PROPERTIES OF ANALYTIC AND HARMONIC FUNCTIONS

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ABSTRACT

The starlikeness of integral transform $V_\lambda f(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt$ was first studied by Fournier and Ruscheweyh in 1994. This investigation is extended to starlikeness and convexity of order δ where $\delta \in (0, \frac{1}{2}]$. Making use the class introduced by Rosihan et al. (2012b), we determine the starlikeness and convexity of order δ for the integral transforms $V_\lambda f$ using the concept of duality and Herglotz formula. In addition, a sufficient condition for $V_\lambda f$ to be starlike and convex functions order δ and some applications of certain operators are also looked at. Furthermore, these properties are obtained for the integral transform $\mathcal{V}_\lambda f(z) = z\int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * f(z) \quad (\rho < 1)$ using the similar manner.

Recently, using the theory of differential subordination, properties of $1+\beta zp'(z)$, $1+\beta z\frac{p'(z)}{p(z)}$ and $1+\beta z\frac{p'(z)}{p^2(z)}$ subordinated to certain classes have been studied by many authors. In 1996, Sokól and Stankiewicz introduced the class $SL^\star$ which contains function associated with the right-half of the lemniscate of Bernoulli. By considering the class of Janowski starlike functions and a class defined via the Cassinian curve, a class closely related to the $SL^\star$, we obtain conditions on β using the differential subordination concept. Furthermore, the Briot-Bouquet differential subordination is used in obtaining the inclusion theorems for classes defined by Dziok-Srivastava operator and generalised multiplier transformations.

The extremal problems of multivalent and univalent harmonic functions have been discussed intensively by numerous authors. Motivated by Ahuja and Jahangiri (2001), new classes are introduced using certain operators. Coefficient conditions, extreme points, convex combination and distortion upper and lower bounds are determined for each of the classes.

Finally, some miscellaneous problems were also investigated which include preservation of certain operators to be in Hardy space as well as the preservation of the Jung-Kim-Srivastava operators for a new introduced class of functions. Suggestion problems for future research are also discussed.

ABSTRAK

Penjelmaan kamiran bak-bintang $V_\lambda f(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt$ mula diselidiki oleh Fournier dan Ruscheweyh pada tahun 1994. Kajian ini telah diperkembangkan kepada sifat bak-bintang dan sifat cembung peringkat δ dengan $\delta \in (0, \frac{1}{2}]$. Menggunakan kelas yang diperkenalkan oleh Rosihan et al. (2012b), sifat bak-bintang dan cembung peringkat δ bagi penjelmaan kamiran $V_\lambda f$ dikaji dengan menggunakan konsep kedualan dan formula Herglotz. Di samping itu, syarat cukup bagi penjelmaan kamiran $V_\lambda f$ supaya menjadi fungsi bak-bintang dan fungsi cembung peringkat δ dan beberapa penggunaan operator-operator tertentu yang berkaitan dengan penjelmaan kamiran $V_\lambda f$ juga dilihat. Selanjutnya, sifat-sifat tersebut turut diperolehi untuk penjelmaan kamiran $\mathcal{V}_\lambda f(z) = z\int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * f(z)$ $(\rho < 1)$ dengan menggunakan kaedah yang serupa.

Akhir-akhir ini, dengan menggunakan teori subordinasi kebezaan, sifat-sifat bagi $1 + \beta zp'(z), 1 + \beta z\frac{p'(z)}{p(z)}$ dan $1 + \beta z\frac{p'(z)}{p^2(z)}$ subordinat kepada kelas-kelas tertentu telah diteliti oleh kebanyakan pengkaji. Pada tahun 1996, Sokół dan Stankiewicz telah memperkenalkan kelas $SL^\star$ yang terkandung di dalam lemniskat Bernoulli bahagian kanan. Dengan mempertimbangkan kelas fungsi bak-bintang Janowski dan kelas yang berada di dalam lengkuk Cassini yang mempunyai kaitan rapat dengan kelas $SL^\star$, syarat β diperolehi dengan menggunakan konsep subordinasi kebezaan. Seterusnya, subordinasi kebezaan Briot-Bouquet digunakan untuk memperolehi teorem-teorem rangkuman bagi kelas-kelas yang ditakrifkan oleh operator Dziok-Srivastava dan penjelmaan pengganda teritlak.

Masalah ekstremum bagi fungsi harmonik multivalen dan univalen telah dibincangkan secara intensif oleh para penyelidik. Bermotivasikan Ahuja dan Jahangiri (2001), kelas-kelas baru diperkenalkan dengan mempertimbangkan beberapa operator tertentu. Syarat-syarat pekali, titik-titik ekstrem, kombinasi cembung serta batas atas dan bawah herotan ditentukan untuk setiap kelas.

Tesis ini diakhiri dengan penyelidikan terhadap pelbagai masalah yang melibatkan pengawetan operator-operator tertentu dalam ruang Hardy dan juga pengawetan bagi operator Jung-Kim-Srivastava dalam kelas yang baru diperkenalkan. Cadangan masalah-masalah untuk kajian selanjutnya juga turut dibincangkan.

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SYMBOLS

$\mathbf{C}$	complex plane
$\mathbf{R}$	set of all real numbers
$\mathbf{D}$	open unit disk
Im	imaginary part of a complex number
Re	real part of a complex number
$\kappa(z)$	Koebe function
$M_o(z)$	Möbius function
$\mathcal{A}$	class of analytic functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ $(z \in \mathbf{D})$
$\mathcal{A}_0$	$\{g : g(z) = \frac{f(z)}{z}, f \in \mathcal{A}\}, g(0) = 1$
$\mathcal{P}$	Carath*é*odory class
$\mathcal{R}$	class of analytic functions whose derivative has a positive real part
$\mathcal{S}$	class of all normalised univalent functions on $\mathbf{D}$
$S^{\star}$	class of starlike functions on $\mathbf{D}$
$S^{\star}(\gamma)$	class of functions starlike of order γ on $\mathbf{D}$
$\mathcal{C}$	class of convex functions on $\mathbf{D}$
$\mathcal{C}(\gamma)$	class of functions convex of order γ on $\mathbf{D}$
$\mathcal{K}$	class of close-to-convex functions on $\mathbf{D}$
$SS^{\star}(\beta)$	class of functions strongly starlike of order β on $\mathbf{D}$
S_H	class of univalent harmonic functions
$S_H^{\star}(\gamma)$	class of univalent harmonic functions starlike of order γ
$S_H(p)$	class of multivalent harmonic functions
$S_H^{\star}(p, \gamma)$	class of multivalent harmonic functions starlike of order γ
$S_S^{\star}$	class of functions starlike with respect to symmetric points
$SL^{\star}$	class of Sok*ó*l-Stankiewicz starlike functions

$S^{\star}[A,B]$	class of Janowski starlike functions
$S^{\star}(\phi),\ \mathcal{C}(\phi),\ \mathcal{K}(\phi)$	subclasses of Ma-Minda
$*$	convolution
$\prec$	subordinate to
$H^{l,m}[\alpha_1]f$	Dziok-Srivastava operator
$H_{a,b,c}f$	Hohlov operator
$L(b,c)f$	Carlson-Shaffer operator
$D^{\mu}f$	Ruscheweyh derivative operator
$F_c f$	Bernardi integral operator
$I_{\mu}f$	Noor integral operator
$I_{\mu,\xi}f$	Choi-Saigo-Srivastava operator
$H_{\lambda}^{l,m}[\alpha_1]f$	Kwon-Cho operator
$P^{\nu}f,\ J_{\nu}f,\ \ell_{\mu}^{\nu}f$	Jung-Kim-Srivastava operator
$D_{\lambda}^{k}f$	Al-Oboudi operator
$I(k,\lambda,c)f$	Cătaş multiplier transformations

CHAPTER 1

PRELIMINARIES

1.1 Introduction

Complex analysis and geometric function theory play an important role in Mathematics and Physics. Complex analysis or better known as the theory of functions of a complex variable, was rigorously investigated in the 19^{th} century. It deals with the study of the sets and functions in the complex plane. It is particularly concerned with the analytic functions of complex variables and the theory of conformal mappings. These theories are useful in many branches of mathematics such as number theory and applied mathematics, as well as in physics including hydrodynamic, thermodynamics, eletrical engineering and others. Geometric properties of analytic functions are studied in geometric function theory and Riemann mapping theorem is the fundamental result of this theory.

Since analytic functions are the central components in complex analysis due to their interesting properties, we give its definition first.

Definition 1.1. *(Duren, 1983) A complex-valued function f of a complex variable z is differentiable at a point $z_0 \in \mathbf{C}$ if $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists at z_0. A function f is analytic at z_0 if it is differentiable at every point in some neighborhood of z_0.*

A power series represents an analytic function of z in its region of convergence. A complex-valued function f of a complex variable has a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} A_n(z-z_0)^n$ where $A_n = \frac{f^{(n)}(z_0)}{n!}$ and is convergent in some open disk centered at z_0.

Univalent function theory is classified under the broader area of geometric function theory. One of the basic result in the theory of univalent functions in one variable is the Riemann mapping theorem. As early as 1851, the Riemann mapping theorem initiated by Bernhard Riemann(1826-1866) states that for every simply connected domain G which is proper subset of the complex plane $\mathbf{C}$, can be mapped conformally onto the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Moreover if z_0 is a given point in G, then there exist a unique function f which maps G conformally onto $\mathbf{D}$ such that $f(z_0) = 0$ and $f'(z) > 0$ (Graham and Kohr, 2003), (Duren, 1983) and (Pommerenke, 1975). In view of the Riemann mapping theorem, it suffices to consider the unit disk $\mathbf{D}$ rather than a general simply connected domain.

Definition 1.2. *(Duren, 1983) A function f is said to be univalent (schlicht or one-to-one) in a domain $\mathbf{E} \subset \mathbf{C}$ if the conditions $f(z_1) = f(z_2)$ for all points z_1 and z_2 in $\mathbf{E}$ imply that $z_1 = z_2$. The function f is locally univalent at a point $z_0 \in \mathbf{E}$ if it is univalent in some neighborhood of z_0.*

The definition for p-valent (multivalent of order p) in $\mathbf{D}$ is given as follows:

Definition 1.3. *(Goodman, 1983) A function f is p-valent (multivalent of order p) in $\mathbf{D}$ if for each w_0 (infinity included) the equation $f(z) = w_0$ has at most p roots in $\mathbf{D}$ (the roots are counted with their multiplicities) and there are some w_1 so that $f(z) = w_1$ has exactly p roots in $\mathbf{D}$.*

The inverse mapping theorem implies that if f is locally univalent then $f'(z) \neq 0$. There are univalent functions but not analytic, for example $f(z) = \frac{1}{z}$. Conversely, the function $f(z) = e^z$ is an analytic function but not univalent. We are interested in univalent functions that are also analytic. An analytic univalent function is called a conformal mapping because of its angle preserving property. The theory of conformal mapping is a shift of emphasis from the function theoretical to the geometric side of problem.

Consider a Maclaurin series expansion that is convergent in $\mathbf{D}$,

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z^1 + a_2 z^2 + \cdots \\ f(z) &= \frac{g(z) - a_0}{a_1} = z + a_2 z^2 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \qquad a_1 \neq 0. \end{aligned} \tag{1.1}$$

The above process is called the normalisation of function with $f(0) = 0$ and $f'(0) = 1$. The reason for introducing this normalisation is to eliminate unnecessary parameters, which simplifies the statement of the results (Pommerenke, 1975). Observe that if g is univalent in $\mathbf{D}$ then the translation of the image domain of $g(z) - a_0$ is univalent in $\mathbf{D}$. Since g is univalent in $\mathbf{D}$ then $a_1 = g'(0) \neq 0$. Thus we may divide by a_1 so that gives $f(z) = \frac{g(z)-a_0}{a_1}$. Since multiplication by $\frac{1}{a_1}$ rotates and stretches the image domain, then for g is univalent in $\mathbf{D}$ would imply the function f is univalent in $\mathbf{D}$.

Definition 1.4. *(Goodman, 1983, p. 15) A function of the form* (1.1) *is said to be normalised. If $f(z)$ is univalent and has the form* (1.1), *it is called a normalised univalent function.*

Let $\mathcal{A}$ denote the class of all analytic functions f in the form (1.1) and normalised by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of functions f that are analytic and normalised univalent functions in the unit disk $\mathbf{D}$. One of the most important example of a function in S is the Koebe function

$$\kappa(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots = \sum_{n=1}^{\infty} nz^n \ , \ z \in \mathbf{D}$$

which we can also write as

$$\kappa(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right].$$

In Figure 1.1 (Goodman, 1983), the sequence of mappings used in building the Koebe functions is shown.The function $w = \frac{1+z}{1-z}$ maps $\mathbf{D}$ univalently onto the right halfplane $Re\{w\} > 0$. Then the function $g(z) = w^2(z)$ takes this half-plane onto the entire plane minus the part of the negative real axis from $\frac{-1}{4}$ to infinity so that the Koebe function $\kappa(z) = \frac{1}{4}(g(z) - 1)$ is established.

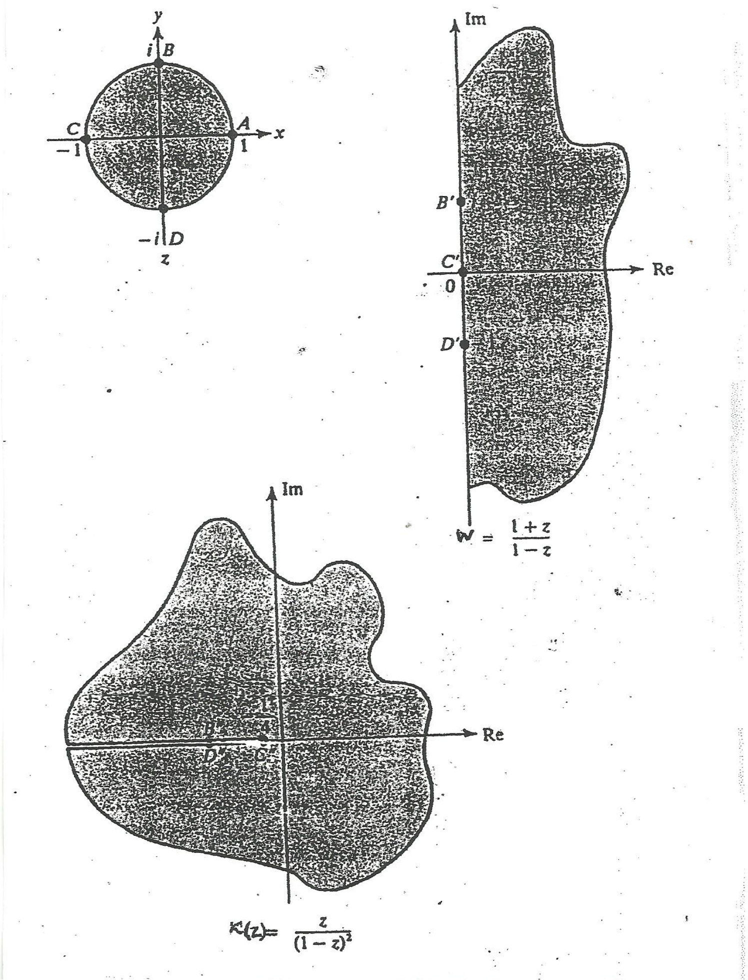

Figure 1.1: The mapping of Koebe functions

Ludwig Bieberbach (1916) proved that the coefficient $|a_2| \leq 2$ if f in S. Since the equality occurs for $a_2 = 2$, the Koebe function is called an extremal function of S where the extremal function refers to a function for which equality occurs.

1.2 Subclasses of univalent functions

Due to Koebe function, most researchers are motivated to study and introduce subclasses of S. The well known subclasses of S are classes of starlike functions, $S^\star$, convex functions, $\mathcal{C}$ and close-to-convex functions, $\mathcal{K}$.

The class of starlike functions in $\mathbf{D}$ was presented by Alexander (1915), and studied by Nevanlinna in 1921.

Definition 1.5. *(Duren, 1983) A set $\mathbf{E} \subset \mathbf{C}$ is said to be starlike with respect to a point $w_0 \in \mathbf{E}$ if the linear segment joining w_0 to every other point $w \in \mathbf{E}$ lies entirely in $\mathbf{E}$. If a function f maps $\mathbf{D}$ onto a domain that is starlike with respect to w_0, then f is called a starlike function with respect to w_0. For $w_0 = 0$, the function f is said a starlike function.*

The Koebe function is a starlike function and the domain $\kappa(\mathbf{D})$ is starlike with respect to each $w_0 > -\frac{1}{4}$. Figure 1.2 (Graham and Kohr, 2003) describes the image for a starlike function.

Alexander (1915) also presented the class of convex functions in $\mathbf{D}$ and later studied by Gronwall (1916) and Löwner (1917).

Definition 1.6. *(Duren, 1983) The set $\mathbf{E} \subset \mathbf{C}$ is said to be convex if the linear segment joining any two points of $\mathbf{E}$ lies entirely in $\mathbf{E}$. If a function f maps $\mathbf{D}$ onto a convex domain then f is said to be a convex function.*

The image of a convex function is shown in Figure 1.3 (Graham and Kohr, 2003).

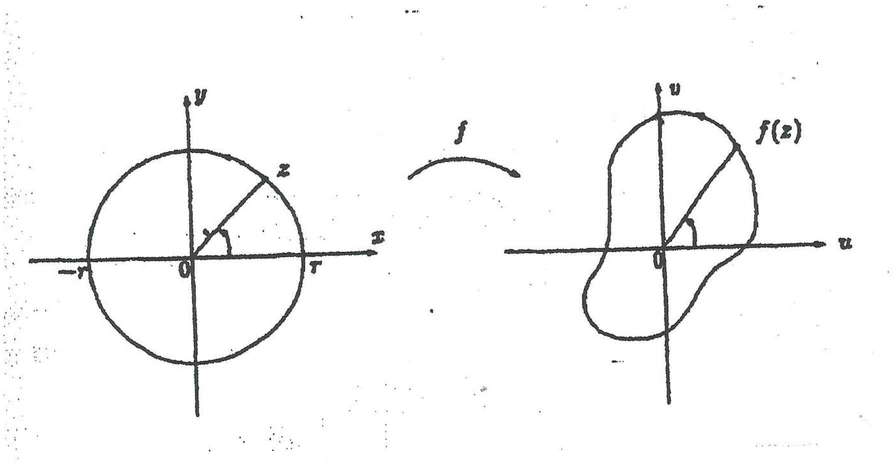

Figure 1.2: The image of starlikeness

The Möbius function $M_o(z) \equiv \frac{1+z}{1-z}$ is a convex function because it maps $\mathbf{D}$ onto a half-plane. Any circular disk or any half-plane is a convex set (Goodman, 1983).

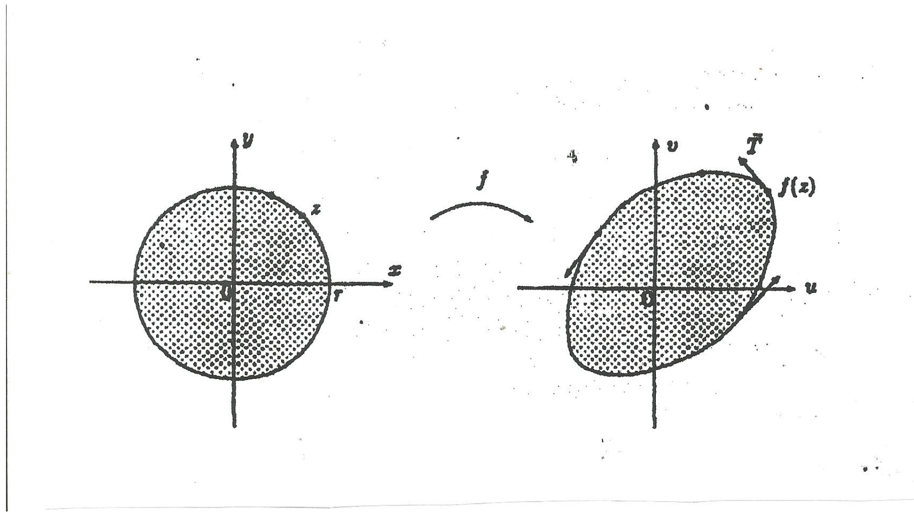

Figure 1.3: The image of convexity

The following theorems give an analytic description of starlike and convex functions.

Theorem 1.1. *(Duren, 1983) Let f be analytic in $\mathbf{D}$ with $f(0) = 0$, $f'(0) = 1$. Then a function $f \in S^\star$ if and only if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, (z \in \mathbf{D})$$

Further, a function $f \in \mathcal{C}$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0, (z \in \mathbf{D}).$$

The concepts of functions starlike of order $\gamma, S^*(\gamma)$ and convex of order $\gamma, \mathcal{C}(\gamma)$ were introduced by Robertson in 1936. The theorem of these classes are given below.

Theorem 1.2. *$f \in S^*(\gamma)$ if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in \mathbf{D}, 0 \leq \gamma < 1),$$

and a function $f \in \mathcal{C}(\gamma)$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \gamma \quad (z \in \mathbf{D}, 0 \leq \gamma < 1).$$

The relationship between starlike and convex function was first noticed by Alexander in 1915.

Theorem 1.3. *Suppose that $f'(z) \neq 0$ in $\mathbf{D}$. Then f is convex in $\mathbf{D}$ if and only if $zf'(z)$ is starlike in $\mathbf{D}$.*

As an example, consider a function

$$f(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$$

which maps $\mathbf{D}$ onto the half-plane $Re\ w > -\frac{1}{2}$. Thus the function f is a convex function in $\mathbf{D}$ which implies the function

$$zf'(z) = \frac{z}{(1-z)^2} \tag{1.2}$$

is starlike in $\mathbf{D}$. Recognize that the right side of (1.2) is the Koebe function.

The definition of the close-to convex functions was given due to Kaplan (1952).

Definition 1.7. *Let $f \in S$. The function f is said close-to convex function in $\mathbf{D}$ if there exists a convex function g in $\mathbf{D}$ such that*

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0, \quad z \in \mathbf{D}$$

or

$$Re\left\{\frac{zf'(z)}{h(z)}\right\} > 0, \quad z \in \mathbf{D}$$

where h is a starlike function in $\mathbf{D}$.

Most of these subclasses have both an analytic and a geometric characterization. These classes are related with functions of positive real part in $\mathbf{D}$ which is called Carathéodory class and denoted by $\mathcal{P}$.

Definition 1.8. *(Goodman, 1983) The set $\mathcal{P}$ is the set of all functions of the form $f(z) = 1+\sum_{n=1}^{\infty} a_n z^n$ that are analytic in $\mathbf{D}$ and such that for $z \in \mathbf{D}$, $Re\{f(z)\} > 0$.*

In the class $\mathcal{P}$, the Möbius function

$$M_o(z) \equiv \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots = 1 + 2\sum_{n=1}^{\infty} z^n$$

plays a central role as the Koebe function for the class S. This function is analytic and univalent in $\mathbf{D}$, it maps $\mathbf{D}$ onto the half-plane.

The class $\mathcal{R}$ is the class of analytic functions whose derivative has a positive real part. Properties for this class have been obtained by many and in particular introduced and studied by MacGregor (1962). Formally, we have

Definition 1.9. *Let $f \in \mathcal{S}$ and $z \in \mathbf{D}$, f is said to be in $\mathcal{R}$ if $Re\{f'(z)\} > 0$ and if f satisfying $Re\{f'(z)\} > \gamma, (0 \leq \gamma < 1)$ then $f \in \mathcal{R}(\gamma)$.*

1.3 Generalised hypergeometric functions, convolution and operators

The Bieberbach conjecture remained open for a long time and was surprisingly proved by Louis de Branges in 1984 using hypergeometric functions (de Branges, 1985). The implication of this discovery, theory on hypergeometric functions was developed [Koepf (2007) and Shanmugam (2007)] and various properties of classes were obtained via operators and generalised hypergeometric functions.

Representation of a function $g(z) = \frac{1}{(1-z)^a}$, $a \in \mathbf{C}$ as a geometric series

$$\frac{1}{(1-z)^a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

leads us to define a function

$$\begin{aligned} {}_1F_1(b, c; z) &= 1 + \frac{bz}{c} + \frac{b(b+1)z^2}{c(c+1)2!} + \frac{b(b+1)(b+2)z^3}{c(c+1)(c+2)3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n n!} z^n \end{aligned} \tag{1.3}$$

where b and c are complex numbers with $c \neq 0, -1, -2, \ldots$ and $(\lambda)_n$ is the Pochhammer symbol defined, in terms of gamma function by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \quad n = 0, \ \lambda \neq 0 \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & , \quad n = 1, 2, 3, \ldots \end{cases}$$

The function (1.3) is called a confluent (Kummer) hypergeometric function. This function is analytic and satisfies Kummer's differential equation

$$zw''(z) + [c - z]w'(z) - aw(z) = 0.$$

Further, (1.3) can be generalised as

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{abz}{c} + \frac{a(a+1)b(b+1)z^2}{c(c+1)2!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \ , \qquad a,b,c \in \mathbf{C}(c \neq 0,-1,-2,\ldots) \end{aligned}$$

and is called the Gaussian hypergeometric function which is analytic and satisfies the hypergeometric differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - abw(z) = 0.$$

More generally, for complex or real parameters

$$\alpha_i (i = 1,2,\ldots,l) \quad \text{and} \quad \beta_j \in \mathbf{C}\backslash\{0,-1,-2,\ldots\} \ (j=1,2,\ldots,m),$$

the generalised hypergeometric function ${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)$ is given as

$$\begin{aligned} {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n n!} z^n \\ (l \leq m+1; \ l,m \in \mathbf{N}_0 &:= \mathbf{N} \cup \{0\}; z \in \mathbf{D}). \end{aligned}$$

An operator is a type of function which acts on functions to produce other functions. In calculus, there are three typical types of operators: integral operators, differential operators and convolution. The convolution is a mathematical operation on two functions φ and ψ in order to produce a third function.

Definition 1.10. *Let $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic functions, then the convolution of these functions is*

$$\varphi(z) * \psi(z) = \sum_{n=0}^{\infty} a_n b_n z^n = \psi(z) * \varphi(z).$$

The convolution is also called the Hadamard product.

Corresponding to the generalised hypergeometric functions and using the convolution, for f in the form (1.1), Dziok and Srivastava (1999) initiated to introduce the operator

$$\begin{aligned} H^{l,m}\left(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m\right)f(z) &= z\ {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)*f(z) \\ &= z+\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}(n-1)!}a_nz^n \\ &= z+\sum_{n=2}^{\infty}\phi_n[\alpha_1]a_nz^n \end{aligned}$$

where $\phi_n[\alpha_1]=\frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}(n-1)!}$,

$$\alpha_i(i=1,2,\ldots,l)\quad\text{and}\quad\beta_j\in\mathbf{C}\backslash\{0,-1,-2,\ldots\}(j=1,2,\ldots,m)$$

are complex or real parameters. For convenience we write

$$H^{l,m}\left[\alpha_1\right]f(z):=H^{l,m}\left(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m\right)f(z).$$

The Dziok-Srivastava operator includes well known operators such as:

i) Hohlov operator (Hohlov, 1978)

$$H^{2,1}(a,b;c)f(z)\equiv H_{a,b,c}f(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_nz^n$$

where $a,b,c,\in\mathbf{C}$ and $c\neq 0,-1,-2,\ldots$.

ii) Carlson-Shaffer operator (Carlson & Shaffer, 1984)

$$\begin{aligned} H^{2,1}(b,1;c)f(z) &\equiv L(b,c)f(z) \\ &= z+\sum_{n=2}^{\infty}\frac{(b)_{n-1}}{(c)_{n-1}(n-1)!}a_nz^n,\quad b,c\in\mathbf{C},c\neq\ 0,-1,-2,\ldots \end{aligned}$$

iii) Ruscheweyh derivative operator (Ruscheweyh, 1975b)

$$H^{2,1}(\mu+1,1;1)f(z)\equiv D^{\mu}f(z)=z+\sum_{n=2}^{\infty}\frac{(\mu+1)_{n-1}}{(n-1)!}a_nz^n\ ,\quad(\mu\geq-1).$$

iv) Bernardi integral operator (Generalised Bernardi-Libera-Livington integral operator) [(Bernardi, 1969), (Libera, 1965) and (Livington, 1966)]

$$H^{2,1}(c+1,1;c+2)f(z) \equiv F_c(z) = z + \sum_{n=2}^{\infty} \frac{(c+1)_{n-1}}{(c+2)_{n-2}} a_n z^n \quad , (c > -1).$$

Motivated by Ruscheweyh derivative operator, Noor (1999) introduced new operator by setting

$$f_\mu(z) = \frac{z}{(1-z)^{\mu+1}}, \quad (\mu \in \mathbf{N}_0)$$

and defining $f_\mu^\ddagger$ in terms of the convolution as

$$f_\mu(z) * f_\mu^\ddagger(z) = \frac{z}{(1-z)^2}, \quad (z \in \mathbf{D}).$$

Then, Noor operator is defined by $I_\mu f(z) = \left(f_\mu^\ddagger * f\right)(z)$. Choi, Saigo and Srivastava (2002) generalised the operator I_μ for $\xi > 0, \ \mu > -1$ and obtained $I_{\mu,\xi} f(z) = \left(f_{\mu,\xi} * f\right)(z)$. Then, using the generalised hypergeometric functions, Kwon and Cho (2007) established an operator using the similar manner as Noor (1999) and Choi et al. (2002). The Kwon-Cho operator is given as

$$H_\lambda^{l,m}\left[\alpha_1\right] f(z) \quad = \quad \sum_{n=0}^{\infty} \frac{(\lambda)_n (\beta_1)_n \ldots (\beta_m)_n}{(\alpha_1)_n \ldots (\alpha_l)_n n!} a_{n+1} z^{n+1} \quad , \quad (a_1 = 1 \ , \ \lambda > 0).$$

For special cases, the Noor operator, Choi-Saigo-Srivastava operator and Kwon-Cho operator are equivalent to the Dziok-Srivastava operator.

i) $H^{2,1}(2,1;\mu+1)f(z) \equiv I_\mu f(z) = z + \sum_{n=2}^{\infty} \frac{(2)_{n-1}}{(\mu+1)_{n-1}} a_n z^n$

ii) $H^{2,1}(\xi,1;\mu+1)f(z) \equiv I_{\mu,\xi} f(z) = z + \sum_{n=2}^{\infty} \frac{(\xi)_{n-1}}{(\mu+1)_{n-1}} a_n z^n$

iii) For $l = m+1, \lambda = 1, \beta_1 = 1$ and $\beta_2 = \alpha_1, \ldots \beta_{m+1} = \alpha_l$,

$$H^{l,m}\left[\alpha_1\right] f(z) \equiv H_\lambda^{l,m}\left[\alpha_1\right] f(z).$$

Besides that, we provide some definitions of other operators that are used in this study.

i) Jung-Kim-Srivastava operator (Jung et al.,1993)

$$J_\nu f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\nu+1}{\nu+n} \right) a_n z^n, \quad \nu > -1$$

and

$$\ell_\mu^\nu f(z) = z + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)} \right) a_n z^n, \quad \nu > 0, \ \mu > -1.$$

ii) Generalised Sălăgean operator (Al-Oboudi, 2004)

Genaralised Sălăgean operator was introduced by Al-Oboudi (2004) as follows:

$$D_\lambda^k f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k \, a_n z^n, \lambda \geq 0, k \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}.$$

For $\lambda = 1$, the Al-Oboudi operator is reduced to Sălăgean operator (Sălăgean, 1983)

iii) Cătaş multiplier transformations (Cătaş, 2008)

For any real numbers k and λ where $k \geq 0, \lambda \geq 0, c \geq 0$, Cătaş defined the multiplier transformations $I(k, \lambda, c)f(z)$ by the following series:

$$I(k, \lambda, c)f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + \lambda(n-1) + c}{1 + c} \right]^k a_n z^n. \tag{1.4}$$

1.4 Duality principle

The concept of duality associated with convolution for a function f in $\mathcal{A}$ was developed by Ruscheweyh (1975a) and the basic results of Ruscheweyh's duality theory can be found in the book (Ruscheweyh, 1982). Here, some basic concepts and results from this theory are given. Let $\mathcal{A}_0 = \{g : g(z) = \frac{f(z)}{z}, f \in \mathcal{A}\}, \ g(0) = 1$ and for a subset $\mathcal{B} \subset \mathcal{A}_0$, the dual set is defined as

$$\mathcal{B}^* = \{g \in \mathcal{A}_0 : (f * g)(z) \neq 0, z \in \mathbf{D}, \quad \text{for all} \quad f \in \mathcal{B}\}$$

and $(\mathcal{B}^*)^* = \mathcal{B}^{**}$ is called the second dual or dual hull of $\mathcal{B}$.

Theorem 1.4. *(Ruscheweyh, 1982) Let*

$$\mathcal{B} = \left\{ \frac{1 + xz}{1 + yz} : |x| = |y| = 1 \right\}.$$

Then $\mathcal{B}^{**} = H$ *where*

$$H = \left\{ g \in \mathcal{A}_0 : \exists \phi \in \mathbf{R} \quad \textit{such that} \quad Re\ e^{i\phi}[g(z)] > 0, \ z \in \mathbf{D} \right\}.$$

Theorem 1.5. *(Ruscheweyh, 1975a) Let*

$$\mathcal{B} = \left\{ \beta + (1 - \beta) \left(\frac{1 + xz}{1 + yz} \right) : |x| = |y| = 1 \right\} \quad , \quad \beta \in \mathbf{R}, \ \beta \neq 1.$$

Then

(i) $\mathcal{B}^{**} = \{g \in \mathcal{A}_0 : \exists \phi \in \mathbf{R}$ *such that* $Re\ e^{i\phi}[g(z) - \beta] > 0, \ z \in \mathbf{D}$.

(ii) If Γ_1 *and* Γ_2 *are two continuous linear functionals on* $\mathcal{B}$ *with* $0 \notin \Gamma_2(\mathcal{B})$*, then for every* $g \in \mathcal{B}^{**}$ *we can find* $\Theta \in \mathcal{B}$ *such that* $\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(\Theta)}{\Gamma_2(\Theta)}$.

These theorems have many applications to classes of functions which are defined using properties like bounded real part, starlikeness, convexity, close-to-convexity, univalence and other properties. The application of Herglotz formula [(Herglotz, 1911), (Rudin, 1973), (Hallenbeck and MacGregor, 1974)] is involved in proving the theorems. The definition of Herglotz formula is stated for all $f \in \mathcal{P}$, there exists a probability measure ξ on the interval$[0, 2\pi]$ so that

$$f(z) = 1 + 2 \int_0^{2\pi} \sum_{n=1}^{\infty} z^n e^{-in\theta} d\zeta(\theta) \quad , \quad \int_0^{2\pi} d\zeta(\theta) = 1.$$

The extreme points of $\mathcal{P}$ are the function $f(z) = \frac{1+z}{1-z}$. The class $\mathcal{P} \subset \mathcal{A}_0$ is defined by

$$\mathcal{P} = \{g \in A_0 : Re[g(z)] > 0, z \in \mathbf{D}\}.$$

The result from Ruscheweyh (1982, p. 23) is stated in the following lemma.

Lemma 1.1. *(Duality Theorem) The dual of the class $\mathcal{P}$ is*

$$\mathcal{P}^* = \left\{g \in A_0 : Re[g(z)] > \frac{1}{2}, z \in \mathbf{D}\right\}.$$

1.5 Differential subordination

In the theory of complex-valued functions, there are numerous characterisation of function which are determined by a differential condition. As a simple example, the Noshiro-Warschawski theorem stated:

Theorem 1.6. *If f is analytic in the unit disk $\mathbf{D}$, then $Re[f'(z)] > 0$ implies f is univalent in $\mathbf{D}$.*

The real-valued techniques were used in the complex plane since most of the known differential implications dealt with real-valued inequalities. Alternatively, a differential inequality in a real variable concept was replaced with its complex analogue by Miller and Mocanu in 1981 and called the differential subordination.

Definition 1.11. *An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)(z \in \boldsymbol{D})$, if there exists an analytic function w in $\boldsymbol{D}$ such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if g is univalent in $\boldsymbol{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\boldsymbol{D}) \subset g(\boldsymbol{D})$.*

The subordination can be illustrated as in Figure 1.4 (Graham and Kohr, 2003).

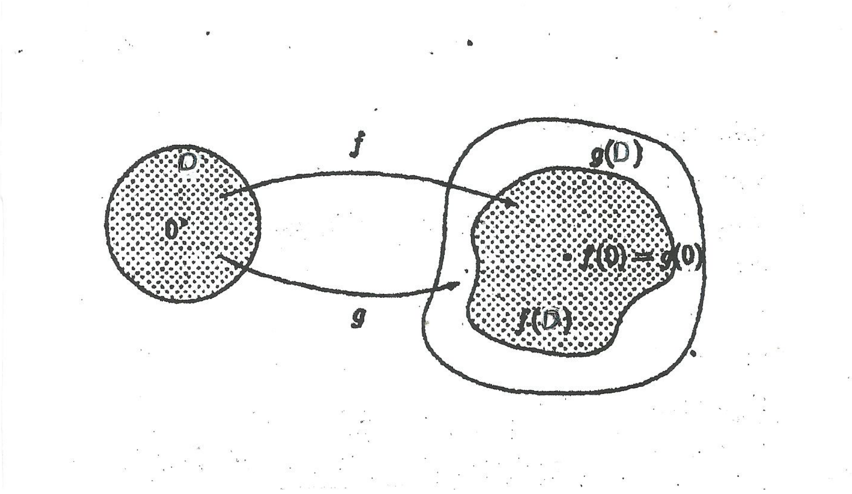

Figure 1.4: Subordination of f and g

The subordination principle can be used to characterize classes of analytic functions. For examples:

i) $f \in \mathcal{P}$ if and only if $f(z) \prec \dfrac{1+z}{1-z}$,

because the function $\frac{1+z}{1-z}$ is a univalent function maps $\mathbf{D}$ conformally onto the right-half plane.

ii) $f \in S^{\star} \Leftrightarrow \dfrac{zf'(z)}{f(z)} \in \mathcal{P} \Leftrightarrow \dfrac{zf'(z)}{f(z)} \prec \dfrac{1+z}{1-z}$.

iii) $f \in \mathcal{C} \Leftrightarrow 1 + \dfrac{zf''(z)}{f'(z)} \in \mathcal{P} \Leftrightarrow \dfrac{zf''(z)}{f'(z)} \prec \dfrac{2z}{1-z}$.

iv) $f \in S^{\star}(\gamma) \Leftrightarrow \dfrac{zf'(z)}{f(z)} \prec \dfrac{1+(1-2\gamma)z}{1-z}$, $(0 \leq \gamma < 1)$.

v) $f \in \mathcal{C}(\gamma) \Leftrightarrow \dfrac{zf''(z)}{f'(z)} \prec \dfrac{2(1-\gamma)z}{1-z}$, $(0 \leq \gamma < 1)$.

Furthermore, we give some definitions of other classes which are used in our study. Firstly, we denote $S^{\star}[A, B]$ as the class of Janowski starlike functions by Janowski (1973) consisting of functions $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1).$$

For $c \in (0, 1]$, Aouf et al. (2011) defined the class $S^\star(q_c)$ as:

$$S^\star(q_c) = \left\{ f \in \mathcal{A} : \left| \left[\frac{zf'(z)}{f(z)} \right]^2 - 1 \right| < c, \quad z \in \mathbf{D} \right\}.$$

It can be established that

$$f \in S^\star(q_c) \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1 + cz} \quad (z \in \mathbf{D}).$$

Denote Θ_c as the set of all points on the right half-plane such that the product of the distances from each point to the focuses -1 and 1 is less than c;

$$\Theta_c := \left\{ w \in C : Re\ w > 0, \left| w^2 - 1 \right| < c \right\}$$

thus the boundary $\partial\Theta_c$ is the right loop of the Cassinian ovals. Geometrically, a function $f \in S^\star(q_c)$ if $w = \frac{zf'(z)}{f(z)}$ is in the interior of the right half of the Cassinian ovals $(x^2 + y^2)^2 - 2(x^2 - y^2) = c^2 - 1$. Particularly, for $c = 1$, the right half of the lemniscate of Bernoulli $(x^2+y^2)^2-2(x^2-y^2) = 0$ is obtained and $S^\star(q_1) \equiv SL^\star$. The class of $SL^\star$ was introduced by Sokól and Stankiewicz in 1996 which is consisting normalised analytic functions f in $\mathbf{D}$ satisfying the condition $\left| \left[\frac{zf'(z)}{f(z)} \right]^2 - 1 \right| < 1, z \in \mathbf{D}$. a function $f \in SL^\star$ if $\frac{zf'(z)}{f(z)}$ is in the interior of the right half of the lemniscate of Bernoulli. The illustration of this class is shown in Figure 1.5 (Sokól, 2009b). A function in the class $SL^\star$ is called Sokól-Stankiewicz starlike function. Alternatively, we can also write

$$f \in SL^\star \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1 + z}\ .$$

Some properties of functions in class $SL^\star$ have been studied by (Rosihan et al., 2012c), (Sokól, 2009a), (Sokól, 2008) and (Sokól, 2007).

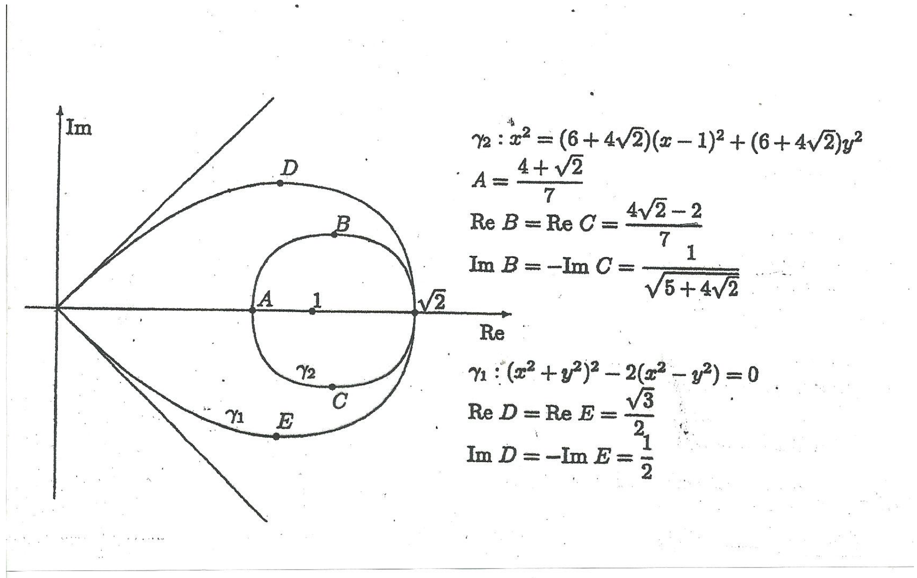

Figure 1.5: The graph of the interior of the right half of the lemniscate of Bernoulli.

Let N be the class of all analytic and univalent functions ϕ in $\mathbf{D}$ and for which $\phi(\mathbf{D})$ is convex with $\phi(0) = 1$ and $Re\,\{\phi(z)\} > 0$ for $z \in \mathbf{D}$. For $\phi, \psi \in N$, Ma and Minda (Ma and Minda, 1992) studied the subclasses $S^{\star}(\phi), \mathcal{C}(\phi)$ and $\mathcal{K}(\phi, \psi)$ of the class $\mathcal{A}$. These classes are defined using the principle of subordination as follows:

$$S^{\star}(\phi) := \left\{ f : f \in \mathcal{A} \;\; \text{and} \;\; \frac{zf'(z)}{f(z)} \prec \phi(z) \;\; \text{in} \;\; \mathbf{D} \right\}$$

$$\mathcal{C}(\phi) := \left\{ f : f \in \mathcal{A} \;\; \text{and} \;\; 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \;\; \text{in} \;\; \mathbf{D} \right\}$$

$$\mathcal{K}(\phi, \psi) := \left\{ f : f \in \mathcal{A} \;\; \text{and} \;\; \exists g \in S^{\star}(\phi) \;\; \text{such that} \;\; \frac{zf'(z)}{g(z)} \prec \psi(z) \;\; \text{in} \;\; \mathbf{D} \right\}$$

Obviously, we have the following relationships for special choices ϕ and ψ:

$$S^{\star}\left(\frac{1+z}{1-z}\right) = S^{\star}, \quad \mathcal{C}\left(\frac{1+z}{1-z}\right) = \mathcal{C},$$

$$\mathcal{K}\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = \mathcal{K}, \quad S^{\star}\left(\frac{1+Az}{1+Bz}\right) = S^{\star}[A, B].$$

1.6 Harmonic functions

A continuous function $f = u + iv$ is said to be a complex-valued harmonic function in a complex domain $\mathbf{E} \subset \mathbf{C}$ if both u and v are real harmonic in $\mathbf{E}$. A real-valued function $u(x, y)$ is harmonic if satisfies the Laplace equation $\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}\right) u = 0$ .There is an interrelation between harmonic functions and analytic functions. In any simply connected domain $\mathbf{E}$ we write $f = h + \bar{g}$ where h and g are analytic in $\mathbf{E}$. Respectively, h and g are called the analytic part and co-analytic part of f. The function $f = h + \bar{g}$ is said to be harmonic univalent in $\mathbf{D}$ if the mapping $z \to f(z)$ is orientation preserving, harmonic and univalent in $\mathbf{D}$. This mapping is orientation preserving and locally univalent in $\mathbf{D}$ if and only if the Jacobian of f, $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ in $\mathbf{D}$ (Lewy, 1936).

From the perspective of geometric function theory, Clunie and Sheil-Small (1984) initiated the study on these functions by introducing the class S_H consisting of normalised complex-valued harmonic univalent functions f defined on the open unit disk $\mathbf{D} = \{z : z \in \mathbf{C}, |z| < 1\}$. Necessary and sufficient conditions for f to be locally univalent and sense-preserving in $\mathbf{D}$ were obtained. Coefficient bounds for functions in S_H were determined. Since then, various subclasses of S_H were investigated by several authors [for examples see (Al-Shaqsi and Maslina, 2008), (Chandrashekar et.al, 2009), (Jahangiri,1999), (Murugusundaramoorthy et.al, 2009), (Murugusundaramoorty, 2003) and (Rosy et.al, 2001)]. Note that the class S_H reduces to the class of normalised analytic univalent functions if the co-analytic part of f is identically to zero($g \equiv 0$). Duren(2004) gives an informed literature on harmonic mappings. Let S_H denote the class of univalent harmonic functions $f = h + \bar{g}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.5}$$

and $S_H^{\star}(\gamma)$ denote the class of univalent harmonic starlike functions of order γ $(0 \leq \gamma < 1)$. The function f of the form (1.5) is in $S_H^{\star}(\gamma)$ if (Sheil-Small, 1990)

$$\frac{\partial}{\partial\theta}\left(argf\left(re^{i\theta}\right)\right) = Im\left\{\frac{\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right\} = Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq \gamma. \tag{1.6}$$

Multivalent harmonic functions in the unit disk $\mathbf{D}$ were investigated by Duren, Hengartner and Laugesen (1996) via the argument principle. For $p \geq 1$, let $S_H(p)$ denote the class of multivalent harmonic functions $f = h + \bar{g}$ where

$$h(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1} \quad , \quad g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1} \tag{1.7}$$

Lastly, let $S_S^{\star}$ denote the class of starlike functions with respect to symmetric points. This class was introduced by Sakaguchi (1959) where f satisfying

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathbf{D}.$$

Then, Ahuja and Jahangiri (2004) studied the class of harmonic starlike functions of order γ with respect to symmetric points, $S_{HS}^{\star}(\gamma)$ and satisfying the condition

$$Im\left\{\frac{2\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right) - f\left(-re^{i\theta}\right)}\right\} = Re\left\{\frac{2\left[zh'(z) - \overline{zg'(z)}\right]}{h(z) + \overline{g(z)}}\right\} \geq \gamma \quad (0 \leq \gamma < 1). \tag{1.8}$$

1.7 Scope of thesis

This thesis has six chapters. The basic concepts and some known results are given in chapter one. The subsections are designed to prepare the back ground for requirement in the subsequent chapters of the thesis.

The main part of this thesis is treated in chapter two. Using duality concept, the integral transform $V_\lambda f$ and $\mathcal{V}_\lambda f$ are considered as an important components. The starlikeness and convexity of order δ for integral transform $V_\lambda f$ and $\mathcal{V}_\lambda f$ are determined for f in a class $\mathcal{W}_\beta(\alpha, \gamma)$. This result is used in obtaining a sufficient condition for $V_\lambda f$ which leads to several applications for specific choices of λ.

The purpose of chapter three is to discuss some applications of differential subordination for certain classes. This chapter gives a combination treatment of results concerning the right-half of the lemniscate of Bernoulli and generalisation of multiplier transformations. The Dziok-Srivastava operator is also considered in getting some inclusion theorems.

Some extremal problems such as coefficient bounds, extreme points, convex combination and distortion upper and lower bounds for multivalent and univalent harmonic functions are obtained in chapter four. New classes are established using a generalisation of certain operator. A starlike function with respect to symmetric points is also studied.

In chapter five, we investigate some miscellaneous problems such as a preservation of certain operators for a class of Hardy space. Besides that, we use the convex null sequence in showing the preservation of these operators. Lastly, in chapter six we suggest some problems for future research.

CHAPTER 2

STARLIKENESS AND CONVEXITY OF INTEGRAL TRANSFORMS USING DUALITY

In the field of geometric function theory, there have been many approaches to solving research problems and obtaining results. Ruscheweyh (1975a) introduced the concept of duality which since then has been progressively utilised to establish results. Recently, this utilisation is much more intensive [see (Rosihan et al., 2012a), (Rosihan et al., 2012b) and (Ponnusamy and Ronning, 2008)]. The duality technique is a powerful method and is widely used in getting results on starlikeness, convexity as well as other properties where other methods have failed. This chapter focuses on starlike and convex properties of certain integral transforms using the duality technique for analytic functions in a certain class of analytic functions.

2.1 Introduction

For some $\phi \in \mathbf{R}$, a class $\mathcal{W}_\beta(\alpha,\gamma)$ where $\alpha \geq 0, \gamma \geq 0$ and $\beta < 1$ was given by Rosihan et al. (2012b) as:

$$\mathcal{W}_\beta(\alpha,\gamma) := \left\{ f \in \mathcal{A} : Re\ e^{i\phi} \left[(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta \right] > 0 \right\}$$

$$(z \in \mathbf{D})$$

and unified the following classes:

(i) $\alpha = 1, \gamma = 0,$

$$\mathcal{W}_\beta(1,0) \equiv \mathcal{P}(\beta) := \{ f \in \mathcal{A} : Re\ e^{i\phi}\, [f'(z) - \beta] > 0, z \in \mathbf{D} \}$$

(Fournier and Ruscheweyh, 1994).

(ii) $\alpha \geq 0, \gamma = 0,$

$$\mathcal{W}_\beta(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta) := \left\{ f \in \mathcal{A} : Re\ e^{i\phi} \left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta \right] > 0 \right\}$$

(Kim and Ronning, 2001).

(iii) $\alpha = 1 + 2\gamma, \gamma \geq 0,$

$$\mathcal{W}_\beta(1+2\gamma, \gamma) \equiv \mathcal{R}_\gamma(\beta) := \{ f \in \mathcal{A} : Re\ e^{i\phi} \left[f'(z) + \gamma z f''(z) - \beta \right] > 0 \}$$

(Ponnusamy and Ronning, 2008).

In 1994, for $f \in \mathcal{A}$, Fournier and Ruscheweyh introduced the integral operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt \tag{2.1}$$

where λ is a nonnegative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t) dt = 1$. The integral transform $V_\lambda(f)$ in the form (2.1) reduces to various well-known operators for specific choices of λ. For examples:

(i) $\lambda(t) := (1+c)t^c, c > -1$ gives the Bernardi integral operator.

(ii) $\lambda(t) := \frac{(a+1)^p}{\Gamma(p)} t^a \left(log\frac{1}{t} \right)^{p-1}, a > -1, p \geq 0$ gives the Komatu operator (Komatu, 1990),

$$K(z) = \frac{(a+1)^p}{\Gamma(p)} \int_0^1 t^{a-1} \left(log\frac{1}{t} \right)^{p-1} f(tz) dt.$$

In fact, for $p = 1$ the Komatu operator becomes the Bernardi operator.

(iii) The integral transform $V_\lambda(f)$ can be expressed as [see (Kiryakova et al., 1998), (Kim and Ronning, 2001)]:

$$
\begin{aligned}
V_\lambda(f) &= H_{(a,b,c)}(f)(z) \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \int_0^1 t^{b-1}(1-t)^{c-a-b} \\
&\qquad\qquad \times \;\; {}_2F_1(c-a, 1-a, c-a-b+1; 1-t)\frac{f(tz)}{t}dt
\end{aligned}
$$

$$(a > 0, b > 0 \;\text{ and }\; c > a + b - 1)$$

where

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1}(1-t)^{c-a-b}{}_2F_1(c-a, 1-a; c-a-b+1; 1-t)$$

and $H_{(a,b,c)}(f)$ is the Hohlov operator.

For operators other than (2.1), a long list of references can be found in the monograph of Miller and Mocanu (2000). Problems on the integral transform $V_\lambda(f)$ for special choices of λ have been recognized in a number of earlier papers by various authors [see e.g. (Rosihan, 1994), (Mocanu, 1986), (Nunokawa, 1991), (Nunokawa and Thomas, 1992), (Ponnusamy, 1994), (Raghavendar and Swaminathan, 2012), (Singh and Singh, 1989), (Singh and Singh, 1982) and (Singh and Singh, 1981)].

There are authors [see (Rosihan and Singh, 1995), (Anbu Durai and Parvatham, 2005) and (Balasubramanian et al., 2007b)] who studied the integral transform $\mathcal{V}_\lambda f(z)$ defined as:

$$\mathcal{V}_\lambda f(z) = z \int_0^1 \lambda(t) \frac{1-\rho tz}{1-tz} dt * f(z) \quad (\rho < 1).$$

The integral transform $\mathcal{V}_\lambda f(z)$ can be written as a generalisation of $V_\lambda f(z)$ as $\mathcal{V}_\lambda f(z) = \rho z + (1-\rho)V_\lambda f(z)$.

The theory of subordination has been used as the typical method in handling problems of starlikeness and convexity of the integral transforms $V_\lambda f$ and $\mathcal{V}_\lambda f$. However the method of differential subordination does not give sharp results. Since the duality technique seems to work best in the sense that it gives sharp estimates of the parameter β, starlikeness of the integral transform $V_\lambda(f)$ was first studied by Fournier and Ruscheweyh (1994) for a function f in the class defined by

$$\mathcal{P}(\beta) := \{g \in A_0 : \exists \phi \in R \text{ such that } Re\ e^{i\phi}\,[g(z) - \beta] > 0\}.$$

Then, these properties were extended to starlikeness of order $\delta, \delta \in (0, \frac{1}{2}]$ by Ponnusamy and Ronning (1997). Furthermore, the starlikeness of integral transform $V_\lambda(f)$ for a function f in the classes $\mathcal{P}_\alpha(\beta)$ and $\mathcal{R}_\gamma(\beta)$ respectively were studied in (Kim and Ronning, 2001) and (Ponnusamy and Ronning, 2008). The convexity of this integral transform over $\mathcal{P}(\beta)$ was investigated by Rosihan and Singh in 1995. For the class $\mathcal{P}_\alpha(\beta)$, the property of convexity was discussed by Choi, Kim and Saigo (2002) and was extended to convexity of order δ $(0 \leq \delta \leq \frac{1}{2})$ by Balasubramanian et al. (2007b). Since, there have been integral transforms being investigated using the duality technique [see (Aghalary et al., 2008), (Sokól, 2010)].

Rosihan et. al. studied the starlikeness and convexity of $V_\lambda(f)$ in (Rosihan et al., 2012b) and (Rosihan et al., 2012a) using duality concept for a function f in the class $\mathcal{W}_\beta(\alpha, \gamma)$. In this section, we continue the study of the integral transform of the form (2.1) to be starlike and convex of order δ for $f \in \mathcal{W}_\beta(\alpha, \gamma)$ where conditions on β and λ are determined for this to be true. Besides that, a sufficient condition for the integral transform $V_\lambda(f)$ to be starlike and convex of order δ is obtained . In addition, the starlikeness and convexity of order δ for the integral transform $\mathcal{V}_\lambda f$ are also investigated.

2.2 Starlikeness of order δ for integral transforms $V_\lambda(f)$

In Rosihan et al. (2012b), the following properties have been obtained. These properties are needed in the sequel.

Let $\mu \geq 0$ and $\nu \geq 0$ satisfying $\mu + \nu = \alpha - \gamma$ and $\mu\nu = \gamma$. When $\gamma = 0$, then μ is chosen to be 0, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (i) for $\gamma > 0$ then choosing $\mu = 1, \nu = \gamma$ and (ii) for $\gamma = 0$ then $\mu = 0, \nu = \alpha = 1$.

Let g be the solution of the initial-value problem satisfying $g(0) = 1$ and

$$\frac{d}{dt} t^{\frac{1}{\nu}} \frac{[1 + g(t)]}{2} = \frac{t^{\frac{1}{\nu}-1}}{\mu\nu} \int_0^1 \frac{s^{\frac{1}{\mu}-1}[1 - \delta(1 + st)]}{(1 - \delta)(1 + st)^2} ds \quad , \quad \gamma > 0 \tag{2.2}$$

It is easy to verify that the solution is given by

$$g(t) = \frac{2t^{-\frac{1}{\nu}}}{\mu\nu(1 - \delta)} \int_0^t \int_0^1 \frac{s^{\frac{1}{\mu}-1} w^{\frac{1}{\nu}-1}[1 - \delta(1 + sw)]}{(1 + sw)^2} dsdw \quad - 1 \tag{2.3}$$

and can be expressed in series form as follows:

$$g(t) = 1 + \frac{2}{(1 - \delta)} \sum_{n=1}^{\infty} \frac{(-1)^n (n + 1 - \delta) t^n}{[1 + n\mu][1 + n\nu]}. \tag{2.4}$$

Remark 2.1. *For $\gamma = 0, \alpha \geq 0$, (Balasubramaniam et al., 2004)*

$$\frac{d}{dt} t^{\frac{1}{\nu}} \frac{[1 + g(t)]}{2} = \frac{t^{\frac{1}{\alpha}-1}[1 - \delta(1 + t)]}{\alpha(1 - \delta)(1 + t)^2}$$

and the solution g is given as:

$$g(t) = \frac{2t^{-\frac{1}{\alpha}}}{\alpha(1 - \delta)} = \int_0^t u^{\frac{1}{\alpha}-1} \frac{1 - \delta(1 + \delta)}{(1 + u)^2} du - 1.$$

Using the function g given in (2.2) and condition β, we obtain the first result so that the integral transform $V_\lambda(f)$ is starlike of order δ.

Theorem 2.1. *Let $\mu \geq 0, \nu \geq 0$ and $\beta < 1$ be constants such that $\mu + \nu = \alpha - \gamma, \quad \mu\nu = \gamma$ and*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)g(t)dt \tag{2.5}$$

where g is defined in (2.2). *Assume the functions*

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{\frac{1}{\nu}}}dx \ , \quad \nu > 0 \tag{2.6}$$

and

$$\Pi_{\mu,\nu}(t) = \begin{cases} \int_t^1 \Lambda_\nu(x)x^{\frac{1}{\nu}-\frac{1}{\mu}-1}dx & \gamma > 0 \ (\mu > 0, \ \nu > 0) \\ \\ \Lambda_\alpha(t) \ , & \gamma = 0, \ (\mu = 0, \ \nu = \alpha > 0) \end{cases} \tag{2.7}$$

satisfying the conditions $t^{\frac{1}{\nu}}\Lambda_\nu(t) \to 0$ and $t^{\frac{1}{\mu}}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. For $f \in \mathcal{W}_\beta(\alpha,\gamma)$ and

$$h(z) = \frac{z\left(1+\frac{\epsilon+2\delta-1}{2-2\delta}z\right)}{(1-z)^2} \ , \quad |\epsilon| = 1, \tag{2.8}$$

the integral transform $F(z) = V_\lambda(f)(z) \in S^\star(\delta), 0 \leq \delta \leq \frac{1}{2}$ if and only if

$$\begin{cases} Re\int_0^1 \Pi_{\mu,\nu}(t)t^{\frac{1}{\mu}-1}\left[\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2}\right]dt \geq 0 \ , & \gamma > 0 \\ \\ Re\int_0^1 \Lambda_\alpha(t)t^{\frac{1}{\alpha}-1}\left[\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2}\right]dt \geq 0 \ , & \gamma = 0. \end{cases}$$

The value of β is sharp.

Proof. Since the case $\gamma = 0$ ($\mu = 0, \nu = \alpha$) corresponds to Theorem 1.2 in (Balasubramaniam et al., 2004), it is sufficient to consider the case $\gamma > 0$.

For $f \in \mathcal{W}_\beta(\alpha,\gamma)$, we define

$$H(z) = (1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z).$$

Then for some $\phi \in \mathbf{R}$ we have $Re\ e^{i\phi}\left[\frac{H(z)-\beta}{1-\beta}\right] > 0$ and

$$\frac{(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta}{1-\beta} \in \mathcal{P}$$

which implies there is a probability measure ζ on $[0, 2\pi]$ so that with the Herglotz representation, we can write

$$\frac{H(z)-\beta}{1-\beta} = 1 + 2\int_0^{2\pi}\sum_{n=1}^{\infty} z^n e^{-in\theta} d\zeta(\theta) \quad , \qquad \int_0^{2\pi} d\zeta(\theta) = 1.$$

$$H(z) = 1 + 2(1-\beta)\int_0^{2\pi}\sum_{n=1}^{\infty} z^n e^{-in\theta} d\zeta(\theta).$$

It can also be shown that $H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1}(n\nu+1)(n\mu+1)z^n$. Thus

$$1+\sum_{n=2}^{\infty} a_n[(n-1)\nu+1][(n-1)\mu+1]z^{n-1} = 1+2(1-\beta)\int_0^{2\pi}\sum_{n=2}^{\infty} z^{n-1}e^{-i(n-1)\theta}d\zeta(\theta).$$

Note that $a_n = \frac{2(1-\beta)}{[(n-1)\nu+1][(n-1)\mu+1]}\int_0^{2\pi} e^{-i(n-1)\theta}d\zeta(\theta)$ and hence

$$f(z) = z + \sum_{n=2}^{\infty}\frac{2(1-\beta)z^n}{[(n-1)\nu+1][(n-1)\mu+1]}\int_0^{2\pi} e^{-i(n-1)\theta}d\zeta(\theta) \tag{2.9}$$

$$\frac{f(z)}{z} = 1 + \sum_{n=2}^{\infty}\frac{2(1-\beta)z^{n-1}}{[(n-1)\nu+1][(n-1)\mu+1]}\int_0^{2\pi} e^{-i(n-1)\theta}d\zeta(\theta). \tag{2.10}$$

A well-known result from the theory of convolution in (Ruscheweyh, 1982):

$$F \in \mathcal{S}^\star(\delta) \quad \text{if and only if} \quad \frac{F(z)}{z} * \frac{h(z)}{z} \neq 0,\ z \in \mathbf{D}$$

where h is given by (2.8). Hence $F \in \mathcal{S}^\star(\delta)$ if and only if

$$0 \neq \frac{1}{z}\int_0^1 \lambda(t)\frac{f(tz)}{t}dt * \frac{h(z)}{z}$$

$$= \int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * \frac{h(z)}{z}$$

$$= \int_0^1 \frac{\lambda(t)}{1-tz}dt * 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)z^{n-1}}{[(n-1)\nu+1][(n-1)\mu+1]} \int_0^{2\pi} e^{-i(n-1)\theta} d\zeta(\theta) * \frac{h(z)}{z}$$

$$= \int_0^1 \lambda(t)\frac{h(tz)}{tz}dt * 1 + \sum_{n=1}^{\infty} \frac{(1-\beta)z^n}{(n\nu+1)(n\mu+1)} * 1 + 2\int_0^{2\pi} \sum_{n=1}^{\infty} z^n e^{-in\theta} d\zeta(\theta).$$

Since $1 + 2\int_0^{2\pi}\sum_{n=1}^{\infty} z^n e^{-in\theta}d\zeta(\theta) \in \mathcal{P}$, application of the Duality Theorem in Ruscheweyh (1982) gives

$$Re\left[\int_0^1 \lambda(t)\frac{h(tz)}{tz}dt * 1 + \sum_{n=1}^{\infty} \frac{(1-\beta)z^n}{(n\nu+1)(n\mu+1)}\right] > \frac{1}{2}$$

$$Re\left[\int_0^1 (1-\beta)\lambda(t)\frac{h(tz)}{tz}dt + \beta - \frac{1}{2} * 1 + \sum_{n=1}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)}\right] > 0.$$

Note that since $(1-\beta)\int_0^1 \lambda(t)[1-g(t)]dt = 1$, we obtain

$$Re \int_0^1 (1-\beta)\lambda(t)\frac{h(tz)}{tz}dt + \beta - \frac{(1-\beta)\int_0^1 \lambda(t)[1-g(t)]dt}{2} *$$
$$1 + \sum_{n=1}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} > 0$$

$$Re(1-\beta)\int_0^1 \lambda(t)\left[\frac{h(tz)}{tz}-\frac{1+g(t)}{2}\right]dt * 1+\sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}>0$$

$$Re\int_0^1 \lambda(t)\left[\sum_{n=0}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} * \frac{h(tz)}{tz}-\frac{1+g(t)}{2}\right]dt>0$$

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{d\eta d\xi}{1-\eta^\nu\xi^\mu z} * \frac{h(tz)}{tz}-\frac{1+g(t)}{2}\right]dt>0$$

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{h(tz\eta^\nu\xi^\mu)d\eta d\xi}{tz\eta^\nu\xi^\mu}-\frac{1+g(t)}{2}\right]dt>0.$$

Making the change of variables $u=\eta^\nu$ and $v=\xi^\mu$, the inequality reduces to

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{h(tzuv)}{tzuv}\frac{u^{\frac{1}{\nu}-1}v^{\frac{1}{\mu}-1}}{\mu\nu}dudv-\frac{1+g(t)}{2}\right]dt>0$$

and by letting $tu=w$, we have

$$Re\int_0^1 \frac{\lambda(t)}{t^{\frac{1}{\nu}}}\left[\int_0^t\int_0^1 \frac{h(wzv)}{wzv}w^{\frac{1}{\nu}-1}v^{\frac{1}{\mu}-1}dvdw-\frac{\mu\nu t^{\frac{1}{\nu}}[1+g(t)]}{2}\right]dt>0.$$

In view of the fact that $\Lambda_\nu'(t)=-\frac{\lambda(t)}{t^{\frac{1}{\nu}}}$, integrating by parts with respect to t and using (2.2) we obtain

$$Re\int_0^1 \Lambda_\nu(t)\left[\int_0^1 \frac{h(tzv)}{tzv}t^{\frac{1}{\nu}-1}v^{\frac{1}{\mu}-1}dv-t^{\frac{1}{\nu}-1}\int_0^1 \frac{s^{\frac{1}{\mu}-1}[1-\delta(1+st)]}{(1-\delta)(1+st)^2}ds\right]dt\geq 0.$$

The change of variables $tv=w$ and $st=\eta$ reduces the inequality to

$$Re\int_0^1 \Lambda_\nu(t)t^{\frac{1}{\nu}-\frac{1}{\mu}-1}\left[\int_0^t \frac{h(wz)}{wz}w^{\frac{1}{\mu}-1}dw-\int_0^t \frac{\eta^{\frac{1}{\mu}-1}[1-\delta(1+\eta)]}{(1-\delta)(1+\eta)^2}d\eta\right]dt\geq 0.$$

Integration by parts with respect to t yields

$$Re\int_0^1 \Pi_{\mu,\nu}(t)t^{\frac{1}{\mu}-1}\left[\frac{h(tz)}{tz}-\frac{1-\delta(1+t)}{(1-\delta)(1+t)^2}\right]dt \geq 0.$$

Next, we proceed to verify the sharpness. Suppose $f_0 \in \mathcal{W}_\beta(\alpha,\gamma)$ is the solution of

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)=\beta+(1-\beta)\frac{1+z}{1-z},$$

from (2.9), we obtain

$$f_0(z)=z+\sum_{n=2}^{\infty}\frac{2(1-\beta)z^n}{[(n-1)\mu+1][(n-1)\nu+1]}.$$

Thus

$$F(z)=V_\lambda(f)(z)=z+2(1-\beta)\sum_{n=2}^{\infty}\frac{\tau_n\ z^n}{[(n-1)\mu+1][(n-1)\nu+1]}$$

where $\tau_n=\int_0^1\lambda(t)t^{n-1}dt.$

Substituting (2.4) into (2.5), we have

$$\frac{\beta}{1-\beta}=-\int_0^1\lambda(t)\left\{1+\frac{2}{(1-\delta)}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(n-\delta)t^{n-1}}{[(n-1)\mu+1][(n-1)\nu+1]}\right\}dt$$

$$\frac{\beta}{1-\beta}=-1-\frac{2}{(1-\delta)}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(n-\delta)\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]}$$

$$\frac{\delta-1}{2(1-\beta)}=\sum_{n=2}^{\infty}\frac{n(-1)^{n-1}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]}-\delta\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]}.$$

Hence

$$\sum_{n=2}^{\infty} \frac{n(-1)^{n-1}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]} = \frac{\delta-1}{2(1-\beta)} + \delta \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]}. \tag{2.11}$$

We see that for $z=-1$,

$$F(-1) = -1 + 2(1-\beta) \sum_{n=2}^{\infty} \frac{(-1)^{n}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]}$$

and

$$F'(-1) = 1 + 2(1-\beta) \sum_{n=2}^{\infty} \frac{n\tau_n \ (-1)^{n-1}}{[(n-1)\mu+1][(n-1)\nu+1]}. \tag{2.12}$$

Substituting RHS of (2.11) into (2.12), we have

$$F'(-1) = 1 + 2(1-\beta) \left[\frac{\delta-1}{2(1-\beta)} + \delta \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]} \right]$$

$$= \delta - 2\delta(1-\beta) \sum_{n=2}^{\infty} \frac{(-1)^{n}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]}$$

$$= -\delta \left[-1 + 2(1-\beta) \sum_{n=2}^{\infty} \frac{(-1)^{n}\tau_n}{[(n-1)\mu+1][(n-1)\nu+1]} \right]$$

$$= -\delta F(-1)$$

therefore $\frac{zF'(z)}{F(z)}$ at $z = -1$ equals δ. This implies that the result is sharp for the order of starlikeness. This completes the proof. $\square$

In the next result, we determine a sufficient condition for $V_\lambda(f)$ to be a starlike function of order δ.

Theorem 2.2. *Let $\Pi_{\mu,\nu}$ and Λ_ν be as given in Theorem 2.1. Assume that both $\Pi_{\mu,\nu}$ and Λ_ν are integrable on $[0,1]$ and positive on $(0,1)$. Furthermore, for $\mu \geq 1$, the function*

$$\frac{\Pi_{\mu,\nu}(t)}{(1+t)(1-t)^{1+2\delta}} \qquad (0 \leq \delta \leq \frac{1}{2}) \tag{2.13}$$

is decreasing on $(0,1)$. If β satisfies (2.5) and $f \in \mathcal{W}_\beta(\alpha,\gamma)$ then $V_\lambda(f) \in S^\star(\delta)$.

Proof. To verify the theorem above, we use a result earlier obtained by Ponnusamy and Ronning (1997, p. 281). This together with (2.13) and the fact that $t^{\frac{1}{\mu}-1}$ is decreasing on $(0,1)$ for $\mu \geq 1$, suggest that

$$Re \int_0^1 \Pi_{\mu,\nu}(t) t^{\frac{1}{\mu}-1} \left[\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2} \right] dt \geq 0.$$

The desired conclusion now follows from Theorem 2.1. $\square$

Our next result is an application of Theorem 2.2. First, we establish that the function

$$p(t) = \frac{\Pi_{\mu,\nu}(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing in the interval $(0,1)$. Note that $p'(t) \leq 0$ for $t \in (0,1)$ is equivalent to the inequality

$$q(t) = \Pi_{\mu,\nu}(t) - \frac{(1-t^2)\Lambda_\nu(t)t^{\frac{1}{\nu}-\frac{1}{\mu}-1}}{2[t+\delta(1+t)]} \leq 0$$

and $q(1) = 0$.

We note that if $q(t)$ is increasing on $(0,1)$ then $p(t)$ is decreasing on $(0,1)$. Therefore, we obtain

$$q'(t) = -\frac{t^{\frac{1}{\nu}-\frac{1}{\mu}-1}(1+t)}{2[t+\delta(1+t)]^2}\Delta(t)$$

where

$$\Delta(t) = -[t+\delta(1+t)](1-t)\lambda(t)t^{-\frac{1}{\nu}}$$

$$+\Lambda_\nu(t)\left[[t+\delta(1+t)]\frac{(1-t)}{t}\left(\frac{1}{\nu}-\frac{1}{\mu}-1\right) - [1-t-\delta(1+t)](1+2\delta)\right]$$

Our aim is to show that if $\Delta(t) \leq 0$ then $q'(t) \geq 0$. Let

$$A(t) = \lambda(t)t^{-\frac{1}{\nu}}$$

$$X(t) = [t+\delta(1+t)](1-t)$$

$$Y(t) = X(t)\left(\frac{1}{\nu}-\frac{1}{\mu}-1\right) + Z(t)$$

$$Z(t) = -t[1-t-\delta(1+t)](1+2\delta);$$

then $\Delta(t) = -A(t)X(t) + \frac{Y(t)}{t}\int_t^1 A(y)dy$.

(i) If $Y(t) \leq 0$ on $(0,1)$ then $\Delta(t) \leq 0$ on $(0,1)$ thus the result follows.

(ii) If $Y(t) \geq 0$, $\Delta(t) = \frac{Y(t)}{t}B(t)$ where

$$B(t) = -A(t)X(t)\frac{t}{Y(t)} + \int_t^1 A(y)dy \quad \text{and} \quad B(1) = 0.$$

In this case, we propose to show $\Delta(t) \leq 0$. It suffices to show that $B(t)$ is an increasing function of t. Simple calculation shows that

$$B'(t) = -\lambda(t)t^{-\frac{1}{\nu}}\left\{\left[-\frac{1}{\nu} + \frac{t\lambda'(t)}{\lambda(t)}\right]\frac{X(t)}{Y(t)} + \left[\frac{tX(t)}{Y(t)}\right]' + 1\right\}$$

$B'(t) \geq 0$ means

$$\left[-\frac{1}{\nu} + \frac{t\lambda'(t)}{\lambda(t)}\right]\frac{X(t)}{Y(t)} + \left[\frac{tX(t)}{Y(t)}\right]' + 1 \leq 0$$

hence the following expression is obtained:

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \frac{1}{\nu} - \frac{Y(t)}{X(t)}\left(\left[\frac{tX(t)}{Y(t)}\right]' + 1\right). \tag{2.14}$$

Now, we determine conditions on ν and μ using the inequality (2.14) so that

$\Delta(t) \le 0$ for $0 \le \delta \le \frac{1}{2}$. In the first case, considering $\delta = 0$, we have

$$X(t) = t(1-t)$$

$$Y(t) = t(1-t)\left(\frac{1}{\nu} - \frac{1}{\mu} - 2\right).$$

$Y(t) \le 0$ if $\frac{1}{\nu} - \frac{1}{\mu} - 2 \le 0$ or $\nu \ge \frac{\mu}{2\mu+1}$.

If $0 \le \nu < \frac{\mu}{2\mu+1}$ then $Y(t) > 0$ on $(0,1)$ and $\frac{t\lambda'(t)}{\lambda(t)} \le 1 + \frac{1}{\mu}$ which leads to the result of Rosihan et al. (2012b).

Next, for $0 < \delta \le \frac{1}{2}$, we derive the inequality (2.14) as

$$Y^2(t)\left[\frac{Y(t)\left(tX'(t) + X(t)\right) - tX(t)Y'(t)}{Y^2(t)} + 1\right] \le \left(\frac{1}{\nu} - \frac{t\lambda'(t)}{\lambda(t)}\right) X(t)Y(t)$$

and

$$Y(t)\left[tX'(t) + X(t)\right] - tX(t)Y'(t) + Y^2(t) \le \left(\frac{1}{\nu} - \frac{t\lambda'(t)}{\lambda(t)}\right) X(t)Y(t). \quad (2.15)$$

Since $Y(t) = X(t)\left(\frac{1}{\nu} - \frac{1}{\mu} - 1\right) + Z(t)$, substituting $Y(t)$ into (2.15) we have

$$\begin{aligned}&\left(\frac{1}{\nu} - \frac{1}{\mu} - 1\right) X(t)\left[X(t) + Z(t)\right] \\ &\quad - \left[\left(1 + \frac{1}{\mu}\right) - \frac{t\lambda'(t)}{\lambda(t)}\right] X(t)\left[X(t)\left(\frac{1}{\nu} - \frac{1}{\mu} - 1\right) + Z(t)\right] \\ &\quad \le tX(t)Z'(t) - Z(t)[tX(t)]' - Z^2(t). \end{aligned} \quad (2.16)$$

Set $D(t) = t(1+\delta) - (1-\delta) = [t + \delta(1+t) - 1]$ and write

$$X(t) = (1-t)[D(t) + 1]$$

$$Z(t) = tD(t)(1 + 2\delta).$$

Simple computation shows that $tX(t)Z'(t) - Z(t)[tX(t)]' - Z^2(t) = 2\delta t^2(1 + 2\delta)(1 - D^2(t))$ is nonnegative on $(0, 1)$. Similarly, it can be verified that $D^2(t) \leq 1$ and also that $X(t)$ and $[X(t) + Z(t)]$ are nonnegative on the interval $(0, 1)$.

Thus, the inequality (2.16) holds if $\nu \geq \frac{\mu}{\mu+1}$ where $Y(t) > 0$. With the restriction $\nu \geq \frac{\mu}{\mu+1}$ for $0 < \delta \leq \frac{1}{2}$, we obtain the condition $\frac{t\lambda'(t)}{\lambda(t)} \leq 1 + \frac{1}{\mu}$.

Remark 2.2. *The problem concerning the condition on λ for $0 < \nu < \frac{\mu}{\mu+1}$ remains open since the calculations become more complicated.*

Theorem 2.3 states the conclusion of the above discussion and gives the following result.

Theorem 2.3. *Let λ be a nonnegative real-valued integrable function on $[0, 1]$. Let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ with $\nu \geq \frac{\mu}{\mu+1}$ and $\beta < 1$ satisfying*

$$\frac{\beta}{1 - \beta} = -\int_0^1 \lambda(t) g(t) dt$$

where g is defined by (2.2) and $\delta \in (0, \frac{1}{2}]$. If λ satisfies

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 1 + \frac{1}{\mu} \qquad (\mu \geq 1, \gamma > 0) \tag{2.17}$$

then $F(z) = V_\lambda(f)(z) \in S^\star(\delta)$.

Remark 2.3. *Taking $\alpha = 1 + 2\gamma, \gamma > 0$ and $\mu = 1$ in Theorem 2.3 yields Theorem 3.1 in (Balasubramaniam et al., 2007a) and for $\mu < 1$, the conditions obtained will be complicated.*

2.3 Applications to certain integral transforms

In this subsection, we present results for various cases of λ. As examples, we consider three operators defined by Bernardi (1969), Komatu (1990) and Hohlov (1978).

i) Bernardi integral operator

Theorem 2.4. *Let* $\nu \geq \frac{\mu}{\mu+1}, \delta \in (0, \frac{1}{2}]$. *Let* $c > -1$ *and* $\beta < 1$ *satisfy*

$$\frac{\beta}{1-\beta} = -(c+1)\int_0^1 t^c g(t)dt$$

where g *is defined by* (2.3). *Let* $f \in \mathcal{W}_\beta(\alpha, \gamma)$ *and*

$$F_c[f(z)] = V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt;$$

if $c \leq 1 + \frac{1}{\mu}$ $(\mu \geq 1, \gamma > 0)$ *then* $F_c f \in S^\star(\delta)$.

Proof. Since $\lambda(t) := (1+c)t^c$, simple computation gives $\frac{t\lambda'(t)}{\lambda(t)} = c$. By hypothesis of Theorem 2.4, $c \leq 1 + \frac{1}{\mu}$ thus the result follows from Theorem 2.3. $\square$

Remark 2.4. *When* $\alpha = 1 + 2\gamma, \gamma > 0, \mu = 1$, *Theorem 2.4 yields [Corollary 4.1, Balasubramaniam et al. (2007a)]*

ii) Komatu operator

Theorem 2.5. *Let* $-1 < a,\ p \geq 1,\ \alpha > 0,\ \nu \geq \frac{\mu}{\mu+1}$ *and* $\beta < 1$ *satisfy*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)}\int_0^1 t^a \left(log\frac{1}{t}\right)^{p-1} g(t)d(t)$$

where g *is given by* (2.3). *Let* $f \in \mathcal{W}_\beta(\alpha, \gamma)$ *and* $\delta \in (0, \frac{1}{2}]$. *Then the function* *defined by*

$$K(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(log\frac{1}{t} \right)^{p-1} t^{a-1} f(tz)dt$$

belongs to $S^\star(\delta)$ *if* $a \leq 1 + \frac{1}{\mu}$ $(\mu \geq 1,\ \gamma > 0)$.

Proof. To prove Theorem 2.5, it suffices to verify inequality (2.17). Since $\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{p-1}{log\frac{1}{t}}$ the inequality (2.17) is equivalent to

$$a - \frac{p-1}{log\frac{1}{t}} \leq 1 + \frac{1}{\mu}$$

which implies

$$0 \leq \left(1 + \frac{1}{\mu} - a\right) + \frac{p-1}{log\frac{1}{t}}. \tag{2.18}$$

Since $p \geq 1$ and $log\left(\frac{1}{t}\right) > 0$ for $t \in (0,1)$, the inequality (2.18) holds by the hypothesis. $\square$

Now, define Φ as $\Phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n,\ b_n \geq 0$ for $n \geq 1$ with

$$\lambda(t) = \kappa t^{b-1}(1-t)^{c-a-b}\Phi(1-t)$$

where κ is a constant chosen such that $\int_0^1 \lambda(t)dt = 1$. Finally, we give our last theorem.

iii) General result for certain operators

Theorem 2.6. *Let* $a, b, c, \alpha, \gamma > 0, \nu \geq \frac{\mu}{\mu+1}$ *and* $\beta < 1$ *satisfy*

$$\frac{\beta}{1-\beta} = -\kappa \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)g(t)dt$$

where κ is a constant such that $\kappa\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)dt = 1$ and g is given by (2.3). For $\delta \in (0, \frac{1}{2}]$, if $f \in \mathcal{W}_\beta(\alpha,\gamma)$ then the function

$$V_\lambda(f)(z) = \kappa \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}dt$$

belongs to $S^\star(\delta)$ whenever a, b, c are related by the conditions $b \leq 2+\frac{1}{\mu}$ and $c \geq a+b$.

Proof. It can be verified that

$$\frac{t\lambda'(t)}{\lambda(t)} = -t\frac{\Phi'(1-t)}{\Phi(1-t)} - \frac{t(c-a-b)}{1-t} + b - 1.$$

Thus, the inequality (2.17) is equivalent to

$$1 + \frac{1}{\mu} - b + 1 + \frac{t(c-a-b)}{1-t} + t\frac{\Phi'(1-t)}{\Phi(1-t)} \geq 0$$

$$\left[\left(2+\frac{1}{\mu}\right) - b\right] + \frac{t(c-a-b)}{1-t} + t\frac{\Phi'(1-t)}{\Phi(1-t)} \geq 0$$

and holds true using the hypothesis. □

Remark 2.5. *For a special case by choosing $\Phi(1-t) = F(c-a, 1-a, c-a-b+1; 1-t)$ and $\kappa = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}$ where $c+1-a-b > 0$, the integral operator $V_\lambda(f)(z)$ reduces to the Hohlov operator.*

2.4 Convexity of order δ for $V_\lambda f$

We start our discussion by considering the function q which is the solution of the initial-value problem satisfying $q(0) = 1$ and

$$\frac{d}{dt}t^{\frac{1}{\nu}}q(t) = \frac{t^{\frac{1}{\nu}}-1}{\mu\nu}\int_0^1 s^{\frac{1}{\mu}-1}\frac{[(1-\delta)-(1+\delta)st]}{(1-\delta)(1+st)^3}ds\ ,\ \gamma \geq 0. \tag{2.19}$$

It can be verified that the solution q is given by

$$q(t) = \frac{t^{-\frac{1}{\nu}}}{\mu\nu(1-\delta)} \int_0^t \int_0^1 s^{\frac{1}{\mu}-1} w^{\frac{1}{\nu}-1} \frac{[(1-\delta)-(1+\delta)sw)]}{(1+sw)^3} ds\, dw. \tag{2.20}$$

Our next result determines condition on the convexity of $V_\lambda f$.

Theorem 2.7. *Let $\mu \geq 0, \nu \geq 0$ and $\beta < 1$ be constants such that $\mu + \nu = \alpha - \gamma, \quad \mu\nu = \gamma$ and*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t) q(t) dt \tag{2.21}$$

where q is defined in (2.20). Assume the functions $\Lambda_\nu(t)$ and $\Pi_{\mu,\nu}(t)$ which are respectively defined in (2.6) and (2.7) and satisfy the conditions $t^{\frac{1}{\nu}}\Lambda_\nu(t) \to 0$ and $t^{\frac{1}{\mu}}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. For $f \in \mathcal{W}_\beta(\alpha,\gamma)$ and h is given by (2.8), the integral transform $F(z) = V_\lambda f(z) \in \mathcal{C}(\delta), \quad 0 \leq \delta \leq \frac{1}{2}$ if and only if

$$Re \int_0^1 \Pi_{\mu,\nu}(t) t^{\frac{1}{\mu}-1} \left[h'(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt \geq 0 \quad , \quad \gamma \geq 0.$$

Proof. The case $\gamma = 0$ ($\mu = 0, \nu = \alpha$) has been discussed by Balasubramanian et al. (2007b). Thus, it is sufficient to consider the case $\gamma > 0$.

It is well known that (Robertson, 1936), $F \in \mathcal{C}(\delta)$ if and only if $zF' \in \mathcal{S}^\star(\delta)$. Thus

$$0 \neq \frac{1}{z} \left[zF'(z) * h(z) \right]$$

where h is defined by (2.8). Let $f \in \mathcal{W}_\beta(\alpha,\gamma)$. Then using the theory of convolution

(Ruscheweyh, 1982) and $\frac{f(z)}{z}$ is given by (2.10),

$$0 \neq \frac{1}{z}\left[F(z) * zh'(z)\right]$$

$$= \frac{1}{z}\left[\int_0^1 \lambda(t)\frac{f(tz)}{t}dt * zh'(z)\right]$$

$$= \int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * h'(z)$$

$$= \int_0^1 \frac{\lambda(t)}{1-tz}dt * 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)z^{n-1}}{[(n-1)\nu+1][(n-1)\mu+1]} \int_0^{2\pi} e^{-i(n-1)\theta} d\zeta(\theta) * h'(z)$$

$$= \int_0^1 \lambda(t)h'(tz)dt * 1 + \sum_{n=1}^{\infty} \frac{(1-\beta)z^n}{(n\nu+1)(n\mu+1)} * 1 + 2\int_0^{2\pi} \sum_{n=1}^{\infty} z^n e^{-in\theta} d\zeta(\theta).$$

Since $1+2\int_0^{2\pi}\sum_{n=1}^{\infty} z^n e^{-in\theta}d\zeta(\theta) \in \mathcal{P}$, application of the Duality Theorem (Ruscheweyh, 1982) gives

$$Re\left[\int_0^1 \lambda(t)h'(tz)dt * 1 + \sum_{n=1}^{\infty}\frac{(1-\beta)z^n}{(n\nu+1)(n\mu+1)}\right] > \frac{1}{2}$$

$$Re\left[\left\{\int_0^1 (1-\beta)\lambda(t)h'(tz)dt + \beta - \frac{1}{2}\right\} * \left\{1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}\right\}\right] > 0$$

$$Re\left[\left\{\int_0^1 \lambda(t)h'(tz)dt + \frac{\beta-\frac{1}{2}}{1-\beta}\right\} * \left\{1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}\right\}\right] > 0$$

$$Re\left[\left\{\int_0^1 \lambda(t)h'(tz)dt - \int_0^1 \lambda(t)q(t)dt\right\} * \left\{1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}\right\}\right] > 0$$

$$Re\int_0^1 \lambda(t)\left[\sum_{n=0}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} * h'(z) - q(t)\right]dt > 0$$

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{d\eta d\xi}{1-\eta^\nu\xi^\mu z} * h'(z) - q(t)\right]dt > 0$$

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 h'(tz\eta^\nu\xi^\mu)d\eta d\xi - q(t)\right]dt > 0.$$

Making the change of variables $u = \eta^\nu$ and $v = \xi^\mu$, the inequality reduces to

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 h'(tzuv)\frac{u^{\frac{1}{\nu}-1}v^{\frac{1}{\mu}-1}}{\mu\nu}dudv - q(t)\right]dt > 0$$

and by letting $tu = w$, we have

$$Re \int_0^1 \frac{\lambda(t)}{t^{\frac{1}{\nu}}} \left[\int_0^t \int_0^1 h'(wzv) w^{\frac{1}{\nu}-1} v^{\frac{1}{\mu}-1} dv dw - \mu\nu t^{\frac{1}{\nu}} q(t) \right] dt > 0.$$

Using integration by parts with respect to t and (2.19) we have

$$Re \int_0^1 \Lambda_\nu(t) \left[\int_0^1 h'(tzv) t^{\frac{1}{\nu}-1} v^{\frac{1}{\mu}-1} dv - t^{\frac{1}{\nu}-1} \int_0^1 \frac{s^{\frac{1}{\mu}-1}[(1-\delta)-(1+\delta)st]}{(1-\delta)(1+st)^3} ds \right] dt \geq 0.$$

The change of variables $tv = w$ and $st = \eta$ reduces the inequality to

$$Re \int_0^1 \Lambda_\nu(t) t^{\frac{1}{\nu}-\frac{1}{\mu}-1} \left[\int_0^t h'(wz) w^{\frac{1}{\mu}-1} dw - \int_0^t \frac{\eta^{\frac{1}{\mu}-1}[(1-\delta)-(1+\delta)\eta]}{(1-\delta)(1+\eta)^3} d\eta \right] dt \geq 0.$$

Integration by parts with respect to t yields

$$Re \int_0^1 \Pi_{\mu,\nu}(t) t^{\frac{1}{\mu}-1} \left[h'(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt \geq 0.$$

$\square$

Before we proceed to the next result, we provide the following lemma given by Ponnusamy and Ronning in 1997 [Theorem 2.3, p. 268].

Lemma 2.1. *Assume* Λ *is integrable on* $[0,1]$ *and positive on* $(0,1)$. *Assume further that*

$$\frac{\Lambda(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing on $(0,1)$*. Then* $L_\Lambda(\mathcal{K}_\delta)=0$ *for* $0\le\delta\le\frac{1}{2}$ *where*

$$L_\Lambda(h)=inf\int_0^1\Lambda(t)\left[Re\frac{h(tz)}{tz}-\frac{1-\delta(1+t)}{(1-\delta)(1+t)^2}\right]dt\quad(z\in\boldsymbol{D})$$

and $L_\Lambda(\mathcal{K}_\delta)=inf\ L_\Lambda(h)\quad(h\in\mathcal{K}_\delta)$*.*

Theorem 2.8. *Let* $\Pi_{\mu,\nu}$ *and* Λ_ν *be as given in Theorem 2.1 where both* $\Pi_{\mu,\nu}$ *and* Λ_ν *are integrable on* $[0,1]$ *and positive on* $(0,1)$*. Furthermore, for* $\mu\ge 1$*, the function*

$$\frac{\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)+\Lambda_\nu(t)t^{\frac{1}{\nu}-\frac{1}{\mu}}}{(1+t)(1-t)^{1+2\delta}}\qquad(0\le\delta\le\frac{1}{2})$$

is decreasing on $(0,1)$*. If* β *satisfies* (2.21) *and* $f\in\mathcal{W}_\beta(\alpha,\gamma)$ *then* $V_\lambda f\in\mathcal{C}(\delta)$*.*

Proof. It can be verified that

$$Re\int_0^1\Pi_{\mu,\nu}(t)t^{\frac{1}{\mu}-1}\left[h'(tz)-\frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3}\right]dt$$

$$=Re\int_0^1\Pi_{\mu,\nu}(t)t^{\frac{1}{\mu}-1}\frac{d}{dt}\left[\frac{h(tz)}{z}-\frac{t[1-\delta(1+t)]}{(1-\delta)(1+t)^2}\right]dt$$

then integration by parts gives

$$Re\int_0^1\Pi_{\mu,\nu}(t)t^{\frac{1}{\mu}-1}\frac{d}{dt}\left[\frac{h(tz)}{z}-\frac{t[1-\delta(1+t)]}{(1-\delta)(1+t)^2}\right]dt$$

$$=Re\int_0^1\Pi_{\mu,\nu}(t)t^{\frac{1}{\mu}-1}\left\{\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)+\Lambda_\nu(t)t^{\frac{1}{\nu}-\frac{1}{\mu}}\right\}\left[\frac{h(tz)}{tz}-\frac{1-\delta(1+t)}{(1-\delta)(1+t)^2}\right].$$

Since $t^{\frac{1}{\mu}-1}$ is decreasing on $(0,1)$ for $\mu\ge1$, together with the hypotheses of Theorem 2.8 and Lemma 2.1 give

$$Re\int_0^1\Pi_{\mu,\nu}(t)t^{\frac{1}{\mu}-1}\left[h'(tz)-\frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3}\right]dt\ge0.$$

Hence, the conclusion follows from Theorem 2.7. □

2.5 Starlikeness and convexity of order δ for integral transform $\mathcal{V}_\lambda f$

First, we determine the starlikeness of order δ for the integral transform $\mathcal{V}_\lambda f$ where $f \in \mathcal{W}_\beta(\alpha, \gamma)$.

Theorem 2.9. *Let $\delta \in [0, \frac{1}{2}]$, $\rho < 1$ and $\beta < 1$ be constants. Let*

$$F(z) = \mathcal{V}_\lambda f(z) = z \int_0^1 \lambda(t) \frac{1 - \rho t z}{1 - t z} dt * f(z)$$

and

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t) \left(\frac{1 - g(t)}{2} \right) dt$$

where g is defined by (2.3)*. For $f \in \mathcal{W}_\beta(\alpha, \gamma)$, $\mathcal{V}_\lambda f \in S^\star(\delta)$ if and only if*

$$Re \int_0^1 \Pi_{\mu,\nu}(t) t^{\frac{1}{\mu}-1} \left[\frac{h(tz)}{tz} - \frac{1 - \delta(1+t)}{(1-\delta)(1+t)^2} \right] dt \geq 0 \quad , \quad \gamma > 0.$$

Proof. By the theory of convolution given by Ruscheweyh (1982), together with (2.10) we have

$$F(z) \in S^\star(\delta) \Leftrightarrow \frac{F(z)}{z} * \frac{h(z)}{z} \neq 0,$$

where h is given in (2.8). Thus

$$0 \neq \int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * \frac{f(z)}{z} * \frac{h(z)}{z}$$

$$= \int_0^1 \lambda(t)(1-\rho tz)\frac{h(tz)}{tz}dt *$$
$$1+\sum_{n=2}^{\infty}\frac{2(1-\beta)z^{n-1}}{[(n-1)\nu+1][(n-1)\mu+1]}\int_0^{2\pi} e^{-i(n-1)\theta}d\zeta(\theta)$$

$$= \int_0^1 \lambda(t)(1-\beta)(1-\rho tz)\frac{h(tz)}{tz}dt + \beta * 1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}$$
$$* 1 + 2\int_0^{2\pi}\sum_{n=1}^{\infty} z^n e^{-in\theta}d\zeta(\theta)$$

$$= \int_0^1 \lambda(t)\left\{(1-\beta)(1-\rho)\frac{h(tz)}{tz} + 1 - (1-\beta)(1-\rho)\right\}dt$$
$$* 1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} * 1 + 2\int_0^{2\pi}\sum_{n=1}^{\infty} z^n e^{-in\theta}d\zeta(\theta).$$

Application of the Herglotz formula and the Duality Theorem give

$$Re \quad \int_0^1 \lambda(t)\left\{(1-\beta)(1-\rho)\frac{h(tz)}{tz} + 1 - (1-\beta)(1-\rho)\right\}dt *$$
$$1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} > \frac{1}{2}$$

$$Re \quad \int_0^1 \lambda(t)(1-\beta)(1-\rho)\left\{\frac{h(tz)}{tz} + \frac{1}{(1-\beta)(1-\rho)} - 1 - \frac{1}{2(1-\beta)(1-\rho)}\right\}dt$$
$$* 1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} > 0$$

$$Re \quad \int_0^1 \lambda(t)\left\{\frac{h(tz)}{tz} - \left(1 - \frac{1}{2(1-\beta)(1-\rho)}\right)\right\}dt *$$
$$1 + \sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} > 0$$

$$Re \quad \int_0^1 \lambda(t)\left\{\frac{h(tz)}{tz}-\left(1-\frac{\int_0^1 \lambda(t)[1-g(t)]dt}{2}\right)\right\} dt*$$

$$1+\sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}>0$$

$$Re\left[\int_0^1 \lambda(t)\left\{\frac{h(tz)}{tz}-\left(\frac{1+g(t)}{2}\right)\right\} dt*1+\sum_{n=1}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}\right]>0$$

$$Re\int_0^1 \lambda(t)\left[\sum_{n=0}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)}*\frac{h(tz)}{tz}-\frac{1+g(t)}{2}\right]dt>0$$

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{d\eta d\xi}{1-\eta^\nu\xi^\mu z}*\frac{h(tz)}{tz}-\frac{1+g(t)}{2}\right]dt>0$$

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{h(tz\eta^\nu\xi^\mu)d\eta d\xi}{tz\eta^\nu\xi^\mu}-\frac{1+g(t)}{2}\right]dt>0.$$

By letting $u=\eta^\nu$ and $v=\xi^\mu$, the inequality is reduced to

$$Re\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{h(tzuv)}{tzuv}\frac{u^{\frac{1}{\nu}-1}v^{\frac{1}{\mu}-1}}{\mu\nu}dudv-\frac{1+g(t)}{2}\right]dt>0$$

and for $tu=w$, the inequality becomes

$$Re\int_0^1 \frac{\lambda(t)}{t^{\frac{1}{\nu}}}\left[\int_0^t\int_0^1 \frac{h(wzv)}{wzv}w^{\frac{1}{\nu}-1}v^{\frac{1}{\mu}-1}dvdw-\frac{\mu\nu t^{\frac{1}{\nu}}[1+g(t)]}{2}\right]dt>0.$$

Using integration by parts with respect to t and by substituting (2.2), we obtain

$$Re\int_0^1 \Lambda_\nu(t)\left[\int_0^1 \frac{h(tzv)}{tzv}t^{\frac{1}{\nu}-1}v^{\frac{1}{\mu}-1}dv-t^{\frac{1}{\nu}-1}\int_0^1 \frac{s^{\frac{1}{\mu}-1}[1-\delta(1+st)]}{(1-\delta)(1+st)^2}ds\right]dt\geq 0.$$

Making the change of variables $tv = w,\ st = \eta$ and integrating by parts with respect to t we have

$$Re \int_0^1 \Lambda_\nu(t) t^{\frac{1}{\nu}-\frac{1}{\mu}-1} \left[\int_0^t \frac{h(wz)}{wz} w^{\frac{1}{\mu}-1} dw - \int_0^t \frac{\eta^{\frac{1}{\mu}-1}[1-\delta(1+\eta)]}{(1-\delta)(1+\eta)^2} d\eta \right] dt \geq 0.$$

and

$$Re \int_0^1 \Pi_{\mu,\nu}(t) t^{\frac{1}{\mu}-1} \left[\frac{h(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2} \right] dt \geq 0.$$

□

Next, let $f \in \mathcal{W}_\beta(\alpha,\gamma)$ then we obtain the convexity of order δ for $\mathcal{V}_\lambda f$.

Theorem 2.10. *Let $\delta \in [0, \frac{1}{2}],\ \rho < 1$ and $\beta < 1$ be constants. Let*

$$F(z) = \mathcal{V}_\lambda f(z) = z \int_0^1 \lambda(t) \frac{1-\rho t z}{1-tz} dt * f(z)$$

and

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\left[1 - q(t)\right] dt \tag{2.22}$$

where q is defined by (2.20)*. For $f \in \mathcal{W}_\beta(\alpha,\gamma),\ \mathcal{V}_\lambda f \in \mathcal{C}(\delta)$ if and only if*

$$Re \int_0^1 \Pi_{\mu,\nu}(t) t^{\frac{1}{\mu}-1} \left[h'(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt \geq 0 \quad , \quad \gamma > 0.$$

Proof. The idea of the proof is similar as in Theorem 2.7. Using $F(z) = \mathcal{V}_\lambda f(z)$, it suffices to verify

$$Re \left[\int_0^1 \lambda(t) \left\{ h'(tz) - \left(1 - \frac{1}{2(1-\beta)(1-\rho)} \right) \right\} dt * 1 + \sum_{n=1}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} \right] > 0$$

Then, by substituting (2.22) together with the change of some variables and using integration by parts, we obtain the condition

$$Re \int_0^1 \Pi_{\mu,\nu}(t) t^{\frac{1}{\mu}-1} \left[h'(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt \geq 0 \quad , \quad \gamma > 0$$

as a result. □

CHAPTER 3

PROPERTIES OF FUNCTIONS FOR CLASSES DEFINED BY SUBORDINATION

In this chapter, properties of functions using differential subordination method associated with certain classes are presented. Ali et al. (2012c) studied conditions on β such that $1 + \beta z p'(z) \prec \sqrt{1+z}$ implies $p(z) \prec \sqrt{1+z}$. Similar results are also obtained for expressions of the form $1 + \frac{\beta z p'(z)}{p(z)}$ and $1 + \frac{\beta z p'(z)}{p^2(z)}$. We determine these properties for classes involving the Janowski starlike functions and the Cassini curve in obtaining the condition on β. Furthermore, for $f \in \mathcal{A}$ to be in the class of Sok*ó*l-Stankiewicz starlike functions if f satisfying $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in \mathbf{D})$. Based on this condition, we extend to α-convex and convex classes. Some applications for these classes are considered. In addition, some inclusion results for classes defined by the Dziok-Srivastava operator and the generalised multiplier transformations are determined using the Briot-Bouquet differential subordinations.

3.1 Properties of Janowski starlike functions

Quite a number of authors [see Nunokawa et al. (1997), Nunokawa et al. (2003), Obradovic and Owa (1988), Ravichandran and Maslina (2003) and Ravichandran et al. (2005)] have intensively studied properties of functions involving the expression $\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)\left(\frac{f(z)}{zf'(z)}\right)\right]$ which can also be expressed as $\left[\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right]$. The properties of starlikeness for functions in the class

$$G_b := \left\{ f \in \mathcal{A} : \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < b\ , \quad 0 < b \leq 1,\ z \in \mathbf{D} \right\}$$

has been obtained by Silverman (1999). Obradovič and Tuneski (2000) showed that $G_b \subset S^\star[0,-b] \subset S^\star\left(\frac{2}{1+\sqrt{1+8b}}\right)$ and the inclusion conditions of $G_b \subset S^\star[A,B]$ was obtained by Tuneski (2003). Rosihan et al. (2007) unified the properties obtained in Tuneski (2003) by considering the general analytic function $p(z)$ as $\frac{zf'(z)}{f(z)}$. The condition is given as follows:

$$1 + \frac{zp'(z)}{p^2(z)} \prec 1 + bz \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}.$$

Also, the authors considered the expression given by Frasin and Maslina (2001);

$$\frac{[zf(z)]''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{2-\alpha} \Rightarrow \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1-\alpha.$$

and further generalised in the following manner: $1 + \frac{\beta zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}$ for $p(z) = \frac{z^2 f'(z)}{[f(z)]^2}, f \in \mathcal{A}$. Another special case of the above implications can be found in Ponnusamy and Rajasekaran (1995).

There are several interesting results related to the above implications. Previously, the criterion for a normalised analytic function to be univalent using the result $1 + zp'(z) \prec 1 + z$ implies $p(z) \prec 1 + z$ where $p(z)$ is analytic in $\mathbf{D}$ and $p(0) = 1$ has been studied by Nunokawa et al. in 1989. Then in 2007, Rosihan et al. determined conditions A, B, D and E so that when $1 + \beta zp'(z)$, $1 + \frac{\beta zp'(z)}{p(z)}$ and $1 + \frac{\beta zp'(z)}{p^2(z)}$ are subordinated to $\frac{1+Dz}{1+Ez}$, the relation $p(z) \prec \frac{1+Az}{1+Bz}$ holds true. Some applications using these properties have been obtained for analytic functions in the class of Janowski starlike functions. Recently, Rosihan et al. (2012c) considered the class of Sokól and Stankiewicz starlike functions to obtain conditions on β. Motivated by these studies, we determine conditions on β by considering the classes of Janowski starlike functions associated with the class of Sokól-Stankiewicz starlike functions.

The following lemma is needed in proving our results.

Lemma 3.1. *(Miller & Mocanu, 2000: p. 135) Let q be univalent in $\boldsymbol{D}$ and let φ be analytic in a domain containing $q(\boldsymbol{D})$. Let $zq'(z)\varphi[q(z)]$ be starlike. If p is analytic in $\mathbf{D}$, $p(0) = q(0)$ and satisfies $zp'(z)\varphi[p(z)] \prec zq'(z)\varphi[q(z)]$ then $p \prec q$ and q is the best dominant.*

We now derive some theorems as the results.

Theorem 3.1. *Let p be an analytic function on $\boldsymbol{D}$ and $p(0) = 1$.*

Let $\beta_0 = \frac{2\sqrt{2}|D-E|}{(1-|E|)}$ where $-1 < E < 1, |D| \leq 1$ and $D \neq E$.

If

$$1 + \beta z p'(z) \prec \frac{1+Dz}{1+Ez} \quad (\beta \geq \beta_0)$$

then

$$p(z) \prec \sqrt{1+z}\ .$$

Proof. Let $q(z) = \sqrt{1+z}$ with $q(0) = 1, q : \mathbf{D} \to \mathbf{C}$. $q(\mathbf{D})$ is a convex set and hence q is a convex function. Thus $zq'(z)$ is starlike with respect to 0.

From Lemma 3.1,

$$1 + \beta z p'(z) \prec 1 + \beta z q'(z) \Rightarrow p(z) \prec q(z).$$

To prove our result, it suffices to show

$$s(z) = \frac{1+Dz}{1+Ez} \prec 1 + \beta z q'(z) = 1 + \frac{\beta z}{2\sqrt{1+z}} = h(z).$$

Since $s^{-1}(w) = \frac{w-1}{D-Ew}$,

$$s^{-1}[h(z)] = \frac{\beta z}{2\sqrt{1+z}(D-E) - \beta E z}\ .$$

For $z = e^{i\theta}, \theta \in [-\pi, \pi]$,

$$|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]|$$

$$= \frac{\beta}{|2\sqrt{1+e^{i\theta}}(D-E) - \beta E e^{i\theta}|}$$

$$\geq \frac{\beta}{2|\sqrt{1+e^{i\theta}}||(D-E)| + \beta|E|}$$

$$= \frac{\beta}{2\sqrt{2|cos\frac{\theta}{2}|}|(D-E)| + \beta|E|}.$$

It can be shown that the above expression is minimum when $\theta = 0$.

Thus

$$|s^{-1}[h(z)]| \geq \frac{\beta}{2\sqrt{2}|(D-E)| + \beta|E|} \geq 1$$

for $\beta \geq \frac{2\sqrt{2}|(D-E)|}{(1-|E|)}$. Therefore $\mathbf{D} \subset s^{-1}[h(\mathbf{D})]$ or $s(\mathbf{D}) \subset h(\mathbf{D})$ implies $s(z) \prec h(z)$ and proves the result. $\square$

The above result is applied to determine sufficient condition for $f \in \mathcal{A}$ to satisfy the condition $\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1$.

Corollary 3.1. *Let* $\beta_0 = \frac{2\sqrt{2}|D-E|}{(1-|E|)}$ *where* $-1 < E < 1$, $|D| \leq 1$, $D \neq E$ *and* $f \in \mathcal{A}$.

i) If f satisfies the following

$$1 + \beta\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right) \prec \frac{1+Dz}{1+Ez} \qquad (\beta \geq \beta_0)$$

then $f \in SL^{\star}$.

ii) If $1+\beta z f''(z) \prec \frac{1+Dz}{1+Ez} \quad (\beta \geq \beta_0)$ *then* $f'(z) \prec \sqrt{1+z}$.

Proof. Define $p(z) = \frac{zf'(z)}{f(z)}$ and using Theorem 3.1, the first part of Corollary 3.1 is proved. The second part of our results in Corollary 3.1 can be derived by taking $p(z) = f'(z)$. □

Using a similar manner, we obtain the second theorem.

Theorem 3.2. *Let p be an analytic function in **D** and $p(0) = 1$. Let* $\beta_0 = \frac{4|D-E|}{(1-|E|)}$, $-1 < E < 1,\ |D| \leq 1$ *and* $D \neq E$.

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \sqrt{1+z} \qquad (\beta \geq \beta_0).$$

Proof. Let $q(z) = \sqrt{1+z}$, $q(0) = 1$. Elementary calculation will show that $\frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1+z)}$ is starlike. Thus, Lemma 3.1 can be applied as

$$1+\beta\frac{zp'(z)}{p(z)} \prec 1+\beta\frac{zq'(z)}{q(z)} \Rightarrow p(z) \prec q(z).$$

Next, we prove the subordination

$$s(z) = \frac{1+Dz}{1+Ez} \prec 1+\beta\frac{zq'(z)}{q(z)} = 1+\frac{\beta z}{2(1+z)} = h(z).$$

$$s^{-1}[h(z)] = \frac{\beta z}{2(1+z)(D-E)-\beta E z} \quad .$$

For $z = e^{i\theta}, \theta \in [-\pi, \pi]$,

$$|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]|$$

$$= \frac{\beta}{|2(1+e^{i\theta})(D-E) - \beta E e^{i\theta}|}$$

$$\geq \frac{\beta}{|2(1+e^{i\theta})||(D-E)| + \beta |E|}$$

$$= \frac{\beta}{4|cos\frac{\theta}{2}||(D-E)| + \beta |E|}.$$

A straight forward computation verifies that the above expression is minimum when $\theta = 0$.

Then

$$|s^{-1}[h(z)]| \geq \frac{\beta}{4|(D-E)| + \beta |E|} \geq 1$$

for $\beta \geq \frac{4|(D-E)|}{(1-|E|)}$. Hence $s(\mathbf{D}) \subset h(\mathbf{D})$ implies $s(z) \prec h(z)$. □

Corollary 3.2. *Let* $\beta_0 = \frac{4|D-E|}{(1-|E|)}$, $-1 < E < 1, |D| \leq 1$ *and* $D \neq E$,

i)

$$1 + \beta \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] \prec \frac{1+Dz}{1+Ez} \Rightarrow f \in SL^{\star} \qquad (\beta \geq \beta_0).$$

ii)

$$1 + \beta \left[\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right] \prec \frac{1+Dz}{1+Ez} \Rightarrow \frac{z^2 f'(z)}{f^2(z)} \prec \sqrt{1+z} \qquad (\beta \geq \beta_0).$$

Proof. Letting $p(z) = \frac{zf'(z)}{f(z)}$ in (i) and $p(z) = \frac{z^2 f'(z)}{f^2(z)}$ in (ii) and applying Theorem 3.2, the results are proved. □

Theorem 3.3. *Let* $\beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$, $-1 < E < 1,\ |D| \leq 1$ *and* $D \neq E$.

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \sqrt{1+z} \qquad (\beta \geq \beta_0).$$

Proof. Let $q(z) = \sqrt{1+z}$, which implies $\frac{zq'(z)}{q^2(z)}$ is starlike.

Using Lemma 3.1,

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{zq'(z)}{q^2(z)} \Rightarrow p(z) \prec q(z) \ .$$

Next, let $h(z) = 1 + \beta \frac{zq'(z)}{q^2(z)} = 1 + \frac{\beta z}{2(1+z)^{\frac{3}{2}}}$

$$s^{-1}[h(z)] = \frac{\beta z}{2(1+z)^{\frac{3}{2}}(D-E) - \beta E z} \quad .$$

For $z = e^{i\theta}, \theta \in [-\pi, \pi]$,

$$|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]|$$

$$= \frac{\beta}{|2(1+e^{i\theta})^{\frac{3}{2}}(D-E) - \beta E e^{i\theta}|}$$

$$\geq \frac{\beta}{|2(1+e^{i\theta})^{\frac{3}{2}}||(D-E)| + \beta|E|}$$

$$= \frac{\beta}{2|(2cos\frac{\theta}{2})^{\frac{3}{2}}||(D-E)| + \beta|E|}.$$

As in previous case, the above expression is minimum when $\theta = 0$.

Then

$$|s^{-1}[h(z)]| \geq \frac{\beta}{4\sqrt{2}|(D-E)| + \beta|E|} \geq 1$$

for $\beta \geq \frac{4\sqrt{2}|(D-E)|}{(1-|E|)}$. Hence $\mathbf{D} \subset s^{-1}[h(\mathbf{D})]$ implies $s(z) \prec h(z)$. $\square$

Corollary 3.3. *Let* $\beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$, $-1 < E < 1,\ |D| \leq 1,\ D \neq E$ *and* $f \in \mathcal{A}$,

$$1-\beta+\beta\left[\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right] \prec \frac{1+Dz}{1+Ez} \Rightarrow f \in SL^{\star} \qquad (\beta \geq \beta_0).$$

Proof. By taking $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 3.3, the result is obtained. $\square$

Since $SL^{\star} \subset SS^{\star}(\frac{1}{2})$, the last theorem gives a result for $\left(\frac{1+Az}{1+Bz}\right)^{\alpha}$.

Theorem 3.4. *Let p be an analytic function in* $\boldsymbol{D}$ *and* $p(0) = 1$. *Let* $\beta_0 = \frac{|1+A||1+B||D-E|}{\alpha|A-B|(1-|E|)}$, $-1 < E < 1,\ |D| \leq 1,\ D \neq E$ *and* $-1 \leq B < A \leq 1$.

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \qquad (\beta \geq \beta_0\ ,\ 0 < \alpha \leq 1).$$

Proof. Let $q(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, then

$$\frac{\beta zq'(z)}{q(z)} = \frac{\beta\alpha z(A-B)}{(1+Az)(1+Bz)} = Q(z).$$

It can easily be verified that $Q(z)$ is starlike. By Lemma 3.1, we prove the subordination

$$s(z) = \frac{1+Dz}{1+Ez} \prec 1+\beta\frac{zq'(z)}{q(z)} = 1+\frac{\beta\alpha z(A-B)}{(1+Az)(1+Bz)} = h(z).$$

Since $s^{-1}(w) = \frac{w-1}{D-Ew}$

$$|s^{-1}[h(z)]| = \left|\frac{\beta\alpha z(A-B)}{[(1+Az)(1+Bz)(D-E)]-\beta\alpha zE(A-B)}\right|$$

$$\geq \frac{|\beta\alpha z(A-B)|}{|[(1+Az)(1+Bz)(D-E)]|+|\beta\alpha zE(A-B)|}\ .$$

For $z = e^{i\theta}, \theta \in [-\pi, \pi]$,

$$|s^{-1}[h(e^{i\theta})]| \geq \frac{\beta\alpha|(A-B)|}{|[(1+Ae^{i\theta})(1+Be^{i\theta})(D-E)]| + \beta\alpha|E(A-B)|}$$

with minimum value being attained at $\theta = 0$.

Hence

$$|s^{-1}[h(e^{i\theta})]| \geq \frac{\beta\alpha|(A-B)|}{|[(1+A)(1+B)(D-E)]| + \beta\alpha|E(A-B)|} \geq 1$$

for $\beta \geq \frac{|[(1+A)(1+B)(D-E)]|}{\alpha|A-B|(1-|E|)}$ implies $s(z) \prec h(z)$ and the result is obtained. □

Remark 3.1. *Theorem 3.4 is reduced to Theorem 3.2 when $\alpha = \frac{1}{2}$, $A = 1$ and $B = 0$.*

Finally, we state the next obvious result.

Corollary 3.4. *Let $\beta_0 = \frac{|1+A||1+B||D-E|}{\alpha|A-B|(1-|E|)}$, $-1 < E < 1$, $|D| \leq 1$, $D \neq E$ and $-1 \leq B < A \leq 1$. Then*

$$1 + \beta\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec \frac{1+Dz}{1+Ez} \Rightarrow \frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha} .$$

3.2 Properties of functions involving Cassini curve

With a similar method as in section 3.1, we investigate the lower bound of β for a function in the region of Cassini curve.

Theorem 3.5. *Let p be an analytic function on $\boldsymbol{D}$, $p(0) = 1$ and $\beta_0 = 2\sqrt{2}(\sqrt{c+1} - 1)$ where $c \in (0, 1]$. If the function p satisfies the subordination*

$$1 + \beta zp'(z) \prec \sqrt{1+cz} \qquad (\beta \geq \beta_0)$$

then

$$p(z) \prec \sqrt{1+z}\ .$$

The lower bound β_0 is best possible.

Proof. Define the function $q : \mathbf{D} \to \mathbf{C}$ by $q(z) = \sqrt{1+z}$ with $q(0) = 1$. Since $q(\mathbf{D}) = \{w : |w^2 - 1| < 1\}$ is the interior of the right half of the lemniscate of Bernoulli, $q(\mathbf{D})$ is a convex set and hence q is a convex function. The Alexander's Theorem showed that $zq'(z)$ is starlike function with respect to 0. It follows from Lemma 3.1 that

$$1 + \beta z p'(z) \prec 1 + \beta z q'(z) \Rightarrow p(z) \prec q(z).$$

To prove our result, it suffices to show

$$s(z) = \sqrt{1+cz} \prec 1 + \beta z q'(z) = 1 + \frac{\beta z}{2\sqrt{1+z}} = h(z).$$

Suppose $s(z) = \sqrt{1+cz} = w$. Then we have

$$s^{-1}(w) = \frac{w^2 - 1}{c}$$

$$s^{-1}[h(z)] = \frac{\left[1 + \frac{\beta z}{2\sqrt{1+z}}\right]^2 - 1}{c}$$

$$= \frac{1 + 2\left(\frac{\beta z}{2\sqrt{1+z}}\right) + \left(\frac{\beta z}{2\sqrt{1+z}}\right)^2 - 1}{c}$$

$$= \frac{1}{c}\left[\frac{\beta z}{2\sqrt{1+z}}\right]\left[2 + \frac{\beta z}{2\sqrt{1+z}}\right].$$

For $z = e^{i\theta}$ $(\theta \in (-\pi, \pi))$, we obtain

$$|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]| = \left|\frac{1}{c}\left[\frac{\beta e^{i\theta}}{2\sqrt{1+e^{i\theta}}}\right]\left[2 + \frac{\beta e^{i\theta}}{2\sqrt{1+e^{i\theta}}}\right]\right|.$$

Since the above expression is minimum when $\left|2\sqrt{1+e^{i\theta}}\right| = 2\sqrt{2\ cos\frac{\theta}{2}}$ is maximum and this occurs at $\theta = 0$,

$$|s^{-1}[h(z)]| \geq \frac{1}{c}\left[\frac{\beta}{2\sqrt{2}}\right]\left[2 + \frac{\beta}{2\sqrt{2}}\right] = \frac{1}{c}\left[\left(1 + \frac{\beta}{2\sqrt{2}}\right)^2 - 1\right] \geq 1$$

for $\beta \geq 2\sqrt{2}(\sqrt{c+1} - 1)$. Hence $\mathbf{D} \subset s^{-1}[h(\mathbf{D})]$ or $s(\mathbf{D}) \subset h(\mathbf{D})$ which implies $s(z) \prec h(z)$ and this proves the result. $\square$

By taking $p(z) = \frac{zf'(z)}{f(z)}$ and $p(z) = f'(z)$, we have the following result using Theorem 3.5.

Corollary 3.5. *Let $\beta_0 = 2\sqrt{2}(\sqrt{c+1} - 1)$ where $c \in (0, 1]$ and $f \in \mathcal{A}$.*

i) If f satisfies the following

$$1 + \beta\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right) \prec \sqrt{1+cz} \quad (\beta \geq \beta_0)$$

then $f \in SL^\star$.

ii) If $1 + \beta zf''(z) \prec \sqrt{1+cz}$ $(\beta \geq \beta_0)$ then $f'(z) \prec \sqrt{1+z}$.

Theorem 3.6. *Let $\beta_0 = 4(\sqrt{c+1} - 1)$ and $c \in (0, 1]$. If*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \sqrt{1+cz} \text{ then } p(z) \prec \sqrt{1+z} \quad (\beta \geq \beta_0),$$

then the lower bound β_0 is best possible.

Proof. Let $q(z) = \sqrt{1+z}$, $q(0) = 1$. Elementary calculation will show that $\frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1+z)}$ is starlike. Application of Lemma 3.1 will deduce

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)} \Rightarrow p(z) \prec q(z),$$

provided we show that

$$s(z) = \sqrt{1+cz} \prec 1 + \beta\frac{zq'(z)}{q(z)} = 1 + \frac{\beta z}{2(1+z)} = h(z).$$

Easily

$$s^{-1}[h(z)] = \frac{\left[1 + \frac{\beta z}{2(1+z)}\right]^2 - 1}{c}$$

$$= \frac{1}{c}\left[\frac{2\beta z}{2(1+z)} + \left(\frac{\beta z}{2(1+z)}\right)^2\right].$$

For $z = e^{i\theta}, \theta \in (-\pi, \pi)$,

$$|s^{-1}[h(e^{i\theta})]| = \left|\frac{1}{c}\left[\frac{2\beta e^{i\theta}}{2(1+e^{i\theta})} + \left(\frac{\beta e^{i\theta}}{2(1+e^{i\theta})}\right)^2\right]\right|$$

$$= \left|\frac{1}{c}\left[\frac{\beta e^{i\theta}}{2(1+e^{i\theta})}\right]\left[2 + \frac{\beta e^{i\theta}}{2(1+e^{i\theta})}\right]\right|.$$

Since $\max\left|1 + e^{i\theta}\right| = 2\cos\frac{\theta}{2}$, this implies the minimum of the above expression is attained at $\theta = 0$.

Then $|s^{-1}[h(z)]| \geq \frac{\beta}{4c}\left[2 + \frac{\beta}{4}\right] = \frac{1}{c}\left[\left(1 + \frac{\beta}{4}\right)^2 - 1\right] \geq 1$ for $\beta \geq 4(\sqrt{c+1} - 1)$. Hence $s(z) \prec h(z)$ $\square$

Applying Theorem 3.6 and letting $p(z) = \frac{zf'(z)}{f(z)}$ in (i) and $p(z) = \frac{z^2 f'(z)}{f^2(z)}$ in (ii), the following results are obtained.

Corollary 3.6. *Let* $\beta_0 = 4(\sqrt{c+1} - 1)$ *and* $f \in \mathcal{A}$. *i)*

$$1 + \beta \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec \sqrt{1 + cz} \Rightarrow f \in SL^{\star} \qquad (\beta \geq \beta_0).$$

ii)

$$1 + \beta \left[\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right] \prec \sqrt{1 + cz} \Rightarrow \frac{z^2 f'(z)}{f^2(z)} \prec \sqrt{1 + z} \qquad (\beta \geq \beta_0).$$

Theorem 3.7. *Let* $\beta_0 = 4\sqrt{2}(\sqrt{c+1} - 1)$. *If*

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec \sqrt{1 + cz} \Rightarrow p(z) \prec \sqrt{1 + z} \quad (\beta \geq \beta_0),$$

then the lower bound β_0 *is best possible.*

Proof. For $q(z) = \sqrt{1+z}$, $\frac{zq'(z)}{q^2(z)}$ is starlike. Lemma 3.1 can then imply the following relation

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{zq'(z)}{q^2(z)} \Rightarrow p(z) \prec q(z) \; .$$

Writing $h(z) = 1 + \beta \frac{zq'(z)}{q^2(z)} = 1 + \frac{\beta z}{2(1+z)^{\frac{3}{2}}}$

we then have

$$s^{-1}[h(z)] = \frac{\left[1 + \frac{\beta z}{2(1+z)^{\frac{3}{2}}}\right]^2 - 1}{c}$$

$$= \frac{1}{c}\left[\frac{\beta z}{2(1+z)^{\frac{3}{2}}}\left(2 + \frac{\beta z}{2(1+z)^{\frac{3}{2}}}\right)\right].$$

In a similar manner to previous cases, for $z = e^{i\theta}, \theta \in (-\pi, \pi)$;

$|s^{-1}[h(z)]| \geq \frac{1}{c}\left[\left(1+\frac{\beta}{4\sqrt{2}}\right)^2 - 1\right] \geq 1$ for $\beta \geq 4\sqrt{2}(\sqrt{c+1}-1)$. Hence $\mathbf{D} \subset s^{-1}[h(\mathbf{D})]$ implies $s(z) \prec h(z)$. $\square$

By setting $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 3.7, we have the following corollary.

Corollary 3.7. *Let* $\beta_0 = 4\sqrt{2}(\sqrt{c+1}-1)$ *and* $f \in \mathcal{A}$ *,*

$$1-\beta+\beta\left[\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right] \prec \sqrt{1+cz} \Rightarrow f \in SL^{\star} \qquad (\beta \geq \beta_0).$$

Remark 3.2. *For the special case of* $c = 1$*, all the above theorems and corollaries are reduced to the results obtained by Rosihan et al. (2012c).*

3.3 Properties of certain analytic classes

Let $\mathcal{SL}(\alpha)$ and $\mathcal{SL}^c$ denote the classes of α-convex and convex functions which respectively satisfy $|\,[J(\alpha, f(z))]^2 - 1| < 1$ and $\left|\left[1+\frac{zf''(z)}{f'(z)}\right]^2 - 1\right| < 1$ $(z \in \mathbf{D})$ where

$$J(\alpha, f(z)) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1+\frac{zf''(z)}{f'(z)}\right], \alpha \geq 0.$$

It is obvious that $f \in \mathcal{SL}(\alpha) \Leftrightarrow J(\alpha, f(z)) \prec \sqrt{1+z}$ and $f \in \mathcal{SL}^c \Leftrightarrow 1+\frac{zf''(z)}{f'(z)} \prec \sqrt{1+z}$. These classes are generalised from the definition of the class $SL^{\star}$. Using results given in section 3.1, properties of functions in the classes $\mathcal{SL}(\alpha)$ and $\mathcal{SL}^c$ are obtained.

Corollary 3.8. *Let* $\beta_0 = \frac{2\sqrt{2}(|D-E|)}{1-|E|}$, $|E| < 1$, $|D| \leq 1$, $D \neq E$, $\beta \geq \beta_0$ *and* $f \in \mathcal{A}$. *If* f *satisfies*

(i)

$$1+\beta\frac{zf''(z)}{f'(z)}\left\{\frac{z[f''(z)]'}{f''(z)} - \frac{zf''(z)}{f'(z)} + 1\right\} \prec \frac{1+Dz}{1+Ez}$$

then $f \in \mathcal{SL}^c$; *and*

(ii)

$$1+\beta\left\{(1-\alpha)\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+1\right)+\alpha\frac{zf''(z)}{f'(z)}\left(\frac{z[f''(z)]'}{f''(z)}-\frac{zf''(z)}{f'(z)}+1\right)\right\} \prec \frac{1+Dz}{1+Ez}$$

then $f \in \mathcal{SL}(\alpha)$.

Proof. (i) Let $p(z) = 1+\frac{zf''(z)}{f'(z)}$. Then

$$zp'(z) = \frac{z^2[f''(z)]'}{f'(z)}+\frac{zf''(z)}{f'(z)}-\frac{z^2[f''(z)]^2}{[f'(z)]^2}$$

$$= \frac{zf''(z)}{f'(z)}\left\{\frac{z[f''(z)]'}{f''(z)}+1-\frac{zf''(z)}{f'(z)}\right\},$$

and applying Theorem 3.1 gives $1+\frac{zf''(z)}{f'(z)} \prec \frac{1+Dz}{1+Ez}$, hence $f \in \mathcal{SL}^c$.

(ii) With $p(z) = (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]+\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]$, we have

$$p'(z) = (1-\alpha)\left\{\frac{f(z)[zf''(z)+f'(z)]-z[f'(z)]^2}{[f(z)]^2}\right\} + \alpha\left\{\frac{f'(z)[z(f''(z))'+f''(z)]-z[f''(z)]^2}{[f'(z)]^2}\right\}$$

$$= (1-\alpha)\frac{f'(z)}{f(z)}\left\{\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right\} + \alpha\frac{f''(z)}{f'(z)}\left\{\frac{z[f''(z)]'}{f''(z)}+1-\frac{zf''(z)}{f'(z)}\right\}$$

and from Theorem 3.1, the result implies $f \in \mathcal{SL}(\alpha)$. □

Corollary 3.9. *Let* $\beta_0 = \frac{4|D-E|}{(1-|E|)}$ *and* $f \in \mathcal{A}$.

(i) If

$$1+\beta\frac{zf''(z)}{[f'(z)+zf''(z)]}\left\{\frac{z[f''(z)]'}{f''(z)}-\frac{zf''(z)}{f'(z)}+1\right\} \prec \frac{1+Dz}{1+Ez}$$

then $f \in \mathcal{SL}^c$.

(ii) If

$$1 + \beta \left\{ \frac{(1-\alpha)zf'(z)\left[1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right]}{(1-\alpha)zf'(z) + \alpha f(z)\left[1 + \frac{zf''(z)}{f'(z)}\right]} + \frac{\alpha zf''(z)\left[1 - \frac{zf''(z)}{f'(z)} + \frac{z[f''(z)]'}{f''(z)}\right]}{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]} \right\}$$

$$\prec \frac{1+Dz}{1+Ez}$$

then $f \in \mathcal{SL}(\alpha)$.

Proof. (i) For $p(z) = 1 + \frac{zf''(z)}{f'(z)}$,

$$\frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)}\left[\frac{f'(z) - zf''(z)}{f'(z)}\right]\left[\frac{f'(z)}{f'(z) + zf''(z)}\right] + \left[\frac{z^2\left[f''(z)\right]'}{f'(z)}\right]\left[\frac{f'(z)}{f'(z) + zf''(z)}\right]$$

$$= \frac{zf''(z)}{[f'(z) + zf''(z)]} - \frac{[zf''(z)]^2}{f'(z)[f'(z) + zf''(z)]} + \frac{z^2[f''(z)]'}{[f'(z) + zf''(z)]}$$

$$= \frac{zf''(z)}{[f'(z) + zf''(z)]}\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}$$

and based on Theorem 3.2, $f \in \mathcal{SL}^c$.

(ii) Let $p(z) = (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right] + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]$.

$$\frac{p'(z)}{p(z)} = \frac{(1-\alpha)\frac{f'(z)}{f(z)}\left\{\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)}\right\} + \alpha\frac{f''(z)}{f'(z)}\left\{\frac{z[f''(z)]'(z)}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{(1-\alpha)\left[\frac{zf'(z)}{f(z)}\right] + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]}$$

$$= \frac{(1-\alpha)f'(z)\left\{zf''(z) + f'(z) - \frac{zf'(z)f'(z)}{f(z)}\right\}}{f'(z)\left\{(1-\alpha)zf'(z) + \alpha f(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}}$$

$$+ \frac{\alpha f''(z)f(z)\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{f(z)\left\{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]\right\}}$$

$$= \frac{(1-\alpha)f'(z)\left\{\frac{zf''(z)}{f'(z)} + \frac{f'(z)}{f'(z)} - \frac{z[f'(z)]^2}{f'(z)f(z)}\right\}}{(1-\alpha)zf'(z) + \alpha f(z)\left[1 + \frac{zf''(z)}{f'(z)}\right]} + \frac{\frac{\alpha f''(z)f(z)}{f(z)}\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]}$$

$$= \frac{(1-\alpha)f'(z)\left\{\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)]}{f(z)}\right\}}{(1-\alpha)zf'(z) + \alpha f(z)\left[1 + \frac{zf''(z)}{f'(z)}\right]} + \frac{\alpha f''(z)\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]}.$$

Applying Theorem 3.2, $f \in \mathcal{SL}(\alpha)$. □

Corollary 3.10. *Let* $\beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$ *and* $f \in \mathcal{A}$.

(i)

$$1 + \beta\frac{zf'(z)f''(z)}{[f'(z) + zf''(z)]^2}\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\} \prec \frac{1+Dz}{1+Ez} \Rightarrow f \in \mathcal{SL}^c.$$

(ii)

$1+\beta z$

$$\left\{\frac{(1-\alpha)[f'(z)]^3 f(z)\left[\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right]+\alpha[f(z)]^2 f'(z)f''(z)\left[\frac{z[f''(z)]'}{f''(z)}+1-\frac{zf''(z)}{f'(z)}\right]}{[(1-\alpha)f'(z)zf'(z)+\alpha f(z)[f'(z)+zf''(z)]]^2}\right\}$$

$$\prec \frac{1+Dz}{1+Ez} \Rightarrow f \in \mathcal{SL}(\alpha).$$

Proof. (i) Using Theorem 3.3 with $p(z)=1+\frac{zf''(z)}{f'(z)}$, we obtain the result.

(ii) Similarly, set $p(z)=(1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]+\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]$. $\square$

Remark 3.3. *For $\alpha=0$ and $\alpha=1$, Corollary 3.8 (ii), Corollary 3.9 (ii) and Corollary 3.10 (ii) give the results $f \in SL^\star$ and $f \in \mathcal{SL}^c$.*

Our discussion is continued by introducing new classes defined by Dziok-Srivastava operator and generalised multiplier transformations. Inclusion theorems are determined using Briot-Bouquet differential subordinations. There were some authors introduced new classes using certain operators and Briot-Bouquet differential subordinations method [see Kanas (1995), Choi et al. (2002a), Cho and Kim (2006) and Kwon and Cho (2007)].

3.4 Classes of function defined by Dziok-Srivastava operator

Using the Dziok-Srivastava operator, we generalise new classes from Sokόl-Stankiewicz and Janowski strongly starlike functions. Classes denoted by $SL^*[\alpha_1], H(\alpha_1; A, B; \lambda)(\lambda \in (0,1])$ and $\mathcal{SL}^c[\alpha_1]$ are introduced and defined below:

$$SL^*[\alpha_1] := \left\{ f : f \in S, \quad \frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)} \prec \sqrt{1+z}\ , \quad z \in \mathbf{D} \right\},$$

$$H(\alpha_1; A, B; \lambda) := \left\{ f : f \in S, \quad \frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^\lambda , \quad z \in \mathbf{D} \right\},$$

$$\mathcal{SL}^c[\alpha_1] := \left\{ f : f \in S, \quad 1+\frac{z\left[H^{l,m}[\alpha_1]f(z)\right]''}{\left[H^{l,m}[\alpha_1]f(z)\right]'} \prec \sqrt{1+z}\ , \quad z \in \mathbf{D} \right\}.$$

Quite trivially, the Alexander's Theorem is also observed for the $\mathcal{SL}^c[\alpha_1]$ and $SL^\star[\alpha_1]$

$$f \in \mathcal{SL}^c[\alpha_1] \Leftrightarrow zf'(z) \in SL^*[\alpha_1]. \tag{3.1}$$

In proving our results, the following lemmas will be required.

Lemma 3.2. *(Enigenberg et al., 1983) Let h be convex in $\boldsymbol{D}$, with Re $[\beta h(z) + \gamma] > 0$. If p is analytic in $\boldsymbol{D}$ with $p(0) = h(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 3.3. *(Kanas, 1995) Let $\lambda \in (0, 1]$ be fixed, $\beta, \sigma \in \boldsymbol{C}$, $Arg\ \beta \in \left(\frac{(\lambda-1)\pi}{2}, \frac{(1-\lambda)\pi}{2}\right)$ and Re $\sigma \geq 0$. Let p be analytic function such that $p(0) = 1$ and $p(z) \neq \frac{-\sigma}{\beta}$ $(z \in \boldsymbol{D})$. If*

$$\left| Arg \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \sigma} \right\} \right| < \frac{\lambda\pi}{2},$$

then $|Arg\ p(z)| < \frac{\lambda\pi}{2}$.

We prove now the inclusion theorem for the class $SL^\star[\alpha_1]$ using the Dziok-Srivastava operator. It is easily to verify that

$$\alpha_1 H^{l,m}[\alpha_1 + 1]f(z) = z\left[H^{l,m}[\alpha_1]f(z)\right]' + (\alpha_1 - 1)H^{l,m}[\alpha_1]f(z). \tag{3.2}$$

Theorem 3.8. *Let $\alpha_1 \geq 1$ and Re$\{(\alpha_1 - 1) + \sqrt{(1+z)}\} > 0$. Then $SL^\star[\alpha_1 + 1] \subset SL^\star[\alpha_1]$.*

Proof. If $f \in SL^\star[\alpha_1 + 1]$ then

$$\frac{z\left[H^{l,m}[\alpha_1 + 1]f(z)\right]'}{H^{l,m}[\alpha_1 + 1]f(z)} \prec \sqrt{1+z}$$

and from (3.2) we have

$$\frac{\alpha_1 H^{l,m}[\alpha_1+1]f(z)}{H^{l,m}[\alpha_1]f(z)} = \frac{z\left[H^{l,m}[\alpha_1 f(z)]\right]'}{H^{l,m}[\alpha_1]f(z)} + (\alpha_1 - 1).$$

After differentiating this equation and then rearranging it, we have

$$\frac{z\left[H^{l,m}[\alpha_1+1]f(z)\right]'}{H^{l,m}[\alpha_1+1]f(z)} = \frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)} + \frac{z\left(\frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)}\right)'}{\frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)} + \alpha_1 - 1}. \tag{3.3}$$

Letting $p(z) = \frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)}$ and $h(z) = \sqrt{1+z}$, it is clear that h is convex in $\mathbf{D}$ and $p(0) = h(0)$.

Since $\frac{z\left[H^{l,m}[\alpha_1+1]f(z)\right]'}{H^{l,m}[\alpha_1+1]f(z)} \prec \sqrt{1+z}$, applying Lemma 3.2 with $\beta = 1$ and $\gamma = \alpha_1 - 1$ proves $\frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)} \prec \sqrt{1+z}$. Thus $f \in SL^{\star}[\alpha_1]$. $\square$

Next, Bernardi operator and Jung-Kim-Srivastava operators are shown to be preserved for the class $SL^{\star}[\alpha_1]$. The results are stated in Theorem 3.9-3.11.

Theorem 3.9. *If for $z \in \boldsymbol{D}$, $Re\left\{c + \sqrt{1+z}\right\} > 0$ and $f \in SL^{\star}[\alpha_1]$ then $F_c f \in SL^{\star}[\alpha_1]$.*

Proof. Since the Bernardi operator satisfies the relation $cF_c[f(z)] + z\left(F_c[f(z)]\right)' = (c+1)f(z)$, it can be established that

$$c\left(H^{l,m}[\alpha_1]F_c[f(z)]\right) + z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)' = (c+1)H^{l,m}[\alpha_1]f(z) \tag{3.4}$$

which upon rewriting gives

$$\frac{(c+1)H^{l,m}[\alpha_1]f(z)}{H^{l,m}[\alpha_1]F_c[f(z)]} = \frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]} + c.$$

Differentiating both sides in the above equation and using the hypothesis that $f \in$

$SL^{\star}[\alpha_1]$, we have the following relation

$$\frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)} = \frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]} + \frac{z\left(\frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]}\right)'}{\frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]}+c} \prec \sqrt{1+z}. \tag{3.5}$$

As before, by letting $p(z) = \frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]}$ and $h(z) = \sqrt{1+z}$. Lemma 3.2 implies $\frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]} \prec \sqrt{1+z}$ and this completes the proof. □

Recall that two of Jung-Kim-Srivastava operators are defined as:

$$P^{\nu}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\nu} a_n z^n$$

and

$$\ell_{\mu}^{\nu}f(z) = z + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)}\sum_{n=2}^{\infty}\left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)}\right) a_n z^n.$$

Using these operators, we obtain the next results.

Theorem 3.10. *Suppose for $z \in \boldsymbol{D}$, $Re\{1+\sqrt{1+z}\} > 0$ and $P^{\nu-1}f(z) \in SL^{\star}[\alpha_1]$. Then $P^{\nu}f(z) \in SL^{\star}[\alpha_1]$ $(\nu > 1)$.*

Proof. It can be derived

$$z\left[H^{l,m}[\alpha_1]P^{\nu}f(z)\right]' = 2\left[H^{l,m}[\alpha_1]P^{\nu-1}f(z)\right] - H^{l,m}[\alpha_1]P^{\nu}f(z)$$

and rearranging the equation gives

$$\frac{z\left[H^{l,m}[\alpha_1]P^{\nu}f(z)\right]'}{H^{l,m}[\alpha_1]P^{\nu}f(z)} = \frac{2\left[H^{l,m}[\alpha_1]P^{\nu-1}f(z)\right]}{H^{l,m}[\alpha_1]P^{\nu}f(z)} - 1.$$

Differentiating both sides, we obtain

$$\frac{z\left[H^{l,m}[\alpha_1]P^{\nu-1}f(z)\right]'}{H^{l,m}[\alpha_1]P^{\nu-1}f(z)} = p(z) + \frac{zp'(z)}{p(z)+1} \prec \sqrt{1+z}$$

where $p(z) = \frac{z\left[H^{l,m}[\alpha_1]P^{\nu}f(z)\right]'}{H^{l,m}[\alpha_1]P^{\nu}f(z)}$. Using Lemma 3.2 with $\beta = \gamma = 1$ implies the result. □

Theorem 3.11. *Let $Re\{(\nu+\mu-1)+\sqrt{1+z}\} > 0$ for $\nu > 1$ and $\mu > -1$. If $\ell_\mu^{\nu-1} f(z) \in SL^\star[\alpha_1]$ then $\ell_\mu^{\nu} f(z) \in SL^\star[\alpha_1]$.*

Proof. Since $z\left[H^{l,m}[\alpha_1]\ell_\mu^\nu f(z)\right]' = (\nu+\mu)\left[H^{l,m}[\alpha_1]\ell_\mu^{\nu-1} f(z)\right] - (\nu+\mu-1)H^{l,m}[\alpha_1]\ell_\mu^\nu f(z)$,

$$\frac{z\left[H^{l,m}[\alpha_1]\ell_\mu^\nu f(z)\right]'}{H^{l,m}[\alpha_1]\ell_\mu^\nu f(z)} = \frac{(\nu+\mu)\left[H^{l,m}[\alpha_1]\ell_\mu^{\nu-1} f(z)\right]}{H^{l,m}[\alpha_1]\ell_\mu^\nu f(z)} - (\nu+\mu-1).$$

From the differentiation of the above equation, we have

$$\frac{z\left[H^{l,m}[\alpha_1]\ell_\mu^{\nu-1} f(z)\right]'}{H^{l,m}[\alpha_1]\ell_\mu^{\nu-1} f(z)} = p(z) + \frac{zp'(z)}{p(z)+(\nu+\mu-1)}.$$

The hypothesis of the theorem and Lemma 3.2 give the result by letting $p(z) = \frac{z\left[H^{l,m}[\alpha_1]\ell_\mu^\nu f(z)\right]'}{H^{l,m}[\alpha_1]\ell_\mu^\nu f(z)}$ with $\beta = 1$ and $\gamma = \nu+\mu-1$. □

Furthermore, inclusion theorems and preservation properties for the classes $H(\alpha_1; A, B; \lambda)$ and $\mathcal{SL}^c[\alpha_1]$ are shown in the following results.

Theorem 3.12. *Let $\lambda \in (0,1]$ and $Re\,(\alpha_1 - 1) \geq 0$, $H(\alpha_1+1; A, B; \lambda) \subset H(\alpha_1; A, B; \lambda)$.*

Proof. The proof is trivial. Since the Dziok-Srivastava operator satisfies (3.3) and $f \in H(\alpha_1+1; A, B; \lambda)$,

$$\left|Arg\left(\frac{z\left[H^{l,m}[\alpha_1+1]f(z)\right]'}{H^{l,m}[\alpha_1+1]f(z)}\right)\right| < \frac{\lambda\pi}{2}.$$

Therefore applying Lemma 3.3 with $p(z) = \frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)}$, $\beta = 1$ and $\sigma = \alpha_1 - 1$ gives

$$|Arg\; p(z)| = \left|Arg\left(\frac{z\left[H^{l,m}[\alpha_1]f(z)\right]'}{H^{l,m}[\alpha_1]f(z)}\right)\right| < \frac{\lambda\pi}{2}.$$

Hence $f \in H(\alpha_1; A, B; \lambda)$. □

Theorem 3.13. *Suppose $\lambda \in (0,1]$ and $Re\ c \geq 0$. If $f \in H(\alpha_1; A, B; \lambda)$ then $F_c f \in H(\alpha_1; A, B; \lambda)$.*

Proof. The proof follows easily since $F_c f$ satisfies the equation (3.4) and with $p(z) = \frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]}$ in Lemma 3.3 ($\beta = 1$ and $\sigma = c$) results

$$|Arg\ p(z)| = \left|Arg\left(\frac{z\left(H^{l,m}[\alpha_1]F_c[f(z)]\right)'}{H^{l,m}[\alpha_1]F_c[f(z)]}\right)\right| < \frac{\lambda\pi}{2}.$$

□

Remark 3.4. *In a similar manner as in previous theorems, it can easily be shown that*

$$P^{\nu-1}f(z) \in H(\alpha_1; A, B; \lambda) \Rightarrow P^{\nu}f(z) \in H(\alpha_1; A, B; \lambda)$$

and

$$\ell_{\mu}^{\nu-1}f(z) \in H(\alpha_1; A, B; \lambda) \Rightarrow \ell_{\mu}^{\nu}f(z) \in H(\alpha_1; A, B; \lambda).$$

Remark 3.5. *For $\lambda = \frac{1}{2}, A = 1$ and $B = 0$, Theorem 3.12 and Theorem 3.13 reduce to Theorem 3.8 and Theorem 3.9.*

Theorem 3.14. *Let $\alpha_1 \geq 1$. Then $\mathcal{SL}^c[\alpha_1 + 1] \subset \mathcal{SL}^c[\alpha_1]$.*

Proof. Using (3.1) and Theorem 3.8, we can easily deduce our result.

$$\begin{aligned} f(z) \in \mathcal{SL}^c[\alpha_1+1] &\Leftrightarrow zf'(z) \in SL^\star[\alpha_1+1] \\ &\Rightarrow zf'(z) \in SL^\star[\alpha_1] \Leftrightarrow H^{l,m}[\alpha_1]\,[zf(z)]' \in SL^\star \\ &\Leftrightarrow z\left[H^{l,m}[\alpha_1]f(z)\right]' \in SL^\star \Leftrightarrow H^{l,m}[\alpha_1]f(z) \in \mathcal{SL}^c \\ &\Leftrightarrow f \in \mathcal{SL}^c[\alpha_1]. \end{aligned}$$

□

Theorem 3.15. *If $f \in \mathcal{SL}^c[\alpha_1]$ then $F_c f \in \mathcal{SL}^c[\alpha_1]$.*

Proof. By applying Theorem 3.9, it follows that

$$\begin{aligned} f \in \mathcal{SL}^c[\alpha_1] &\Leftrightarrow zf'(z) \in SL^\star[\alpha_1] \\ &\Rightarrow F_c[zf'(z)] \in SL^\star[\alpha_1] \Leftrightarrow z\left(F_c[f(z)]\right)' \in SL^\star[\alpha_1] \\ &\Leftrightarrow F_c f \in \mathcal{SL}^c[\alpha_1]. \end{aligned}$$

□

3.5 Subclasses of analytic functions associated with generalised multiplier transformations

Recently, some properties of functions using the multiplier transformations have been studied by Cătaş et al. (2008), Cătaş (2009), Cho and Noor (2012), El-Ashwah et al. (2010) and Lupaş (2010). Using the convolution, we extend the multiplier transformation in (1.4) to be a unified operator. The approach used is similar to Noor's (Noor, 1999), except we generalise and extend to include powers and use the multiplier Cătaş as basis instead of the Ruscheweyh operator.

Set the function

$$f_{k,c}(z) = z + \sum_{n=2}^{\infty} \left[\frac{1+c}{1+\lambda(n-1)+c}\right]^k z^n \qquad (k, \lambda \in \mathbf{R}, k \geq 0, \lambda \geq 0, c \geq 0)$$

and note that for $\lambda = 1$, $f_{k,c}(z)$ is the generalised polylogarithm functions discussed in Mondal and Swaminathan (2010). A new function $f^{\mu}_{k,c}(z)$ is defined in terms of the Hadamard product(or convolution) as follows:

$$f_{k,c}(z) * f^{\mu}_{k,c}(z) = \frac{z}{(1-z)^{\mu}} \qquad (\mu > 0).$$

Motivated by Cho and Kim (2006), Choi et al. (2002a), Kwon and Cho (2007) and analogous to (1.4), the following operator is introduced:

$$\begin{aligned} I_c^k(\lambda,\mu)f(z) &= f_{k,c}^{\mu} * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\mu)_{n-1}}{(n-1)!} \left[\frac{1+\lambda(n-1)+c}{1+c}\right]^k a_n z^n. \end{aligned}$$

The operator $I_c^k(\lambda,\mu)f$ unifies other previously defined operators. For examples;

(i) $I_c^k(\lambda,1)f$ is the $I_1(\delta,\lambda,l)f$ given in Cătaş (2008)

(ii) $I_c^k(1,1)f$ is the $I_c^k f$ given in Cho and Srivastava (2003).

Also, for any integer k,

(iii) $I_0^k(\lambda,1)f(z) \equiv D_\lambda^k f(z)$ given in Al-Oboudi (2004)

(iv) $I_0^k(1,1)f(z) \equiv D^k f(z)$ given in Sălăgean (1983)

(v) $I_1^k(1,1)f(z) \equiv I_k f(z)$ given in Uralegaddi and Somanatha (1992).

The following relations are easily derived using the definition:

$$(1+c)I_c^{k+1}(\lambda,\mu)f(z) = (1-\lambda+c)I_c^k(\lambda,\mu)f(z) + \lambda z\left[I_c^k(\lambda,\mu)f(z)\right]' \tag{3.6}$$

and

$$\mu I_c^k(\lambda,\mu+1)f(z) = z\left[I_c^k(\lambda,\mu)f(z)\right]' + (\mu-1)I_c^k(\lambda,\mu)f(z). \tag{3.7}$$

Using Ma and Minda (1992) classes and the generalised multiplier transformations $I_c^k(\lambda,\mu)f$, new classes $S_c^k(\lambda,\mu;\phi), C_c^k(\lambda,\mu;\phi)$ and $K_c^k(\lambda,\mu;\phi,\psi)$ are introduced and defined as below:

$$S_c^k(\lambda,\mu;\phi) := \left\{ f \in A : I_c^k(\lambda,\mu) f(z) \in S^\star(\phi) \right\}$$

$$C_c^k(\lambda,\mu;\phi) := \left\{ f \in A : I_c^k(\lambda,\mu) f(z) \in \mathcal{C}(\phi) \right\}$$

$$K_c^k(\lambda,\mu;\phi,\psi) := \left\{ f \in A : I_c^k(\lambda,\mu) f(z) \in \mathcal{K}(\phi,\psi) \right\}.$$

It can be shown easily that

$$f(z) \in C_c^k(\lambda,\mu;\phi) \Leftrightarrow zf'(z) \in S_c^k(\lambda,\mu;\phi). \tag{3.8}$$

Lemma 3.2 and the following lemma are needed in proving results;

Lemma 3.4. *(Miller and Mocanu, 1981) Let ϕ be convex univalent in $\boldsymbol{D}$ and ω be analytic in $\boldsymbol{D}$ with $Re\{\omega(z)\} \geq 0$. If p is analytic in $\boldsymbol{D}$ and $p(0) = \phi(0)$, then*

$$p(z) + \omega(z) z p'(z) \prec \phi(z) \Rightarrow p(z) \prec \phi(z).$$

3.5.1 Inclusion properties involving $I_c^k(\lambda,\mu)f$

Some results on inclusion theorems are given;

Theorem 3.16. *For any real numbers k and λ where $k \geq 0,\ \lambda \geq 0$ and $c \geq 0$. Let $\phi \in N$ and $Re\left\{\phi(z) + \frac{1-\lambda+c}{\lambda}\right\} > 0$. Then $S_c^{k+1}(\lambda,\mu;\phi) \subset S_c^k(\lambda,\mu;\phi) \quad (\mu > 0)$.*

Proof. Let $f \in S_c^{k+1}(\lambda,\mu;\phi)$ and set $p(z) = \frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)f(z)}$ where p is analytic in $\mathbf{D}$ with $p(0) = 1$. Rearranging (3.6), we have

$$\frac{(1+c) I_c^{k+1}(\lambda,\mu) f(z)}{I_c^k(\lambda,\mu) f(z)} = (1-\lambda+c) + \frac{\lambda z \left[I_c^k(\lambda,\mu) f(z)\right]'}{I_c^k(\lambda,\mu) f(z)}. \tag{3.9}$$

Next, differentiating (3.9) and multiplying by z gives

$$\frac{z\left[I_c^{k+1}(\lambda,\mu)f(z)\right]'}{I_c^{k+1}(\lambda,\mu)f(z)} = \frac{z\left[I_c^{k}(\lambda,\mu)f(z)\right]'}{I_c^{k}(\lambda,\mu)f(z)} + \frac{z\left(\frac{z\left[I_c^{k}(\lambda,\mu)f(z)\right]'}{I_c^{k}(\lambda,\mu)f(z)}\right)'}{\frac{z\left[I_c^{k}(\lambda,\mu)f(z)\right]'}{I_c^{k}(\lambda,\mu)f(z)} + \frac{(1-\lambda+c)}{\lambda}}$$

$$= p(z) + \frac{zp'(z)}{p(z) + \frac{(1-\lambda+c)}{\lambda}}.$$

Since $\frac{z\left[I_c^{k+1}(\lambda,\mu)f(z)\right]'}{I_c^{k+1}(\lambda,\mu)f(z)} \prec \phi(z)$ and applying Lemma 3.2, it follows that $p \prec \phi$. Thus $f \in S_c^k(\lambda,\mu;\phi)$. $\square$

Theorem 3.17. *Let $k,\lambda \in \boldsymbol{R}, k \geq 0, \lambda \geq 0$ and $\mu \geq 1$. Then $S_c^k(\lambda,\mu+1;\phi) \subset S_c^k(\lambda,\mu;\phi) \quad (c \geq 0; \phi \in N)$.*

Proof. Let $f \in S_c^k(\lambda,\mu+1;\phi)$. From (3.7) we obtain

$$\frac{\mu I_c^k(\lambda,\mu+1)f(z)}{I_c^k(\lambda,\mu)} = \frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)} + (\mu-1). \tag{3.10}$$

Making use of the differentiation on both sides in (3.10) and setting $p(z) = \frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)f(z)}$, we get

$$\frac{z\left[I_c^k(\lambda,\mu+1)f(z)\right]'}{I_c^k(\lambda,\mu+1)f(z)} = p(z) + \frac{zp'(z)}{p(z)+(\mu-1)} \prec \phi(z).$$

Since $\mu \geq 1$ and $Re\{\phi(z)+(\mu-1)\} > 0$, using Lemma 3.2 we conclude that $f \in S_c^k(\lambda,\mu;\phi)$. $\square$

Corollary 3.11. *Let $\lambda \geq 0, \mu \geq 1$ and $-1 \leq B < A \leq 1$. Then $S^{\star}_{k+1,c}\,[\mu;A,B] \subset S^{\star}_{k,c}\,[\mu;A,B]$ and $S^{\star}_{k,c}\,[\mu+1;A,B] \subset S^{\star}_{k,c}\,[\mu;A,B]$.*

Next, we obtain inclusion theorems for class of convex functions defined by generalised multiplier transformations using the result of Theorem 3.16 and Theorem 3.17.

Theorem 3.18. *Let $\lambda \geq 0$. Then $C_c^{k+1}(\lambda,\mu;\phi) \subset C_c^k(\lambda,\mu;\phi)$ and $C_c^k(\lambda,\mu+1;\phi) \subset C_c^k(\lambda,\mu;\phi)$.*

Proof. Using (3.8) and Theorem 3.16, we observe that

$$\begin{aligned} f(z) \in C_c^{k+1}(\lambda,\mu;\phi) &\Leftrightarrow zf'(z) \in S_c^{k+1}(\lambda,\mu;\phi) \\ &\Rightarrow zf'(z) \in S_c^k(\lambda,\mu;\phi) \\ &\Leftrightarrow I_c^k(\lambda,\mu)zf'(z) \in S^\star(\phi) \\ &\Leftrightarrow z\left[I_c^k(\lambda,\mu)f(z)\right]' \in S^\star(\phi) \\ &\Leftrightarrow I_c^k(\lambda,\mu)f(z) \in C(\phi) \\ &\Leftrightarrow f \in C_c^k(\lambda,\mu;\phi). \end{aligned}$$

To prove the second part of Theorem, using the similar manner and applying Theorem 3.17 the result is obtained. □

Finally, we use Lemma 3.4 to prove the following theorem;

Theorem 3.19. *Let $\lambda \geq 0,\ c \geq 0$ and $Re\{\frac{1-\lambda+c}{\lambda}\} > 0$. Then $K_c^{k+1}(\lambda,\mu;\phi,\psi) \subset K_c^k(\lambda,\mu;\phi,\psi)$ and $K_c^k(\lambda,\mu+1;\phi,\psi) \subset K_c^k(\lambda,\mu;\phi,\psi) \quad (\phi,\psi \in N)$.*

Proof. Let $f \in K_c^{k+1}(\lambda,\mu;\phi,\psi)$. In view of the definition of the class $K_c^{k+1}(\lambda,\mu;\phi,\psi)$, there is a function $g \in S_c^{k+1}(\lambda,\mu;\phi)$ such that

$$\frac{z\left[I_c^{k+1}(\lambda,\mu)f(z)\right]'}{I_c^{k+1}(\lambda,\mu)g(z)} \prec \psi(z).$$

Apply Theorem 3.16, we have $g \in S_c^k(\lambda,\mu;\phi)$. Let $q(z) = \frac{z\left[I_c^k(\lambda,\mu)g(z)\right]'}{I_c^k(\lambda,\mu)g(z)} \prec \phi(z)$.

Letting the analytic function p with $p(0) = 1$ as:

$$p(z) = \frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)g(z)}. \tag{3.11}$$

Then, rearranging and differentiating (3.11) we have

$$\frac{\left[I_c^k(\lambda,\mu)zf'(z)\right]'}{I_c^k(\lambda,\mu)g(z)} = \frac{p(z)\left[I_c^k(\lambda,\mu)g(z)\right]'}{I_c^k(\lambda,\mu)g(z)} + p'(z). \tag{3.12}$$

Making use of (3.6), (3.11), (3.12) and $q(z)$, we obtain

$$\begin{aligned}
\frac{z\left[I_c^{k+1}(\lambda,\mu)f(z)\right]'}{I_c^{k+1}(\lambda,\mu)g(z)} &= \frac{\left[I_c^{k+1}(\lambda,\mu)zf'(z)\right]}{I_c^{k+1}(\lambda,\mu)g(z)} \\
&= \frac{(1-\lambda+c)I_c^k(\lambda,\mu)zf'(z) + \lambda z\left[I_c^k(\lambda,\mu)zf'(z)\right]'}{(1-\lambda+c)I_c^k(\lambda,\mu)g(z) + \lambda z\left[I_c^k(\lambda,\mu)g(z)\right]'} \\
&= \frac{\frac{(1-\lambda+c)I_c^k(\lambda,\mu)zf'(z)}{I_c^k(\lambda,\mu)g(z)} + \frac{\lambda z\left[I_c^k(\lambda,\mu)zf'(z)\right]'}{I_c^k(\lambda,\mu)g(z)}}{(1-\lambda+c) + \frac{\lambda z\left[I_c^k(\lambda,\mu)g(z)\right]'}{I_c^k(\lambda,\mu)g(z)}} \\
&= \frac{(1-\lambda+c)p(z) + \lambda\left[p(z)q(z)+p'(z)\right]}{(1-\lambda+c)+\lambda q(z)} \\
&= p(z) + \frac{zp'(z)}{q(z)+\frac{(1-\lambda+c)}{\lambda}} \prec \psi(z).
\end{aligned}$$

Since $q(z) \prec \phi(z)$ and $Re\{\frac{1-\lambda+c}{\lambda}\} > 0$, $Re\{q(z)+\frac{(1-\lambda+c)}{\lambda}\} > 0$. Using Lemma 3.4, we conclude that $p(z) \prec \psi(z)$ and thus $f \in K_c^k(\lambda,\mu;\phi,\psi)$. By using a similar manner and (3.7), we obtain the second result. □

3.5.2 Inclusion properties involving $F_c f$

In this section, we determine properties of Bernardi operator and satisfies the following:

$$cI_c^k(\lambda,\mu)F_c[f(z)] + z\left[I_c^k(\lambda,\mu)F_c[f(z)]\right]' = (c+1)I_c^k(\lambda,\mu)f(z). \tag{3.13}$$

Theorem 3.20. *If $f \in S_c^k(\lambda,\mu;\phi)$ then $F_c f \in S_c^k(\lambda,\mu;\phi)$.*

Proof. Let $f \in S_c^k(\lambda,\mu;\phi)$ then $\frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)f(z)} \prec \phi(z)$.

Taking the differentiation on both sides of (3.13) and multiplying by z, we obtain

$$\frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)f(z)} = \frac{z\left[I_c^k(\lambda,\mu)F_c[f(z)]\right]'}{I_c^k(\lambda,\mu)F_c[f(z)]} + \frac{z\left(\frac{z\left[I_c^k(\lambda,\mu)F_c[f(z)]\right]'}{I_c^k(\lambda,\mu)F_c[f(z)]}\right)'}{\frac{z\left[I_c^k(\lambda,\mu)F_c[f(z)]\right]'}{I_c^k(\lambda,\mu)F_c[f(z)]}+c}.$$

Setting $p(z) = \frac{z\left[I_c^k(\lambda,\mu)F_c[f(z)]\right]'}{I_c^k(\lambda,\mu)F_c[f(z)]}$, we have

$$\frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)f(z)} = p(z) + \frac{zp'(z)}{p(z)+c}.$$

Lemma 3.2 implies $\frac{z\left[I_c^k(\lambda,\mu)F_c[f(z)]\right]'}{I_c^k(\lambda,\mu)F_c[f(z)]} \prec \phi(z)$. Hence $F_cf \in S_c^k(\lambda,\mu;\phi)$. □

Theorem 3.21. *If $f \in C_c^k(\lambda,\mu;\phi)$ then $F_cf \in C_c^k(\lambda,\mu;\phi)$.*

Proof. By using (3.8) and Theorem 3.20, we have

$$f \in C_c^k(\lambda,\mu;\phi) \Leftrightarrow zf'(z) \in S_c^k(\lambda,\mu;\phi) \Rightarrow F_c[zf'(z)] \in S_c^k(\lambda,\mu;\phi)$$

$$\Leftrightarrow z\left[F_c[f(z)]\right]' \in S_c^k(\lambda,\mu;\phi) \Leftrightarrow F_c[f(z)] \in C_c^k(\lambda,\mu;\phi).$$

□

Theorem 3.22. *If $\phi,\psi \in N$ and $f \in K_c^k(\lambda,\mu;\phi,\psi)$ then $F_cf \in K_c^k(\lambda,\mu;\phi,\psi)$.*

Proof. If$f \in K_c^k(\lambda,\mu;\phi,\psi)$ then there exists function $g \in S_c^k(\lambda,\mu;\phi)$ such that $\frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)g(z)} \prec \psi(z)$. Since $g \in S_c^k(\lambda,\mu;\phi)$, from Theorem 3.20, $F_c[f(z)] \in S_c^k(\lambda,\mu;\phi)$.

Then let

$$q(z) = \frac{z\left[I_c^k(\lambda,\mu)F_c[g(z)]\right]'}{I_c^k(\lambda,\mu)F_c[g(z)]} \prec \phi(z). \tag{3.14}$$

Set

$$p(z) = \frac{z\left[I_c^k(\lambda,\mu)F_c[f(z)]\right]'}{I_c^k(\lambda,\mu)F_c[g(z)]}. \tag{3.15}$$

By rearranging and differentiating (3.15), we obtain

$$\frac{\left[I_c^k(\lambda,\mu)F_c[zf'(z)]\right]'}{I_c^k(\lambda,\mu)F_c[g(z)]} = \frac{p(z)\left[I_c^k(\lambda,\mu)F_c[g(z)]\right]'}{I_c^k(\lambda,\mu)F_c[g(z)]} + \frac{\left[I_c^k(\lambda,\mu)F_c[g(z)]\right]p'(z)}{I_c^k(\lambda,\mu)F_c[g(z)]}.$$

Making use of (3.13), (3.15) and (3.14), it can be derived that

$$\frac{z\left[I_c^k(\lambda,\mu)f(z)\right]'}{I_c^k(\lambda,\mu)g(z)} = p(z) + \frac{zp'(z)}{c+q(z)}.$$

Hence, applying Lemma 3.4 we conclude that $p(z) \prec \psi(z)$ and it follows that $F_c f \in K_c^k(\lambda,\mu;\phi,\psi)$. □

CHAPTER 4

MULTIVALENT AND UNIVALENT HARMONIC FUNCTIONS

Ahuja and Jahangiri (2001) discussed and studied the class of multivalent harmonic functions and multivalent harmonic functions starlike order γ, $S_H^*(p,\gamma)$, $p \geq 1$ where $0 \leq \gamma < 1$. Since then, there are authors [see (Ahuja et al., 2009), (Jahangiri et al., 2009), (Rosihan et al., 2009), (Subramanian et al., 2012) and (Sharma and Khan, 2009)] introduced subclasses of multivalent harmonic functions using linear operators. For univalent harmonic functions, new subclasses defined by linear operators were obtained by Dixit et al. (2009), Murugusundaramoorthy et al. (2009) and Rosy et al. (2001). Furthermore, subclasses of univalent harmonic functions with respect to symmetric points were studied by Murugusundaramoorty et al. (2011) and Guney (2007). In this chapter, new subclasses of multivalent and univalent harmonic functions starlike of order γ using certain operators are introduced. Properties of functions in these classes are studied. Generally, we determine the extremal problems via coefficient conditions in all sections.

4.1 Multivalent harmonic functions defined by Dziok-Srivastava operator

Al-Kharsani and Al-Khal (2007) introduced a class of univalent harmonic functions starlike of order γ using the Dziok-Srivastava operator and studied some extremal problems. Now, we define subclasses of multivalent harmonic functions starlike of order γ using the same operator. We determine a sufficient condition bound, convolution condition, extreme points, convex combination and distortion bounds.

First, we define the Dziok-Srivastava operator for multivalent harmonic functions $f = h + \bar{g}$ given by (1.7) as follows:

$$H_p^{l,m}[\alpha_1] f(z) = H_p^{l,m}[\alpha_1] h(z) + \overline{H_p^{l,m}[\alpha_1] g(z)}$$

where

$H_p^{l,m}[\alpha_1] h(z) = z^p + \sum_{n=2}^{\infty} \phi_n a_{n+p-1} z^{n+p-1}$, $H_p^{l,m}[\alpha_1] g(z) = \sum_{n=1}^{\infty} \phi_n b_{n+p-1} z^{n+p-1}$

and

$$\phi_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}(n-1)!}. \tag{4.1}$$

$\alpha_1, ..., \alpha_l, \beta_1, ..., \beta_m$ are positive real numbers such that $l \leq m + 1$.

Denote by $S_H^*(p, \alpha_1, \gamma)$, the class of multivalent harmonic functions satisfying

$$Re \left\{ \frac{z \left(H_p^{l,m}[\alpha_1] h(z)\right)' - \overline{z \left(H_p^{l,m}[\alpha_1] g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1] h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1] g(z)\right)}} \right\} \geq p\gamma \tag{4.2}$$

for $p \geq 1, 0 \leq \gamma < 1, |z| = r < 1$.

Note that $S_H^*(1, \alpha_1, \gamma) \equiv S_H^*(\alpha_1, \gamma)$ is the class defined by Al-Kharsani and Al-Khal (2007). In the case of $l = m + 1$ and $\alpha_2 = \beta_1, \ldots, \alpha_l = \beta_m$, $S_H^\star(p, 1, \gamma) \equiv S_H^\star(p, \gamma)$ is investigated in Ahuja and Jahangiri (2001) and $S_H^\star(1, 1, \gamma) \equiv S_H^\star(\gamma)$ is the class introduced by Jahangiri (1999).

Further $T_H^*(p, \alpha_1, \gamma)$, $p \geq 1$ denotes the class of functions $f = h + \bar{g} \in S_H^*(p, \alpha_1, \gamma)$ where h and g are functions of the form

$$h(z) = z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} \quad , \quad g(z) = \sum_{n=1}^{\infty} |b_{n+p-1}| z^{n+p-1}. \tag{4.3}$$

Necessary coefficient conditions for the harmonic starlike functions and harmonic convex functions can be found in Clunie and Sheil-Small (1984) and Sheil-Small (1990). Now we derive sufficient coefficient bound for the class $S_H^*(p, \alpha_1, \gamma)$.

Theorem 4.1. *Let* $f = h + \bar{g}$ *be given by* (1.7) *and* $\prod_{i=1}^{l} (\alpha_i)_{n-1} \geq \prod_{j=1}^{m} (\beta_j)_{n-1} (n-1)!$. *If*

$$\sum_{n=2}^{\infty} \left\{ \frac{n + p\ (1-\gamma) - 1}{p\ (1-\gamma)} |a_{n+p-1}| + \frac{n + p\ (1+\gamma) - 1}{p\ (1-\gamma)} |b_{n+p-1}| \right\} |\phi_n| \leq 1 - \frac{1+\gamma}{1-\gamma} |b_p| \tag{4.4}$$

where $|b_p| < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$ *and* ϕ_n *is given by* (4.1) *then the harmonic function* f *is orientation preserving in* D *and* $f \in S_H^*(p, \alpha_1, \gamma)$.

Proof. The inequality $|h'(z)| \geq |g'(z)|$ is enough to show that f is orientation preserving. Note that

$$\begin{aligned}
|h'(z)| &\geq p\ |z|^{p-1} - \sum_{n=2}^{\infty} (n+p-1) |a_{n+p-1}| |z|^{n+p-2} \\
&= p|z|^{p-1} \left\{ 1 - \sum_{n=2}^{\infty} \frac{(n+p-1)}{p} |a_{n+p-1}| |z|^{n-1} \right\} \\
&\geq p|z|^{p-1} \left\{ 1 - \sum_{n=2}^{\infty} \frac{(n+p-1)}{p} |a_{n+p-1}| \right\} \\
&\geq |z|^{p-1} \left\{ 1 - \sum_{n=2}^{\infty} \frac{(n + p\ (1-\gamma) - 1)}{p\ (1-\gamma)} |\phi_n| |a_{n+p-1}| \right\}
\end{aligned}$$

By the hypothesis, since $|\phi_n| \geq 1$ and by (4.4),

$$\begin{aligned}
|h'(z)| &\geq |z|^{p-1} \left\{ \frac{1+\gamma}{1-\gamma}|b_p| + \sum_{n=2}^{\infty} \frac{(n+p\ (1+\gamma)-1)}{p\ (1-\gamma)} |\phi_n||b_{n+p-1}| \right\} \\
&= |z|^{p-1} \left\{ \sum_{n=1}^{\infty} \frac{(n+p\ (1+\gamma)-1)}{p\ (1-\gamma)} |\phi_n||b_{n+p-1}| \right\} \\
&\geq |z|^{p-1} \left\{ \sum_{n=1}^{\infty} (n+p-1)|b_{n+p-1}| \right\} \\
&\geq |z|^{p-1} \left\{ \sum_{n=1}^{\infty} (n+p-1)|b_{n+p-1}||z|^{n-1} \right\} \\
&= \sum_{n=1}^{\infty} (n+p-1)|b_{n+p-1}||z|^{n+p-2} \\
&= |g'(z)|.
\end{aligned}$$

Thus, f is orientation preserving in D.

Next, we prove $f \in S_H^*(p, \alpha_1, \gamma)$ by establishing the equation (4.2). First, let

$$w(z) = \frac{z\left(H_p^{l,m}[\alpha_1]\, h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\, g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]\, h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]\, g(z)\right)}} = \frac{A(z)}{B(z)}$$

where

$$\begin{aligned}
A(z) &= z\left(H_p^{l,m}[\alpha_1]\, h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\, g(z)\right)'} \\
B(z) &= \left(H_p^{l,m}[\alpha_1]\, h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]\, g(z)\right)}.
\end{aligned}$$

Now

$$
\begin{aligned}
&|A(z)+p\ (1-\gamma)B(z)|-|A(z)-p\ (1+\gamma)B(z)|\\
&\geq (2p-p\gamma)|z^p|-\sum_{n=2}^{\infty}(n+2p-p\gamma-1)|\phi_n a_{n+p-1}z^{n+p-1}|\\
&\quad -\sum_{n=1}^{\infty}(n+p\gamma-1)\overline{|\phi_n b_{n+p-1}z^{n+p-1}|}-p\gamma|z^p|\\
&\qquad\qquad -\sum_{n=2}^{\infty}(n-p\gamma-1)|\phi_n a_{n+p-1}z^{n+p-1}|\\
&\qquad\qquad -\sum_{n=1}^{\infty}(n+2p+p\gamma-1)\overline{|\phi_n b_{n+p-1}z^{n+p-1}|}
\end{aligned}
$$

$$
\begin{aligned}
&=2p\ (1-\gamma)|z^p|-\sum_{n=2}^{\infty}(2n+2p-2p\gamma-2)|\phi_n||a_{n+p-1}||z^{n+p-1}|\\
&\quad -\sum_{n=1}^{\infty}(2n+2p+2p\gamma-2)|\overline{\phi}_n||\overline{b}_{n+p-1}||\overline{z}^{n+p-1}|
\end{aligned}
$$

$$
=2p\ (1-\gamma)|z^p|
$$

$$
\left\{1-\sum_{n=2}^{\infty}\frac{(n+p-p\gamma-1)}{p\ (1-\gamma)}|\phi_n||a_{n+p-1}||z^{n-1}|-\sum_{n=1}^{\infty}\frac{(n+p+p\gamma-1)}{p\ (1-\gamma)}|\phi_n||b_{n+p-1}||z^{n-1}|\right\}
$$

$$
\geq 2p\ (1-\gamma)|z^p|
$$

$$
\left\{1-\sum_{n=2}^{\infty}\frac{(n+p-p\gamma-1)}{p\ (1-\gamma)}|\phi_n||a_{n+p-1}|-\sum_{n=1}^{\infty}\frac{(n+p+p\gamma-1)}{p\ (1-\gamma)}|\phi_n||b_{n+p-1}|\right\}
$$

$$
=2p\ (1-\gamma)|z^p|
$$

$$
\left\{1-\frac{1+\gamma}{1-\gamma}|b_p|-\left(\sum_{n=2}^{\infty}\left[\frac{(n+p-p\gamma-1)}{p\ (1-\gamma)}|a_{n+p-1}|+\frac{(n+p+p\gamma-1)}{p\ (1-\gamma)}|b_{n+p-1}|\right]|\phi_n|\right)\right\}
$$

The last expression is non-negative by (4.4). Since $Re\ w\geq p\gamma$ if and only if $|A(z)+p\ (1-\gamma)B(z)|\geq|A(z)-p\ (1+\gamma)B(z)|$, $f\in S_H^*(p,\alpha_1,\gamma)$. □

For $\sum_{n=1}^{\infty}(|x_{n+p-1}|+|\bar{y}_{n+p-1}|)=1$ and $x_p=0$, the function

$$f_1(z)=z^p+\sum_{n=2}^{\infty}\frac{p\ (1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|}x_{n+p-1}z^{n+p-1}+ \sum_{n=1}^{\infty}\frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|}\bar{y}_{n+p-1}\bar{z}^{n+p-1} \tag{4.5}$$

shows equality in the coefficient bound given by (4.4) is attained. For the function f_1 defined in (4.5) the coefficients are

$a_{n+p-1}=\frac{p\ (1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|}x_{n+p-1}$ and $b_{n+p-1}=\frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|}\bar{y}_{n+p-1}$,

and since the equation (4.4) holds, this implies $f_1\in S_H^*(p,\alpha_1,\gamma)$.

To show that the converse need not be true, consider the function

$$f(z)=z^p+\frac{p(1-\gamma)}{[1+p(1-\gamma)]\phi_2}z^{p+1}+\frac{\gamma-1}{2(1+\gamma)}\bar{z}^p.$$

It can be shown that

$$Re\left\{\frac{z\left[z^p+\frac{p(1-\gamma)}{[1+p(1-\gamma)]}z^{p+1}\right]'-\bar{z}\left[\frac{(\gamma-1)}{2(1+\gamma)}\bar{z}^p\right]'}{z^p+\frac{p(1-\gamma)}{[1+p(1-\gamma)]}z^{p+1}+\frac{(\gamma-1)}{2(1+\gamma)}\bar{z}^p}\right\}\geq p\gamma$$

$(p\geq 1, 0\leq\gamma<1)$

thus $f\in S_p^*(p,\alpha_1,\gamma)$ but

$$\sum_{n=2}^{\infty}\frac{n+p(1-\gamma)-1}{p(1-\gamma)}|a_{n+p-1}||\phi_n|+\sum_{n=1}^{\infty}\frac{n+p(1+\gamma)-1}{p(1-\gamma)}|b_{n+p-1}||\phi_n|$$
$$=\frac{1+p(1-\gamma)}{p(1-\gamma)}\left|\frac{p(1-\gamma)}{[1+p(1-\gamma)]\phi_2}\right||\phi_2|+\frac{1+\gamma}{1-\gamma}\left|\frac{\gamma-1}{2(1+\gamma)}\right|>1.$$

The next result provides a convolution condition for f to be in the class $S_H^*(p,\alpha_1,\gamma)$.

Theorem 4.2. *$f\in S_H^*(p,\alpha_1,\gamma)$ if and only if*

$$H_p^{l,m}[\alpha_1]\,h(z)*\left[\frac{2p(1-\gamma)z^p+(\xi-2p+2p\gamma+1)z^{p+1}}{(1-z)^2}\right]$$
$$-\ \overline{H_p^{l,m}[\alpha_1]\,g(z)}*\left[\frac{2p(\xi+\gamma)\bar{z}^p+(\xi-2p\xi-2p\gamma+1)\bar{z}^{p+1}}{(1-\bar{z})^2}\right]\neq 0, |\xi|=1, z\in\mathbf{D}.$$

Proof. A necessary and sufficient condition for $f \in S_H^*(p,\alpha_1,\gamma)$ is given by (4.2) and we have

$$Re\left\{\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}[\alpha_1]\,h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\,g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]\,h(z)\right)+\overline{\left(H_p^{l,m}[\alpha_1]\,g(z)\right)}}-p\gamma\right]\right\}\geq 0.$$

Since

$$\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}[\alpha_1]\,h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\,g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]\,h(z)\right)+\overline{\left(H_p^{l,m}[\alpha_1]\,g(z)\right)}}-p\gamma\right]$$
$$=1$$

at $z=0$, the above required condition is equivalent to

$$\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}[\alpha_1]\,h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\,g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]\,h(z)\right)+\overline{\left(H_p^{l,m}[\alpha_1]\,g(z)\right)}}-p\gamma\right]\neq\frac{\xi-1}{\xi+1}, \tag{4.6}$$
$$|\xi|=1,\ \xi\neq -1,\ 0<|z|<1.$$

Simple algebraic manipulation in (4.6) yields

$$0\neq(\xi+1)$$
$$\left\{z\left(H_p^{l,m}[\alpha_1]\,h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\,g(z)\right)'} - p\gamma H_p^{l,m}[\alpha_1]\,h(z) - p\gamma\overline{H_p^{l,m}[\alpha_1]\,g(z)}\right\}$$
$$-\quad(\xi-1)p(1-\gamma)H_p^{l,m}[\alpha_1]\,h(z)-(\xi-1)p(1-\gamma)\overline{H_p^{l,m}[\alpha_1]\,g(z)}$$

$$=H_p^{l,m}[\alpha_1]\,h(z)*\left\{(\xi+1)\left(\frac{z^p}{(1-z)^2}-\frac{(1-p)z^p}{(1-z)}\right)-\frac{(2p\gamma+p\xi-p)z^p}{(1-z)}\right\}$$
$$-\quad\overline{H_p^{l,m}[\alpha_1]\,g(z)}*\left\{\overline{(\bar{\xi}+1)\left(\frac{z^p}{(1-z)^2}-\frac{(1-p)z^p}{(1-z)}\right)}+\overline{\frac{(2p\gamma+p\bar{\xi}-p)z^p}{(1-z)}}\right\}$$

$$= H_p^{l,m}[\alpha_1]\,h(z) * \left[\frac{2p(1-\gamma)z^p + (\xi - 2p + 2p\gamma + 1)z^{p+1}}{(1-z)^2}\right]$$

$$- \quad \overline{H_p^{l,m}[\alpha_1]\,g(z)} * \left[\frac{2p(\xi+\gamma)\bar{z}^p + (\xi - 2p\xi - 2p\gamma + 1)\bar{z}^{p+1}}{(1-\bar{z})^2}\right].$$

$\square$

The coefficient bound for the class $T_H^*(p, \alpha_1, \gamma)$ is determined in the following theorem. Furthermore, we use the coefficient condition to obtain extreme points, convex combination and distortion upper and lower bounds.

Theorem 4.3. *Let $f = h + \bar{g}$ be given by (4.3). Then $f \in T_H^*(p, \alpha_1, \gamma)$ if and only if*

$$\sum_{n=2}^{\infty} \left\{ \frac{n + p\,(1-\gamma) - 1}{p\,(1-\gamma)} |a_{n+p-1}| + \frac{n + p\,(1+\gamma) - 1}{p\,(1-\gamma)} |b_{n+p-1}| \right\} |\phi_n| \le 1 - \frac{1+\gamma}{1-\gamma}|b_p| \tag{4.7}$$

where $|b_p| < \frac{1-\gamma}{1+\gamma}$, $0 \le \gamma < 1$ and ϕ_n is given by (4.1).

Proof. Since $T_H^\star(p, \alpha_1, \gamma) \subset S_H^\star(p, \alpha_1, \gamma)$, sufficiency part follows from Theorem 4.1. To prove the necessity part, suppose that $f \in T_H^\star(p, \alpha_1, \gamma)$. Then we obtain

$$Re\left\{ \frac{1}{p(1-\gamma)} \left[\frac{z\left(H_p^{l,m}[\alpha_1]\,h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]\,g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]\,h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]\,g(z)\right)}} - p\gamma \right] \right\}$$

$$= Re\left\{ \frac{z^p - \sum_{n=2}^{\infty} \frac{(n+p(1-\gamma)-1)}{p(1-\gamma)} |a_{n+p-1}| \phi_n z^{n+p-1} - \sum_{n=1}^{\infty} \frac{n+p(1+\gamma)-1)}{p(1-\gamma)} |\bar{b}_{n+p-1}| \bar{\phi}_n \bar{z}^{n+p-1}}{z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| \phi_n z^{n+p-1} + \sum_{n=1}^{\infty} |\bar{b}_{n+p-1}| \bar{\phi}_n \bar{z}^{n+p-1}} \right\}$$

≥ 0

The condition must hold for all values of $z, |z| = r < 1$. Choosing the values of z on the positive specific values, $0 \leq z = r < 1$, and ϕ_n is real, we have

$$\frac{1 - \left(\sum_{n=2}^{\infty} \frac{(n+p(1-\gamma)-1)}{p(1-\gamma)} |a_{n+p-1}| \phi_n r^{n-1} + \sum_{n=1}^{\infty} \frac{n+p(1+\gamma)-1)}{p(1-\gamma)} |b_{n+p-1}| \phi_n r^{n-1}\right)}{1 - \sum_{n=2}^{\infty} |a_{n+p-1}| \phi_n r^{n-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| \phi_n r^{n-1}} \geq 0 \tag{4.8}$$

Letting $r \to 1^-$ and if the condition (4.7) does not hold, then the numerator in (4.8) is negative. There exists a $z_0 = r_0 \in (0, 1)$ such that (4.8) is negative and this contradicts the required condition for $f \in T_H^*(p, \alpha_1, \gamma)$. $\square$

Let *clco* $T_H^*(p, \alpha_1, \gamma)$ denotes the closed convex hull of $T_H^*(p, \alpha_1, \gamma)$. Now we determine the extreme points of *clco* $T_H^*(p, \alpha_1, \gamma)$.

Theorem 4.4. *Let f be given by* (4.3)*. Then $f \in$ clco $T_H^*(p, \alpha_1, \gamma)$ if and only if f can be expressed in the form*

$$f = \sum_{n=1}^{\infty} (X_{n+p-1} h_{n+p-1} + Y_{n+p-1} g_{n+p-1}) \tag{4.9}$$

where

$$h_p = z^p, \quad h_{n+p-1}(z) = z^p - \frac{p(1-\gamma)}{[n + p(1-\gamma) - 1]|\phi_n|} z^{n+p-1} \quad (n = 2, 3, ...),$$

$$g_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{[n + p(1+\gamma) - 1]|\phi_n|} \overline{z}^{n+p-1} \quad (n = 1, 2, 3, ...),$$

ϕ_n is given by (4.1) *and $\sum_{n=1}^{\infty} (X_{n+p-1} + Y_{n+p-1}) = 1$, with $X_{n+p-1} \geq 0, Y_{n+p-1} \geq 0$. In particular, the extreme points of $T_H^*(p, \alpha_1, \gamma)$ are h_{n+p-1} and g_{n+p-1}.*

Proof. Let f be of the form (4.9). Then we have

$$f(z) = X_p h_p + \sum_{n=2}^{\infty} X_{n+p-1}\left(z^p - \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} z^{n+p-1}\right)$$
$$+ \sum_{n=1}^{\infty} Y_{n+p-1}\left(z^p + \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} \bar{z}^{n+p-1}\right)$$

$$f(z) = z^p - \sum_{n=2}^{\infty} \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} X_{n+p-1} z^{n+p-1}$$
$$+ \sum_{n=1}^{\infty} \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} Y_{n+p-1} \bar{z}^{n+p-1}.$$

Furthermore, let

$|a_{n+p-1}| = \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|} X_{n+p-1}$ and $|b_{n+p-1}| = \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|} Y_{n+p-1}$.

Then

$$\sum_{n=2}^{\infty} \frac{[n+p\ (1-\gamma)-1]\,|\phi_n|}{p\ (1-\gamma)} |a_{n+p-1}| + \sum_{n=1}^{\infty} \frac{[n+p\ (1+\gamma)-1]\,|\phi_n|}{p\ (1-\gamma)} |b_{n+p-1}|$$

$$= \sum_{n=2}^{\infty} \frac{[n+p\ (1-\gamma)-1]\,|\phi_n|}{p\ (1-\gamma)} \left(\frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} X_{n+p-1}\right)$$
$$+ \sum_{n=1}^{\infty} \frac{[n+p\ (1+\gamma)-1]\,|\phi_n|}{p\ (1-\gamma)} \left(\frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} Y_{n+p-1}\right)$$

$$= \sum_{n=2}^{\infty} X_{n+p-1} + \sum_{n=1}^{\infty} Y_{n+p-1}$$

$$= 1 - X_p \leq 1.$$

Thus $f \in clco\ T_H^*(p, \alpha_1, \gamma)$.

Conversely, suppose that $f \in clco\ T_H^*(p, \alpha_1, \gamma)$. Set

$$X_{n+p-1} = \frac{[n+p\ (1-\gamma)-1]|\phi_n||a_{n+p-1}|}{p\ (1-\gamma)} \quad (n = 2, 3, ...),$$

$$Y_{n+p-1} = \frac{[n+p\ (1+\gamma)-1]|\phi_n||b_{n+p-1}|}{p\ (1-\gamma)} \quad (n=1,2,...),$$

and define $X_p = 1 - \sum_{n=2}^{\infty} X_{n+p-1} - \sum_{n=1}^{\infty} Y_{n+p-1}$. Then

$$f(z) = z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| \bar{z}^{n+p-1}$$

$$= z^p - \sum_{n=2}^{\infty} \frac{p\ (1-\gamma) X_{n+p-1}}{[n+p\ (1-\gamma)-1]|\phi_n|} z^{n+p-1} + \sum_{n=1}^{\infty} \frac{p\ (1-\gamma) Y_{n+p-1}}{[n+p\ (1+\gamma)-1]|\phi_n|} \bar{z}^{n+p-1}$$

$$= X_p z^p + \sum_{n=2}^{\infty} X_{n+p-1} \left(z^p - \frac{p\ (1-\gamma)}{[n+p\ (1-\gamma)-1]|\phi_n|} z^{n+p-1} \right) + \sum_{n=1}^{\infty} Y_{n+p-1} \left(z^p + \frac{p\ (1-\gamma)}{[n+p\ (1+\gamma)-1]|\phi_n|} \bar{z}^{n+p-1} \right)$$

$$= \sum_{n=1}^{\infty} (X_{n+p-1} h_{n+p-1} + Y_{n+p-1} g_{n+p-1})$$

as required. □

Theorem 4.5. *The class $T_H^*(p, \alpha_1, \gamma)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$, suppose that $f_i(z) \in T_H^*(p, \alpha_1, \gamma)$ where f_i is given by

$$f_i(z) = z^p - \sum_{n=2}^{\infty} |a_{i_{n+p-1}}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{i_{n+p-1}}| \bar{z}^{n+p-1}$$

By Theorem 4.3,

$$\sum_{n=2}^{\infty} \frac{n+p\ (1-\gamma)-1}{p\ (1-\gamma)} |\phi_n||a_{i_{n+p-1}}| + \sum_{n=1}^{\infty} \frac{n+p\ (1+\gamma)-1}{p\ (1-\gamma)} |\phi_n||b_{i_{n+p-1}}| \leq 1. \quad (4.10)$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^p - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_{n+p-1}}| z^{n+p-1} \right) + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_{n+p-1}}| \bar{z}^{n+p-1} \right).$$

Then, by (4.10)

$$\sum_{n=2}^{\infty}\left(\frac{[n+p\ (1-\gamma)-1]|\phi_n|}{p\ (1-\gamma)}\right)\left(\left|\sum_{i=1}^{\infty} t_i|a_{i_{n+p-1}}|\right|\right)$$
$$+\sum_{n=1}^{\infty}\left(\frac{[n+p\ (1+\gamma)-1]|\phi_n|}{p\ (1-\gamma)}\right)\left(\left|\sum_{i=1}^{\infty} t_i|b_{i_{n+p-1}}|\right|\right)$$
$$=\sum_{i=1}^{\infty} t_i$$
$$\left\{\sum_{n=2}^{\infty}\frac{[n+p\ (1-\gamma)-1]|\phi_n|}{p\ (1-\gamma)}|a_{i_{n+p-1}}|+\sum_{n=1}^{\infty}\frac{[n+p\ (1+\gamma)-1]]|\phi_n|}{p\ (1-\gamma)}|b_{i_{n+p-1}}|\right\}$$
$$\leq \sum_{i=1}^{\infty} t_i\,(1)=1.$$

Hence, $\sum_{i=1}^{\infty} t_i f_i(z) \in T_H^*(p,\alpha_1,\gamma)$. □

In the last theorem below we give distortion inequalities for f in the class $T_H^*(p,\alpha_1,\gamma)$.

Theorem 4.6. *If $f \in T_H^*(p,\alpha_1,\gamma)$ with $\phi_n \geq \phi_2$, then for $|z|=r<1$,*

$$|f(z)| \leq (1+|b_p|)\,r^p + r^{p+1}\left\{\frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|}-\frac{p\ (1+\gamma)|b_p|}{[p\ (1-\gamma)+1]|\phi_2|}\right\}$$

and

$$|f(z)| \geq (1-|b_p|)\,r^p - r^{p+1}\left\{\frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|}-\frac{p\ (1+\gamma)|b_p|}{[p\ (1-\gamma)+1]|\phi_2|}\right\}.$$

Proof. Since

$$\frac{p\ (1-\gamma)+1}{p\ (1-\gamma)}|\phi_2|\sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)$$
$$\leq \sum_{n=2}^{\infty}\frac{n+p\ (1-\gamma)-1}{p\ (1-\gamma)}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)|\phi_n|$$
$$\leq \sum_{n=2}^{\infty}\left(\frac{n+p\ (1-\gamma)-1}{p\ (1-\gamma)}|a_{n+p-1}|+\frac{n+p\ (1+\gamma)-1}{p\ (1-\gamma)}|b_{n+p-1}|\right)|\phi_n|,$$

the result of Theorem 4.3 gives

$$\sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)\leq \frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|}\left\{1-\frac{1+\gamma}{1-\gamma}|b_p|\right\}. \tag{4.11}$$

Next, again since $f \in T_H^*(p, \alpha_1, \gamma)$, we have from (4.11) and $|z| = r$ that

$$
\begin{aligned}
|f(z)| &= \left| z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| \bar{z}^{n+p-1} \right| \\
&\leq |z^p| + \sum_{n=2}^{\infty} |a_{n+p-1}|\, |z|^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}|\, |\bar{z}|^{n+p-1} \\
&= r^p + \sum_{n=2}^{\infty} |a_{n+p-1}| r^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| r^{n+p-1} \\
&\leq (1 + |b_p|) r^p + \left(\sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \right) r^{p+1} \\
&\leq (1 + |b_p|) r^p + r^{p+1} \left\{ \frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|} - \frac{p\ (1+\gamma)|b_p|}{[p\ (1-\gamma)+1]|\phi_2|} \right\}
\end{aligned}
$$

which gives the first result.

In a similar manner, we obtain the following lower bound.

$$
\begin{aligned}
|f(z)| &\geq r^p - \sum_{n=2}^{\infty} |a_{n+p-1}| r^{n+p-1} - \sum_{n=1}^{\infty} |b_{n+p-1}| r^{n+p-1} \\
&= (1 - |b_p|) r^p - \sum_{n=2}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|)\, r^{n+p-1} \\
&\geq (1 - |b_p|) r^p - r^{p+1} \left\{ \frac{p\ (1-\gamma)}{[p\ (1-\gamma)+1]|\phi_2|} - \frac{p\ (1+\gamma)|b_p|}{[p\ (1-\gamma)+1]|\phi_2|} \right\}.
\end{aligned}
$$

□

4.2 Subclasses of univalent harmonic functions

In 1983, Sălăgean introduced an operator $D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n,\ \ k = 0, 1, 2, \ldots$.

Then, Al-Oboudi (2004) derived the generalised Sălăgean operator as:

$$
D_\lambda^k f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k\, a_n z^n
$$

where $\lambda \geq 0, k \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Combining the generalised hypergeometric functions and the generalised Sălăgean operator, we establish the operator $\mathbf{H}_\lambda^k$ defined as below:

$$\begin{aligned}\mathbf{H}_\lambda^k(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) &= z{}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) * D_\lambda^k f(z)\\ &= z+\sum_{n=2}^{\infty}\Phi_{n,\lambda}^k a_n z^n\end{aligned}$$

where

$$\Phi_{n,\lambda}^k = \frac{[1+(n-1)\lambda]^k\,(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}(n-1)!}, \tag{4.12}$$

$\alpha_1,...,\alpha_l,\beta_1,...,\beta_m$ are positive real numbers such that $l \leq m+1$. For convenience we write $\mathbf{H}_\lambda^k(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) = \mathbf{H}_\lambda^k f(z)$.

Observe that when $k=0$, the linear operator $\mathbf{H}_\lambda^k f(z)$ reduces to the Dziok-Srivastava operator (Dziok and Srivastava, 1999) which includes well known operators such as the Hohlov operator (Hohlov, 1978), Carlson-Shaffer operator (Carlson and Shaffer, 1984), Ruscheweyh derivative operator (Ruscheweyh, 1975b) and the generalised Bernadi-Libera-Livington integral operator (Bernadi, 1969), (Libera, 1965), (Livington, 1966).

Also for the case $k=0$, $\mathbf{H}_\lambda^k(1,2;\mu+1) \equiv I_\mu f(z)$[Noor operator (Noor, 1999)], $\mathbf{H}_\lambda^k(1,\zeta;\mu+1) \equiv I_{\mu,\zeta} f(z)$[Choi-Saigo-Srivastava operator (Choi et al., 2002)] and $\mathbf{H}_\lambda^k(\mu,1,\beta_1,\ldots,\beta_m;\alpha_1,\ldots,\alpha_l) \equiv H_\mu^{l,m}[\alpha_1]\, f(z)$[Kwon-Cho operator (Kwon and Cho, 2007)].

Furthermore, in the case $\alpha_2=\beta_1,\ldots,\alpha_l=\beta_m$, the operator $\mathbf{H}_\lambda^k f(z)$ reduces to the Sălăgean operator ($\lambda=\alpha_1=1$) (Salagean, 1983), generalising Sălăgean operator ($\alpha_1=1$) (Al-Oboudi, 2004) and Ruscheweyh derivative operator ($k=1,\alpha_1=\mu+1$)

(Maslina and Al-Shaqsi, 2006).

The operator $\mathbf{H}_\lambda^k f(z)$ for harmonic functions $f = h + \bar{g}$ given by (1.5) is defined as

$$\mathbf{H}_\lambda^k f(z) = \mathbf{H}_\lambda^k h(z) + \overline{\mathbf{H}_\lambda^k g(z)}$$

where $\mathbf{H}_\lambda^k h(z) = z + \sum_{n=2}^{\infty} \Phi_{n,\lambda}^k a_n z^n$ and $\mathbf{H}_\lambda^k g(z) = \sum_{n=1}^{\infty} \Phi_{n,\lambda}^k b_n z^n$.

Two classes using the operator $\mathbf{H}_\lambda^k f$ are introduced. For $0 \leq \gamma < 1, \lambda \geq 0$, let $S_H^\star(\lambda, k, \alpha_1, \gamma)$ denote the class of univalent harmonic functions starlike of order γ satisfying

$$Re\left\{\frac{z\left(H_\lambda^k f(z)\right)'}{H_\lambda^k f(z)}\right\} = Re\left\{\frac{z\left(\mathbf{H}_\lambda^k h(z)\right)' - \overline{z\left(\mathbf{H}_\lambda^k g(z)\right)'}}{\mathbf{H}_\lambda^k h(z) + \overline{\mathbf{H}_\lambda^k g(z)}}\right\} \geq \gamma$$

where $\left[\mathbf{H}_\lambda^k f(z)\right]' = \frac{\partial}{\partial\theta}\left[\mathbf{H}_\lambda^k f(re^{i\theta})\right]$.

Further denote $T_H^\star(\lambda, k, \alpha_1, \gamma)$ as the class of functions $f = h+\bar{g} \in S_H^\star(\lambda, k, \alpha_1, \gamma)$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n. \tag{4.13}$$

Theorem 4.7. *If f is of the form* (1.5) *and*

$$[1 + (n-1)\lambda]^k \prod_{i=1}^{l} (\alpha_i)_{n-1} \geq \prod_{j=1}^{m} (\beta_j)_{n-1} (n-1)!.$$

If

$$\sum_{n=2}^{\infty}\left\{\frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n|\right\}|\Phi_{n,\lambda}^k| \leq 1 - \frac{1+\gamma}{1-\gamma}|b_1| \tag{4.14}$$

where $|b_1| < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$ and $\Phi_{n,\lambda}^k$ is given by (4.12),

then the harmonic function f is orientation preserving in $\mathbf{D}$ and $f \in S_H^\star(\lambda, k, \alpha_1, \gamma)$.

Proof. By differentiation of h and the hypothesis of the theorem, $|\Phi_{n,\lambda}^k| \geq 1$, we obtain

$$\begin{aligned}
|h'(z)| &= \left|1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right| \\
&\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \\
&\geq 1 - \sum_{n=2}^{\infty} \frac{n-\gamma}{1-\gamma}|a_n||\Phi_{n,\lambda}^k| \\
&\geq \sum_{n=2}^{\infty} \frac{n+\gamma}{1-\gamma}|b_n||\Phi_{n,\lambda}^k| \\
&\geq \sum_{n=1}^{\infty} n b_n z^{n-1} \\
&= |g'(z)|
\end{aligned}$$

which implies f is orientation preserving in $\mathbf{D}$.

To prove $f \in S_H^{\star}(\lambda, k, \alpha_1, \gamma)$, let

$$w = \frac{z\left(\mathbf{H}_\lambda^k h(z)\right)' - \overline{z\left(\mathbf{H}_\lambda^k g(z)\right)'}}{\mathbf{H}_\lambda^k h(z) + \overline{\mathbf{H}_\lambda^k g(z)}} = \frac{A(z)}{B(z)},$$

where

$$A(z) = z + \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^k a_n z^n - \sum_{n=1}^{\infty} \overline{n\Phi_{n,\lambda}^k b_n z^n}$$

and

$$B(z) = z + \sum_{n=2}^{\infty} \Phi_{n,\lambda}^k a_n z^n + \sum_{n=1}^{\infty} \overline{\Phi_{n,\lambda}^k b_n z^n}.$$

Notice that $Re\ w \geq \gamma$ if and only if $|A(z) + (1-\gamma)B(z)| \geq |A(z) - (1+\gamma)B(z)|$.

Thus

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)|$$

$$\geq (2-\gamma)|z| - \sum_{n=2}^{\infty}(n+1-\gamma)|\Phi_{n,\lambda}^{k}||a_n||z^n| - \sum_{n=1}^{\infty}(n-1+\gamma)|\Phi_{n,\lambda}^{k}||b_n||z^n| - \gamma|z|$$
$$- \sum_{n=2}^{\infty}(n-1-\gamma)|\Phi_{n,\lambda}^{k}||a_n||z^n| - \sum_{n=1}^{\infty}(n+1+\gamma)|\Phi_{n,\lambda}^{k}||b_n||z^n|$$

$$= 2(1-\gamma)|z| - \sum_{n=2}^{\infty}(2n-2\gamma)|\Phi_{n,\lambda}^{k}||a_n||z^n| - \sum_{n=1}^{\infty}(2n+2\gamma)|\Phi_{n,\lambda}^{k}||b_n||z^n|$$

$$= 2(1-\gamma)|z| - \sum_{n=2}^{\infty}\frac{2(n-\gamma)(1-\gamma)}{1-\gamma}|\Phi_{n,\lambda}^{k}||a_n||z^n| - \sum_{n=1}^{\infty}\frac{2(n+\gamma)(1-\gamma)}{1-\gamma}|\Phi_{n,\lambda}^{k}||b_n||z^n|$$

$$= 2(1-\gamma)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}||a_n||z^{n-1}| - \sum_{n=1}^{\infty}\frac{n+\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}||b_n||z^{n-1}|\right\}$$

$$\geq 2(1-\gamma)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}||a_n| - \sum_{n=1}^{\infty}\frac{n+\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}||b_n|\right\}$$

$$= 2(1-\gamma)|z|\left\{1 - \frac{1+\gamma}{1-\gamma}|b_1| - \sum_{n=2}^{\infty}\left(\frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n|\right)|\Phi_{n,\lambda}^{k}|\right\}$$

This last expression is non-negative by (4.14), and thus $f \in S_H^{\star}(\lambda, k, \alpha_1, \gamma)$. $\square$

The result on distortion bounds for f in the class $S_H^{\star}(\lambda, k, \alpha_1, \gamma)$ is given in the following theorem.

Theorem 4.8. *If $f \in S_H^{\star}(\lambda, k, \alpha_1, \gamma)$ with $\Phi_{n,\lambda}^{k} \geq \Phi_{2,\lambda}^{k}$ then for $|z| = r < 1$,*

$$|f(z)| \leq (1+|b_1|)\,r + r^2\left\{\frac{1-\gamma}{(2-\gamma)|\Phi_{2,\lambda}^{k}|} - \frac{(1+\gamma)|b_1|}{(2-\gamma)|\Phi_{2,\lambda}^{k}|}\right\}$$

and

$$|f(z)| \geq (1-|b_1|)\,r - r^2\left\{\frac{1-\gamma}{(2-\gamma)|\Phi_{2,\lambda}^{k}|} - \frac{(1+\gamma)|b_1|}{(2-\gamma)|\Phi_{2,\lambda}^{k}|}\right\}$$

Proof. Since

$$\frac{2-\gamma}{1-\gamma}\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|\Phi_{2,\lambda}^{k}| \leq \sum_{n=2}^{\infty}\frac{n-\gamma}{1-\gamma}\left(|a_n|+|b_n|\right)|\Phi_{n,\lambda}^{k}|$$

$$\leq \sum_{n=2}^{\infty}\frac{n-\gamma}{1-\gamma}|a_n||\Phi_{n,\lambda}^{k}|+\frac{n+\gamma}{1-\gamma}|b_n||\Phi_{n,\lambda}^{k}|$$

$$\leq \left\{1-\frac{1+\gamma}{1-\gamma}|b_1|\right\},$$

for $0<|z|=r<1$, we prove the result by considering the above inequality.

$$\begin{aligned}
|f(z)| &= \left|z+\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^n\right| \\
&\leq |z|+\sum_{n=2}^{\infty}|a_n|\,|z|^n+\sum_{n=1}^{\infty}|b_n|\,|\bar{z}|^n \\
&= (1+|b_1|)|z|+\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|z|^n \\
&\leq (1+|b_1|)|z|+\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|z|^2 \\
&= (1+|b_1|)r+r^2\left\{\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)\right\} \\
&\leq (1+|b_1|)r+r^2\left\{\frac{1-\gamma}{(2-\gamma)|\Phi_{2,\lambda}^{k}|}-\frac{1+\gamma}{(2-\gamma)|\Phi_{2,\lambda}^{k}|}|b_1|\right\}.
\end{aligned}$$

The lower bound of f can be derived using a similar manner.

$$\begin{aligned}
|f(z)| &\geq |z|-\sum_{n=2}^{\infty}|a_n|\,|z|^n-\sum_{n=1}^{\infty}|b_n|\,|z|^n \\
&\geq (1-|b_1|)|z|-\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|z|^2 \\
&\geq (1-|b_1|)r-r^2\left\{\frac{1-\gamma}{(2-\gamma)|\Phi_{2,\lambda}^{k}|}-\frac{1+\gamma}{(2-\gamma)|\Phi_{2,\lambda}^{k}|}|b_1|\right\}.
\end{aligned}$$

□

Next, we prove the hypothesis in Theorem 4.7 is a necessary and sufficient condition for f to be in the class $T_H^{\star}(\lambda,k,\alpha_1,\gamma)$.

Theorem 4.9. *Let $f = h + \bar{g}$ be given by (4.13), $f \in T_H^{\star}(\lambda, k, \alpha_1, \gamma)$ if and only if*

$$\sum_{n=2}^{\infty} \left\{ \frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n| \right\} |\Phi_{n,\lambda}^k| \leq 1 - \frac{1+\gamma}{1-\gamma}|b_1| \tag{4.15}$$

where $|b_1| < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$ and $\Phi_{n,\lambda}^k$ is given by (4.12).

Proof. In view of the fact that $T_H^{\star}(\lambda, k, \alpha_1, \gamma) \subset S_H^{\star}(\lambda, k, \alpha_1, \gamma)$, the 'if' part follows from Theorem 4.7. For 'only if' part, assume that $f \in T_H^{\star}(\lambda, k, \alpha_1, \gamma)$. Therefore, we have

$$Re \left\{ \frac{z\left(\mathbf{H}_\lambda^k h(z)\right)' - \overline{z\left(\mathbf{H}_\lambda^k g(z)\right)'}}{\mathbf{H}_\lambda^k h(z) + \overline{\mathbf{H}_\lambda^k g(z)}} - \gamma \right\}$$

$$= Re \left\{ \frac{1}{1-\gamma} \left(\frac{z - \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^k |a_n| z^n - \sum_{n=1}^{\infty} \overline{n\Phi_{n,\lambda}^k |b_n| z^n}}{z - \sum_{n=2}^{\infty} \Phi_{n,\lambda}^k |a_n| z^n + \sum_{n=1}^{\infty} \overline{\Phi_{n,\lambda}^k |b_n| z^n}} - \gamma \right) \right\}$$

$$= Re \left\{ \frac{z - \sum_{n=2}^{\infty} \frac{n-\gamma}{1-\gamma} \Phi_{n,\lambda}^k |a_n| z^n - \sum_{n=1}^{\infty} \frac{n+\gamma}{1-\gamma} \overline{\Phi_{n,\lambda}^k |b_n| z^n}}{z - \sum_{n=2}^{\infty} \Phi_{n,\lambda}^k |a_n| z^n + \sum_{n=1}^{\infty} \overline{\Phi_{n,\lambda}^k |b_n| z^n}} \right\}$$

$$\geq 0.$$

The same condition as in previous case, the above inequality reduces to

$$\frac{1 - \sum_{n=2}^{\infty} \frac{n-\gamma}{1-\gamma} \Phi_{n,\lambda}^k |a_n| r^{n-1} - \sum_{n=1}^{\infty} \frac{n+\gamma}{1-\gamma} \overline{\Phi_{n,\lambda}^k |b_n| r^{n-1}}}{1 - \sum_{n=2}^{\infty} \Phi_{n,\lambda}^k |a_n| r^{n-1} + \sum_{n=1}^{\infty} \overline{\Phi_{n,\lambda}^k |b_n| r^{n-1}}} \geq 0$$

and the result follows by letting $r \to 1^-$ along real axis. □

The following result gives extreme points of *clco* $T_H^{\star}(\lambda, k, \alpha_1, \gamma)$ where *clco* $T_H^{\star}(\lambda, k, \alpha_1, \gamma)$ denotes the closed convex hull of $T_H^{*}(\lambda, k, \alpha_1, \gamma)$.

Theorem 4.10. *Let $f = h + \bar{g}$ be given by (4.13). Then $f \in$ clco $T_H^{\star}(\lambda, k, \alpha_1, \gamma)$ if and only if f can be expressed in the form*

$$f = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \tag{4.16}$$

where

$$h_1(z) = z, \quad h_n(z) = z - \frac{1-\gamma}{(n-\gamma)|\Phi^k_{n,\lambda}|} z^n \quad (n = 2, 3, ...),$$

$$g_n(z) = z + \frac{1-\gamma}{(n+\gamma)|\Phi^k_{n,\lambda}|} \bar{z}^n \quad (n = 1, 2, 3, ...),$$

$\Phi^k_{n,\lambda}$ *is given by* (4.12) *and* $\sum_{n=1}^{\infty}(X_n + Y_n) = 1$, *with* $X_n \geq 0, Y_n \geq 0$. *In particular the extreme points of* $T^{\star}_H(\lambda, k, \alpha_1, \gamma)$ *are* h_n *and* g_n.

Proof. Let f be of the form (4.16). Then we have

$$f(z) = \sum_{n=1}^{\infty}(X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\gamma}{(n-\gamma)|\Phi^k_{n,\lambda}|} X_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{(n+\gamma)|\Phi^k_{n,\lambda}|} Y_n \bar{z}^n$$

$$= z - \sum_{n=2}^{\infty} \frac{1-\gamma}{(n-\gamma)|\Phi^k_{n,\lambda}|} X_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{(n+\gamma)|\Phi^k_{n,\lambda}|} Y_n \bar{z}^n.$$

Furthermore, let $|a_n| = \frac{1-\gamma}{(n-\gamma)|\Phi^k_{n,\lambda}|} X_n$ and $|b_n| = \frac{1-\gamma}{(n+\gamma)|\Phi^k_{n,\lambda}|} Y_n$.

Applying Theorem 4.9, gives

$$\sum_{n=2}^{\infty} \frac{(n-\gamma)|\Phi^k_{n,\lambda}|}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{(n+\gamma)|\Phi^k_{n,\lambda}|}{1-\gamma} |b_n|$$

$$= \sum_{n=2}^{\infty} \frac{(n-\gamma)|\Phi^k_{n,\lambda}|}{1-\gamma} \frac{1-\gamma}{(n-\gamma)|\Phi^k_{n,\lambda}|} X_n + \sum_{n=1}^{\infty} \frac{(n+\gamma)|\Phi^k_{n,\lambda}|}{1-\gamma} \frac{1-\gamma}{(n+\gamma)|\Phi^k_{n,\lambda}|} Y_n$$

$$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n$$

$$= 1 - X_1 \leq 1.$$

Thus $f \in clco\ T_H^{\star}(\lambda, k, \alpha_1, \gamma)$.

Conversely, let $f \in clco\ T_H^{\star}(\lambda, k, \alpha_1, \gamma)$. Setting

$$X_n = \frac{(n-\gamma)|\Phi_{n,\lambda}^k||a_n|}{1-\gamma} \quad (n = 2, 3, ...),$$

$$Y_n = \frac{(n+\gamma)|\Phi_{n,\lambda}^k||b_n|}{1-\gamma} \quad (n = 1, 2, ...)$$

and define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$.

Thus,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(1-\gamma)X_n}{(n-\gamma)|\Phi_{n,\lambda}^k|} z^n + \sum_{n=1}^{\infty} \frac{(1-\gamma)Y_n}{(n+\gamma)|\Phi_{n,\lambda}^k|} \bar{z}^n$$

$$f(z) = X_1 z + \sum_{n=2}^{\infty} X_n \left\{ z - \frac{(1-\gamma)z^n}{(n-\gamma)|\Phi_{n,\lambda}^k|} \right\} + \sum_{n=1}^{\infty} Y_n \left\{ z + \frac{(1-\gamma)\bar{z}^n}{(n+\gamma)|\Phi_{n,\lambda}^k|} \right\}$$

$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$ as required. □

Theorem 4.11. *The class $T_H^{\star}(\lambda, k, \alpha_1, \gamma)$ is closed under convex combination.*

Proof. Suppose that for $i = 1, 2, 3, ..., f_i(z) \in T_H^{\star}(\lambda, \alpha_1, \gamma)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=1}^{\infty} |b_{n,i}| \bar{z}^n.$$

From Theorem 4.9,

$$\sum_{n=2}^{\infty} \left(\frac{n-\gamma}{1-\gamma} |\Phi_{n,\lambda}^k||a_{n,i}| \right) + \sum_{n=1}^{\infty} \left(\frac{n+\gamma}{1-\gamma} |\Phi_{n,\lambda}^k||b_{n,i}| \right) \leq 1. \qquad (4.17)$$

For $\sum_{i=1}^{\infty} t_i = 1$ where $\forall i,\ 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| z^n \right) + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \bar{z}^n \right).$$

Then, by (4.17)

$$\sum_{n=2}^{\infty}\left(\frac{n-\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}|\right)\left(\left|\sum_{i=1}^{\infty}t_i|a_{n,i}|\right|\right)+\sum_{n=1}^{\infty}\left(\frac{n+\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}|\right)\left(\left|\sum_{i=1}^{\infty}t_i|b_{n,i}|\right|\right)$$

$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{n=2}^{\infty}\left(\frac{n-\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}||a_{n,i}|\right)+\sum_{n=1}^{\infty}\left(\frac{n+\gamma}{1-\gamma}|\Phi_{n,\lambda}^{k}||b_{n,i}|\right)\right\}$$

$$\leq\sum_{i=1}^{\infty}t_i\,(1)=1.$$

Hence, $\sum_{i=1}^{\infty}t_if_i(z)\in T_H^{\star}(\lambda,k,\alpha_1,\gamma)$. $\square$

4.3 Subclasses of harmonic functions with respect to symmetric points

The class of analytic univalent functions in the unit disk which are starlike with respect to symmetrical points was first introduced by Sakaguchi (1959). Since then, some authors [for examples see Guney (2007), Aini et al. (2008), Aini and Suzeini (2009) and Murugusundaramoorthy et al. (2011)] have studied the classes of harmonic starlike and convex functions with respect to symmetrical points motivated by Jahangiri (1999) and Ahuja and Jahangiri (2004). In (Ahuja and Jahangiri, 2004), for $0\leq\gamma<1$, the authors introduced the class $S_{HS}^{\star}(\gamma)$ which denote the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1.5) and satisfying condition

$$Im\left\{\frac{2\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})-f(-re^{i\theta})}\right\}=Re\left\{\frac{2\left[zh'(z)-\overline{zg'(z)}\right]}{h(z)+\overline{g(z)}}\right\}\geq\gamma.$$

Using the operator $\mathbf{H}_{\lambda}^{k}f$ defined in section 4.2 and for $0\leq\gamma<1$, $S_{HS}^{\star}(\lambda,k,\alpha_1,\gamma)$ denote the class of harmonic univalent functions starlike of order γ with respect to symmetric points. The function $f \in S^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$ is satisfying

$$
\begin{aligned}
&Re\left\{\frac{2z\left[\mathbf{H}^k_\lambda f(z)\right]'}{\left[\mathbf{H}^k_\lambda f(z)-\mathbf{H}^k_\lambda f(-z)\right]}\right\}\\
&= Re\left\{\frac{2\left[z\left(\mathbf{H}^k_\lambda h(z)\right)'-\overline{z\left(\mathbf{H}^k_\lambda g(z)\right)'}\right]}{\left(\left[\mathbf{H}^k_\lambda h(z)-\mathbf{H}^k_\lambda h(-z)\right]+\left[\overline{\mathbf{H}^k_\lambda g(z)-\mathbf{H}^k_\lambda g(-z)}\right]\right)}\right\}\geq\gamma
\end{aligned}
\tag{4.18}
$$

where $\left[\mathbf{H}^k_\lambda f(z)\right]' = \frac{\partial}{\partial\theta}\left[\mathbf{H}^k_\lambda f(re^{i\theta})\right]$.

Further denote $T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$ as the class of functions $f = h+\bar{g} \in S^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$ such that h and g are of the form (4.13).

Theorem 4.12. *Let $f = h + \bar{g}$ be given by* (1.5) *and* $[1 + (n-1)\lambda]^k \prod_{i=1}^{l}(\alpha_i)_{n-1} \geq \prod_{j=1}^{m}(\beta_j)_{n-1}(n-1)!$. *If*

$$
\sum_{n=2}^{\infty}\left\{\frac{2n-\gamma\,\left[1-(-1)^n\right]}{2\,(1-\gamma)}|a_n| + \frac{2n+\gamma\,\left[1-(-1)^n\right]}{2\,(1-\gamma)}|b_n|\right\}|\Phi^k_{n,\lambda}| \leq 1 - \frac{1+\gamma}{1-\gamma}|b_1|
\tag{4.19}
$$

where $|b_1| < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$ *and* $\Phi^k_{n,\lambda}$ *is given by* (4.12) *then the harmonic function* f *is orientation preserving in* $\mathbf{D}$ *and* $f \in S^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$.

Proof. To verify that f is orientation preserving, we show $|h'(z)| \geq |g'(z)|$.

$$
\begin{aligned}
|h'(z)| &= |1 + \sum_{n=2}^{\infty} n a_n z^{n-1}| \\
&\geq 1 - |\sum_{n=2}^{\infty} n|a_n||z|^{n-1} \\
&\geq 1 - \sum_{n=2}^{\infty} n|a_n|.
\end{aligned}
$$

By the hypothesis of the theorem, $|\Phi^k_{n,\lambda}| \geq 1$ and by (4.19) give

$$\begin{aligned}
&\geq 1 - \sum_{n=2}^{\infty} \frac{2n - \gamma[1-(-1)^n]}{2\ (1-\gamma)} |\Phi^k_{n,\lambda}||a_n| \\
&\geq \sum_{n=1}^{\infty} \frac{2n + \gamma[1-(-1)^n]}{2\ (1-\gamma)} |\Phi^k_{n,\lambda}||b_n| \\
&\geq \sum_{n=1}^{\infty} n|b_n| \\
&\geq \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \\
&= |g'(z)|.
\end{aligned}$$

Thus, f is orientation preserving in $\mathbf{D}$.

Next, we prove $f \in S^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$. It suffices to show that the condition (4.18) is satisfied. Then, let

$$w = \frac{2\left[z\left(\mathbf{H}^k_\lambda h(z)\right)' - \overline{z\left(\mathbf{H}^k_\lambda g(z)\right)'}\right]}{\left(\left[\mathbf{H}^k_\lambda h(z) - \mathbf{H}^k_\lambda h(-z)\right] + \left[\overline{\mathbf{H}^k_\lambda g(z) - \mathbf{H}^k_\lambda g(-z)}\right]\right)} = \frac{A(z)}{B(z)}$$

where $\quad A(z) = 2z + \sum_{n=2}^{\infty} 2n\Phi^k_{n,\lambda} a_n z^n - \sum_{n=1}^{\infty} 2n\overline{\Phi^k_{n,\lambda} b_n z^n}\quad$,

and $\quad B(z) = 2z + \sum_{n=2}^{\infty}[1-(-1)^n]\Phi^k_{n,\lambda} a_n z^n + \sum_{n=1}^{\infty}[1-(-1)^n]\overline{\Phi^k_{n,\lambda} b_n z^n}$.

Since $Re\ w \geq \gamma$ if and only if $|A(z) + (1-\gamma)B(z)| \geq |A(z) - (1+\gamma)B(z)|$, thus the result is achieved by showing $|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0$.

Consider

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|$$

$$\geq (4-2\gamma)|z| - \sum_{n=2}^{\infty} |\Phi^k_{n,\lambda}||a_n||z^n|\left(2n+(1-\gamma)[1-(-1)^n]\right)$$

$$-\sum_{n=1}^{\infty} |\Phi^k_{n,\lambda}||b_n||z^n|\left(2n-(1-\gamma)[1-(-1)^n]\right) - 2\gamma|z|$$

$$-\sum_{n=2}^{\infty} |\Phi^k_{n,\lambda}||a_n||z^n|\left(2n-(1+\gamma)[1-(-1)^n]\right)$$

$$-\sum_{n=1}^{\infty} |\Phi^k_{n,\lambda}||b_n||z^n|\left(2n+(1+\gamma)[1-(-1)^n]\right)$$

$$= 4(1-\gamma)|z| - \sum_{n=2}^{\infty} |\Phi^k_{n,\lambda}||a_n||z^n|(4n-2\gamma[1-(-1)^n])$$

$$-\sum_{n=1}^{\infty} |\Phi^k_{n,\lambda}||b_n||z^n|(4n+2\gamma[1-(-1)^n])$$

$$\geq 4(1-\gamma)|z|$$

$$\left\{1-\sum_{n=2}^{\infty}|\Phi^k_{n,\lambda}||a_n|\left(\frac{2n-\gamma[1-(-1)^n]}{2(1-\gamma)}\right)-\sum_{n=1}^{\infty}|\Phi^k_{n,\lambda}||b_n|\left(\frac{2n+\gamma[1-(-1)^n]}{2(1-\gamma)}\right)\right\}$$

$$= 4(1-\gamma)|z|$$

$$\left\{1-\frac{1+\gamma}{1-\gamma}|b_1|-\left(\sum_{n=2}^{\infty}\left[\frac{2n-\gamma[1-(-1)^n]}{2(1-\gamma)}|a_n|+\frac{2n+\gamma[1-(-1)^n]}{2(1-\gamma)}|b_n|\right]|\Phi^k_{n,\lambda}|\right)\right\}.$$

This last expression is non-negative by (4.19), and thus $f \in S^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$. $\square$

For $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|\bar{y}_n| = 1$, the functions

$$f_1(z) = z + \sum_{n=2}^{\infty}\frac{2\,(1-\gamma)}{(2n-\gamma[1-(-1)^n])|\Phi^k_{n,\lambda}|}x_n z^n + \sum_{n=1}^{\infty}\frac{2(1-\gamma)}{(2n+\gamma[1-(-1)^n])|\Phi^k_{n,\lambda}|}\bar{y}_n\bar{z}^n$$

shows the equality in the coefficient bound given by (4.19) is attained.

The following result proves the hypothesis in Theorem 4.12 is a necessary and sufficient condition for f to be in the class $T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$.

Theorem 4.13. *Let $f = h + \bar{g}$ be given by (4.13), $f \in T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$ if and only if*

$$\sum_{n=2}^{\infty} \left\{ \frac{2n - \gamma\ [1-(-1)^n]}{2\ (1-\gamma)} |a_n| + \frac{2n + \gamma\ [1-(-1)^n]}{2\ (1-\gamma)} |b_n| \right\} |\Phi^k_{n,\lambda}| \le 1 - \frac{1+\gamma}{1-\gamma}|b_1| \tag{4.20}$$

where $|b_1| < \frac{1-\gamma}{1+\gamma}$, $0 \le \gamma < 1$ and $\Phi^k_{n,\lambda}$ is given by (4.12).

Proof. Since $f \in T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma) \subset S^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$, sufficiency part follows from Theorem 4.12. To prove the necessity part, assume that $f \in T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$. For functions f of the form (4.13), the condition (4.18) is equivalent to

$$Re \left\{ \frac{2\left[z\left(\mathbf{H}^k_\lambda h(z)\right)' - \overline{z\left(\mathbf{H}^k_\lambda g(z)\right)'}\right]}{\left(\left[\mathbf{H}^k_\lambda h(z) - \mathbf{H}^k_\lambda h(-z)\right] + \overline{\left[\mathbf{H}^k_\lambda g(z) - \mathbf{H}^k_\lambda g(-z)\right]}\right)} - \gamma \right\}$$

$$= Re \left\{ \frac{2z - \sum_{n=2}^{\infty} 2n\Phi^k_{n,\lambda} a_n z^n - \sum_{n=1}^{\infty} 2n\overline{\Phi^k_{n,\lambda} b_n z^n}}{2z - \sum_{n=2}^{\infty} \Phi^k_{n,\lambda} a_n z^n [1-(-1)^n] + \sum_{n=1}^{\infty} \overline{\Phi^k_{n,\lambda} b_n z^n}[1-(-1)^n]} - \gamma \right\}$$
$$\ge 0.$$

The condition should hold for all values of $z, |z| = r < 1$. Choosing the values of z on the real positive axis, $0 \le z = r < 1$, and $\Phi^k_{n,\lambda}$ is real, we have

$$\left\{ \frac{2(1-\gamma) - \sum_{n=2}^{\infty} \Phi^k_{n,\lambda} a_n r^{n-1}\left(2n - \gamma[1-(-1)^n]\right)}{2 - \sum_{n=2}^{\infty} \Phi^k_{n,\lambda} a_n r^{n-1}[1-(-1)^n] + \sum_{n=1}^{\infty} \Phi^k_{n,\lambda} b_n r^{n-1}[1-(-1)^n]} \right\}$$
$$- \left\{ \frac{\sum_{n=1}^{\infty} \Phi^k_{n,\lambda} b_n r^{n-1}\left(2n + \gamma[1-(-1)^n]\right)}{2 - \sum_{n=2}^{\infty} \Phi^k_{n,\lambda} a_n r^{n-1}[1-(-1)^n] + \sum_{n=1}^{\infty} \Phi^k_{n,\lambda} b_n r^{n-1}[1-(-1)^n]} \right\}$$
$$\ge 0. \tag{4.21}$$

Letting $r \to 1^-$ and if the condition (4.20) does not hold, then the numerator in (4.21) is negative. Thus the coefficient bound inequality (4.21) holds true when $f \in T^*_H(p, k, \alpha_1, \gamma)$. This completes the proof of Theorem 4.13. $\square$

Denote *clco* $T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$ as the closed convex hull of $T^{*}_{HS}(\lambda, k, \alpha_1, \gamma)$. The following result gives extreme points of *clco* $T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$.

Theorem 4.14. *$f = h + \bar{g} \in$ clco $T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$ if and only if f can be expressed in the form*

$$f = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \tag{4.22}$$

where

$$h_1(z) = z, \quad h_n(z) = z - \frac{2(1-\gamma)}{(2n - \gamma[1-(-1)^n])\,|\Phi^k_{n,\lambda}|} z^n \quad (n = 2, 3, ...),$$

$$g_n(z) = z + \frac{2(1-\gamma)}{(2n + \gamma[1-(-1)^n])\,|\Phi^k_{n,\lambda}|} \bar{z}^n \quad (n = 1, 2, 3, ...),$$

$\Phi^k_{n,\lambda}$ is given by (4.12) and $\sum_{n=1}^{\infty}(X_n + Y_n) = 1$, with $X_n \geq 0, Y_n \geq 0$. In particular the extreme points of $T^{\star}_{HS}(\lambda, k, \alpha_1, \gamma)$ are h_n and g_n.

Proof. Let f be of the form (4.22). Then we have

$$f(z) = \sum_{n=1}^{\infty} (X_n + Y_n)\, z - \sum_{n=2}^{\infty} \frac{2(1-\gamma)}{(2n - \gamma[1-(-1)^n])\,|\Phi^k_{n,\lambda}|} X_n z^n + \sum_{n=1}^{\infty} \frac{2(1-\gamma)}{(2n + \gamma[1-(-1)^n])\,|\Phi^k_{n,\lambda}|} Y_n \bar{z}^n$$

$$= z - \sum_{n=2}^{\infty} \frac{2(1-\gamma)}{(2n - \gamma[1-(-1)^n])\,|\Phi^k_{n,\lambda}|} X_n z^n + \sum_{n=1}^{\infty} \frac{2(1-\gamma)}{(2n + \gamma[1-(-1)^n])\,|\Phi^k_{n,\lambda}|} Y_n \bar{z}^n.$$

Furthermore, let $|a_n| = \frac{2(1-\gamma)}{(2n-\gamma[1-(-1)^n])|\Phi^k_{n,\lambda}|} X_n$ and $|b_n| = \frac{2(1-\gamma)}{(2n+\gamma[1-(-1)^n])|\Phi^k_{n,\lambda}|} Y_n$.

Applying Theorem 4.13, gives

$$\sum_{n=2}^{\infty} \frac{(2n-\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|}{2(1-\gamma)}|a_n| + \sum_{n=1}^{\infty} \frac{(2n+\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|}{2(1-\gamma)}|b_n|$$

$$= \sum_{n=2}^{\infty} \frac{(2n-\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|}{2(1-\gamma)} \frac{2(1-\gamma)}{(2n-\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|} X_n$$
$$+ \sum_{n=1}^{\infty} \frac{(2n+\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|}{2(1-\gamma)} \frac{2(1-\gamma)}{(2n+\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|} Y_n$$

$$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n$$

$$= 1 - X_1 \le 1.$$

Thus $f \in clco\ T_{HS}^{\star}(\lambda, k, \alpha_1, \gamma)$.

Conversely, let $f \in clco\ T_{HS}^{\star}(\lambda, k, \alpha_1, \gamma)$. Set

$$X_n = \frac{(2n-\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k||a_n|}{2(1-\gamma)} \quad (n = 2, 3, ...),$$

$$Y_n = \frac{(2n+\gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k||b_n|}{2(1-\gamma)} \quad (n = 1, 2, ...)$$

and define $X_1 = 1 - \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n$.

Therefore,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

$$f(z) = z - \sum_{n=2}^{\infty} \frac{2(1-\gamma)X_n}{(2n - \gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|} z^n + \sum_{n=1}^{\infty} \frac{2(1-\gamma)Y_n}{(2n + \gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|} \bar{z}^n$$

$$\begin{aligned} f(z) = X_1 z + \sum_{n=2}^{\infty} X_n \left\{ z - \frac{2(1-\gamma)z^n}{(2n - \gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|} \right\} \\ + \sum_{n=1}^{\infty} Y_n \left\{ z + \frac{2(1-\gamma)\bar{z}^n}{(2n + \gamma[1-(-1)^n])\,|\Phi_{n,\lambda}^k|} \right\} \end{aligned}$$

$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$ as required. □

Theorem 4.15. *The class $T_{HS}^{\star}(\lambda, k, \alpha_1, \gamma)$ is closed under convex combination.*

Proof. Suppose that for $i = 1, 2, 3, ..., f_i(z) \in T_{HS}^{\star}(\lambda, k, \alpha_1, \gamma)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=1}^{\infty} |b_{n,i}| \bar{z}^n.$$

From Theorem 4.13,

$$\sum_{n=2}^{\infty} \left(\frac{2n - \gamma\,[1-(-1)^n]}{2\,(1-\gamma)} |\Phi_{n,\lambda}^k||a_{n,i}| \right) + \sum_{n=1}^{\infty} \left(\frac{2n + \gamma\,[1-(-1)^n]}{2\,(1-\gamma)} |\Phi_{n,\lambda}^k||b_{n,i}| \right) \leq 1. \tag{4.23}$$

For $\sum_{i=1}^{\infty} t_i = 1$ where $\forall i,\ 0 \leq t_i \leq 1$, the convex combination of f_i may be written as,

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| z^n \right) + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \bar{z}^n \right).$$

Then, by (4.23)

$$\sum_{n=2}^{\infty}\left(\frac{2n-\gamma\left[1-(-1)^{n}\right]}{2\left(1-\gamma\right)}|\Phi_{n,\lambda}^{k}|\right)\left(\left|\sum_{i=1}^{\infty}t_i|a_{n,i}|\right|\right)$$
$$+\sum_{n=1}^{\infty}\left(\frac{2n+\gamma\left[1-(-1)^{n}\right]}{2\left(1-\gamma\right)}|\Phi_{n,\lambda}^{k}|\right)\left(\left|\sum_{i=1}^{\infty}t_i|b_{n,i}|\right|\right)$$

$$=\sum_{i=1}^{\infty}t_i$$

$$\left\{\sum_{n=2}^{\infty}\left(\frac{2n-\gamma\left[1-(-1)^{n}\right]}{2\left(1-\gamma\right)}|\Phi_{n,\lambda}^{k}||a_{n,i}|\right)+\sum_{n=1}^{\infty}\left(\frac{2n+\gamma\left[1-(-1)^{n}\right]}{2\left(1-\gamma\right)}|\Phi_{n,\lambda}^{k}||b_{n,i}|\right)\right\}$$

$$\leq\sum_{i=1}^{\infty}t_i\left(1\right)=1.$$

This is the condition required by (4.23) and hence, $\sum_{i=1}^{\infty}t_if_i(z)\in T_{HS}^{\star}(\lambda,k,\alpha_1,\gamma)$.

□

In the theorem below we give distortion bounds for f in the class $T_{HS}^{\star}(\lambda,k,\alpha_1,\gamma)$

Theorem 4.16. *If $f\in T_{HS}^{\star}(\lambda,k,\alpha_1,\gamma)$ with $\Phi_{n,\lambda}^{k}\geq\Phi_{2,\lambda}^{k}$ then for $|z|=r<1$,*

$$|f(z)|\leq\left(1+|b_1|\right)r+r^2\left\{\frac{(1-\gamma)}{2|\Phi_{2,\lambda}^{k}|}-\frac{(1+\gamma)|b_1|}{2|\Phi_{2,\lambda}^{k}|}\right\}$$

and

$$|f(z)|\geq\left(1-|b_1|\right)r-r^2\left\{\frac{(1-\gamma)}{2|\Phi_{2,\lambda}^{k}|}-\frac{(1+\gamma)|b_1|}{2|\Phi_{2,\lambda}^{k}|}\right\}.$$

Proof. Since

$$\frac{2}{(1-\gamma)}\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|\Phi_{2,\lambda}^{k}|$$

$$\leq \sum_{n=2}^{\infty}\left(\frac{2n-\gamma[1-(-1)^n]}{2(1-\gamma)}\right)\left(|a_n|+|b_n|\right)|\Phi_{n,\lambda}^{k}|$$

$$\leq \sum_{n=2}^{\infty}\left(\frac{2n-\gamma[1-(-1)^n]}{2(1-\gamma)}|a_n|+\frac{2n+\gamma[1-(-1)^n]}{2(1-\gamma)}|b_n|\right)|\Phi_{n,\lambda}^{k}|$$

$$\leq 1-\frac{1+\gamma}{1-\gamma}|b_1|.$$

Thus using the result of Theorem 4.13, the above gives

$$\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)\leq \frac{(1-\gamma)}{2|\Phi_{2,\lambda}^{k}|}-\frac{1+\gamma}{2|\Phi_{2,\lambda}^{k}|}|b_1|. \tag{4.24}$$

Next, since $f\in T_{HS}^{\star}(\lambda,k,\alpha_1,\gamma)$ and for $0<|z|=r<1$, we have using (4.24)

$$\begin{aligned}
|f(z)| &= \left|z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^n\right|\\
&\leq |z|+\sum_{n=2}^{\infty}|a_n|\,|z|^n+\sum_{n=1}^{\infty}|b_n|\,|\bar{z}|^n\\
&= (1+|b_1|)|z|+\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|z|^n\\
&\leq (1+|b_1|)|z|+\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|z|^2\\
&= (1+|b_1|)r+r^2\left\{\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)\right\}\\
&\leq (1+|b_1|)r+r^2\left\{\frac{(1-\gamma)}{2|\Phi_{2,\lambda}^{k}|}-\frac{1+\gamma}{2|\Phi_{2,\lambda}^{k}|}|b_1|\right\}
\end{aligned}$$

which gives the first result.

In a similar manner, we derive the following lower bound.

$$
\begin{aligned}
|f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n|\,|z|^n - \sum_{n=1}^{\infty} |b_n|\,|z|^n \\
&= (1-|b_1|)|z| - \sum_{n=2}^{\infty} \left(|a_n| + |b_n|\right)|z|^n \\
&\geq (1-|b_1|)|z| - \sum_{n=2}^{\infty} \left(|a_n| + |b_n|\right)|z|^2 \\
&= (1-|b_1|)r - r^2 \left\{ \sum_{n=2}^{\infty} \left(|a_n| + |b_n|\right) \right\} \\
&\geq (1-|b_1|)r - r^2 \left\{ \frac{(1-\gamma)}{2|\Phi_{2,\lambda}^{k}|} - \frac{1+\gamma}{2|\Phi_{2,\lambda}^{k}|}|b_1| \right\}.
\end{aligned}
$$

$\square$

CHAPTER 5

PRESERVATION OF CERTAIN OPERATORS

Jung, Kim and Srivastava (1993) introduced the families of integral operators and investigated the preservation of these operators in Hardy space of analytic functions. In this chapter, preservation of the Carlson-Shaffer operator, Hohlov operator, Dziok-Srivastava operator and Kwon-Cho operator are shown to be preserved in the same class. Beside that, class $R(\lambda, k, \alpha_1, \gamma)$ is defined using the operator $\mathbf{H}_\lambda^k f$ as introduced in section 4.2. Using the condition of convex null sequences, we show the preservation of Jung-Kim-Srivastava operators in this class.

5.1 Preservation on hardy space

Recently, some authors (Raina, 2009), (Raina and Srivastava, 1999) and (Jung et al., 1993) have considered relationships between certain families of integral operators and the Hardy space of analytic functions. For our case, we consider Carlson-Shaffer operator, Hohlov operator, Dziok-Srivastava operator and Kwon-Cho operator. For $f \in \mathcal{R}$, these operators will be shown to be preserved in Hardy space.

First, let $H^P(0 < P \leq \infty)$ denotes the Hardy space of analytic functions in $\mathbf{D}$. The integral mean $M_P(r, f)$ is defined as follows:

Definition 5.1.

$$M_P(r, f) = \begin{cases} (\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}|^P \quad d\theta)^{\frac{1}{P}} & , \quad 0 < P < \infty \\ \\ \max_{|z|\leq r} \|f(z)\| & , \qquad P = \infty \end{cases} \tag{5.1}$$

An analytic function f in $\mathbf{D}$ belongs to the Hardy space H^P $(0 < P \leq \infty)$ if $\lim_{r\to 1^-} M_P(r,f) < \infty$. H^P is a Banach space for $1 \leq P \leq \infty$ with norm

$$\|f\|_P = \lim_{r\to 1^-} M_P(r,f) \tag{5.2}$$

whilst H^∞ is the class of bounded analytic functions in $\mathbf{D}$.

Next, we list some prior results which will be used in establishing our results.

Lemma 5.1. *(Jung et al., 1993) If $f \in \mathcal{R}$ then $f \in H^P (0 < P < \infty)$.*

Lemma 5.2. *(Libera, 1965) If M and N are regular in $\mathbf{D}, N(0) = M(0) = 0$, N maps $\mathbf{D}$ onto a many sheeted region which is starlike with respect to the origin, if $\frac{M'}{N'} \in \mathcal{P}$ then $\frac{M}{N} \in \mathcal{P}$.*

Therefore if $Re\left\{\frac{M'(z)}{N'(z)}\right\} > 0$ then $Re\left\{\frac{M(z)}{N(z)}\right\} > 0$.

Proposition 5.1. *(Duren, 1970: p. 42) A function $f(z)$ which is analytic in $|z| < 1$ is continuous in $|z| \leq 1$ and absolutely continuous on $|z| = 1$ if and only if $f' \in H^1$.*

For the first theorem, we show the preservation of Carlson-Shaffer operator, $L(b,c)f$.

Theorem 5.1. *Given $Re\ b > 0, Re\ c > 0$ and $f \in \mathcal{R}$ then $L(b,c)f \in H^P$ $(0 < P < \infty)$ and $L(b,c)f \in H^\infty$.*

Proof. The Euler representation of Carlson-Shaffer operator is

$$Ł(b,c)f(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{1}{t}f(tz)\ dt \tag{5.3}$$

with $Re\ c > 0, Re\ b > 0$.

Differentiating with respect to z, (5.3) becomes

$$\frac{d}{dz}[L(b,c)]f(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{1}{t}f'(tz)\,t\ dt$$

and taking the real part gives,

$$Re\left(\frac{d}{dz}[L(b,c)]f(z)\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,Re\,[f'\,(tz)]\;\,dt.$$

Since $f \in \mathcal{R}$, $Re[f'(tz)] > 0$ where $t \in (0,1)$.

Hence, $Re\left(\frac{d}{dz}[L(b,c)]f(z)\right) > 0$ and thus $[L(b,c)]f \in \mathcal{R}$. Lemma 5.1 implies $[L(b,c)]f \in H^P$.

To prove $L(b,c)f \in H^{\infty}$, it can be shown that

$$z\left\{\frac{d}{dz}\,[L(b,c)]\,f(z)\right\} = b[L(b+1,c)]f(z) - (b-1)[L(b,c)]f(z), \quad b > 0$$

which gives

$$\left|\frac{d}{dz}[L(b,c)]f(z)\right|^P = \left|\frac{b}{z}[L(b+1,c)]f(z) - \frac{(b-1)}{z}[L(b,c)]f(z)\right|^P \quad (0 < P < \infty).$$

Furthermore, using

$$max\left\{A^P, B^P\right\} \leq (A+B)^P \leq 2^P(A^P + B^P) \tag{5.4}$$

we can easily derive

$$\left|\frac{d}{dz}[L(b,c)]f(z)\right|^P \leq \frac{2^P}{|z|^P}\left\{|\ b[L(b+1,c)]f(z)\ |^P + |\ (b-1)[L(b,c)]f(z)\ |^P\right\}.$$

From Proposition 5.1, we obtain that if $\frac{d}{dz}[L(b,c)]f(z) \in H^1$ then $[L(b,c)]\,f$ is continuous in $\overline{\mathbf{D}} = \{z : z \in \mathbf{C} \text{ and } |z| \leq 1\}$.

With $P = 1,\ |z| = r < 1$,

$$\left|\frac{d}{dz}\,[L(b,c)]\,f(z)\right|^1 \leq \frac{2}{|z|}\left\{|\ b\,[L(b+1,c)]\,f(z)\ |^1 + |\ (b-1)\,[L(b,c)]\,f(z)\ |^1\right\}$$

$$\left|\frac{d}{dz}\,[L(b,c)]\,f(z)\right|^1 \leq \frac{2}{r}\left\{|\ b\,[L(b+1,c)]\,f(z)\ |^1 + |\ (b-1)\,[L(b,c)]\,f(z)\ |^1\right\}. \tag{5.5}$$

Making use of (5.1) and (5.2), the inequality (5.5) yields

$$M_1\left(r, \frac{d}{dz}\left[L(b,c)\right] f(z)\right)$$
$$\leq \frac{2}{r}\{\ |b|\ \ M_1(r, [L(b+1,c)]\, f(z) + |\ (b-1)|\ \ M_1(r, [L(b,c)]\, f(z))\}$$

and

$$\left\| \frac{d}{dz}\left[L(b,c)\right] f(z) \right\|_1 \leq 2|\ b\ |\ \|\ \ [L(b+1,c)]\, f(z)\ \|_1 + 2|\ (b-1)\ |\ \|\ \ [L(b,c)]\, f(z)\ \|_1 . \tag{5.6}$$

Since $[L(b,c)]\, f \in H^P\ (0 < P < \infty)$, $[L(b,c)]\, f(z) \in H^1$, $[L(b+1,c)]\, f \in H^1$ and the equation (5.6) implies $\frac{d}{dz}\left[L(b,c)\right] f \in H^1$.

Upon application of Proposition 5.1, $[L(b,c)]f$ is continuous in $\overline{\mathbf{D}} = \mathbf{D} + \delta\mathbf{D}$. $\overline{\mathbf{D}}$ is compact(closed and bounded). Therefore, $[L(b,c)]f$ is a bounded analytic function in $\mathbf{D}$. Hence $[L(b,c)]\, f \in H^\infty$. □

Next, the result for Hohlov operator, $H_{a,b\ ,c}\, f$ is obtained.

Theorem 5.2. *If $f \in \mathcal{R}$ then $H_{a,b\ ,c}\, f \in H^P\ \ (0 < P < \infty)$ and $H_{a,b\ ,c}\, f \in H^\infty$.*

Proof. Hohlov operator in terms of Euler presentation is given as

$$H_{a,b\ ,c}\, f(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{(1-t)^{c-b-1}}{\Gamma(c-a-b+1)} t^{b-2} F(c-a, 1-a; c-a-b+1; 1-t) f\,(tz)\ dt \tag{5.7}$$

where $(c-a+1) > b > 0$.

Differentiation of (5.7) gives

$$\frac{d}{dz}\left[H_{a,b\ ,c}\right] f(z)$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{t^{b-2}\,(1-t)^{c-a-b}}{\Gamma(c-a-b+1)} F(c-a, 1-a; c-a-b+1; 1-t) f'\,(tz)\, t\ dt$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-a-b}}{\Gamma(c-a-b+1)} F(c-a, 1-a; c-a-b+1; 1-t) f'(tz) \ dt.$$

Taking the real part

$$Re\left(\frac{d}{dz}\left[H_{a,b\ ,c}\right] f(z)\right)$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-a-b}}{\Gamma(c-a-b+1)} F(c-a, 1-a; c-a-b+1; 1-t) Re f'(tz) \ dt$$

and since $f \in \mathcal{R}$, $Re\left[f'(tz)\right] > 0, \ t \in (0,1)$. Then

$$Re\left(\frac{d}{dz}\left[H_{a,b\ ,c}\right] f(z)\right) > 0$$

Thus $H_{a,b\ ,c} f \in \mathcal{R}$. From Lemma 5.1, $H_{a,b\ ,c} f \in H^P \quad (0 < P < \infty)$. The second result is obtained by using a similar manner as in Theorem 5.1. $\square$

We now present two results concerning inclusion theorems of Dziok-Srivastava operator and Kwon-Cho operator in Hardy space using Libera's Lemma.

Theorem 5.3. *Let* $H^{l,m}\left[\alpha_1\right] f(z) = \sum_{n=0}^{\infty} \phi_n \left[\alpha_1\right] a_{n+1} z^{n+1}$ *where* $z \in \mathbf{D}$, $\quad \phi_n\left[\alpha_1\right] = \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n n!}$ *and* $\alpha_1 < 1$, $a_1 = 1$

$\alpha_i > 0 \ (i = 1, 2, \ldots)$ *and* $\beta_j \neq 0, -1, -2, \ldots \ (j = 1, 2, \ldots)$ *are real parameters with* $\prod_{i=1}^{l} (\alpha_i)_n \geq \prod_{j=1}^{m} (\beta_j)_n n!$ *and* $(\alpha_1 + 1)_n \prod_{i=2}^{l} (\alpha_i)_n \geq \prod_{j=1}^{m} (\beta_j)_n n!$.

If $f \in \mathcal{R}$ *then* $H^{l,m}\left[\alpha_1\right] f \in H^P \quad (0 < P < \infty)$ *and* $H^{l,m}\left[\alpha_1\right] f \in H^{\infty}$.

Proof. Easily, it can be shown that the differentiation of Dziok-Srivastava operator gives

$$\frac{d}{dz} H^{l,m}\left[\alpha_1\right] f(z) = \frac{1}{z} \left\{\alpha_1 H^{l,m}\left[\alpha_1 + 1\right] f(z) + (1 - \alpha_1) H^{l,m}\left[\alpha_1\right] f(z)\right\}$$

and

$$
\begin{aligned}
&Re\left\{\frac{d}{dz}H^{l,m}\left[\alpha_1\right]f(z)\right\}\\
&\quad= Re\left\{\alpha_1\sum_{n=0}^{\infty}\frac{\phi_n\left[\alpha_1+1\right]a_{n+1}z^{n+1}}{z}+(1-\alpha_1)\sum_{n=0}^{\infty}\frac{\phi_n\left[\alpha_1\right]a_{n+1}z^{n+1}}{z}\right\}\\
&\quad= Re\left\{\sum_{n=0}^{\infty}\frac{a_{n+1}z^{n+1}}{z}\left(\alpha_1\phi_n\left[\alpha_1+1\right]+(1-\alpha_1)\phi_n\left[\alpha_1\right]\right)\right\}\\
&\quad\geq Re\left\{\sum_{n=0}^{\infty}\frac{a_{n+1}z^{n+1}}{z}\right\}\\
&\quad= Re\left\{\frac{f(z)}{z}\right\}
\end{aligned}
$$

Since $f \in \mathcal{R}$, using Lemma 5.2 results in $Re\left\{\frac{f(z)}{z}\right\} > 0$ and Lemma 5.1 implies $H^{l,m}\left[\alpha_1\right]f \in H^P \quad (0 < P < \infty)$.

Next, consider

$$
\left|\frac{d}{dz}H^{l,m}\left[\alpha_1\right]f(z)\right|^P \quad = \quad \left|\frac{1}{z}\left\{\alpha_1 H^{l,m}\left[\alpha_1+1\right]f(z)+(1-\alpha_1)H^{l,m}\left[\alpha_1\right]f(z)\right\}\right|^P .
$$

Using (5.4)the following inequality is easily derived.

$$
\begin{aligned}
&\left|\frac{d}{dz}H^{l,m}\left[\alpha_1\right]f(z)\right|^P\\
&\leq \left(\frac{2}{|z|}\right)^P\left\{\left|\alpha_1 H^{l,m}\left[\alpha_1+1\right]f(z)\right|^P+\left|(1-\alpha_1)H^{l,m}\left[\alpha_1\right]f(z)\right|^P\right\}.
\end{aligned}
$$

From Proposition 5.1, we have

$$
\begin{aligned}
&\left|\frac{d}{dz}H^{l,m}\left[\alpha_1\right]f(z)\right|^1\\
&\leq \left(\frac{2}{r}\right)\left\{\left|\alpha_1 H^{l,m}\left[\alpha_1+1\right]f(z)\right|^1+\left|(1-\alpha_1)H^{l,m}\left[\alpha_1\right]f(z)\right|^1\right\}. \qquad (5.8)
\end{aligned}
$$

Making use of (5.1) and (5.2), the inequality (5.8) yields

$$M_1\left(r, \frac{d}{dz}H^{l,m}\left[\alpha_1\right]f(z)\right)$$
$$\leq \frac{2}{r}\left\{ \ \mid \alpha_1 \mid \ M_1\left(r, H^{l,m}\left[\alpha_1+1\right]f(z)\right) + \mid (1-\alpha_1)) \mid \ M_1\left(r, H^{l,m}\left[\alpha_1\right]f(z)\right) \ \right\}$$

and

$$\left\|\frac{d}{dz}H^{l,m}\left[\alpha_1\right]f(z)\right\|_1$$
$$\leq 2 \mid \alpha_1 \mid \ \left\| \ H^{l,m}\left[\alpha_1+1\right]f(z) \ \right\|_1 + 2 \mid (1-\alpha_1) \mid \ \left\| \ H^{l,m}\left[\alpha_1\right]f(z) \ \right\|_1. \quad (5.9)$$

Since, we have established earlier that $H^{l,m}\left[\alpha_1\right]f \in H^P \ \ (0 < P < \infty)$, $H^{l,m}\left[\alpha_1\right]f \in H^1$ and $H^{l,m}\left[\alpha_1+1\right]f \in H^1$ thus (5.9) implies $\frac{d}{dz}H^{l,m}\left[\alpha_1\right]f \in H^1$. Applying Proposition 5.1, $H^{l,m}\left[\alpha_1\right]f$ is continuous in $\overline{\mathbf{D}} = \mathbf{D} + \delta\mathbf{D}$. $\overline{\mathbf{D}}$ is compact(closed and bounded). Therefore, $H^{l,m}\left[\alpha_1\right]f$ is a bounded analytic function in $\mathbf{D}$. Hence $H^{l,m}\left[\alpha_1\right]f \in H^\infty$. □

Lastly, the Kwon-Cho operator defined by Kwon and Cho (2007) as $H_\lambda^{l,m}\left[\alpha_1\right]f(z)$ is used in obtaining the following theorem.

Theorem 5.4. *Let* $H_\lambda^{l,m}\left[\alpha_1\right]f(z) = \sum_{n=0}^{\infty}\Psi_n\left[\lambda\right]a_{n+1}z^{n+1}$ *where* $z \in \mathbf{D}$, $\Psi_n\left[\lambda\right] = \frac{(\lambda)_n(\beta_1)_n\cdots(\beta_m)_n}{(\alpha_1)_n\cdots(\alpha_l)_n n!}$, $0 < \lambda < 1, a_1 = 1, \ \alpha_i > 0 \ (i = 1, 2, \ldots)$ *and* $\beta_j \neq 0, -1, -2, \ldots \ (j = 1, 2, \ldots)$ *are real parameters with* $(\lambda)_n\prod_{j=1}^{m}(\beta_j)_n \geq \prod_{i=1}^{l}(\alpha_i)_n n!$ *and* $(\lambda+1)_n\prod_{j=1}^{m}(\beta_j)_n \geq \prod_{i=1}^{l}(\alpha_i)_n n!$. *If* $f \in \mathcal{R}$ *then* $H_\lambda^{l,m}\left[\alpha_1\right]f \in H^P \ \ (0 < P < \infty)$.

Proof. Differentiation of $H_{\lambda}^{l,m}[\alpha_1] f$ results in

$$\begin{aligned}
z\left\{\frac{d}{dz}H_{\lambda}^{l,m}[\alpha_1] f(z)\right\} &= \lambda H_{\lambda+1}^{l,m}[\alpha_1+1] f(z) - (\lambda-1)H_{\lambda}^{l,m}[\alpha_1] f(z)\\
\frac{d}{dz}H_{\lambda}^{l,m}[\alpha_1] f(z) &= \frac{1}{z}\left\{\lambda H_{\lambda+1}^{l,m}[\alpha_1+1] f(z) + (1-\lambda)H_{\lambda}^{l,m}[\alpha_1] f(z)\right\}\\
\frac{d}{dz}H_{\lambda}^{l,m}[\alpha_1] f(z) &= \left\{\lambda\sum_{n=0}^{\infty}\frac{\Psi_n[\lambda+1]\, a_{n+1}z^{n+1}}{z} + (1-\lambda)\sum_{n=0}^{\infty}\frac{\Psi_n[\lambda]\, a_{n+1}z^{n+1}}{z}\right\}\\
Re\left\{\frac{d}{dz}H_{\lambda}^{l,m}[\alpha_1] f(z)\right\} &= Re\left\{\sum_{n=0}^{\infty}\frac{a_{n+1}z^{n+1}}{z}\left(\lambda\Psi_n[\lambda+1] + (1-\lambda)\Psi_n[\lambda]\right)\right\}\\
&\geq Re\left\{\sum_{n=0}^{\infty}\frac{a_{n+1}z^{n+1}}{z}\right\}\\
&= Re\left\{\frac{f(z)}{z}\right\}.
\end{aligned}$$

The hypothesis $f \in \mathcal{R}$ implies $Re\left\{\frac{f(z)}{z}\right\} > 0$. Applying Lemma 5.1, $H_{\lambda}^{l,m}[\alpha_1] f \in H^P \ (0 < P < \infty)$. □

Remark 5.1. *$H_{\lambda}^{l,m}[\alpha_1] f \in H^{\infty}$ can be shown using similar method as in Theorem 5.3.*

Remark 5.2. *The notation of Kwon-Cho operator can be written as $H^{m+1,l}[\lambda]f(z)$, hence Theorem 5.4 is a special case of Theorem 5.3.*

5.2 Preservation using convex null sequences

There are some classes were introduced using various operators and the class $\mathcal{R}(\gamma)$ [for examples see (Al-Oboudi, 2004) and (Murugusundaramoorthy, 2003)]. The operator $\mathbf{H}_{\lambda}^{k}f$ defined in section 4.2 is considered to introduce a class $R(\lambda, k, \alpha_1, \gamma)$ which satisfies the condition $Re\left[\mathbf{H}_{\lambda}^{k}f(z)\right]' > \gamma \ (0 \leq \gamma < 1, \lambda \geq 0, k \in N_0 = \{0, 1, 2, \ldots\})$.

The Jung-Kim-Srivastava operators:

$$J_\nu f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\nu+1}{\nu+n} \right) a_n z^n, \quad \nu > -1$$

and

$$\ell_\mu^\nu f(z) = z + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)} \right) a_n z^n, \quad \nu > 0, \ \mu > -1$$

are shown to be preserved in the class $R(\lambda, k, \alpha_1, \gamma)$ using condition of convex null sequence. Some papers on convex null sequences can be found in (Al-Oboudi, 2004) and (Babalola, 2009).

Throughout this section, these two lemmas are used in proving our results.

Lemma 5.3. *(Goodman, 1983) If $p(z)$ is analytic in the unit disc $\mathbf{D}$, $p(0) = 1$ and $Re[p(z)] > \frac{1}{2}, z \in \mathbf{D}$, then for any analytic function q in $\mathbf{D}$, the function $p * q$ takes its values in the convex hull of $q(\mathbf{D})$.*

Definition 5.2. *A sequence $c_0, c_1, \ldots, c_n, \ldots$ of nonnegative numbers is called a convex null sequence if $c_n \to 0$ as $n \to \infty$ and $c_0 - c_1 \geq c_1 - c_2 \geq \ldots \geq c_n - c_{n+1} \geq \ldots \geq 0$.*

Lemma 5.4. *(Fejèr, 1925) Let $\{c_n\}_{n=0}^{\infty}$ be a convex null sequence. Then the function $s(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n$ is analytic and $Re[s(z)] > 0$ in $\mathbf{D}$.*

Theorem 5.5. *If $f \in R(\lambda, k, \alpha_1, \gamma)$ then $\mathrm{J}_\nu f \in R(\lambda, k, \alpha_1, \gamma)$.*

Proof. Let $f \in R(\lambda, k, \alpha_1, \gamma)$ then

$$Re\left\{ z + \sum_{n=2}^{\infty} \Phi_{n,\lambda}^k a_n z^n \right\}' = Re\left\{ 1 + \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^k a_n z^{n-1} \right\} > \gamma.$$

Let $q(z) = 1 + \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^k a_n z^{n-1}$.

Considering

$$Re\left\{z+\sum_{n=2}^{\infty}\Phi_{n,\lambda}^{k}\left(\frac{\nu+1}{\nu+n}\right)a_nz^n\right\}' = Re\left\{1+\sum_{n=2}^{\infty}n\Phi_{n,\lambda}^{k}\left(\frac{\nu+1}{\nu+n}\right)a_nz^{n-1}\right\}.$$

Let

$$P(z)=1+\sum_{n=2}^{\infty}n\Phi_{n,\lambda}^{k}\left(\frac{\nu+1}{\nu+n}\right)a_nz^{n-1}$$

$$=\left(1+\sum_{n=2}^{\infty}n\Phi_{n,\lambda}^{k}a_nz^{n-1}\right)*\left(1+\sum_{n=2}^{\infty}\left(\frac{\nu+1}{\nu+n}\right)a_nz^{n-1}\right)$$

$$=q(z)*p(z).$$

Applying Lemma 5.4, let $c_n=\frac{\nu+1}{\nu+n+1}$ $(\nu>-1)$ with $c_0=1$. Then

i) c_n nonnegative numbers.

ii) $c_n\to 0$ as $n\to\infty$.

iii)

$$\frac{c_n+c_{n+2}}{2c_{n+1}}=\frac{\nu+1}{\nu+n+1}+\frac{\nu+1}{\nu+n+3}\div\frac{2(\nu+1)}{\nu+n+2}$$

$$=\frac{(\nu+n+2)^2}{(\nu+n+1)(\nu+n+3)}$$

$$=1+\frac{1}{(\nu+n+1)(\nu+n+3)}$$

$$\geq 1.$$

Therefore $\{c_n\}_{n=0}^{\infty}$ is a convex null sequence. Thus the function $s(z)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_nz^n$

is analytic and $Re[s(z)] > 0$. Using the above c_n, we have

$$s(z) = \frac{1}{2} + \frac{\nu+1}{\nu+2}z + \frac{\nu+1}{\nu+3}z^2 + \cdots$$

$$s(z) - \frac{1}{2} = \frac{\nu+1}{\nu+2}z + \frac{\nu+1}{\nu+3}z^2 + \cdots .$$

The function

$$\begin{aligned} p(z) &= 1 + \sum_{n=2}^{\infty} \left(\frac{\nu+1}{\nu+n}\right) a_n z^{n-1} \\ p(z) &= 1 + s(z) - \frac{1}{2} = \frac{1}{2} + s(z) \\ Re[p(z)] &= Re[\frac{1}{2} + s(z)] > \frac{1}{2}. \end{aligned}$$

Using Lemma 5.3 we obtain

$Re[P(z)] = Re\left[1 + \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^{k} \left(\frac{\nu+1}{\nu+n}\right) a_n z^{n-1}\right] > \gamma$, hence $J_\nu f \in R(\lambda, k, \alpha_1, \gamma)$. $\square$

Theorem 5.6. *If $f \in R(\lambda, k, \alpha_1, \gamma)$ then $\ell_\mu^\nu f \in R(\lambda, k, \alpha_1, \gamma)$.*

Proof. Let $f \in R(\lambda, k, \alpha_1, \gamma)$ then

$$Re\left\{z + \sum_{n=2}^{\infty} \Phi_{n,\lambda}^{k} a_n z^n\right\}' = Re\left\{1 + \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^{k} a_n z^{n-1}\right\} > \gamma.$$

Let $q(z) = 1 + \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^{k} a_n z^{n-1}$.

Considering

$$\begin{aligned} &Re\left\{z + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} \Phi_{n,\lambda}^{k} \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)}\right) a_n z^n\right\}' \\ &= Re\left\{1 + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^{k} \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)}\right) a_n z^{n-1}\right\}. \end{aligned}$$

Let

$$P(z) = 1 + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^{k} \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)} \right) a_n z^{n-1}$$

$$= \left(1 + \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^{k} a_n z^{n-1}\right) * \left(1 + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)} \right) a_n z^{n-1}\right)$$

$$= q(z) * p(z).$$

Applying Lemma 5.4, let $c_n = \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)(\Gamma(\nu+\mu+n+1)}$ with $c_0 = 1$. Then

i) c_n nonnegative numbers.

ii) $c_n \to 0$ as $n \to \infty$.

iii)

$$\frac{c_n + c_{n+2}}{c_{n+1}} = \left\{ \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\nu+\mu+n+1)} + \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+n+3)}{\Gamma(\mu+1)\Gamma(\nu+\mu+n+3)} \right\} \div \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+n+2))}{\Gamma(\mu+1)\Gamma(\nu+\mu+n+2)}$$

$$= \frac{(\nu+\mu+n+1)}{(\mu+n+1)} + \frac{(\mu+n+2)}{(\nu+\mu+n+2)}$$

$$= 2 + \frac{1+\nu^2}{(\mu+n+1)(\nu+\mu+n+2)}$$

$$\geq 2.$$

Therefore $\{c_n\}_{n=0}^{\infty}$ is a convex null sequence. Thus the function $s(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n$

is analytic and $Re[s(z)] > 0$. Using the above c_n, we have

$$s(z) = \frac{1}{2} + \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+2)}{\Gamma(\mu+1)\Gamma(\nu+\mu+2)}z + \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+3)}{\Gamma(\mu+1)\Gamma(\nu+\mu+3)}z^2 + \cdots$$

$$s(z) - \frac{1}{2} = \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+2)}{\Gamma(\mu+1)\Gamma(\nu+\mu+2)}z + \frac{\Gamma(\nu+\mu+1)\Gamma(\mu+3)}{\Gamma(\mu+1)\Gamma(\nu+\mu+3)}z^2 + \cdots .$$

The function

$$\begin{aligned} p(z) &= \left(1 + \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)}\right) a_n z^{n-1}\right) \\ p(z) &= 1 + s(z) - \frac{1}{2} = \frac{1}{2} + s(z) \\ Re[p(z)] &= Re[\frac{1}{2} + s(z)] > \frac{1}{2}. \end{aligned}$$

Using Lemma 5.3 we obtain $Re[P(z)] = 1+\frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \sum_{n=2}^{\infty} n\Phi_{n,\lambda}^k \left(\frac{\Gamma(\mu+n)}{\Gamma(\nu+\mu+n)}\right) a_n z^{n-1} > \gamma$. Hence $\ell_\mu^\nu f \in R(\lambda, \alpha_1, \gamma)$. $\square$

CHAPTER 6

FUTURE RESEARCH

In the last chapter, we discuss open problems for future research. Some topics in this thesis can be extended to generate new results.

Problem I: Necessary and sufficient conditions for the integral transform $V_\lambda f$ to be a convex functions of order δ where f in $\mathcal{W}_\beta(\alpha, \gamma)$ have been investigated in chapter two. This study can be continued to determine the sharpness of β and to find conditions on λ such that the results can be applied for certain integral transforms as Bernardi integral operator, Komatu operator and Hohlov operator.

Problem II: Raghavendar & Swaminathan (2012) studied the combination of starlike and convex functions for the integral transform $V_\lambda f$ where f in certain class.The investigation in chapter two can be extended to obtain new result using combination properties of starlikeness and convexity of order δ for $V_\lambda f$ where $f \in \mathcal{W}_\beta(\alpha, \gamma)$.

Problem III: In chapter three, the conditions on β have been obtained for functions in the classes of Janowski starlike and the Cassini curve. Beside that, the class of uniformly convex can be considered in getting new results on β.

Problem IV: Lastly, the discussion on properties of functions for the subclasses of multivalent and univalent harmonic functions in chapter four can be extended to meromorphic multivalent and univalent harmonic functions. Previously, there are authors [see (Ahuja and Jahangiri, 2002), (Patel and Palit, 2009) and (Wang et al., 2009)] studied meromorphic harmonic functions.

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