t \ge T_2 \ge T$, the condition (19) implies that there exists $T_3 \ge T_2$ such that

$$\int_{T_3}^{t} \left(C_0 q(s) - p(s)\right) ds \ge 0 \qquad for\ t \ge T_3.$$

Multiplying the equation (1.2) by $\rho(t)$ and by condition (12), we have

$$\rho(t)\left(r(t)\Psi(x(t))\dot{x}(t)\right)^{\bullet} + C_0\rho(t)g(x(t))q(t) \le \rho(t)g(x(t))p(t),\ t \ge T_3,$$

where $0 < C_0 = \min_{\omega(t)\in R} \Phi(1, \omega(t))$.

Integrate the last inequality from T_3 to t, we obtain

$$\rho(t)r(t)\Psi(x(t))\dot{x}(t) \le \rho(T_3)r(T_3)\Psi(x(T_3))\dot{x}(T_3) + a_1\int_{T_3}^{t}\dot{\rho}(s)r(s)\dot{x}(s)ds - g(x(t))\int_{T_3}^{t}\rho(s)\big(C_0 q(s) - p(s)\big)ds$$

$$+\int_{T_3}^{t} g'(x(s))\dot{x}(s)\int_{T_3}^{s}\rho(u)\big(Cq(u) - p(u)\big)du\, ds,\ t \ge T_3$$

$$\le \rho(T_3)r(T_3)\Psi(x(T_3))\dot{x}(T_3),\ t \ge T_3.$$

By condition (1), the last inequality divided by $\rho(t)r(t)$ and integrate from T_3 to t, we obtain

$$a_2 x(t) \le a_2 x(T_3) + \rho(T_3)r(T_3)\Psi(x(T_3))\dot{x}(T_3)\int_{T_3}^{t}\frac{ds}{\rho(s)r(s)} \to -\infty,\ as\, t \to \infty,$$

which is a contradiction to the fact that $x(t) > 0$ for $t \ge T.$ Hence the proof is completed.

Example 4.3.7

Consider the following differential equation

$$\left(\frac{x^2(t)+6}{t(x^2(t)+5)}\dot{x}(t)\right)^{\bullet} + \frac{1}{t^2}\frac{x^{35}(t)}{2x^{28}(t) + \left(\frac{x^2(t)+6}{t(x^2(t)+5)}\dot{x}(t)\right)^4}) = -\frac{x^{11}(t)\cos^2(\dot{x}(t))}{x^4(t)+1},\ t > 0.$$

Here we have

$$r(t) = \frac{1}{t},\ h(t) = 0,\ q(t) = \frac{1}{t^2},\ g(x) = x^7,\ \Phi(u,v) = u^5\big/\big(2u^4 + v^4\big), \Psi(x) = \frac{x^2(t)+6}{x^2(t)+5} > 0$$

and $\dfrac{H(t, x(t), \dot{x}(t))}{g\big(x(t)\big)} = -\dfrac{x^4(t)\cos^2(\dot{x}(t))}{x^4(t)+1} \le 0 = p(t)$ *for all* $t > 0$ *and* $x \ne 0.$

Taking $\rho(t) = 4$ and

$$(1)\ \liminf_{t\to\infty}\left\{\int_T^t \Omega_1(s)ds\right\} = \liminf_{t\to\infty}\left\{\int_T^t\left[\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{a_2}{4k}\frac{r(s)}{\rho(s)}\left(\dot{\rho}(s) - \frac{\rho(s)h(s)}{a_1 r(s)}\right)^2\right]ds\right\}$$

$$= \liminf_{t\to\infty}\left\{\int_T^t \frac{4C_0}{s^2}ds\right\} = \frac{4C_0}{T} > 0,\ T_0 > 0.$$

$$(2)\ \lim_{t\to\infty}\int_T^t \frac{1}{\rho(s)r(s)}\int_s^\infty \Omega_1(u)\,duds$$

$$= \lim_{t\to\infty}\int_T^t \frac{1}{\rho(s)r(s)}\int_s^\infty\left[\rho(u)\big(C_0 q(u) - p(u)\big) - \frac{a_2}{4k}\frac{r(u)}{\rho(u)}\left(\dot{\rho}(u) - \frac{\rho(u)h(u)}{a_1 r(u)}\right)^2\right]duds$$

$$= \lim_{t\to\infty}\int_T^t \frac{s}{4}\int_s^\infty\left[\frac{C_0}{u^2}\right]du\,ds = \frac{1}{4}\lim_{t\to\infty}\left[C_0 t - C_0 t_0\right] = \infty.$$

All conditions of Theorem 4.3.7 are satisfied and hence every solution of the given equation is oscillatory. To demonstrate that our result in Theorem 4.3.7 is true, we also find the numerical solution of the given differential equation in Example 4.3.7 using the Runge Kutta method of fourth order.

We have

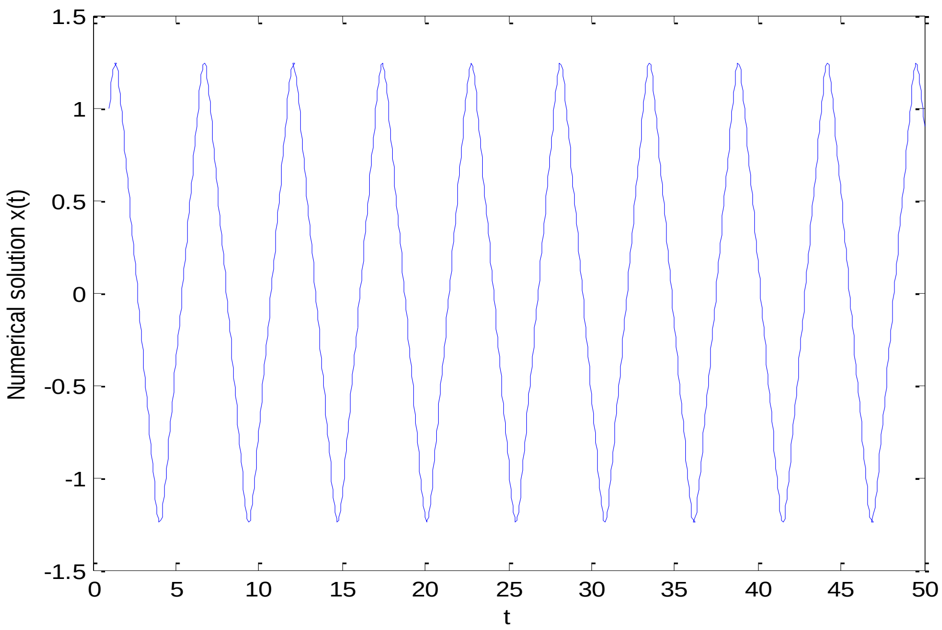
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = -\frac{x^{11}\cos^2(\dot{x}(t))}{x^4 + 1} - \frac{x^{35}}{2x^{28} + \dot{x}^4(t)}$$

with initial conditions $x(1) = 1,\ \dot{x}(1) = 1$ on the chosen interval $[1, 50]$, the function $\Psi(x) \equiv 1$ and finding the values of the functions *r, q* and *f* where we consider $H(t, x, \dot{x}) = f(t)\,l(x, \dot{x})$ at *t*=1, *n*=980 and *h*=0.05.

Table 4.7: Numerical solution of ODE 4.7

k	t_k	$x(t_k)$
1	1	1
81	5	-0.38710915
195	10.75	0.02100274
250	13.45	-0.04535638
321	17	0.94181375
381	20	-1.21862863
461	24	0.11404729
521	27	0.12824333
581	30	-0.55051467
624	31.15	0.01009389
721	37	-0.50610435
781	40	0.17389288
821	42	-0.86814628
921	47	-1.19426955
981	50	0.89817500

Figure 4.7: Solution curve of ODE 4.7



Remark 4.3.6: Theorem 4.3.7 extends Result of Wong and Yeh (1992), Result of Philos (1985), Result of Onose (1975), Result of Philos and Purnaras who have studied the equation (1.2) as $r(t) \equiv 1,\ \Psi(x(t)) \equiv 1,\ h(t) \equiv 0,\ \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$ and Result of Elabbasy, et al. (2005) who have studied the equation (1.1) as $\Psi(x(t)) \equiv 1,\ \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Also, Theorem 4.3.7 extends and improves the results of Greaf, et al. (1978) and Remili (2010) who have considered the equation (1.2) as $\Psi(x(t)) \equiv 1,$ $h(t) \equiv 0$ and $\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv \Phi(t, x(t))$. Our result can be applied on their equations, but their oscillation results cannot be applied on the given equation in Example 4.3.7 because their equations are particular cases of our equation (1.2).

Theorem 4.3.8

Suppose, in addition to the conditions (1), (8) and (12) hold that there exists the function ρ such that $(\rho h)^{\bullet} \leq 0,$ condition (13) holds and

(21) $\dfrac{r(t)}{\rho(t)} \leq \beta t, \beta > 0.$

(22) $\liminf_{t\to\infty} \int_T^t \Omega_1(s)\, ds > -\infty$ for all large t.

(23) $\limsup_{t\to\infty} \dfrac{1}{t}\int_T^t \dfrac{1}{\rho(s)}\int_T^s \Omega_1(u) du ds = \infty.$

Then, every solution of superlinear equation (1.2) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t)>0$ on $[T,\infty)$ for some $T \geq t_0 \geq 0.$

From the inequality (4.3.19) divided by $\rho(t)$, we have

$$\frac{1}{\rho(t)}\int_T^t \Omega_1(s)ds \leq \frac{C_1}{\rho(t)} - \omega(t). \tag{4.3.20}$$

Now, we have three cases for $\dot{x}(t)$.

Case 1: If $\dot{x}(t)$ is oscillatory, then, there exists a sequence τ_n in $[T,\infty)$ with $\lim_{t\to\infty} \tau_n = \infty$ and such that $\dot{x}(\tau_n) = 0$. Then, from the inequality (4.3.20), we have

$$\int_T^{\tau_n} \frac{k\rho(s)}{a_2 r(s)}\eta^2(s)\ ds \leq C_1 - \int_T^{\tau_n} \Omega_1(s)ds.$$

Hence, by the condition (22), we get

$$\int_T^{\infty} \frac{k\rho(s)}{a_2 r(s)}\eta^2(s)\ ds < \infty.$$

This gives, for a positive constant N

$$\int_T^{\infty} \frac{k\rho(s)}{a_2 r(s)}\eta^2(s)\ ds < N \quad \text{for every } t \geq T. \tag{4.3.21}$$

Further, by using the Schwarz's inequality, for $t \geq T,$ we obtain

$$\left|-\int_T^t \eta(s)\ ds\right|^2 = \left|-\int_T^t \sqrt{\frac{k\rho(s)}{a_2 r(s)}}\eta(s)\sqrt{\frac{a_2 r(s)}{k\rho(s)}}\ ds\right|^2 \leq \int_T^t \frac{k\rho(s)}{a_2 r(s)}\eta^2(s)\,ds\int_T^t \frac{a_2 r(s)}{k\rho(s)}ds \leq \frac{a_2 N}{k}\int_T^t \frac{r(s)}{\rho(s)}ds.$$

By the condition (22), the last inequality becomes

$$\left|-\int_T^t \eta(s)\ ds\right|^2 \leq \frac{a_2 N}{k}\beta\int_T^t sds = \frac{a_2 N\beta}{2k}(t^2 - T^2) \leq \frac{a_2 N\beta}{2k}t^2.$$

Then,

$$-\int_T^t \eta(s)\ ds = -\int_T^t \left[\omega(s) - \frac{r(s)\Psi(x(s))}{2k\rho(s)}\left(\dot{\rho}(s) - \frac{\rho(s)h(s)}{r(s)\Psi(x(s))}\right)\right]ds \leq \sqrt{\frac{a_2 N\beta}{2k}}\,t.$$

Thus, for $t \geq T$ we have

$$-\int_T^t \omega(s)\,ds \leq \sqrt{\frac{a_2 N\beta}{2k}}t. \qquad (4.3.22)$$

Integrate the inequality (4.3.20) and from the inequality (4.3.22), we obtain

$$\int_T^t \frac{1}{\rho(s)}\int_T^t \Omega_1(u)duds \leq C_1\int_T^t \frac{ds}{\rho(s)} - \int_T^t \omega(s)ds$$

$$\leq \frac{C_1}{\rho(T)}(t-T) + \sqrt{\frac{a_2 N\beta}{2k}}\,t \leq \left(\frac{C_1}{\rho(T)} + \sqrt{\frac{a_2 N\beta}{2k}}\right)t.$$

Dividing the last inequality by *t* and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_T^t \frac{1}{\rho(s)}\int_T^s \Omega_1(u)duds \leq \frac{C_1}{\rho(T)} + \sqrt{\frac{a_2 N\beta}{2k}} < \infty,$$

which contradicts the condition (23).

Case 2: If $\dot{x}(t) > 0$ *for* $t \ge T_1 \ge T$, then, from (4.3.20), we get

$$\frac{1}{\rho(t)}\int_{T_1}^{t}\Omega_1(s)ds \le \frac{C_1}{\rho(t)}.$$

Integrate the inequality, dividing by t and taking the limit superior on both sides, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_{T_1}^{t}\frac{1}{\rho(s)}\int_{T_1}^{t}\Omega_1(u)duds \le \frac{C_1}{\rho(T_1)} < \infty,$$

which also contradicts the condition (23).

Case 3: If $\dot{x}(t) < 0$ for $t \ge T_2 \ge T$, from inequality (4.3.18), we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \le C_1 - \int_{T_2}^{t}\rho(s)\big(C_0 q(s) - p(s)\big)\,ds - \int_{T_2}^{t}\frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}\,ds$$
$$-\int_{T_2}^{t}\frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds.$$

Since $\rho(t)h(t)$ is non-increasing and by Bonnet's Theorem for a fixed $t \ge T_2$, there exists $\alpha_t \in [T_2, t]$ such that

$$-\int_{T_2}^{t}\rho(s)h(s)\frac{\dot{x}(s)}{g(x(s))}ds = -\rho(T_2)h(T_2)\int_{T_2}^{\alpha_t}\frac{\dot{x}(s)}{g(x(s))}ds$$
$$= -\rho(T_2)h(T_2)\int_{x(T_2)}^{x(\alpha_t)}\frac{du}{g(u)}.$$

Since the equation (1.2) is superlinear, we have

$$\int_{x(T_2)}^{x(\alpha_t)} \frac{du}{g(u)} < \begin{cases} 0 & , if\ x(\alpha_t) < x(T_2) \\ \int_{x(T_2)}^{\infty} \frac{du}{g(u)} & , if\ x(\alpha_t) \geq x(T_2), \end{cases}$$

and $\left(-\rho(t)h(t)\right) \geq 0$, it follows that

$$-\int_{T_2}^{t} \rho(s)h(s) \frac{\dot{x}(s)}{g(x(s))} ds \leq B_3, \text{ where } B_3 = -\rho(T_2)h(T_2) \int_{x(T_2)}^{\infty} \frac{du}{g(u)}.$$

Hence, we have

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq C_1 + B_3 - \int_{T_2}^{t} \rho(s)\left(C_0 q(s) - p(s)\right) ds - \int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))} ds.$$

Now, we have two cases for $\int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))} ds$: If this integral is finite, in this case, we can get a contradiction by the procedure of case (1). If this integral is infinite, from condition (22), we obtain

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq C_1 + B_3 + \lambda - \int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))} ds.$$

Also, from the last inequality, we obtain

$$-\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \geq N^* + \int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))} ds, \tag{4.3.23}$$

where $N^* = -\left(C_1 + B_3 + \lambda\right)$.

We consider a $T_3 \geq T_2$ such that

$$N_1 = N^* + k\int_{T_2}^{T_3} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds > 0.$$

Hence, for all $t \geq T_3$, we get

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq -\left[N^* + \int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds\right].$$

From the last inequality, we get

$$\frac{\rho(t)r(t)\Psi(x(t))g'(x(t))\dot{x}^2(t)}{g^2(x(t))} \Bigg/ \left[N^* + \int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds\right] \leq -\frac{g'(x(t))\dot{x}(t)}{g(x(t))}.$$

Integrate the last inequality from T_3 to t, we have

$$\ln\left\{N^* + \int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds \Bigg/ N_1\right\} \geq \ln\left\{\frac{g(x(T_3))}{g(x(t))}\right\},\ t \geq T_3.$$

Thus,

$$N^* + \int_{T_2}^{t} \frac{\rho(s)r(s)\Psi(x(s))g'(x(s))\dot{x}^2(s)}{g^2(x(s))}\,ds \geq N_1 \frac{g(x(T_3))}{g(x(t))},\ t \geq T_3.$$

From (4.3.23), we have

$$-\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \geq N_1 \frac{g(x(T_3))}{g(x(t))},\ t \geq T_3.$$

By condition (1) and the last inequality, we obtain

$$\dot{x}(t) \le -\frac{N_1 g(x(T_3))}{a_2} \frac{1}{\rho(t) r(t)},\ t \ge T_3.$$

Since $N_1 g(x(T_3)) > 0,$ thus, from the last inequality, we obtain

$$x(t) \le x(T_3) - \frac{N_1 g(x(T_3))}{a_2} \int_{T_3}^{t} \frac{ds}{\rho(s) r(s)},\ t \ge T_3,$$

which leads to $\lim_{t\to\infty} x(t) = -\infty,$ which is a contradiction to the fact that $x(t) > 0$ for $t \ge T.$

Hence the proof is completed.

Example 4.3.8

Consider the following differential equation

$$\left(\frac{t^{3/4}(x^2(t)+4)}{(x^2(t)+3)} \dot{x}(t) \right)^{\bullet} + \left((t^3+2)x^3(t) + \frac{(t^3+2)x^{21}(t)}{2x^{18}(t) + \left(\frac{t^{3/4}(x^2(t)+4)\dot{x}(t)}{(x^2(t)+3)} \right)^6} \right) = \frac{x^3(t)\sin(\dot{x}(t))}{t^8},\ t > 0.$$

We have

$r(t) = t^{3/4},\ h(t) \equiv 0,\ q(t) = t^3 + 2,\ \ g(x) = x^3,\ \Psi(x) = \dfrac{x^2+4}{x^2+3}$ and $1 \le \Psi(x) \le \dfrac{4}{3}$ for all $x \in \mathrm{R}.$ $\Phi(u,v) = u + \dfrac{u^7}{2u^6 + v^6},$

$$\frac{H(t, x(t), \dot{x}(t))}{g(x(t))} = \frac{\sin(\dot{x}(t))}{t^8} \le \frac{1}{t^8} = p(t) \quad \textit{for all } x \ne 0 \textit{ and } t > 0.$$

Taking $\rho(t) = 5$ such that

$$(1)\ \liminf_{t\to\infty}\left\{\int_T^t \Omega_1(s)ds\right\} = \liminf_{t\to\infty}\left\{\int_T^t\left[\rho(s)\left(C_0 q(s) - p(s)\right) - \frac{a_2}{4k}\frac{r(s)}{\rho(s)}\left(\dot{\rho}(s) - \frac{\rho(s)h(s)}{a_1 r(s)}\right)^2\right]ds\right\}$$

$$= 5\liminf_{t\to\infty}\left\{\int_T^t C_0(s^3+2) - \frac{1}{s^8}\,ds\right\} = \infty > -\infty.$$

$$(2)\ \limsup_{t\to\infty}\frac{1}{t}\int_T^t \frac{1}{\rho(s)}\int_T^s \Omega_1(u)\,duds = \limsup_{t\to\infty}\frac{1}{t}\int_T^t \frac{1}{\rho(s)}\int_T^s\left[\left(C_0 q(u) - p(u)\right) - \frac{a_2}{4k}\frac{r(u)}{\rho(u)}\left(\dot{\rho}(u) - \frac{\rho(u)h(u)}{a_1 r(u)}\right)^2\right]duds$$

$$= \limsup_{t\to\infty}\frac{1}{t}\int_T^t\int_T^s\left[C_0(u^3+2) - \frac{1}{u^8}\right]du\,ds = \infty.$$

All conditions of Theorem 4.3.8 are satisfied and hence every solution of the given equation is oscillatory. The numerical solution of the given differential equation using the Runge Kutta method of fourth order (RK4) is as follows:

We have

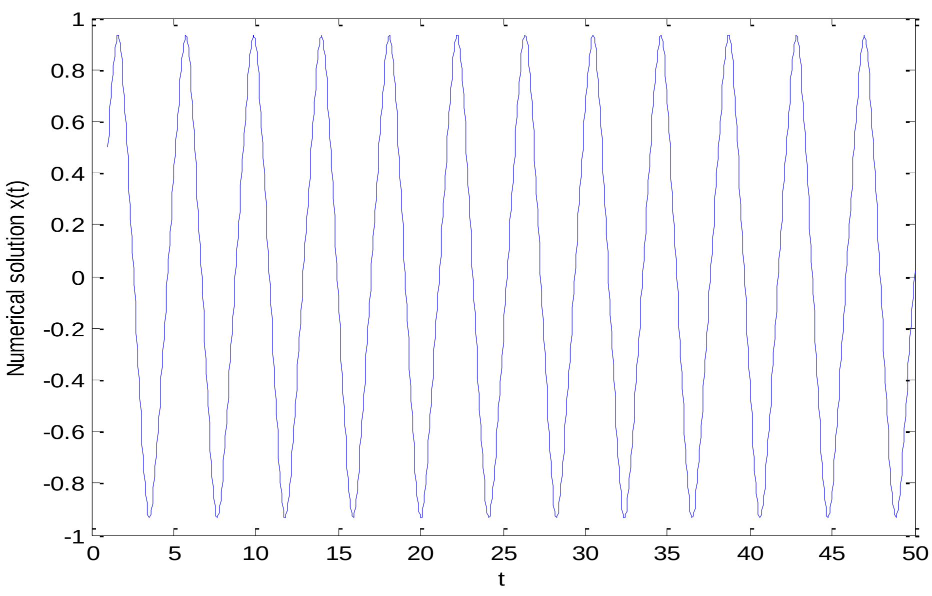
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = x^3(t)\sin(\dot{x}(t)) - 3\left(x^3(t) + \frac{x^{21}(t)}{2x^{18}(t) + \dot{x}^6(t)}\right)$$

with initial conditions $x(1) = 0.5,\ \dot{x}(1) = 1$ on the chosen interval $[1,50]$, the function $\Psi(x) \equiv 1$ and $h(t) \equiv 0$ and finding the values of the functions r, q and f where we consider $H(t, x(t), \dot{x}(t)) = f(t)\,l(x(t), \dot{x}(t))$ at t=1, n=980 and h=0.05.

Table 4.8: Numerical solution of ODE 4.8

k	t_k	$x(t_k)$
1	1	1
71	4.5	-0.13945104
181	10	0.89757769
221	12	-0.84742563
322	17.05	0.05022124
381	20	-0.93139166
461	24	-0.89616464
521	27	0.38338342
581	30	0.64126671
641	33	-0.5028129
734	37.65	0.03915567
781	40	-0.40817452
821	42	0.27436989
881	45	-0.83142722
981	50	0.02214472

Figure 4.8: Solution curve of ODE 4.8



Remark 4.3.7

Theorem 4.3.8 is the extension of the results of Greaf, et al. (1978) and Remili (2010) who have studied the equation (1.2) when $\Psi(x(t)) \equiv 1,\ \ h(t) \equiv 0$ and $\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv \Phi(t, x(t))$. Also, Theorem 4.3.8 extends and improves results of Grace (1992) and Elabbasy, et al. (2005) who have studied the equation (1.2) as $\Psi(x(t)) \equiv 1,\ \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Our result can be applied on their equations, but their oscillation results cannot be applied on the given equation in Example 4.3.8 because their equations are particular cases of our equation (1.2).

Theorem 4.3.9

Suppose, in addition to the conditions (8), (9) and (12) hold that

(24) $\dot{r}(t) \leq 0$ *for all* $t \geq t_0$ *and* $(r(t)q(t))^{\bullet} \geq 0$ *for all* $t \geq t_0$.

(25) $\dot{h}(t) \leq 0$ for all $t > 0$.

(26) $\Phi(1, v) \geq v$ *for all* $v \neq 0$.

(27) $$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds = \infty,$$

where, $p : [t_0, \infty) \to (0, \infty)$, then every solution of superlinear equation (1.2) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t) > 0 \ on \ [T, \infty)$ for some $T \geq t_0 \geq 0$.

By (4.3.17), conditions (8) and (25), we obtain

$$\left(\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \right)^{\bullet} \leq p(t) - \frac{h(t)\dot{x}(t)}{g(x(t))} - q(t)\omega(t).$$

Integrate the last inequality from T to t, we obtain

$$\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq A_1 + \int_T^t p(s)ds - \int_T^t h(s)\frac{\dot{x}(s)}{g(x(s))}ds - \int_T^t r(s)q(s)\frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}ds, \qquad (4.3.24)$$

where $A_1 = \dfrac{r(T)\Psi(x(T))\dot{x}(T)}{g(x(T))}$.

By the Bonnet's theorem, we see that for each $t \geq T$, there exists $\alpha_t \in [T, t]$ such that

$$-\int_T^t \frac{h(s)\dot{x}(s)}{g(x(s))}ds = -h(T)\int_T^{\alpha_t} \frac{\dot{x}(s)}{g(x(s))}ds = -h(T)\int_{x(T)}^{x(\alpha_t)} \frac{du}{g(u)}.$$

Since the equation (1.2) is superlinear, so we have

$$\int_{x(T)}^{x(\alpha_t)} \frac{du}{g(u)} < \begin{cases} 0 & , if \ x(\alpha_t) < x(T) \\ \displaystyle\int_{x(T)}^{\infty} \frac{du}{g(u)} & , \ if \ x(\alpha_t) \geq x(T). \end{cases}$$

$$-\int_T^t \frac{h(s)\dot{x}(s)}{g(x(s))}ds \leq -h(T)A_2 \, , \qquad (4.3.25)$$

where $A_2 = \displaystyle\int_{x(T)}^{\infty} \frac{du}{g(u)}$.

By condition (24) and the Bonnet's Theorem, we see that for each $t \geq T$, there exists $T_2 \in [T,t]$ such that

$$\int_T^t r(s)q(s)\frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}ds = r(t)q(t)\int_{T_2}^t \frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}ds = r(t)q(t)\int_{x(T_2)}^{x(t)} \frac{\Psi(u)du}{g(u)}.$$

Since $r(t)q(t) \geq 0$ and the condition (9), we have

$$\int_T^t r(s)q(s)\frac{\Psi(x(s))\dot{x}(s)}{g(x(s))}ds \geq A_3 r(t)q(t), \tag{4.3.26}$$

where $A_3 = \inf \int_{x(T_2)}^{x(t)} \frac{\Psi(u)du}{g(u)}.$

From inequalities (4.3.25) and (4.3.26), the inequality (4.3.24) becomes

$$\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq A_1 - A_2 h(T) + \int_T^t p(s)ds - A_3 r(t)q(t).$$

Integrate the last inequality from *T* to *t*, we have

$$\int_T^t \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds \leq (A_1 - A_2 h(T))(t-T) - \int_T^t \left[A_3 r(s)q(s) - \int_T^s p(u)du \right] ds.$$

Since $r(t)$ is positive and non-increasing for $t \geq T$, the condition (24) and by Bonnet's Theorem, there exists $\beta_t \in [T,t]$ such that

$$\int_T^t \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds = r(T)\int_T^{\beta_t} \frac{\Psi(x(s))}{g(x(s))}ds = r(T)\int_{x(T)}^{x(\beta_t)} \frac{\Psi(u)du}{g(u)} \geq A_4 r(T),$$

where $A_4 = \inf \int_{x(T)}^{x(\beta_t)} \frac{\Psi(u)du}{g(u)}$.

Thus, for $t \geq T$ we have

$$\int_T^t \left[A_3 r(s)q(s) - \int_T^s p(u)du \right] ds \leq (A_1 - A_2 h(T))(t-T) - A_4 r(T).$$

Dividing the last inequality by t and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_3 r(s)q(s) - \int_T^s p(u)du \right] ds \leq \limsup_{t\to\infty}\ (A_1 - A_2 h(T))(1-\frac{T}{t}) - \limsup_{t\to\infty} \frac{A_4 r(T)}{t} < \infty,$$

as $t \to \infty$, which contradicts to the condition (27). Hence the proof is completed.

Example 4.3.9

Consider the following differential equation

$$\left(\frac{1}{t}\dot{x}(t)\right)^{\bullet} - t\,\dot{x}(t) + t^4 \left(x^5(t) + \frac{x^{15}(t)}{x^{10}(t) + \left(\dot{x}(t)/t\right)^2} \right) = \frac{x^7(t)\cos(\dot{x}(t))}{t^4(x^2(t)+1)},\ t > 0.$$

We note that $r(t) = \frac{1}{t}$, $\Psi(x) = 1$ *for all* $x \in R$, $h(t) = -t$, $q(t) = t^4$ and $g(x) = x^5$.

$\Phi(u,v) = u + u^3/(u^2 + v^2)$, $\frac{H(t, x(t), \dot{x}(t))}{g(x(t))} = \frac{x^2(t)\cos(\dot{x}(t))}{t^4(x^2(t)+1)} \leq \frac{1}{t^4} = p(t)$ for all $t > 0$ and

$x \neq 0$, and

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^s \left[A_3 r(s)q(s) - \int_T^s p(u)du \right] ds = \limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_3 s^3 - 6\int_T^s \frac{du}{u^4} \right] ds = \infty.$$

All conditions of Theorem 4.3.9 are satisfied and hence every solution of the given equation is oscillatory. The numerical solution of the given differential equation using the Runge Kutta method of fourth order (RK4) is as follows:

We have

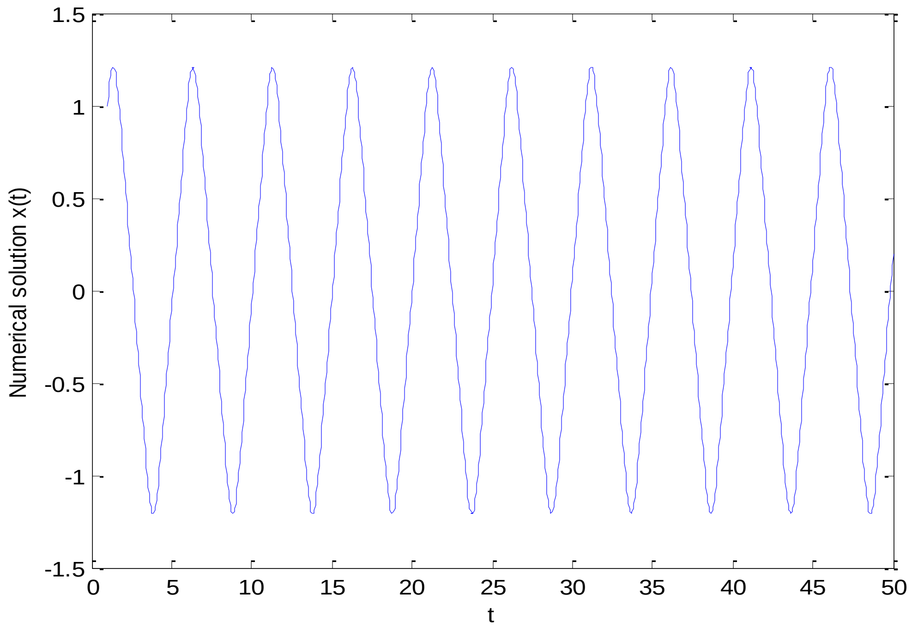
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = \frac{x^7(t)\cos(\dot{x}(t))}{x^2(t)+1} - \left(x^5(t) + \frac{x^{15}(t)}{x^{10}(t) + \dot{x}^2(t)} \right)$$

with initial conditions $x(1) = 1, \dot{x}(1) = 1$ on the chosen interval $[1,50]$, the function $h(t) \equiv 0$ and finding the values of the functions r, q and f where we consider $H(t, x(t), \dot{x}(t)) = f(t)\, l(x(t), \dot{x}(t))$ at t=1, n=980 and h=0.05.

Table 4.9: Numerical solution of ODE 4.9

k	**t_k**	**$x(t_k)$**
1	1	1
81	5	-0.10579952
184	10.15	0.10131997
232	12.55	-0.0022419
321	17	0.60071833
405	21.2	1.20586650
461	24	-1.08911367
521	27	0.53416290
579	29.9	-0.05451221
627	32.3	0.15360619
730	37.45	-0.05346404
781	40	0.12804823
829	42.4	-0.02894594
921	47	0.40081430
981	50	0.19478376

Figure 4.9: Solution curve of ODE 4.9



Remark 4.3.8:

Theorem 4.3.9 is the extension of the results of Bihari (1963), Kartsatos (1968), who have studied the equation (1.2) when $r(t) \equiv 1$, $\Psi(x(t)) \equiv 1$, $h(t) \equiv 0$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$ and results of Kamenev (1978) and Wintiner (1949) who have studied the equation (1.2) as $r(t) \equiv 1$, $\Psi(x(t)) \equiv 1$, $h(t) \equiv 0$, $g(x(t)) \equiv x(t)$, $\Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Our result can be applied on their equations, but their oscillation results cannot be applied on the given equation in Example 4.3.9 because their equations are particular cases of our equation (1.2).

Theorem 4.3.10

Suppose the conditions (1), (4) and (8) hold. Moreover, assume that there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$, $(\rho h)^{\bullet}(t) \le 0$ for $t \ge t_0$ and the continuous functions $h, H : D = \{(t,s) : t \ge s \ge t_0\} \to$ R, *H* has a continuous and non-positive partial derivative on *D* with respect to the second variable such that

$$H(t,t) = 0 \quad \textit{for } t \ge t_o,\ H(t,s) > 0 \quad \textit{for } t > s \ge t_0.$$

$$-\frac{\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D.$$

If

(28) $$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s)\sigma^2(t,s)ds < \infty,$$

where $\sigma(t,s) = \left[h(t,s) - \frac{\overset{\bullet}{\rho}(s)}{\rho(s)} \sqrt{H(t,s)} \right]$.

(29) $$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds = \infty,$$

where $p : [t_0, \infty) \to [0, \infty)$ and C_0 is a positive constant, then, every solution of superlinear equation (1.2) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution *x*(*t*) of equation (1.2) such that $x(t) > 0$ *on* $[T, \infty)$ *for some* $T \ge t_0 \ge 0$. We define the function ω as

$$\omega(t) = \frac{\rho(t) r(t) \Psi(x(t)) \dot{x}(t)}{g\big(x(t)\big)}, \; t \geq T.$$

From $\omega(t)$, Eq. (1.2), condition (1) and since $\Phi(1, \omega(t)/\rho(t)) > 0$, then, there exists a positive constant C_0 such that $\Phi(1, \omega(t)/\rho(t)) > C_0$, we have

$$\dot{\omega}(t) \leq \rho(t) p(t) - \frac{\rho(t) h(t) \dot{x}(t)}{g(x(t))} - C_0 \rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} \omega(t) - \frac{k}{a_2 \rho(t) r(t)} \omega^2(t), \; t \geq T.$$

Integrating the last inequality multiplied by $H(t,s)$ from *T* to *t*, we have

$$\begin{aligned} \int_T^t H(t,s) \rho(s) \big(C_0 q(s) - p(s)\big) ds \leq & -\int_T^t H(t,s) \dot{\omega}(s) ds - \int_T^t \frac{H(t,s) \rho(s) h(s) \dot{x}(s)}{g(x(s))} ds \\ & + \int_T^t \frac{\dot{\rho}(s)}{\rho(s)} H(t,s) \omega(s) ds - \int_T^t \frac{k H(t,s)}{a_2 \rho(s) r(s)} \omega^2(s) ds. \qquad (4.3.27) \end{aligned}$$

From the first integral in the R. H. S. for $t \geq T$, we have

$$\begin{aligned} -\int_T^t H(t,s) \dot{\omega}(s) ds &= H(t,T) \omega(T) - \int_T^t \left[-\frac{\partial}{\partial t} H(t,s) \right] \omega(s) ds \\ &= H(t,T) \omega(T) - \int_T^t h(t,s) \sqrt{H(t,s)}\, \omega(s) ds, \;\; t \geq T. \qquad (4.3.28) \end{aligned}$$

Since *H* has a continuous and non-positive partial derivative on *D* with respect to the second variable and ρh is non-increasing. The second integral in the R. H. S. is by using the Bonnet's theorem twice as follows: for $t \geq T$, there exists $a_t \in [T, t]$ such that

$$\int_T^t \frac{H(t,s) \rho(s) h(s) \dot{x}(s)}{g(x(s))} ds = H(t,T) \int_T^{a_t} \frac{\rho(s) h(s) \dot{x}(s)}{g(x(s))} ds$$

and $b_t \in [T, a_t]$ such that

$$H(t,T)\int_T^{a_t} \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds = H(t,T)\rho(T)h(T)\int_T^{b_t} \frac{\dot{x}(s)}{g(x(s))}ds$$
$$= H(t,T)\rho(T)h(T)\int_{x(T)}^{x(b_t)} \frac{du}{g(u)}.$$

Since H and $\rho(t)$ are positive functions, by condition (4) and the equation (1.2) is superlinear, we have

$$\int_{x(T)}^{x(b_t)} \frac{du}{g(u)} < \begin{cases} 0 & ,if\ x(b_t) < x(T); \\ \int_{x(T)}^{\infty} \frac{du}{g(u)} & ,if\ x(b_t) \geq x(T). \end{cases}$$

Thus, it follows that

$$\int_T^t \frac{H(t,s)\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds = H(t,T)\int_T^{a_t} \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds \geq A_1 H(t,T)\rho(T)h(T), \qquad (4.3.29)$$

where $A_1 = \inf \int_{x(T)}^{x(b_t)} \frac{du}{g(u)}$.

Thus, from (4.3.28) and (4.3.29), the inequality (4.3.27) becomes

$$\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds \leq H(t,T)\omega(T) - A_1 H(t,T)\rho(T)h(T)$$
$$-\int_T^t \left[\frac{kH(t,s)}{a_2\rho(s)r(s)}\omega^2(s) + \left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)} \right)\sqrt{H(t,s)}\omega(s) \right] ds.$$

Since $A_1 H(t,T)\rho(T)h(T) \geq 0$ and for $t \geq T$, we have

$$\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds \le H(t,T)\omega(T) - \int_T^t \left[\frac{kH(t,s)}{a_2\rho(s)r(s)}\omega^2(s) + \sigma(t,s)\sqrt{H(t,s)}\omega(s) \right] ds,$$

where $\sigma(t,s) = h(t,s) - \dfrac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}.$

Hence, we have

$$\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds \le H(t,T)\omega(T) + \int_T^t \frac{a_2\rho(s)r(s)}{4k}\sigma^2(t,s)ds$$
$$- \int_T^t \left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}}\omega(s) + \frac{1}{2}\sqrt{\frac{a_2\rho(s)r(s)}{k}}\sigma(t,s) \right]^2 ds. \qquad (4.3.30)$$

Then, for $t \ge T,$ we have

$$\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds \le H(t,T)\omega(T) + \frac{a_2}{4k}\int_T^t r(s)\rho(s)\sigma^2(t,s)\,ds, t \ge T.$$

Dividing the last inequality by $H(t,T)$, taking the limit superior and by condition (28), we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t H(t,s)\,\rho(s)\big(C_0 q(s) - p(s)\big)ds \le \omega(T) + \frac{a_2}{4k}\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\sigma^2(t,s)\,ds < \infty,$$

which contradicts to the condition (29). Hence, the proof is completed.

Theorem 4.3.11

Suppose, in addition to the conditions (1), (4), (8) and (28) hold that there exist continuous functions h and H are defined as in Theorem 4.3.10 and

(30) $$0 < \inf_{s \geq t_0} \left[\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty.$$

If there exists a continuous function Ω *on* $[t_0, \infty)$ such that

(31) $$\limsup_{t \to \infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{a_2}{4k} r(s)\rho(s)\sigma^2(t,s) \right] ds \geq \Omega(T)$$

for $T \geq t_0$, where $\sigma(t,s) = h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(t,s)}$, k is a positive constant and a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ and

(32) $$\int_T^\infty \frac{\Omega_+^2(s)}{\rho(s) r(s)} ds = \infty$$

where $\Omega_+(t) = \max\{\Omega(t), 0\}$, then, every solution of superlinear equation (1.2) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t) > 0$ *on* $[T, \infty)$ for some $T \geq t_0 \geq 0$.

Dividing inequality (4.3.30) by $H(t,T)$ and taking the limit superior as $t \to \infty$, we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)\big(C_0q(s)-p(s)\big)-\frac{a_2}{4k}\rho(s)r(s)\sigma^2(t,s)\right]ds\le\omega(T)$$

$$-\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{KH(t,s)}{a_2\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{a_2\rho(s)r(s)}{k}}\sigma(t,s)\right]^2ds$$

$$\le\omega(T)-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{a_2\rho(s)r(s)}{k}}\,\sigma\big(t,s\big)\right]^2ds$$

By condition (31), we get

$$\omega(T)\ge\Omega(T)+\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{a_2\rho(s)r(s)}{k}}\,\sigma(t,s)\right]^2ds.$$

This shows that

$$\omega(T)\ge\Omega(T)\quad for\,every\,t\ge T, \tag{4.3.31}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{a_2\rho(s)r(s)}{k}}\,\sigma(t,s)\right]^2ds<\infty.$$

Hence,

$$\infty>\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{a_2\rho(s)r(s)}{k}}\,\sigma(t,s)\right]^2ds$$

$$\ge\liminf_{t\to\infty}\left[\frac{1}{H(t,t_0)}\int_{t_0}^t\frac{kH(t,s)}{a_2\rho(s)r(s)}\,\omega^2(s)ds+\frac{1}{H(t,t_0)}\int_{t_0}^t\sigma(t,s)\sqrt{H(t,s)}\omega(s)ds\right]. \tag{4.3.32}$$

Define

$$U(t) = \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{kH(t,s)}{\rho(s)r(s)} \omega^2(s)\, ds,\ t \ge t_0$$

and

$$V(t) = \frac{1}{H(t,t_0)} \int_{t_0}^{t} \sigma(t,s)\sqrt{H(t,s)}\omega(s) ds\,,\ t \ge t_0\,.$$

Then, (4.3.32) becomes

$$\liminf_{t\to\infty} \left[U(t) + V(t)\right] < \infty. \qquad (4.3.33)$$

Now, suppose that

$$\int_{t_0}^{\infty} \frac{\omega^2(s)}{\rho(s)r(s)} ds = \infty. \qquad (4.3.34)$$

Then, by condition (30) we can easily see that

$$\lim_{t\to\infty} U(t) = \infty. \qquad (4.3.35)$$

Let us consider a sequence $\{T_n\}_{n=1,2,3,...}$ in $[t_0, \infty)$ *with* $\lim_{n\to\infty} T_n = \infty$ and such that

$$\lim_{n\to\infty} \left[U(T_n) + V(T_n)\right] = \liminf_{t\to\infty} \left[U(t) + V(t)\right].$$

By inequality (4.3.33), there exists a constant *N* such that

$$U(T_n) + V(T_n) \le N,\ n = 1,2,3,... \qquad (4.3.36)$$

From inequality (4.3.35), we have

$$\lim_{n\to\infty} U(T_n) = \infty, \tag{4.3.37}$$

and hence inequality (4.3.36) gives

$$\lim_{n\to\infty} V(T_n) = -\infty. \tag{4.3.38}$$

By taking into account inequality (4.3.37), from inequality (4.3.36), we obtain

$$1 + \frac{V(T_n)}{U(T_n)} \le \frac{N}{U(T_n)} < \frac{1}{2},$$

provided that n is sufficiently large. Thus

$$\frac{V(T_n)}{U(T_n)} < -\frac{1}{2},$$

which by inequality (4.3.38) and inequality (4.3.37) we have

$$\lim_{n\to\infty} \frac{V^2(T_n)}{U(T_n)} = \infty. \tag{4.3.39}$$

On the other hand by Schwarz's inequality, we have

$$\begin{aligned}
V^2(T_n) &= \frac{1}{H^2(T_n,t_0)}\left[\int_{t_0}^{T_n} \sigma(T_n,s)\sqrt{H(T_n,s)}\,\omega(s)ds\right]^2 \\
&\le \left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{a_2\rho(s)r(s)}{k}\sigma^2(T_n,s)ds\right]\times\left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{kH(T_n,s)}{a_2 r(s)\rho(s)}\omega^2(s)ds\right] \\
&= \frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{a_2\rho(s)r(s)}{k}\sigma^2(T_n,s)ds \times U(T_n).
\end{aligned}$$

Thus, we have

$$\frac{V^2(T_n)}{U(T_n)} \le \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} \frac{a_2 \rho(s) r(s)}{k} \sigma^2(T_n,s)ds \text{ for large } n.$$

By inequality (4.3.39), we have

$$\frac{a_2}{k} \lim_{n\to\infty} \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} r(s)\rho(s)\sigma^2(T_n,s)ds = \infty.$$

Consequently,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s)\sigma^2(t,s)ds = \infty,$$

which contradicts to the condition (28), Thus inequality (4.3.34) fails and hence

$$\int_{t_0}^{\infty} \frac{\omega^2(s)}{r(s)\rho(s)} ds < \infty.$$

Hence from inequality (4.3.31), we have

$$\int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{r(s)\rho(s)} ds \le \int_{t_0}^{\infty} \frac{\omega^2(s)}{r(s)\rho(s)} ds < \infty,$$

which, contradicts to the condition (32), hence the proof is completed.

Example 4.3.10

Consider the following differential equation

$$\left(\frac{(x^2(t)+2)}{t^6(x^2(t)+1)}\dot{x}(t)\right)^{\bullet} + \frac{\dot{x}(t)}{t^2} + \frac{1}{t^3}\left(x^7(t) + \frac{x^{133}(t)}{9x^{126}(t) + 6\left(\frac{(x^2(t)+2)}{t^6(x^2(t)+1)}\dot{x}(t)\right)^{18}}\right) = -\frac{x^9(t)\sin^2(\dot{x}(t))}{(x^2(t)+1)}, \; t > 0.$$

Here we note

$$r(t)=\frac{1}{t^6},\Psi(x)=\frac{(x^2(t)+2)}{(x^2(t)+1)}\ \ for\ all\ x\in R,\ h(t)=\frac{1}{t^2},\ q(t)=\frac{1}{t^3},\ g(x)=x^7,\ \Phi(u,v)=u+\frac{u^{19}}{9u^{18}+6v^{18}}$$

and $\dfrac{H(t,x(t),\dot{x}(t))}{g(x(t))}=-\dfrac{x^2(t)\sin^2(\dot{x}(t))}{(x^2(t)+1)}\le -\dfrac{x^2(t)}{(x^2(t)+1)}\le 0=p(t)\quad for\ all\ t\ge t_0.$

Let $H(t,s)=(t-s)^2>0\quad for\ all\ t>s\ge t_0$, thus

$\dfrac{\partial}{\partial s}H(t,s)=-2(t-s)=h(t,s)\sqrt{H(t,s)}$ for all $t\ge t_0>0$. Taking $\rho(t)=6$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\sigma^2(t,s)ds=\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\left(h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds$$

$$=\limsup_{t\to\infty}\frac{24}{(t-T)^2}\int_T^t\frac{1}{s^6}ds=0<\infty,$$

$$\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]=\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{(t-s)^2}{(t-t_0)^2}\right]=\inf_{s\ge t_0}(1)=1,$$

$$thus\ \ 0<\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]<\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)\big(C_0q(s)-p(s)\big)-\frac{a_2r(s)\rho(s)}{4k}\sigma^2(t,s)\right]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t\left[6C_0\frac{(t-s)^2}{s^3}-\frac{12}{ks^6}\right]ds=\frac{3C_0}{T^2}>\frac{3C_0}{4T^2}.$$

Set $\Omega(T)=\dfrac{3C_0}{4T^2}$, then $\Omega_+(T)=\dfrac{3C_0}{4T^2}$ and

$$\int_T^\infty\frac{\Omega^2{}_+(s)}{r(s)\rho(s)}ds=\frac{3C_0^2}{32}\int_T^\infty s^2ds=\infty.$$

All conditions of Theorem 4.3.11 are satisfied, thus, the given equation is oscillatory. We also compute the numerical solution of the given differential equation using the Runge Kutta method of fourth order (RK4). We have

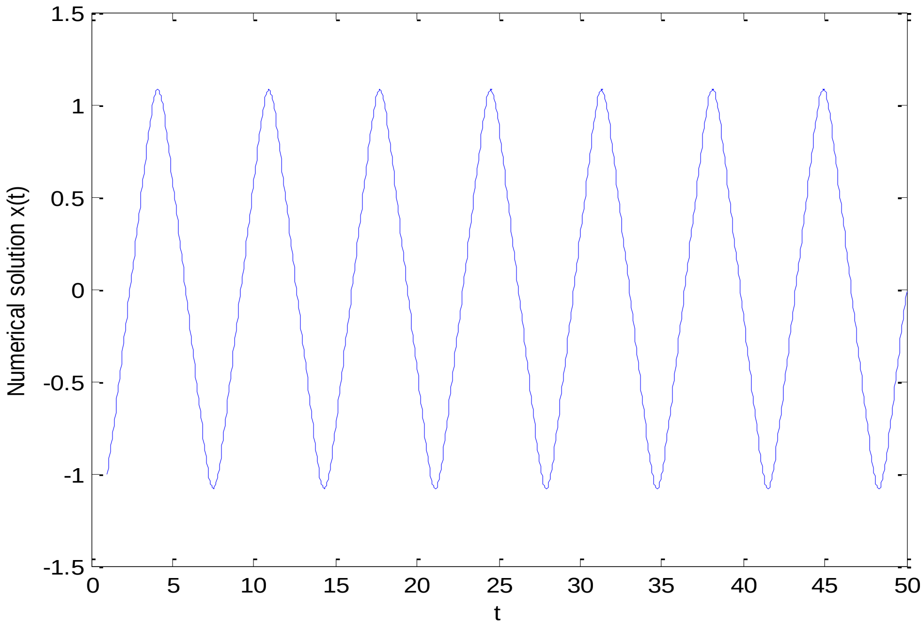
$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = \frac{-x^9(t)\sin^2(\dot{x}(t))}{x^2(t)+1} - \left(x^7(t) + \frac{x^{133}(t)}{9x^{126}(t) + 6\left(\dot{x}(t)\right)^{18}} \right)$$

with initial conditions $x(1) = -1,\ \dot{x}(1) = 0.5$ on the chosen interval $[1,50]$, the functions $\Psi(x) \equiv 1$ and $h(t) \equiv 0$ and finding the values of the functions r, q and f where we consider $H(t, x, \dot{x}) = f(t)l(x, \dot{x})$ at *t=1, n=980* and *h=0.05*.

Table 4.10: Numerical solution of ODE 4.10

k	t_k	$x(t_k)$
1	1	-1
81	5	0.59598105
161	9	-0.15335506
221	12	0.45164375
301	16	-0.00824951
381	20	-0.43550908
461	24	0.87072286
521	27	-0.58206656
581	30	0.28505118
642	33.05	-0.02507562
721	37	0.43312716
781	40	-0.13643410
821	42	-0.89118231
925	74.2	-0.43426037
981	50	-0.01036663

Figure 4.10: Solution curve of ODE 4.10



Remark 4.3.9

Theorem 4.3.10 and Theorem 4.3.11 extend and improve Results of Kamenev (1978), Results of Philos (1989) and Results of Yan (1986) who studied the equation (1.2) as $r(t) \equiv 1, \Psi(x(t)) \equiv 1, h(t) \equiv 0, \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv x(t)$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Our result can be applied on their equation, but their oscillation results cannot be applied on the given equation in Example 4.3.10 because their equation is particular case of our equation (1.2).

We need the following lemma which will significantly simplify the proof of our next Theorem.

Let $D = \{(t,s) : t \geq s \geq t_0\}$, we say that a function $H \in C(D, R)$ belongs to the class W if

(1) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ when $t \neq s$;

(2) $H(t,s)$ has partial derivatives on D such that

$$\frac{\partial}{\partial t}H(t,s) = h_1(t,s)\sqrt{H(t,s)},$$

$$\frac{\partial}{\partial s}H(t,s) = -h_2(t,s)\sqrt{H(t,s)},$$ for all $(t,s) \in D$, and some $h_1, h_2 \in L^1_{loc}(D,R)$.

Lemma 4.3.1: Let $A_0, A_1, A_2 \in C([t_0,\infty),R)$ with $A_2 > 0$ and $z \in C^1([t_0,\infty),R)$. If there exist $(a,b) \subset [t_0,\infty)$ and $c \in (a,b)$ such that

$$z' \le -A_0(s) + A_1(s)z - A_2(s)z^2, s \in (a,b), \tag{4.3.40}$$

then,

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)\rho(s)A_0(s) - \frac{k}{4A_2(s)}\eta_1^2(s,a)\right]ds + \frac{1}{H(b,c)}\int_c^b \left[H(b,c)\rho(s)A_0(s) - \frac{k}{4A_2(s)}\eta_2^2(b,s)\right]ds \le 0, \tag{4.3.41}$$

for all $H \in W$ where

$$\eta_1(s,a) = \left[h_1(s,a) - A_1(s)\sqrt{H(s,a)}\right]$$

and

$$\eta_2(b,s) = \left[h_2(b,s) - A_1(s)\sqrt{H(b,s)}\right].$$

The proof of this lemma is similar to that of Lu and Meng (2007) and hence will be omitted.

Theorem 4.3.12

Suppose in addition to the condition (8) holds that $\Psi(x) \equiv 1$ for $x \in \mathrm{R}$ and assume that there exist $c \in (a,b) \subset (T,\infty)$ and $H \in W$ such that

$$(33) \qquad \frac{1}{H(c,a)}\int_a^c \left[H(s,a)\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{k}{4\rho(s)r(s)}\eta_1^2(s,a) \right] ds$$
$$+ \frac{1}{H(b,c)}\int_c^b \left[H(b,c)\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{k}{4\rho(s)r(s)}\eta_2^2(b,s) \right] ds > 0,$$

where

$$\eta_1(t,a) = \left[h_1(t,a) - \left(\frac{\dot{\rho}(t)}{\rho(t)} - \frac{h(t)}{r(t)} \right)\sqrt{H(t,a)} \right],$$
$$\eta_2(b,t) = \left[h_2(b,t) - \left(\frac{\dot{\rho}(t)}{\rho(t)} - \frac{h(t)}{r(t)} \right)\sqrt{H(b,t)} \right]$$

and the function ρ is defined as in Theorem 4.3.10. Then, every solution of equation (1.2) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t) > 0 \textit{ on } [T,\infty) \textit{ for some } T \geq t_0 \geq 0.$ We define the function ω as

$$\omega(t) = \frac{\rho(t)r(t)\dot{x}(t)}{g\big(x(t)\big)},\ t \geq T.$$

This and (1.2) imply

$$\dot{\omega}(t) \le \rho(t)p(t) - \frac{h(t)}{r(t)}\omega(t) - \rho(t)q(t)\Phi(1, v_1(t)) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{\rho(t)r(t)}\omega^2(t),\ t \ge T,$$

where $v_1(t) = \omega(t)/\rho(t)$.

Since $\Phi(1, v_1(t)) > 0$ then there exists C_0 such that $\Phi(1, v_1(t)) \ge C_0$, we have

$$\dot{\omega}(t) \le -\rho(t)\big(q(t) - p(t)\big) + \left(\frac{\dot{\rho}(t)}{\rho(t)} - \frac{h(t)}{r(t)}\right)\omega(t) - \frac{k}{\rho(t)r(t)}\omega^2(t),\ t \ge T.$$

From the last inequality and by lemma 4.3.1, we conclude that for any $c \in (a,b)$ and $H \in W$

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{k}{4\rho(s)r(s)}\eta_1^2(s,a)\right]ds$$
$$+ \frac{1}{H(b,c)}\int_c^b \left[H(b,c)\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{k}{4\rho(s)r(s)}\eta_2^2(b,s)\right]ds \le 0,$$

which contradicts (33). Thus, the equation (1.2) is oscillatory.

Remark 4.3.10

Theorem 4.3.12 is the extension of results of Lu and Meng (2007) who have studied the equation (1.2) when $\Psi(x(t)) \equiv 1,\ \ \Phi(g(x(t)), r(t)\Psi(x(t))\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t), \dot{x}(t)) \equiv 0.$ Our result can be applied on their equation, but their oscillation results cannot be applied on our equation (1.2) because their equation is a particular case of our equation (1.2).

4.4 Conclusion

In this section, the problem of finding the sufficient conditions for oscillation of solutions of ordinary differential equations of second order with damping term of type (1.2) is considered. We present some oscillation results that contain the sufficient conditions for oscillation of solutions of the equation of type (1.2). These sufficient conditions have been derived by using the generalized Riccati technique. Our results extend and improve many previous results that have been obtained before, for example, such as the works of Fite (1918), Wintner (1949), Atkinson (1955), Bihari (1963), Kartsatos (1968), Greaf, et al. (1978), Grace (1992), Elabbasy et al. (2005), Lu & Meng (2007), Berkani (2008), and Remili (2010). All these previous results have been studied for particular cases of the equation (1.2) whereas our sufficient conditions have been derived for the generalized equation (1.2). A number of theorems and illustrative examples for oscillation differential equation of type (1.2) are given. Further, a number of numerical examples are given to illustrate the theorems which are computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of proposed research study.

CHAPTER FIVE

OSCILLATION OF THIRD ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

5.1 Introduction

In this chapter, we are concerned with the problem of oscillation of third order non-linear ordinary differential equation of the form

$$\left(r(t)\,f(\ddot{x}(t))\right)^{\bullet} + q(t)g_1(\dot{x}(t)) = H(t, x(t), \dot{x}(t), \ddot{x}(t)), \tag{1.3}$$

where q and r are continuous functions on the interval $[t_0, \infty)$, $t_0 \geq 0$, $r(t)$ is a positive function, g_1 is continuously differentiable function on the real line R except possibly at 0 with $yg_1(y) > 0$ and $g_1'(y) \geq k > 0$ for all $y \neq 0$, f is a continuous function on R and $H : [t_0, \infty) \times \mathrm{R} \times \mathrm{R} \times \mathrm{R} \rightarrow \mathrm{R}$ is a continuous function such that $H(t, x, y, z)/g_1(y) \leq p(t)$ for all $y \neq 0$ and $t \geq t_0$.

5.2 Third Order Nonlinear ODE Of Type (1.3)

In this chapter, we present the oscillation results of our study of finding the sufficient conditions for oscillation of solutions of ordinary differential equations of third order of type (1.3). The present oscillation results have among other finding extended and improved many previous oscillation results, for examples, such as the works of Hanan (1961), Lazer (1966), Jones (1973), Mehri (1976), Parhi & Das (1990), Parhi & Das (1993), Adamets & Lomtatidze (2001) and Remili (2007). We have established some new sufficient conditions which guarantee that our differential equations are oscillatory. A number of theorems and an illustrative example for oscillation differential equation of type (1.3) are shown. Further, a numerical example is given to illustrate the theorems. This numerical example is computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of proposed research study.

5.3 Oscillation Theorems

Theorem 5.2.1

Suppose that

(1) $$0 < \int_{\pm\varepsilon}^{\pm\infty} \frac{du}{g_1(u)} < \infty \ \ for\ all\ \ \varepsilon > 0,$$

(2) $$0 < k_1 < \frac{f(z)}{z} < k_2 \ \text{ for all } \ z \neq 0,$$

(3) $$\int_T^{\infty} \frac{ds}{r(s)} = M_1, \ \ for\ \ all\ \ T \geq t_0,$$

(4) $$\limsup_{t\to\infty} \int_T^t \frac{1}{r(s)} \int_T^s \big(q(u) - p(u)\big) du\, ds = \infty,$$

where $p:[t_0,\infty) \to (0,\infty)$. Then, every solution of equation (1.3) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution $x(t) > 0$ of equation (1.3) such that $x(t) > 0$ and $\dot{x}(t) > 0$ on $[T,\infty)$ for some $T \geq t_0 \geq 0$. Define

$$\omega(t) = \frac{r(t) f(\ddot{x}(t))}{g_1(\dot{x}(t))},\ t \geq T.$$

This and (1.3) and condition (2), we have

$$\dot{\omega}(t) = \left(\frac{r(t) f(\ddot{x}(t))}{g_1(\dot{x}(t))} \right)^{\bullet} \leq p(t) - q(t),\ t \geq T.$$

Integrate the last inequality from *T* to *t* and also by condition (2), we have

$$\frac{k_1 r(t) \ddot{x}(t)}{g_1(\dot{x}(t))} \leq \omega(T) - \int_T^t \big(q(s) - p(s)\big) ds,\ t \geq T. \tag{5.3.1}$$

Integrate (5.3.1) divided by $r(t)$ from *T* to *t*, we obtain

$$k_1 \int_T^t \frac{\ddot{x}(s) ds}{g_1(\dot{x}(s))} \leq \omega(T) \int_T^t \frac{ds}{r(s)} - \int_T^t \frac{1}{r(s)} \int_T^s \big(q(u) - p(u)\big) du\, ds,\ t \geq T. \tag{5.3.2}$$

By condition (1), we obtain

$$\int_{\dot{x}(T)}^{\dot{x}(t)} \frac{du}{g(u)} \leq \begin{cases} 0 & ,if\ \dot{x}(t) < x(T) \\ \int_{x(T)}^{\infty} \frac{du}{g(u)} & ,if\ \dot{x}(t) \geq x(T) \end{cases}$$

This follows that

$$\int_T^t \frac{\ddot{x}(s)}{g_1(\dot{x}(s))} ds \geq A_1,$$

where $A_1 = \inf \int_{\dot{x}(T)}^{\infty} \frac{du}{g_1(u)}$.

Thus, the inequality (5.3.2) becomes

$$\int_T^t \frac{1}{r(s)}\int_T^s \big(q(u)-p(u)\big)du\,ds \le \omega(T)\int_T^t \frac{ds}{r(s)} - k_1 A_1,\ t \ge T.$$

By condition (3) and taking the limit superior on both sides, we have

$$\limsup_{t\to\infty}\int_T^t \frac{1}{r(s)}\int_T^s \big(q(u)-p(u)\big)du\,ds \le \limsup_{t\to\infty}\{\omega(T)M_1 - k_1 A_1\} < \infty,$$

as $t \to \infty$, which contradicts to the condition (4). Hence, the proof is completed.

Theorem 5.3.2

Suppose, in addition to the conditions (1) and (2) hold that there exists the differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ and $\dot{\rho} \ge 0$ such that

(5) $$\int_T^\infty \frac{ds}{r(s)\rho(s)} = M_2 \ \ \textit{for all}\ \ T \ge t_0,$$

(6) $$\int_T^\infty \frac{1}{\rho(s)r(s)}\int_T^s \left[\rho(u)\big(q(u)-p(u)\big) - k_3\frac{\dot{\rho}^2(u)r(u)}{\rho(u)}\right]du\,ds = \infty.$$

Then, every solution of equation (1.3) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution $x(t)>0$ of equation (1.3) such that $x(t)>0$ and $\dot{x}(t)>0$ on $[T,\infty)$ for some $T \ge t_0 \ge 0$. Define

$$\omega(t) = \frac{\rho(t) r(t) f(\ddot{x}(t))}{g_1(\dot{x}(t))}, \; t \geq T.$$

This, (1.3) and the condition (2), we have

$$\dot{\omega}(t) \leq \rho(t) p(t) - \rho(t) q(t) + \frac{k_2 \dot{\rho}(t) r(t) \ddot{x}(t)}{g_1(\dot{x}(t))} - \frac{k k_1 \rho(t) r(t) \ddot{x}^2(t)}{g_1^2(\dot{x}(t))}, \; t \geq T.$$

Integrate the last inequality, we get

$$\frac{\rho(t) r(t) f(\ddot{x}(t))}{g_1(\dot{x}(t))} \leq \omega(T) - \int_T^t \rho(s)\big(q(s) - p(s)\big) ds - \int_T^t \left[\frac{k k_1 \rho(s) r(s)}{g_1^2(\dot{x}(s))} \ddot{x}^2(s) - \frac{k_2 \dot{\rho}(s) r(s)}{g_1(\dot{x}(s))} \ddot{x}(s) \right] ds. \quad (5.3.3)$$

From the second integral in the R. H. S., we have

$$- \int_T^t \left[\frac{k k_1 \rho(s) r(s)}{g_1^2(\dot{x}(s))} \ddot{x}^2(s) - \frac{\dot{\rho}(s) r(s)}{g_1(\dot{x}(s))} \ddot{x}(s) \right] ds = - \int_T^t \left(\frac{\sqrt{k k_1 \rho(s) r(s)}}{g_1(\dot{x}(s))} \ddot{x}(s) - \frac{k_2 \dot{\rho}(s) \sqrt{r(s)}}{2\sqrt{k k_1 \rho(s)}} \right)^2 ds$$

$$+ \frac{k_2^2}{4 k k_1} \int_T^t \frac{\dot{\rho}^2(s) r(s)}{\rho(s)} ds$$

$$\leq k_3 \int_T^t \frac{\dot{\rho}^2(s) r(s)}{\rho(s)} d \quad (5.3.4)$$

where $k_3 = \dfrac{k_2^2}{4 k k_1}$.

From (5.3.4) in (5.3.3), we obtain

$$\frac{\rho(t) r(t) f(\ddot{x}(t))}{g_1(\dot{x}(t))} \leq \omega(T) - \int_T^t \left[\rho(s)\big(q(s) - p(s)\big) - k_3 \frac{\dot{\rho}^2(s) r(s)}{\rho(s)} \right] ds.$$

By condition (2) and integrating the last inequality divided by $\rho(t)r(t)$, we obtain

$$k_1 \int_T^t \frac{\ddot{x}(s)}{g_1(\dot{x}(s))} ds \le \omega(T) \int_T^t \frac{ds}{\rho(s)r(s)} - \int_T^t \frac{1}{\rho(s)r(s)} \int_T^s \left[\rho(u)\big(q(u) - p(u)\big) - k_3 \frac{\dot{\rho}^2(u)r(u)}{\rho(u)} \right] du\, ds$$

By using the condition (1) and as in Theorem 5.3.1 the first integral in the L. H. S. is bounded and the condition (5), then, we get

$$\int_T^t \frac{1}{\rho(s)r(s)} \int_T^s \left[\rho(u)\big(q(u) - p(u)\big) - k_3 \frac{\dot{\rho}^2(u)r(u)}{\rho(u)} \right] du\, ds \le \omega(T)M_2 - k_1 A_1 < \infty,$$

as $t \to \infty$, which contradicts to the condition (6). Hence, the proof is completed.

Theorem 5.3.3

Suppose, in addition to the conditions (1) and (2) hold that there exists the differentiable function $\rho : [t_0, \infty) \to (0, \infty)$, $\dot{\rho} \ge 0$, $(\rho r)^{\bullet} \le 0$ and $(\dot{\rho} r)^{\bullet} \le 0$ such that

(7) $\displaystyle \int_T^\infty \frac{ds}{\rho(s)r(s)} = \infty,$

(8) $\displaystyle \liminf_{t\to\infty} \int_T^t \rho(s)\big(q(s) - p(s)\big) ds > -\infty,$

(9) $\displaystyle \limsup_{t\to\infty} \frac{1}{t} \int_T^t \int_T^s \rho(u)\big(q(u) - p(u)\big) du\, ds = \infty,$

then, every solution of equation (1.3) is oscillatory.

Proof: Without loss of generality, we assume that there exists a solution $x(t) > 0$ of equation (1.3) such that $x(t) > 0$ and $\dot{x}(t) > 0$ on $[T, \infty)$ for some $T \geq t_0 \geq 0.$ Define

$$\omega(t) = \frac{\rho(t) r(t) f(\ddot{x}(t))}{g_1(\dot{x}(t))},\ t \geq T.$$

This and the equation (1.3), we have

$$\dot{\omega}(t) \leq \rho(t) p(t) - \rho(t) q(t) + \frac{\dot{\rho}(t) r(t) f(\ddot{x}(t))}{g_1(\dot{x}(t))} - \frac{\rho(t) r(t) f(\ddot{x}(t)) \ddot{x}(t) g_1'(\dot{x}(t))}{g_1^2(\dot{x}(t))},\ t \geq T.$$

By the condition (2), we have

$$\dot{\omega}(t) \leq -\rho(t)\big(q(t) - p(t)\big) + \frac{k_2\, \dot{\rho}(t) r(t) \ddot{x}(t)}{g_1(\dot{x}(t))} - \frac{g_1'(\dot{x}(t))}{k_2 \rho(t) r(t)} \omega^2(t),\ t \geq T. \tag{5.3.5}$$

Integrating (5.3.5) from *T* to *t*, we get

$$\int_T^t \rho(s)\big(q(s) - p(s)\big) ds \leq \omega(T) - \omega(t) + k_2 \int_T^t \frac{\dot{\rho}(s) r(s) \ddot{x}(s)}{g_1(\dot{x}(s))} ds - \frac{k}{k_2} \int_T^t \frac{1}{\rho(s) r(s)} \omega^2(s)\, ds. \tag{5.3.6}$$

Since $\dot{\rho}(t) r(t)$ is a decreasing function, then by the Bonnet's Theorem there exists a $a_t \in [T, t]$ such that the first integral in the R. H. S. becomes

$$k_2 \int_T^t \frac{\dot{\rho}(s) r(s) \ddot{x}(s)}{g_1(\dot{x}(s))} ds = k_2\, \dot{\rho}(T) r(T) \int_{\dot{x}(T)}^{\dot{x}(a_t)} \frac{du}{g_1(u)}.$$

By condition (1), we have

$$k_2 \dot{\rho}(T) r(T) \int_{\dot{x}(T)}^{\dot{x}(a_t)} \frac{du}{g(u)} \le \begin{cases} 0 & , if \ \dot{x}(a_t) < x(T), \\ k_2 \dot{\rho}(T) r(T) \int_{\dot{x}(T)}^{\infty} \frac{du}{g(u)} & , if \ \dot{x}(a_t) \ge x(T). \end{cases}$$

Hence

$$0 < k_2 \dot{\rho}(T) r(T) \int_{\dot{x}(T)}^{\dot{x}(a_t)} \frac{du}{g(u)} \le A_2, \qquad (5.3.7)$$

where $A_2 = k_2 \dot{\rho}(T) r(T) \int_{\dot{x}(T)}^{\infty} \frac{du}{g(u)}$.

From (5.3.7) in (5.3.6), we get

$$\int_T^t \rho(s)\big(q(s) - p(s)\big) ds \le \omega(T) - \omega(t) + A_2 - \frac{k}{k_2} \int_T^t \frac{1}{\rho(s) r(s)} \omega^2(s)\, ds. \qquad (5.3.8)$$

We have two cases for the integral

$$\int_T^t \frac{1}{\rho(s) r(s)} \omega^2(s)\, ds.$$

Case 1:

$$\int_T^\infty \frac{1}{\rho(s) r(s)} \omega^2(s)\, ds \text{ is finite.}$$

Thus, there exists a positive constant B such that

$$\int_T^t \frac{1}{\rho(s) r(s)} \omega^2(s)\, ds \le B \ for\, t \ge T.$$

Thus, (5.3.8) becomes

$$\int_T^t \rho(s)\big(q(s)-p(s)\big)ds \le A_3 - \omega(t),\ t \ge T,$$

where $A_3 = \omega(T) + A_2 + kB/k_2$.

Integrating the last inequality from T to t, we get

$$\int_T^t \int_T^s \rho(u)\big(q(u)-p(u)\big)du\,ds \le A_3(t-T) - \int_T^t \omega(s)\,ds. \tag{5.3.9}$$

By using the Schwarz's inequality, we obtain

$$-\int_T^t \omega(s)\,ds \le \left|\int_T^t \omega(s)\,ds\right|^2 = \left|\int_T^t \frac{\omega(s)}{\sqrt{\rho(s)r(s)}}\sqrt{\rho(s)r(s)}\,ds\right|^2 \le \int_T^t \frac{\omega^2(s)}{\rho(s)r(s)}ds \int_T^t \rho(s)r(s)ds$$
$$\le B\int_T^t \rho(s)r(s)ds. \tag{5.3.10}$$

From (5.3.10) in (5.3.9), we obtain

$$\int_T^t \int_T^s \rho(u)\big(q(u)-p(u)\big)du\,ds \le A_3(t-T) + B\int_T^t \rho(s)r(s)ds$$
$$\le A_3(t-T) + B\rho(T)r(T)(t-T).$$

Dividing the last inequality by t and taking the limit superior on both sides, we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \int_T^s \rho(u)\big(q(u)-p(u)\big)du\,ds \le \limsup_{t\to\infty}\left\{A_3 + B\rho(T)r(T)\left(1-\frac{T}{t}\right)\right\} < \infty,$$

as $t \to \infty$, which contradicts to the condition (9).

Case 2: If the integral

$$\int_T^\infty \frac{1}{\rho(s)r(s)}\omega^2(s)\,ds \quad \text{is infinite,}$$

by condition $g_1'(\dot{x}(t)) \geq k,$ we get

$$\int_T^\infty \frac{g_1'(\dot{x}(s))}{\rho(s)r(s)}\,\omega^2(s)\,ds = \infty. \qquad (5.3.11)$$

Integrating (5.3.5), from (5.3.7) and condition (8), it follows that here exists a constant A_4 such that

$$-\omega(t) \geq A_5 + \frac{1}{k_2}\int_T^t \frac{g_1'(\dot{x}(s))}{\rho(s)r(s)}\,\omega^2(s)\,ds, \qquad (5.3.12)$$

where $A_5 = \omega(T) + A_2 + A_4$.

From (5.3.11), we get $\omega(t)$ is negative on $[T,\infty)$. Furthermore, we choose a $T_1 \geq T$ such that

$$A_6 = A_5 + \frac{1}{k_2}\int_T^{T_1} \frac{g_1'(\dot{x}(s))}{\rho(s)r(s)}\,\omega^2(s)\,ds.$$

Thus, for $t \geq T_1$, we have

$$\frac{g_1'(\dot{x}(t))\omega^2(t)}{k_2\rho(t)r(t)}\left[A_5 + \frac{1}{k_2}\int_T^t \frac{g_1'(\dot{x}(s))}{\rho(s)r(s)}\,\omega^2(s)\,ds\right]^{-1} \geq -\frac{g_1'(\dot{x}(t))\,f(\ddot{x}(t))}{k_2 g_1(\dot{x}(t))}$$

$$\geq -\frac{g_1'(\dot{x}(t))\,\ddot{x}(t)}{g_1(\dot{x}(t))}.$$

Thus

$$\log\left[A_5 + \frac{1}{k_2}\int_T^t \frac{g_1'(\dot{x}(s))}{\rho(s)r(s)}\,\omega^2(s)\,ds\right]\Big/A_6 \geq \log\frac{g_1(\dot{x}(T_1))}{g_1(\dot{x}(t))}.$$

Hence

$$A_5 + \frac{1}{k_2}\int_T^t \frac{g_1'(\dot{x}(s))}{\rho(s)r(s)}\omega^2(s)\,ds \geq A_6\,\frac{g_1(\dot{x}(T_1))}{g_1(\dot{x}(t))}.$$

By (5.3.12), we get

$$-\omega(t) \geq A_6\,\frac{g_1(\dot{x}(T_1))}{g_1(\dot{x}(t))}.$$

From the last inequality and condition (2), we obtain

$$k_1\,\ddot{x}(t) \leq -A_6 g_1(\dot{x}(T_1))\frac{1}{\rho(t)r(t)}.$$

Integrate from T_1 to t and condition (7), we have

$$k_1\,\dot{x}(t) \leq k_1\,\dot{x}(T_1) - A_6 g_1(\dot{x}(T_1))\int_{T_1}^t \frac{ds}{\rho(s)r(s)} \to -\infty, \text{as } t \to \infty,$$

this contradicts to the assumption that $\dot{x}(t) > 0$. This completes the proof of Theorem 5.3.3.

Example 5.3.3

Consider the following differential equation

$$\left(\frac{1}{t^4}\ddot{x}(t)\right)^{\bullet} + t^4\,\dot{x}^5(t) = \frac{\dot{x}^5(t)\sin^2(\ddot{x}(t))}{t^6},\; t > 0.$$

We have $r(t) = \dfrac{1}{t^4}, q(t) = t^4$ and $g(\dot{x}) = \dot{x}^5, \dot{x}\,g(\dot{x}) = \dot{x}^6 > 0$ for all $\dot{x} > 0$ and

$$0 < \int_{\pm\varepsilon}^{\pm\infty} \frac{du}{g(u)} = \frac{1}{4\varepsilon^4} < \infty.$$

(1) $f(\ddot{x}(t)) = \ddot{x}(t)$ and $0 < 1 \le \dfrac{f(\ddot{x}(t))}{\ddot{x}(t)} = 1 < 2$ for all $\ddot{x} \neq 0.$

(2) $H(t, x(t), \dot{x}(t), \ddot{x}(t)) = \dfrac{\dot{x}^5(t)\sin^2(\ddot{x}(t))}{t^6}$ and

$$\frac{H(t, x(t), \dot{x}(t), \ddot{x}(t))}{g_1(\dot{x}(t))} = \frac{\sin^2(\ddot{x}(t))}{t^6} \le \frac{1}{t^6} = p(t) \text{ for all } \dot{x} \neq 0 \text{ and } t > 0.$$

Taking $\rho(t) = t^2, \dot{\rho}(t) = 2t, (\rho r)^\bullet(t) = -\dfrac{1}{t} < 0$ and $(\dot{\rho} r)^\bullet(t) = -\dfrac{1}{t^2} < 0$ for $t > 0$ such that

(3) $\displaystyle\int_T^\infty \frac{ds}{\rho(s)r(s)} = \int_T^\infty s^2 ds = \infty,$

(4) $\displaystyle\liminf_{t\to\infty} \int_T^t \rho(s)\big(q(s) - p(s)\big)ds = \liminf_{t\to\infty} \int_T^t s^2\left(s^4 - \frac{1}{s^6}\right)ds = \infty > -\infty,$

(5) $\displaystyle\limsup_{t\to\infty} \frac{1}{t}\int_T^t\int_T^s \rho(u)\big(q(u) - p(u)\big)du\, ds = \limsup_{t\to\infty} \frac{1}{t}\int_T^t\int_T^s u^2\left(u^4 - \frac{1}{u^6}\right)du\, ds$

$$= \limsup_{t\to\infty} \frac{1}{t}\left[\frac{t^7}{42} - \frac{1}{6t^2} - \left(\frac{T^6}{6} - \frac{1}{3T^3}\right)t - \left(\frac{1}{6T^2} - \frac{6T^7}{7}\right)\right] = \infty,$$

All conditions of Theorem 5.3.3 are satisfied, thus, the given equation is oscillatory. We also compute the numerical solution of the given differential equation using Runge Kutta method of fourth order. We have

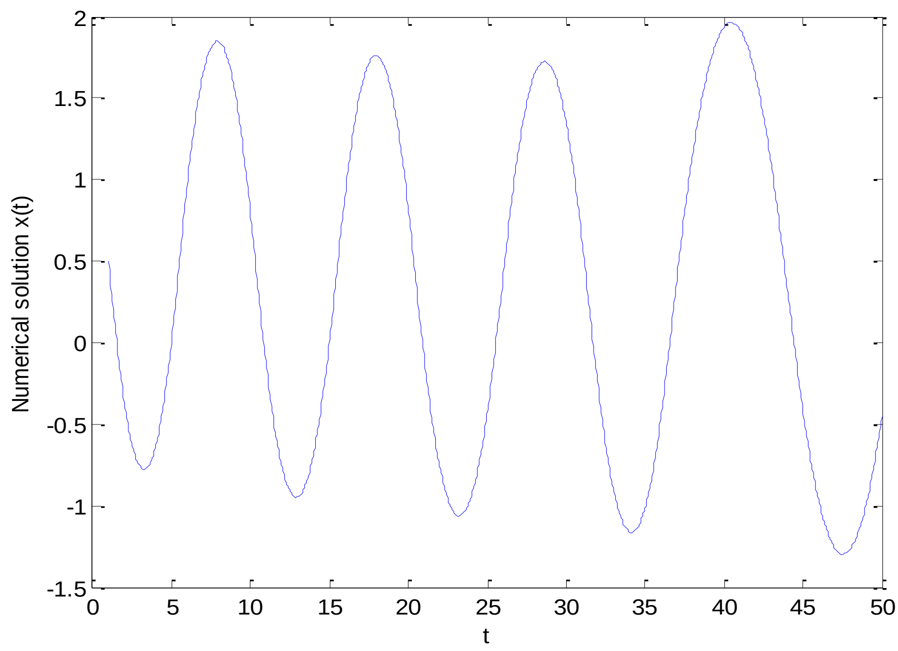
$$\dddot{x}(t) = f(t, x(t), \dot{x}(t), \ddot{x}(t)) = \frac{\dot{x}^5(t)(t)\sin^2(\ddot{x}(t))}{t^2} - \dot{x}^5(t),$$

with initial conditions $x(1)=0.5,\ \dot{x}(1)=-1,\ \ddot{x}(1)=0$ on the chosen interval [1,50] and finding the values of the functions r, q and f where we consider $H(t,x,\dot{x},\ddot{x})=g(t)\,l(x,\dot{x},\ddot{x})$.

Table 5.1: Numerical solution of ODE 5.1

k	t_k	$x(t_k)$
1	1	0.5
80	4.954	-0.01467758
181	10.009	0.76997118
221	12.011	-0.77145176
321	17.016	1.56655689
379	20.92	-0.02744315
461	24.023	-0.93017943
521	27.026	1.21756203
615	31.731	-0.02268732
721	37.036	0.41913726
800	40.99	1.92358434
868	44.444	-0.00644259
929	47.447	-1.29549185
981	50	-0.45092350

Figure 5.1: Solution curve of ODE 5.1



Remark 5.3.1

Our theorems extend and improve the obtained results by Hanan (1961) and Adamets and Lomtatidze (2001) for the equation (2.14), results of Lazer [1966] and Jones (1973) for the equation (2.15) and result of Kiguradze (1992) for the equation (2.18), as mentioned in Chapter Two. Our results can be applied on their equations but, their oscillation results cannot be applied on the equation (1.3) because their equations are special cases of the equation (1.3).

5.4 Conclusion

In this section, we present the oscillation results of our study of finding the sufficient conditions for oscillation of solutions of ordinary differential equations of third order of type (1.3). Some oscillation have been introduced which contain the sufficient conditions for oscillation of solutions of the equation of type (1.3). These sufficient conditions have been established by using the generalized Riccati technique. Our results extend and improve many previous results that have been obtained before, for example, such as the works of Hanan (1961), Lazer (1966), Jones (1973), Mehri (1976), Parhi & Das (1990), Parhi & Das (1993), Adamets & Lomtatidze (2001) and Remili (2007). All these previous results have been studied for particular cases of the equation (1.3) whereas our sufficient conditions have been derived for the generalized equation (1.3). A number of theorems and an illustrative example for oscillation differential equation of type (1.3) are given. Further, the numerical example is given to illustrate the theorem which is computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of proposed research study.

CHAPTER SIX

CONCLUSION AND FUTURE WORK

6.1 Conclusion

Over the past three decades, there have been many studies which have dealt with the oscillatory properties of nonlinear ordinary differential equations. The problem of finding oscillation criteria for second order nonlinear differential equations has received a great deal of attention in the 20 years from the publication of the classic paper by Atkinson (1955). The study of the oscillation of second order nonlinear ordinary differential equations with alternating coefficients is of special interest because of the fact that many physical systems are modeled by second order nonlinear ordinary differential equations.

In this thesis, we are concerned with oscillation behavior of solutions of non-linear ordinary differential equations of second order and third order with variable coefficients. The main results are presented in chapter three, chapter four and five. Oscillation of second order nonlinear differential equation with alternating coefficients of type (1.1) has been investigated in chapter three. The present oscillation results contain the sufficient conditions for oscillation of solutions of the equation of type (1.1) which have been derived by using the generalized Riccati technique. Our results extend and improve many previous results that have been obtained before, for example, such as the works of Fite (1918), Wintner (1949), Philos (1989) for the equation (2.1), Atkinson (1955) for the equation (2.4), Bihari (1963), Kartsatos (1968) for equation (2.5), Elabbasy (1996) for the equation (2.6), and Elabbasy et al. (2005) for the equation (2.9). All these previous results have been studied for particular cases of the equation (1.1) whereas our sufficient conditions have been derived for the generalized equation (1.1). A number of oscillation theorems differential equation of type (1.1) are given. Further, a number of numerical examples are given to illustrate the theorems which are computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of our oscillation results for equation (1.1).

In chapter four, the problem of finding the sufficient conditions for oscillation of solutions of ordinary differential equations of second order with damping term of type (1.2) is considered. The present oscillation results contain the sufficient conditions for oscillation of solutions of the equation of type (1.2). These sufficient conditions have been derived by using the generalized Riccati technique. Our results extend and improve many previous results that have been obtained before, for example, such as the works of Fite (1918), Wintner (1949) for the equation (2.1), Atkinson (1955) for the equation (2.4), Bihari (1963), Kartsatos (1968) for equation (2.5), Greaf, et al. (1978), Grace (1992), Elabbasy et al. (2005), Lu & Meng (2007), Berkani (2008), and Remili (2010). All these previous results have been studied for particular cases of the equation (1.2) (as mentioned in chapter two) whereas our sufficient conditions have been derived for the generalized equation (1.2). A number of theorems and illustrative examples for oscillation differential equation of type (1.2) are given. Further, a number of numerical examples are given to illustrate the theorems which are computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of our oscillation results for the equation (1.2).

Therefore, we present the oscillation results of our study of finding the sufficient conditions for oscillation of solutions of ordinary differential equations of third order of type (1.3) in chapter five. Some oscillation results have been introduced which contain the sufficient conditions for oscillation of solutions of the equation of type (1.3). These sufficient conditions have been established by using the generalized Riccati technique. The present results extend and improve many previous results that have been obtained before, for example, such as the works of Hanan (1961), Lazer (1966), Jones (1973), Mehri (1976), Parhi & Das (1990), Parhi & Das (1993), Adamets & Lomtatidze (2001) and Remili (2007). All these previous results have been studied for particular cases of the equation (1.3) (as mentioned in chapter two) whereas our sufficient conditions have been derived for the generalized equation (1.3). A number of theorems and an illustrative example for oscillation differential equation of type (1.3) are given. Further, the numerical example is given to illustrate the theorem which is computed by using Runge Kutta of fourth order function in Matlab version 2009. The present results are compared with existing results to explain the motivation of proposed research study. We compare our results with other previous oscillation results in the literature to show that our oscillation results are more general where our sufficient conditions are derived to more general equations. These results can be applied to many particular cases of our general equations but many previous oscillation results cannot applied to our equations since all terms of our equations are not included in their studies. This is the main advantage of our research work.

6.2 Future Work

Many phenomena in different branches of sciences are interpreted in terms of second order differential equations and their solutions. In future, this research work will be continued to the study of oscillation behavior of the higher nonlinear ordinary differential equations and also partial differential equations.

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LIST OF PUBLICATIONS

1. List of Papers Submitted and Accepted for Publication

1.1 List of Papers Submitted and Accepted for Publication in Journals

1- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Oscillation of Second Order Nonlinear Ordinary Differential Equation with Alternating Coefficients, Submitted to International Conference on Mathematical Modelling & Scientific Computation 2012, India. Published in Springer Journal, Communications in Computer and Information Science, 283 (2012), 367-373. **(ISI publication).**

2- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Oscillation Theorems for Nonlinear Second Order Equations with Damping. Accepted to publish in the Bulletin of Malaysian Mathematical Sciences Society. **(ISI publication).**

3- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Oscillation Criterion for Second Order Nonlinear Equations With Alternating Coefficients, Accepted to be published in American Institute of Physics. **(ISI publication).**

1.2 List of Papers Submitted and Accepted for Publication at International Conferences

1- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Oscillation of Second Order Nonlinear Ordinary Differential Equation with Alternating Coefficients, Presented at the International Conference on Mathematical Modelling & Scientific Computation 2012, India.

2- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Oscillation Criterion for Second Order Nonlinear Equations With Alternating Coefficients, Accepted to present at The 2013 International Conference on Mathematics and Its Applications (ICMA-2013), Malaysia.

2. Publications Under Review

1- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Oscillation Theorems For Second Order Nonlinear Differential Equations. Submitted to Filomat Journal. (Under review).

2- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Some Oscillation Theorems for Nonlinear Second Order Ordinary Differential Equations with Alternating Coefficients. Submitted to Sains Malaysiana Journal. (Under Review).

3- **M. J. Saad**, N. Kumaresan and Kuru Ratnavelu, Oscillation of Third Order Nonlinear Ordinary Differential Equations. Submitted to Applied and Computational Mathematics Journal. (Under Review).