CHAPTER 2

YANG AND LAUGHLIN'S HAMILTONIANS AND THE ANGULAR MOMENTUM

2.1 Introduction

The Yang potential is discussed in detail so that the ground state of the Laughlin's wave function can be understood. The solutions of the Yang potential are discussed and the Yang-Baxterization is mentioned as a model of phase transition. The BCS-BEC crossover is discussed. Several efforts to study the crossover are reviewed. The Laughlin's wave function and its many varied applications are pointed out. The miscellaneous discussions with several references to the contexts are described. An effort is made to understand the experimental data from the view point of the interpretation.

The Laughlin's wave function [1] is found to be an exact ground state of a Hamiltonian which involves the derivatives of the δ function in the complex Riemann space. It requires the knowledge of Yang's potential to understand the ground state. Such a potential is possible in the case of a phase transition and hence requires the Yang-Baxter model. The discussion of the Laughlin's wave function spans a wide variety of topics in the condensed matter physics. It does not apply immediately to the experimental data of the quantized Hall effect. Recently, Shrivastava reviewed [2] some of the applications of the Laughlin's wave function including some of his theory [3]. The topics which were left out in Shrivastava previous article have now been included in the present work. The Schrödinger equation with the δ function potential is

solvable by using an ansatz of Bethe [4]. In the first instance it appears that it is unlikely to have applications in physics but then it is felt that it is useful for a system which has a phase transition. There is considerable debate on the BCS-BEC crossover. The effort to discuss the Laughlin's wave function continues. Usually, apparently different theories based on quantum mechanics were found to be equivalent. For example, the Heisenberg's approach looks different from that of Schrödinger but they were based on the same principles. Similarly, Feynman, Schwinger and Tomonaga's approaches look different but they were based on the same principles. Shrivastava is theory [3] which also explains the fractional charges seems to be quite different from that of Laughlin but is not equivalent. If any equivalence exists, it will require the wave function to be significantly modified. Hence, we make more extensive effort to understand the Laughlin's wave function and understand the experimental data.

2.2 The Yang Potential

In the case of a many-body system, the potential can be written in the form of a vector δ function which can be solved exactly. In one such example, the system consists of N particles arranged in a line of length L with a repulsive (positive) potential so that the Hamiltonian appears as,

$$\mathcal{H} = -\sum_{1}^{N} \partial^2/\partial\, x_j^2 + 2c \sum_{i<j} \delta\left(x_i - x_j\right) \tag{2.1}$$

where c > 0 with no limitation on the symmetry of the wave function ψ . The Hamiltonian can be solved for the eigen values provided that Bethe's wave function is used. For a given irreducible representation R_ψ of the permutation group S_N of the N coordinate's x_i, we assume the Bethe's

hypothesis [4]. There is a set of N unequal numbers, $p_1 \ldots p_N$. Similarly, for coordinates $0 < x_{Q1} < x_{Q2} <, \ldots < x_{QN} <$ L so that L is the length. The Bethe's wave function is,

$$\psi = \sum_P [Q, P] \exp i\left[p_{P1} x_{Q1} + \cdots + p_{PN} x_{QN}\right] \tag{2.2}$$

where P= [P_1, P_2... P_N] and Q= [Q_1, Q_2... Q_N] are two permutations of the integers 1, 2… N. The [Q, P] can be arranged as an N! × N! matrix. Although, there is no spin in the Hamiltonian, Yang [5] has proposed that we can consider N-M spins up and M spins down so that the total number of spins is N which is the same as the number of particles. In this case it is possible to find the eigen value per unit length as,

$$\mathrm{E/L} = constant + \frac{\mathrm{M}}{\mathrm{L}}\left[cr - \left(\frac{c^2}{2\pi} + 2\pi r^2\right) tan^{-1}\frac{2\pi r}{c}\right] + \cdots \tag{2.3}$$

where r=N/L. Yang [6] continued the study of this problem for a long time. Sutherland [7] has shown that the exact solutions can be obtained from Yang's method. The particles can be permuted by the operator P_{ij} in the representation $\tilde{R}_\psi$=[m_1,m_2,..,m_k]. In the Yang's case [5] $\tilde{R}_\psi$ = [N-M, M] where N is the number of particles and M is the number of vacancies which cannot be zero. In Sutherland's problem, the projection operator P_{ij} is represented by permutations of m_1 identical vacancies and N - m_1 distinguishable particles (so they do not have spin). For k=3 $\tilde{R}_\psi$ = [N-M, M-M_1, M1]. For N distinguishable classical particles on a ring of length L with density N/L=d, the eigen vectors of the Liouville equation and the corresponding eigen values can be identified. Lai [8] has shown that for a one dimensional fermion system with a repulsive δ function interaction, the free energy can be obtained from the solution of a set of coupled equations. It was also found that Fermi distribution can be introduced through the pressure and the δ functioncan be solved for a special ansatz [9]. In case, the sign of c is changed by replacing c in (2.1) by - g, it was shown by Calogero and Degasparis [10] that N particle wave function,

$$\psi_N = c_N \exp\left(-\frac{1}{4} g \sum_{i,j=1}^{N} |x_i - x_j|\right) \tag{2.4}$$

for the attractive system has the ground-state energy,

$$E_N = -\frac{1}{48} g^2 N(N^2 - 1). \tag{2.5}$$

For large N, it collapses to a linear volume of $1/gN$ and the binding energy per particle is proportional to g N^2 Andrei and Lowenstein [11] have considered the Hamiltonian of the type,

$$\mathcal{H} = -i \sum_{i=1}^{N} \beta_i \partial_i - 4g \sum_{i,j} \delta(x_i - x_j) P^{i,j} [\frac{1}{2}(1 - \beta_i \beta_j) \tag{2.6}$$

where $P^{i,j}$ exchanges β_i and β_j which allows nontrivial permutation symmetry. To diagonalize this Hamiltonian we divide the configuration space into regions labeled by permutations $Q \propto S_N$. In the interior of region Q, defined by $0 \leq x_{Q1} \leq x_{Q2} \leq \cdots \leq x_{QN} \leq$ L, the particles are free so that we can write plane waves. This produces the boundary value problem which is solved for the total energy. Gutierrez et al [12] considered a system of fermions with N >> 1 (spin 1/2) in one dimension, placed inside a box of length L with periodic boundary conditions with Hamiltonian,

$$\mathcal{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} - \upsilon_0 \sum_{i<j}^{N} \delta(x_i - x_j) \tag{2.7}$$

with $\upsilon_0 > 0$. The Slater determinant,

$$\phi_0 = (N!)^{-1/2} det \phi_{k_i,\sigma_i}(x_j, \sigma_{3j}) \tag{2.8}$$

is constructed with single particle orbital's ϕ_{k_i,σ_i} . The orbitals determined via a variational procedure are of the form,

$$\phi_{k,\sigma}(x,\sigma_3) = L^{-1/2}F(k,x)e^{ikx}\chi_\sigma(\sigma_z), k \leq k_0 \tag{2.9}$$

where $2k_o$ is the wave vector of the Fermi sea. $F(k,x)$ is a complex function,

$$F(k,x) = u(x,k) - iv(x,k). \tag{2.10}$$

For $\rho = N/L = 0$, the exact energy is given by,

$$E_0(\rho = 0) = -\frac{mv_0}{8\hbar^2}. \tag{2.11}$$

The Hartree-Fock energy can be expressed in terms of this energy. Introducing the parameters,

$$\gamma = \frac{mv_0}{2\pi\hbar^2 k_0} \tag{2.12}$$

the relative energy can be written as,

$$\epsilon = (3\pi^2\gamma^2)^{-1} - 2(3\pi^2\gamma)^{-1}[2 - \mu + (1-\mu)c^2] \tag{2.13}$$

Where $c = [\frac{K(\mu)}{E(\mu)}]^{1/2}$, in which $K(\mu)$ is the complete elliptic integral of the first species and $E(\mu)$ corresponds to the second species. Furuya and Lowenstein [13] have considered the Hamiltonian,

$$\mathcal{H} = -i\sum_{j=1}^{N}\frac{\partial}{\partial x_j} + 2J\sum_{j=1}^{N}\delta(x_j)\,\sigma_j.S \tag{2.14}$$

which is similar to the Kondo problem because $\sigma_j/2$ is the spin of the jth electron and S is that of the impurity which is located at x = 0. The energy of this Hamiltonian is found to be zero in the ground state. An exact solution of the eigen value problem,

$$\{-d^2/dx^2 + \Delta[1 - a_0\rho(x)]\}\psi_n(x) = E_n\psi_n(x) \tag{2.15}$$

with$\rho(x) = \sum_{n=1}^{N}|\psi_n(x)|^2$, $0 < x < L$,$1 \leq n \leq N$ $a_0 = L/N$ is found by Shastry [14] in one dimension with periodic boundary condition $\psi_n(x + L) = \psi_n(x)$. Shastry [15] also found the exact solution of the Heisenberg anti-ferromagnetic chain,

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N-1}\sum_{m=1}^{N} J_n \sigma_m . \sigma_{m+n} \tag{2.16}$$

with $J_n = J_0/sin^2(\frac{n\pi}{N})$. When momenta N - M are occupied by the up electrons and the set of M by the down electrons, it is possible to write the eigen value, E_M in terms of M and the coupling constants which is a ground state for small number of spins. The δ function potential for N=2 and 3 has been discussed by Koltun [16]. Craig et.al [17] have considered a one-dimensional system with a large number of bosons having the Hamiltonian,

$$\mathcal{H}(c, g, N) = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\sum_{i<j}^{N}\left(g + ce_i e_j\right)\delta\left(x_i - x_j\right) \tag{2.17}$$

where g and c > 0 are the constants and the charge is identified by,

$$e_i = +1 \quad if \quad 1 \leq i \leq N/2$$

$$-1 \;\; if \quad (N/2) + 1 \leq i \leq N \tag{2.18}$$

and $-\infty < x_i < \infty$. The ground state energy is,

$$E_0(c, g, N) = \frac{N^2}{L}\, g - \frac{4}{3\pi} N \left(\frac{N}{2L}\right)^{1/2} (c + g)^{1/2} \quad for \;\; (0 < g < c). \tag{2.19}$$

Adding a kinetic term $\pi^2 N/L^2$ for localization and minimizing with respect to the length L, the energy has been obtained. Similarly, a Bogoliubov type transformation gives the approximate unperturbed energy [17].K$\ddot{a}$rtner and Haus [18] have considered a gas of bosons interacting via a δ-function potential,

$$\left[-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+2c\sum_{1\le i\le j\le n}\delta(x_j-x_i)\right]f_n=i\frac{\partial}{\partial x}f_n \tag{2.20}$$

which has bound solutions. By using the Bethe ansatz the eigen states can be found. A subset of these bound states are characterized by one additional quantum number p besides the boson number n given by,

$$f_{n,p}=N_n\exp\left[ip\sum_{j=1}^{n}x_j+\frac{1}{2}c\sum_{i,j}|x_i-x_j|\right] \tag{2.21}$$

where N_n is the proper normalization constant,

$$N_n=\left[\frac{(n-1)!\,|c|^{n-1}}{2\pi}\right]^{1/2} \tag{2.22}$$

The quantum number p is related to the momentum of the centre of n bosons. The wave function decays exponentially with the separation of boson pairs. The function $f_{n,p}$ is the wave function of n bosons, moving with momentum p. The Schrödinger equation is linear in f_n. The energy eigen values of these n bosons are,

$$E_{n,p}=np^2-\frac{|c|^2}{12}n(n^2-1) \tag{2.23}$$

which consists of the sum of kinetic energy of n bosons with momenta p and the binding energy due to the interaction, the Kerr nonlinearity. K$\ddot{a}$rtner and Haus [18] have shown how the nonlinearity is generated and one can apply the present solution to the photon which measures the correlation function of arbitrary order m. The m=1 corresponds to the original Han bury Brown and Twiss effect [19]. The solitonic nonlinear solution is given by Lu-ming and Guan-can [20]. The exact solution of a hard-core system interacting with a single impurity has been obtained by Li and Ma [21]. They consider the Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} - \frac{\hbar^2}{2m'}\frac{\partial^2}{\partial x'^2} + u\sum_{j>i=1}^{N}\delta(x_i - x_j) + v\sum_{i=1}^{N}\delta(x_i - x') \quad (2.24)$$

where x' is the coordinate of the impurity and x_i are those of the i th particle of the system. The total momentum of the system is conserved. We make a scalar transformation $x' \to x_0 = \sqrt{\mu}x'$where $\mu = m'/m$ and $\hbar = m = 1$. We combine the first and the second term of (2.24) so that they become a Laplace operator in an (N+1) dimensional Euclidean space with Cartesian coordinates ($x_0, x_1, x_2 \ldots x_N$). In this case the Schrödinger equation takes the form,

$$\left[-\frac{1}{2}\sum_{l=0}^{N}\frac{\partial^2}{\partial x_l^2} + u\sum_{j>i=1}^{N}\delta(x_i - x_j) + v\sum_{i=1}^{N}\delta\left(x_i - \frac{1}{\sqrt{\mu}}x_0\right)\right]\psi(x_0, x_1, \ldots, x_N) = E\psi(x_0, x_1, \ldots, x_N) \quad (2.25)$$

which requires the use of "Gauss box" that is cut by hyper planes [21]. We write the wave function as,

$$\psi(y_0, y_1, \ldots, y_N) = e^{i\mathrm{K}y_0}\varphi(y_0, y_1, \ldots, y_N) \quad (2.26)$$

where K is a constant, $y_0 = x' = \frac{1}{\sqrt{\mu}}x_0$ and $y_i = x_i - x' = x_i - \frac{1}{\sqrt{\mu}}x_0$. The energy is then of the form,

$$E = \frac{1}{2}\sum_{i=1}^{N}k_i^2 + \frac{1}{2\mu}\lambda^2 \quad (2.27)$$

where $\lambda = \mathrm{K} - \sum_{i=1}^{N}k_i$ so that the energy can be written as an integral. The solution of δ-function boson in one-dimensional potential is also given by Li [22]. For the potential $\sum_{i\neq j}\frac{1}{4}v\delta(x_i - x_j)\left(1 \pm \sigma_i\sigma_j\right)$, the solution was discussed by Ge and Wang [23,24]. The problem of a magnetic impurity interacting with conduction electrons can be written in the form of the potential,

$$\mathcal{H}' = 2c\sum_{i<j}\delta(x_i - x_j) + \sum_{j=1}^{N}[\delta(x_i) + \delta(L - x_i)]\left[J_{\mathcal{T}.j.S} + V\right] \tag{2.28}$$

where c > 0, J describes the Kondo coupling constant and the boundary potential. The δ-function potential has also been applied to dimmers of [3] He, atoms in pores [25] as well as to winding numbers in polynomials [26]. In the case of a one-dimensional Fermi system with the total number of particles, $N = N_{\uparrow} + N_{\downarrow}$ and the contact interaction, the Hamiltonian is given by,

$$\mathcal{H} = N\hbar\omega_{\rho} + \mathcal{H}_{1D}^{0} + \frac{1}{2}\sum_{i=1}^{N} m\,\omega_z^2 z_i^2 \tag{2.29}$$

where

$$\mathcal{H}_{1D}^{0} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{i=1}^{N\uparrow}\sum_{j=1}^{N\downarrow}\delta(x_i - x_j). \tag{2.30}$$

The interaction constant is given in terms of 3-dimensional s-wave scattering length, a_{3D}, as,

$$g_{1D} = \frac{2\hbar^2 a_{3D}}{m a_{\rho}^2}\frac{1}{1 - A\,a_{3D}/a_{\rho}} \tag{2.31}$$

where $a_{\rho} = \sqrt{\hbar/m\omega_{\rho}}$ is the oscillator length in the transverse direction and $A = |\zeta(1/2)|/\sqrt{2} = 1.0326$. The coupling constant of (2.30) can be expressed also through the effective 1D scattering length a_{1D} as $g_{1D} = -2\hbar^2/m a_{1D}$ where,

$$a_{1D} = -a_{\rho}\left(\frac{a_{\rho}}{a_{3D}} - A\right). \tag{2.32}$$

The 1D interaction is repulsive for $g_{1D} > 0$ and $0 < a_{3D} < a_{3D}^{c}$ $(a_{3D}^{c} = 0.9684 a_{\rho})$ and attractive for $g_{1D} < 0$ and $a_{3D} > a_{3D}^{c}$ $(a_{3D} < 0)$. For two fermions with different spins, the Hamiltonian (2.30) supports one bound state with binding energy $\epsilon_{bound} = -\hbar^2/m a_{1D}^2$ and spatial extenta_{1D} $(g_{1D} < 0)$. The molecular state is delocalized as $g_{1D} \to 0^{-}$. The ground state

energy E_{hom} of homogenous $\mathcal{H}_{1D}^{0}$ has been calculated exactly [27] in terms of linear number density$n_{1D} = N/L$ where L is the length of the system as,

$$\frac{E_{hom}}{N} = \frac{\hbar^2 n_{1D}^2}{2m}\, e(\gamma) \tag{2.33}$$

where $e(\gamma)$ is a solution of the coupled equations with $\gamma = mg_{1D}/\hbar^2 n_{1D}$. This predicts a strongly attractive gas region.

2.3 The Yang-Baxter model

We consider the eight-vortex model of Baxter and include the δ-function potential. The problem remains integrable in one dimension that is why it is called the Yang-Baxter model [28]. We denote the two-particle scattering matrix by S_{ij} then,

$$S_{ij}S_{ik}S_{jk} = S_{jk}S_{ik}S_{ij} \tag{2.34}$$

is called the Yang-Baxter equation. The local potential for the interaction of conduction electrons with the spin of an impurity ion is represented by,

$$V_c(x) \simeq [\delta'(x^+) + \delta'(x^-)]x/|x| \tag{2.35}$$

with x=0 as the location of the impurity ion. This potential produces a localized Kondo effect.

The integrable Hamiltonian is given by

$$\mathcal{H} = \sum_i [-\partial_{x_i}^2 + (J\sigma_i.\sigma_0 + J')\delta(x_i)] + \sum_i V_c\,(x_i) + \sum_{i<j} 2c\delta\big(x_j - x_i\big) \tag{2.36}$$

where $2J = -c < 0$ and $J' = -J$ are integrable which permits the calculation of exact ground-state energy in one dimension [29].

2.4 The BCS-BEC crossover

The Bose-Einstein distribution which allows any number of particles to occupy one energy level never crosses the Fermi-Dirac distribution which permits only one particle per level and when spin is considered, two particles per level with one spin up and the other down. Several workers are interested in looking at a pair of electrons in which each electron obeys the anti-commentators and compare them with a pair of bosons [30]. Let us try the Yang potential. For the spin 1/2 Fermi gas interacting via a short range potential, the Hamiltonian is written as,

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_1\sum_{i<j}\delta(x_i - x_j) \tag{2.37}$$

where N is the total number of fermions and m their mass. The dimensionless coupling constant is $\gamma = mg_1/\hbar^2 n$ where n=N/L is the 1-dimensional density. For $\gamma \to 0^-$, the ground state is a BCS like state with Cooper pairs, whose size is much larger than the average inter particle spacing. It never reaches a weakly interacting Bose-Einstein condensation (BEC) as one of the limits of the standard BCS-BEC crossover in 3 dimensions. The two-body potential $g_1\delta(x)$ has a bound state only when $g_1 > 0$ but no bound state when $g_1 < 0$. In the regime, $g_1 < 0$, the ground state of (2.37) is that of repulsive fermions. The continuous evolution from attractive Fermi to a repulsive Bose gas in one dimension is implied in the Bethe ansatz equations of Gaudin-Yang model [31].

2.5 The Laughlin's wave function

In 1983, Laughlin gave a wave function which could give a fractional charge [1]. An exhaustive review of this wave function was presented by Shrivastava [2] along with his theory of fractions relevant to the experimental data on the quantum Hall effect.

Now we wish to further elaborate on the study of Laughlin's wave function, in particular, complete those aspects which were not covered previously [2]. In particular, the relationship of the wave function with the Hamiltonian for which it is a ground state is emphasized. It turns out that the Hamiltonian of the Laughlin's wave function has some remote analogy with the Yang potential while Shrivastava theory is more relevant to the interpretation of the experimental data. Wang et al. [32] have shown that it could give a ground state which is about 91 % of the Wigner value. The effect of impurities has been examined by Hikami [33] who found that the value of the conductivity is slightly modified to $\sigma_{xx} = 1.4e^2/2h\pi^2$ .The particle-hole symmetry is discussed by Girvin [34]. Part of the wave function of Laughlin can be written as,

$$\psi_m(z_1, \dots, z_N) = \prod\nolimits_{i<j} (z_i - z_j)^m. \tag{2.38}$$

The particle-hole conjugate should occur at,

$$\nu = 1 - \frac{1}{m} \quad . \tag{2.39}$$

For example, for m=3, the Laughlin's state gives the charge 1/3 so that according to the above formula, a particle of charge 2/3 should exist so that there is a particle-hole symmetry. It is found that Laughlin's wave function for the charge 1/3 does not predict that there is a particle of charge 2/3 so that it does not have the particle hole symmetry. Since it does not have spin, it does not have a properly defined helicity ,$\vec{p}, \vec{S}$. MacDonald and Aers [35] have pointed out that electron-electron separation becomes comparable to interlayer separation which modifies the Coulomb interaction but does not affect the ground state seriously. There is a small number of electrons in the calculation so that the energy per electron depends on the number of electrons as well as on the size of the cell [36]. There is a small effect of the Landau-level mixing [37]. Although the original Laughlin calculation is limited to the lowest Landau level, excited states including

magneto plasma modes have been described [38]. Haldane and Chen [39] have shown that the quantum Hall effect experiments are performed in two-dimensional conducting Hall surface embedded in a three-dimensional insulating medium in which the electromagnetic fields reside. The charged defects on the Hall surface carry a small non integral material-dependent magnetic flux of the order of $(\alpha')^2$where$\alpha' = (\mu/\epsilon)^{1/2}/2R_H^0$. Here μ and ϵ are the permeability and permittivity of the medium and $R_H^0 = h/e^2$. Although Laughlin's state is incompressible, Haldane and Rezayi [40] have shown that a transition to compressible state is possible. A general discussion did not lead to the evaluation of fractions at which plateaus occur in the conductance [41].A calculation of the dispersion relation for the electron-hole pair shows that the Laughlin's wave function is "degenerate" because different values of m all give a ground state [42]. Another calculation shows that Laughlin state hierarchy is similar to that of classical fluids [43]. The wave function proposed by Tao and Thouless [44] implies long-range correlations. This type of correlations is very different from those of the Laughlin's wave function. A careful study of the eight particle wave function showed some overlap with the Laughlin's wave function. The long-range order is not found in the Laughlin's wave function [45]. A wave vector dependence [46] of the response function shows that $\chi_m(q) \propto q^2$. Rasolt et al. [47] consider the strain energy in the form,

$$\mathcal{H}' = \int d^2r\,(\gamma/2)\left(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2\right) \tag{2.40}$$

where $\psi_1$and ψ_2 are the wave functions in two valeys. When $\gamma \to 0$, there is a Goldstone mode which varies with wave vector so that the response function for two angular momenta states,

$$\chi_{J_-,J_+} \sim q^{2+\epsilon}\ . \tag{2.41}$$

For finite q, which means that there is a phase transition in going from one plateau to another. Oji and MacDonald [48]find that there are poles in the density response function. Fano et al. [49]

have reported the calculation of the wave function for electrons on a sphere exactly. Apparently, the problem is very big even for a reasonable computer to carry out the calculation without approximations but it is solvable in certain symmetries. In another calculation in the Hartree-Fock approximation the ground state is found to change from the charge-density wave to a uniform density state and to a crystal state of holes in the lowest Landau level [50]. The calculation of the density of states for seven electrons shows that the bands are very narrow [51]. A Hartree-Fock calculation claims to find cusps in the energy at odd denominators when the wave vector is equal to the inverse of the magnetic length [52]. The interlayer coupling produces an extra plasmon like mode which for certain wave vectors merges with the continuum. The response function favors a Wigner solid in the multilayer structures [53]. Although Laughlin did not consider spin because all of the electrons were polarized in a single spin state and there was no consideration of spin states separated by a Boltzmann's factor, Haldane and Rezayi [54] considered spin singlet wave functions while maintaining the incompressibility found that half-integral fractions can arise. In particular, the value of 5/2 was discussed even though it is greater than 1. The particle-hole symmetry has been considered by Fano and Ortolani [55] who have slightly improved 95% the original calculation of Laughlin [1]. Yoshioka et al. [56] have given naive ideas of spin and similarities with spin zero problem have been pointed out. Qiu et al [57] have compared the Laughlin state with that of a solid and found that Laughlin state is a liquid state so that it is possible to draw a phase diagram. Bander [58] has calculated the energy for single-particle wave functions for various field in homogeneities. It has been reported [59] that in the case of double quantum well structures the soft modes can destroy the quantum Hall plateaus and the effect of Coulomb interaction can be studied by bringing a sheet of a metal near the two-dimensional electron gas [60]. When a layer separation is larger than a critical value excitonic

charge density waves are favored over the metallic charge-density waves [61]. When there are no transitions between Landaus levels, eigen values from the scattering amplitudes give rise to a charge-density wave .Rezayi and MacDonald [62] present numerical evidence to suggest that there is no pairing at the fractional filling factor $\nu = n/(2n+1)$. Usually, there is no restriction on the number of Landau levels. However, the Laughlin wave function is limited to the lowest Landau level, n=0 in (n+1/2) so that the filling factor is always less than one,$\nu < 1$. If all of the Landau levels are neglected except the n=0, it is equivalent to saying that the Hilbert space is limited to n=0 only. In the Rezayi and MacDonald's calculation, lowest two Landau levels are considered and the cyclotron frequency is an adjustable parameter. At $\hbar\omega_c = 0$, the zero-energy ground state occurs at $\nu = 2/5$. The incompressibility is a fundamental assumption arising from the condition $a_0^2 = 1$ which is independent of the value of $\hbar\omega_c$. However, the expression $\nu = n/(2n+1)$ is the same, as far as the numerical value is concerned, as that of one of the two series of Shrivastava [2] which are, $\nu_- = l/(2l+1)$ and $\nu_+ = (l+1)/(2l+1)$ in which l is surely the angular momentum and spin is 1/2. In the Rezayi and MacDonald's expression ν is independent of angular momentum and electrons do not have spin. It implies that spin is zero as in a paired state. Lutken and Ross [63] have changed the notation to k/(2k+1) and have made effort to use a bifurcation model to obtain the hierarchy of fractions. The scaling theory on the position of delocalization fixed points suggests that there is a phase transition in going from one plateau to another plateau. The projection of the angular momentum on the first Landau level, L_z is found [64] to be the generator of rotation, which corresponds to the lowest energy. Several authors have discussed a variety of problems dealing with edge currents [65], gauge invariance [66] and with mean field theory which show that wave functions of small number of particles at large distances overlap with Laughlin's wave function [67]. The occupation number distribution

function for finite number of electrons in Laughlin states is found to show large oscillations [68]. However, fractional effect is not found in the Aharanov-Bohm phase factors due to neglect of angular momentum effects [69]. The short-range potentials are discussed because of the short-range in the wave function [70]. The general theory of response function has been discussed [71]. The electrons on a sphere without symmetry restrictions show chaotic behavior [72]. The energy gaps have been calculated by Xie [73]. Several calculations of the variational wave functions have been reported [74] which describe the rotations [75], half-filled Landau level [76,77] and edge states [78]. The comparison of the ground state energy with that of the charge-density waves has been discussed [79].A boson to fermion transformation is discussed by Borgh et al. [80]. The orbital angular momentum quantum number $l = 0$ is called a " boson". It is then possible to find an approximate transformation from a many-boson state to a many-fermion state in the lowest Landau level. The overlap between the transformed boson state and the true fermion state for certain a value of the angular momenta turns out to be quite large such as 90 per cent. However, the overlap depends on the value of the angular momenta and for some intermediate values, it is very small. The ground-state energies of $\nu = 1/3$ state for 20 electrons and of $\nu = 5/2$ state for 26 electrons has been calculated by Feiguin et al. [81] by using density matrix renormalization group method but there is no spin in the calculation . Gulacsi [82] has obtained the ground state energy for a half-filled Landau level in two dimensions. The products of three Laughlin type wave functions have been suggested by Halper in as calculated by Deidel and Yang [83]. A nematic state has been found by Doan and Manousakis [84]. The fractional states 2/3 and 1/4 have been calculated [85,86].The response function has been calculated by Ho et al. [87] and a conformal field theory has been suggested [88]. Interference effect across a single incompressible state has been studied [89]. It has been reported [90] that incompressible

states evolve adiabatically which is by means of a second-order phase transition from a plateau to another plateau whereas the compressible states are driven by first-order phase transition. The correlations between electrons have been computed by product states [90,91]. If the wave function fractionalizes the charge, it must represent the (i) ground state of a Coulomb Hamiltonian, (ii) there should be no parameters which can take the blame of fractionalization instead of the charge and (iii) no important component of the mechanics, such as equations, should be left out. We will discuss these three properties of the wave function. It was found that Laughlin's wave function is not the ground state of the Coulomb Hamiltonian. It is the exact ground state of a non-physical Hamiltonian. However, the non-physical Hamiltonian may be passed as a new Hamiltonian which arises in a phase transition. In that case all of the algebraic equations must be solved and the number of equations should match with the number of unknowns. It was also found that incompressibility is needed to obtain the fractional charge, otherwise, the charge will not fractionalize and only the area in which flux is quantized, will be fractionalized. It is possible to find the area of overlap of Laughlin's wave function with that of the exact wave functions for a small, 2 or 3, number of electrons in a given symmetry. If this overlap is of the order of unity, then we can conclude that Laughlin's wave function is quite satisfactory. If the overlap is small, then the Laughlin's wave function is away from the exact solution. The flux quantization condition is $Ba_0^2 = n'hc/e$ which leads to fractionally charged quasiparticles with charge e/m so that$Ba_0'^2 = n'hc/(e/m)$. It is quite possible that the blame of m occurs in the area in which the flux is quantized so that the flux quantization becomes $B(a_0'^2/m) = n'hc/e$ so that charge is not fractionalized and only the area is. The quantity ea_0^2/m does not uniquely determine the fractional charge. We also note that Laughlin does not use a good definition of the angular momentum so that although there is angular momentum, it is

not associated with L, S or J. At that time, it was thought that all N number of electrons are spin polarized so that the magnetic moment is NS. In Laughlin's paper, the spin was completely ignored and there is no regard to Boltzmann factor between states of different spin polarizations. Hence, the ground state property, flux quantum area, a_0^2, and the spin require considerable reconsideration. The ground state energy calculated by Laughlin using the radial distribution function g(r) is quite good but not the minimum. The Wigner crystallization has lower energy than that calculated by Laughlin which may be 90.98 per cent of the lowest energy known. The Wigner energy is about -0.455 whereas Laughlin's value is -0.414 in units of $(e^2/\epsilon a_0)$. A lot of authors consider it as a very good result because the wave function is not too far away from being that of the lowest energy. The good value of the energy is naturally due to a good value of g(r). Wang et al. [32] have compared the energy with those of other calculations and found it to be lower than many other values. It was found that [92] Laughlin's wave function has a very small range so that it is the ground state of a potential $V_2(|r|)$ the range of which tends to zero. Expanding V_2 in powers of range b, the potential obtained is,

$$V_2(r) = \sum_{j=0} c_j \, b^{2j} \nabla^{2j} \delta^2(\vec{r}) \tag{2.42}$$

which is positive (repulsive). As the range tends to zero, $\rightarrow 0$, only leading term contributes to the energy. The average $< c_0 \delta^2(r) >$ vanishes for anti-symmetric wave function so that the pure δ-function never contributes. Rezayi and MacDonald [62] have found that Laughlin's wave function gives the zero-energy ground state exactly with the interaction given by,

$$V(r) = \lambda a_0^2 \nabla^2 \delta(\vec{r}) \tag{2.43}$$

where $a_0 = (\hbar c/eB)^{1/2}$ is the magnetic length. Considerable amount of work has been done in the $\nabla^2\delta(r)$type interactions (for example see Yannouleas and Landman [93]). In the case of half-filled Landau level, the filling factor is $\nu = 1/2$ so that the wave function is written as,

$$\psi_{1/2} = Pf\left(\frac{1}{z_i - Z_j}\right)\psi_{1/2} \tag{2.44}$$

$$\psi_{1/2} = \prod_{i<j}(z_i - z_j)^2 \exp\left[-\frac{eB}{4hc}\sum_j |z_j|^2\right] \tag{2.45}$$

The second wave function, $\psi_{1/2}$, is the Laughlin state which describes a boson for the even denominator in 1/2 and Fermion for odd denominator. Laughlin used hc/eB = 1 as a matter of convenience but it is not applicable. The Pf(M) is a Pfaffian represented by an anti-symmetric matrix M. The $\psi_{1/2}$ can be obtained from the adiabatic evolution of a singular paired state for δ-function attraction described by the Pfaffian factor. The $\psi_{1/2}$ is the exact ground state of a simple local effective Hamiltonian. The wave function has a zero when three particles occur at the same point. The pairing state is the ground state of a Hamiltonian with repulsive three body δ- function interactions [94],

$$V_{i,j,k} = \sum_{triples} \delta^{(2)}(z_i - z_j)\delta^{(2)}(z_i - z_k) \tag{2.46}$$

The product of the Pfaffian and the full Landau-level wave function is actually the lowest degree polynomial. For a finite number N of electrons on a sphere subject to a uniform normal magnetic field with total flux N_ϕ, the incompressible state occurs for,

$$N_\phi = 2N - 3 \tag{2.47}$$

with N=even. The wave function is exact for idealized model Hamiltonians involving three-body interactions. The pseudo potential V_m for relative angular momentum m is obtained by projecting down to the first Landau level as,

$$V_m = \int_0^\infty dq\, exp(-l^2q^2)\, L_m(-l^2q^2)F(q) \tag{2.48}$$

where l is the magnetic length, L_m is the Laguerre polynomial. We can ignore the electromagnetic theory and change $\nu^{-1} \rightarrow \nu^{-1} + 2p$ and $\nu \rightarrow 1 - \nu$ in the filling factor and pretend that the filling factor is given by a continued fraction,

$$\nu = 4[2p_n + 1, 2p_{n-1}, \dots, 2p_l + 1] \tag{2.49}$$

whose first and last entries are odd with all intervening ones even. Thus for n = 2, the filling fraction becomes,

$$\nu = 4\frac{1}{2p_2 + 1 + 1/(2p_1 + 1)} = \frac{4(2p_1 + 1)}{(2p_1 + 1)(2p_2 + 1)} \tag{2.50}$$

For $p_1 = 0$, $\nu = 2/(p_2 + 1)$, i.e., an arbitrary fraction with numerator 2. These give the Laughlin $1/(2p_2 + 2)$ states for the effective bosons. Wen [95] has been pointed out that there are topological orders so that there is shift $\tilde{S}$ in the number of flux quanta, $N_\phi = \nu^{-1}N_e - \tilde{S}$ where N_e is the number of electron. The topological order requires a local Hamiltonian to support the incompressible ground state. The Pfaffian wave function, ψ_{Pf} is an exact ground state of the following three body Hamiltonian,

$$\mathcal{H}_{3b} = -Uc^\dagger(z_1)c^\dagger(z_2)c^\dagger(z_3)\partial z_1^*\delta(z_1 - z_2)\partial z_1\partial^2 z_3^*\delta(z_3 - z_2)\partial^2 z_3^* c(z_3)c(z_2)c(z_1) \tag{2.51}$$

The coefficient U is chosen so that the three electron state,

$$\prod_{i<j}(z_i - z_j)\, exp\left(-\frac{1}{4}\sum_{i=1}^{3}|z_i|^2\right)$$

has unit energy. We introduce the pseudo potentials by the Hamiltonian,

$$\mathcal{H} = \sum_{l=0}^{q-1} V_l P_l^{i,j} \tag{2.52}$$

Where V_l are positive constants and $P_l^{i,j}$ is the projection operator onto the relative angular momentum state of angular momentum$l$ for particles i and j. The densest zero-energy eigen state of $\mathcal{H}$ is the one with the lowest total angular momentum which is the Laughlin state,

$$\psi_L(z_1, \dots, z_N) = \prod_{i<j}(z_i - z_j)^q \exp\left[-\frac{1}{4}\sum|z_i|^2\right] \tag{2.53}$$

If it is multiplied by any symmetric polynomial in the $z_i's$ it will still be a zero-energy state. We consider a microscopic model of a two-dimensional electron gas confined to a disk with a mixed Hamiltonian,

$$\mathcal{H} = \lambda\mathcal{H}_{3B} + (1+\lambda)\mathcal{H}_c \tag{2.54}$$

For, $\lambda = 0$, the above becomes a two-dimensional Coulomb Hamiltonian and for$\lambda = 1$, it become a three-body Hamiltonian [96],

$$\mathcal{H}_{3B} = \sum_{i<j<k} S_{ijk}[\,\nabla_i^2\nabla_j^2(\nabla_i^2 + \nabla_j^2)\delta(r_i - r_j)\delta(r_i - r_k)] \tag{2.55}$$

where S is a symmetrizer, $S_{123}(f_{123}) = f_{123} + f_{231} + f_{312}$ with f symmetric in its first two indices. the wave function

$$\psi_{MR}(z_1, z_2, \dots, z_N) = Pf\frac{1}{z_i - z_j}\prod_{i<j}(z_i - z_j)^2 \exp\left(-\sum_i \frac{|z_i|^2}{4}\right) \tag{2.56}$$

is the exact zero-energy ground state of $\mathcal{H}_{3B}$with smallest total angular momentum $M_0 = N(2N-3)/2$. For fermions, the wave function is,

$$\Psi_M(z_1, z_2, z_3) = B_M(z_1 + z_2 + z_3)^{M-3} J(z_1, z_2, z_3) \tag{2.57}$$

where $J(z_1, z_2, z_3) = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$ with total angular momentum M. Hu et al. [85] have pointed out that part of the Coulomb interaction can be replaced by pseudo potentials in which case Laughlin states become exact ground states for specific pseudo potential Hamiltonians. In case of $\nu = 1/3$, the pseudo potential is $V_m = \delta_{l,m}$. In the absence of confining potential the Laughlin state,

$$\psi_{1/3}(z_1, \dots, z_N) = \prod_{i>j}^{N} (z_i - z_j)^3 \exp\left[-\frac{1}{4}\sum_{i=1}^{N} |z_i|^2\right] \tag{2.58}$$

Is the exact ground state with zero energy which exists in the subspace of total angular momentum $M = 3N_e(N_e - 1)/2$ for N_e electrons in $3N_e - 2$ orbitals. The three-body interaction [97] that makes the MR state, the exact ground state at half filling takes the form,

$$\mathcal{H}_{3B} = \sum_{i<j<k} S_{ijk} \left[\nabla_i^2 \nabla_j^2 \delta(r_i - r_j)\delta(r_i - r_j)\right. . \tag{2.59}$$

For spinless electrons, the MR state is the unique zero-energy ground state at half filling,

$$\psi_{MR} = \left[\prod_{i<j} (z_i - z_j)^2\right] A\left(\frac{1}{(z_1 - z_2)} \dots \frac{1}{(z_{2N-1} - z_{2N})}\right) \tag{2.60}$$

where N is the number of pairs [97]. Chen et al. [98] have discussed the method of calculation of matrix elements from the state of charge e/4 to e/2. Wang et al. [99] have shown that there is a phase transition from the Coulomb ground state to a $\mathcal{H}_{3B}$state. There is a zero-energy state

belonging to special δ-function type localized Hamiltonian. The pseudo potential are needed to reach the ground state. Hence, there is a phase transition from δ-function state to a Coulomb state. The ground state of the Coulomb Hamiltonian is quite far from that of the δ-function Hamiltonian [100]. There exists no transformation from Coulomb interaction to δ-function Hamiltonian. These are two different problems. The overlap with non-physical Hamiltonians is not useful for performing experiments. Similarly, the composite fermion (CF) approach becomes non-physical because flux attachment to the electrons violates the symmetries of the electric and magnetic field vectors of the electromagnetic theory [101].

The most important question about the Laughlin wave function is the "incompressibility". The algebra is not completely solved. The number of variables and the number of equations are not matched. The charge and the area are two variables and there is only one condition of flux quantization. To compress first the area with constant charge or keep the area constant and compress the charge or in selected steps, first press the area by one unit and then change the charge by one unit, then change the area by one more unit and then change the charge by one more unit and so on and so forth. That will create the protocol problem between area and the charge, which one to compress first or in which sequence or steps? Compress the area first and then change the charge, is not the something as compress the charge first and then change the area. From previous discussion it seems that the Laughlin's wave function ground state energy is about 91 per cent of the Wigner value. Hence the value of the ground state energy is very good. It is very clear that Laughlin's wave function does not represent the ground state of the Coulomb Hamiltonian. However, it is the ground state of a Hamiltonian which represents a very short-range δ-function type potential. If it was the Coulomb Hamiltonian, it will be useful for the understanding of the physical properties of solids. However, if it is the ground state of a δ

function Hamiltonian, it will probably not occur in physics. The interesting question is whether the Laughlin's wave function will lead to a fractional charge? Because the 91% of the Wigner value is not sufficient for the ground state. The flux quantization is $Ba_0^2 = n'hc/e$. The density σ_m is given by,

$$\sigma_m = \frac{1}{m}\frac{1}{(2\pi a_0^2)}. \tag{2.61}$$

If the charge is fractionalized, e becomes e/m so that the flux quantization condition becomes,

$$Ba_0^2 = n'\frac{hc}{(e/m)}. \tag{2.62}$$

This can also be written as,

$$B = n'\frac{hc}{e(a_0^2/m)}. \tag{2.63}$$

Hence, charge need not fractionalize. In order to obtain fractional charge, the condition of a_0= constant, for "incompressible" states is required. So that the charge will change to e/m. For compressible states the charge will remain a constant and only the area will change. If m changes from 1 to 3, the value of ma_0^2 will change and the charge need not change.

The Laughlin's wave function is independent of spin. Usually, the spin-orbit interaction will give a small contribution to the ground state energy. However, if the spin occurs in a more complicated way, such as S rather than S_x, S_y and S_z, there will be a serious effect of spin on the unperturbed Hamiltonian. Hence, neglect of spin is not justified. We thus find that the Laughlin's wave function need not give the ground state of the Coulomb Hamiltonian. It need not fractionalize the charge since it can fractionalize the area in which flux is quantized. Apparently, the Laughlin's wave function has a large overlap with that of a short-range potential. There is no way to transform a Coulomb Hamiltonian to a δ-function Hamiltonian. Hence, the overlap between the wave functions does not have a physical interpretation. The δ-function Hamiltonian

is usually not found in physical problems. Hence, it will not help to overlap with non-physical wave function.

2.6 Miscellaneous Discussion

2.6.1 Non-Abelian products.

In an effort to find the wave function for the fractional charge, the non-Abelian products forming the Pfaffian determinants have been used as factors of wave functions. Usually, the probability of non-Abelian dot product is very small compared with that of Abelian dot product. If the potential has a $1/r^2$ term the solutions are known in terms of Jack polynomials. Although it is quite different from the Coulomb potential, under some approximations, it is completely solvable. The solutions are useful for introducing series of fractions [102]. Some of the fractions found in the quantum Hall effect are due to anyons which can be Abelian [103] while some are due to parafermion [104]. Since Pfaffian part of the wave function always has two particles, they may occurrence in different layers so that the interlayer distance becomes important for the stability [105]. It is possible to go from the Abelian to non-Abelian states [106] and non-Abelians have local interactions [107].The numerical estimates of energy from non-Abelian wave functions appear to produce no difficulty [108]. Usually the n=0 Landau level has been used because Laughlin's calculation implied n=0. As a consideration of experimental data, higher values of n have been considered. For example at n=1, non-Abelian wave functions give large values of the fractional filling factors [109]. The fractions 12/5 and 13/5 have been discussed to arise from possibly non-Abelian wave function. Some of the observed fractions require that electron clusters are present in the sample. Hence, clustering properties of the non-Abelian wave functions have been calculated [110]. The collective states of non-Abelian quasiparticles in a

magnetic field have been calculated by Levin and Halperin [111]. In the case of anti-ferromagnetism [112] the non-Abelian statistics is found for finite spin. For S=1 in two dimensions, the spin on (spin only) and the holon (charge only) are decofined and obey non-Abelian statistics. The ultra cold Fermi atoms have been trapped in optical lattices which show non-Abelian statistics [113]. In the case of a Pfaffian factor in the wavefunction, there must be two particles so that braiding properties occur [114]. A spin ½ systems on a honeycomb lattice has a gapless phase with vortex excitations that obey non-Abelian statistics [115]. A chiral spin liquid is found to be the exact ground state for the gapless honeycomb lattice [116,117]. The non-Abelian statistics for complex paired state is discussed by Read [118] and exactly resonant model of non-Abelian states is given by Fiste et al. [119]. The non-Abelian current algebra might describe some of the states of fractional charges [120]. It is found that ground-state wave functions go to zero as various clusters of electrons are brought together. A class of non-Abelian states has been constructed for multilayers of two-dimensional sheets. The possibility of occurring such states in the real experiments has been discussed by Barkeshi and Wen [121]. The string nets and single and double-stranded loop gases have also been considered [122]. The order parameter represents mixed s and p wave symmetries. The wave function of the superconductor can be described by a Pfaffian [123]. The non-Abelian resonant level modes are discussed in the quantum Hall effect [124]. The phase transition from Abelian to non-Abelian states is given by Peterson and Das Sarma [125]. The algebra of non-Abelian states has been described in detail [126]. The interaction of non-Abelian anyons in two-dimensional quantum liquids is described by Ludwig et al [127]. The statistics of the non-Majorana fermions has been discussed by Zhu et al. [128].

2.6.2 Phase transitions

The scaling theory is based on the exponents as given by Stroud and Bergman [129]. When a magnetic field is applied perpendicular to the surface of a metallic film, the normal (phase-I) conductivity is determined by the Ohm's law so that it is proportional to the current,

$$\sigma_N^s = aI \tag{2.64}$$

for the symmetric part, and,

$$\sigma_N^a = b\mathrm{K} \quad ; \quad \mathrm{K} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.65}$$

for the anti-symmetric part. Here, σ_N^s is the symmetric part of the conductivity, I is the current and a is a constant. The σ_N^a is the anti-symmetric part of the normal (phase- I) state conductivity and b is a constant. The fractional volume p in phase-II, as an example, so that 1 -p is a phase-I conducting state with scalar conductivity equal to a. The resistivity is the inverse of conductivity, $\rho_e = \sigma_e^{-1}$. The conductivity is thus,

$$\sigma_e = aI + b\mathrm{K}. \tag{2.66}$$

Usually, there is a divergence in the free energy near p_c at which the volume fraction first forms a connected path across the material. The conductivity diverges at $p = p_c$with a characteristic exponent,

$$\sigma_e = Aa(\Delta p)^{-s} \tag{2.67}$$

where $\Delta p = (p_c - p)/p_c$ and s=1.1-1.3 in two dimensions. If $a = 1/H^2$, $b = 1/H$, then at high fields, the resistivity varies as,

$$\frac{\rho_e(p, H \to \infty)}{\rho_s(p, H \to 0)} \propto (\Delta p)^{-2s} \tag{2.68}$$

Note that flux quantization has been ignored in this theory [130]. For a two-dimensional system, above the percolation threshold,

$$\rho(p, H) \propto (\Delta p)^{-s}. \tag{2.69}$$

The variable-range hopping conductivity has a large range of magnetic field in which it varies as the square root of the magnetic field [131]. Platzman et al. [132] have discussed the conductivity for the magneto-roton system. Chui et al [133] have discussed the long-range order in the magneto-phonon system. The electron gas in a magnetic field forms domains which can go to a vortex state [134]. A transition from two-dimensions to three dimensions is found by the density-matrix method. The interlayer coupling drives the dimensional crossover [135]. It has been reported [209] that there is a large overlap between bosons,

$$\psi_m = \prod_{i<j} (z_i - z_j)^m$$

For m=2 and fermions,

$$\psi_F = \prod_{i<j} (z_i - z_j)\psi_B.$$

For $\nu^B, \nu^F = (1/2\,, 1/3), (2, 2/3)$ etc. consider the two-body interaction Hamiltonian,

$$\mathcal{H} = \sum_{i,j,m} V_m P_m(i,j) \tag{2.70}$$

where P_m is the projection operator onto states with relative angular momentum m. A model Hamiltonian, $\mathcal{H}_n$ is obtained, not by a transformation but by a "truncation" of the Coulomb interaction where pseudo potential parameter $V_m = 0$ for m > n and $V_m = (m = 1,2, \dots, n)$ takes on the same values as in the Coulomb interaction. For n=1, the short range Hamiltonian, $\mathcal{H}_1$, the ground state is exactly the Laughlin wave function,

$$\psi_m = \prod_{i<j} (z_i - z_j)^m \tag{2.71}$$

with m=2 (bosons) and m=3 (fermions). In this situation the mapping,

$$\psi_{fermions} = \prod_{i<j} (z_i - z_j)\, \psi_{boson} \tag{2.72}$$

transforms the boson ground state exactly to the fermion ground state. At the boundaries, a Luttinger liquid may stabilize [137]. Lee et al. [138] consider the space dimension cutoff due to inelastic scattering to vary as, $L_{eff}(T) \propto T^{-p/2}$ and the time direction cutoff to vary as$t \propto 1/k_B\, T$. The transition is affected by varying the magnetic field. At the transition the correlation length diverges, $\xi \propto (\delta B)^{-\nu}$ and the characteristic energy vanishes, $\hbar\Omega \propto \xi^{-z}$,where $\delta B = B - B_c$. The temperature appears through the dependence of the length. There are two dimensionless parameters $\mu_1(T, \delta B) = L_{eff}(T)/\xi$ and $\mu_2 = \hbar\,\Omega/k_B T$. At a finite temperature, the width ΔB of the transition is determined by the greater of the solutions of the condition $\mu_1 \simeq 1$ and $\mu_2 = 1$. The product of the two exponents $\nu z < 2\nu/p$ so that $\Delta B = T^{\rho/2\nu}$. The energy gap vanishes as $\Delta \propto (\delta B)^{\nu z}$ in the temperature range $1 \leq \mu_2 \leq \mu_1$. This result shows that activation energy varies with the magneticfield [139]. The electrons and the holes may phase separate [140] and due to correlations, an insulating phase arises [141]. At high magnetic field, the Wigner solid is formed which can melt to produce a liquid state [142]. D'Agosta et al. [143] found that transverse resistivity varies with current as, $R_{xx}(I, 0) \propto I^{\alpha-2}$ which depends on the interaction showing cross over at $\hbar c/d$ where d is the length of the constriction. Goerbig et al. [144] have considered higher Landau levels with n=1 and 2. They noted that liquid ground state transforms into an insulating state in which the Hall resistance is quantized at integral values. Kettemann [145] also finds that the quantum Hall transition is driven in lowest two Landau levels by a noncritical dimensional crossover of the localization length. As the temperature decreases a crossover from fixed range hopping of the transport to the variable range hopping of the transport in the 2-dimensional electron system may occur [146]. A liquid to solid phase

transformation from Laughlin state to Wigner crystal suggests that Hall plateau should not occur for $\nu < 1/7$. The correlation energy indicates that very low temperature is required to see $\nu = 1/5$ liquid state [147]. Tao [148] has made an effort to develop the theory on very general grounds by assuming $\nu = p/q$ just a fractional number not related to spin.

2.6.3 Statistics

The wave function has the quantity $\left|z_i - z_j\right|^{-\alpha/m_q}$ in which $\alpha = \pm 1$ with + sign for the particle and the negative sign for the hole. We remove the absolute sign and introduce a phase factor. If $m_q \neq 1$, the new wave function is a multivalued function of the position $|R_k|$ and one should consider it as a function defined on the appropriate Riemann surface for $|z_k|$. One could use a single-valued definition and specify discontinuities along cuts in the variable $\left|z_i - z_j\right|$. Now, if we continuously interchange the positions of two particles, the wave function will change by a complex phase factor $(-1)^{\pm 1/m_q}$ with the sign depending on the sense of rotation as the quasiparticles pass by each other. The pseudo wave function is an eigen state of the operator $[\nabla_{\mathrm{k}} \mp \mathrm{iqe}\,\vec{\mathrm{A}}(\mathrm{R_{k)}}/\hbar c]^2$ with special boundary conditions at the point $z_i = z_j$. The wave function having the factor $\prod_{k=1}^{N} \exp\left(-|q||z_k|^2/4a_0^2\right)$ can be described as a general wave function appropriate to a collection of particles of charge $\pm qe$obeying fractional statistics [149]. This problem can also be written as a random matrix [150]. Hershfield [151] has written the average value of an operator in a non-equilibrium situations that non-equilibrium statistics applies. The dimensional regularization technique has been applied to the N-anyons with δ-function potential [152]. For small systems of lattice anyons, the statistics is $\nu = p/q$ with the product, pq=even. The $\nu = 1 - (1/n)$ is also included [153]. A study of the viral expansions for interacting electrons in the lowest Landau level of a two-dimensional electron gas shows that there is a

crossover between the low temperature limit and the high temperature limit. The exclusion statistics description breaks down when the temperature exceeds a small fraction of the gap [154]. The particles in a circle, interacting by short-range attraction, obey fractional statistics [155]. The concept of fractional exclusion statistics is based on the structure of Hilbert space, rather than spin configuration space of the particle assembly. It is not restricted to $d \leq 2$. If ΔD denotes the change in the size of the subset of available states in the Hilbert space corresponding to a change ΔN of the number of particles, we define the ratio, $g = -\Delta D/\Delta N$. Here $g = 0$ for bosons and $g = 1$ for fermions. For other values of g we call exclusion statistics [156] in which n electrons can be distributed on m orbits, e.g., 5 electrons on 6 orbits. The distribution function is given by Wu [157] and by March [158]. The correlation function for this type of statistics is given by Pellegrino et al. [159]. It has been shown that the size and shape of the sample creates a non-Ohmic current, $V \propto I^{\alpha}$, so that voltage, V and the current I describe an exponent α . The area in which flux is quantized may not be preserved so that there are special cases but the algebra cannot resolve the incompressible states [160] from the compressible ones. The chiral fermions have an infinite scaling dimension [160]. The statistical parameter affects the entanglement properties of the anyonic system by renormalizing the effective interaction [161]. The bunching of bosons and the anti-bunching of fermions has been described by Vishveshwara and Cooper [162] who point out the indications of anionic statistics. An effort is made to distinguish the charge-density waves with the fractionally charged states by replacing the distance between electrons by the magnetic length in the Coulomb interactions [163]. A four-qubit system is found to be enough to demonstrate the basic braiding statistics of anyons. The unit qubit can be used to construct the series n/(n + 1) or (n + 1)/n at which fractional charges

occur [164]. Several statistical models have been described which provide fractional statistics [165].

2.6.4 Order parameter

When the area in which flux is quantized, can be fixed, it is possible to define the charge which permits the off-diagonal long-range order in a theoretically incompressible state. In the case of compressible states no off-diagonal long-range order is found. If charge is the only variable, it is possible to get the long-range order. However, if charges as well as the flux area, both are variables, long-range order is not possible in the compressible state [166]. It has been reported [167] that the Laughlin wave function shows no off-diagonal long-range order due to phases of the wave function. The removal of the phases yields a power law decay of the off -diagonal elements of the density matrix. Yang [168] has suggested that if products of three wave functions are used as in the case of Halperin (m,m',m"), a phase coherence can be introduced which can sustain a long-range order. In the case of a bilayers, the fractional filling is given by $\nu = 2/(m + m')$. The particles in one layer haveopposite parity to those in the second layer so that excitons form the quasiparticles in the bilayers system which can have a wave vector representation. In that case, the Laughlin wave function is suppressed and wave vector dependent wave functions dominate so that there is no difficulty to create the spin waves but Laughlin's wave function does not have spin. The off- diagonal long range order of ‘‘imperfect" super fluid between layers is possible [169]. In the case of Laughlin's wave function random (Chern-Simon) type disorders possible [170]. These random numbers are thermodynamically consistent. As the disorder increases towards a critical value, an insulating phase appears.

2.6.5 Topology

Let us describe an anyonic system by a one-component wave function [171]. There is no period smaller than, $\phi_0^* = hc/e^*$. If $e^* = e/3$ larger value 3hc/e will result. If we use a multicomponent wave function, we can derive the condition for a small period which leads to a hidden topological Z_n symmetry. Here n is the smallest integer satisfying, $n(e^*/e)$ =integer. On a torous, gauge invariance implies a relationship between E^* and the statistics θ for an irreducible braid-group representation (BGR) and in thermodynamic limit there are two noncommuting topological symmetries which lead to the ground-state degeneracy. The topological symmetry Z_n supports a broken symmetry for the quantum Hall effect. In the case of a torous, the nyons require a multicomponent wave function so that $(\theta/\pi)\phi_0^*$ is always a period of ϕ . On a torous, besides $\sigma_i = e^{i\theta} I_M$, there are generators $\tau_i, \rho_i (i = 1, \dots, N), \tau_{i+1} = \tau_i e^{-2i\theta}$, $\rho_{i+1} = \rho_i e^{2i\theta}$ and $\tau_i(\rho_i)$ have a factorized $\phi_x(\phi_y)$ dependences in $\rho_j(\phi) =$ exp $[i2\pi e^* \phi/hc] T_j$ where I_M is MXM unit matrix. Changing ϕ_x by $(\theta/\pi)\phi_0^*$ gives rise to a phase factor $exp(2i\theta)$. For irreducible BGR, M=q for $\theta = \pi(p/q)$, with p and q mutually prime, there are no other periods with minimal period ϕ_0^*/q. For gauge invariance $e^*/e = m/q$ with integer m. Usually, there are angular momenta in θ. However, according to Shrivastava, it is proper [2, 3], to introduce $(\theta/\pi)hc/[(1/2)ge]$ so that θ is replaced by $\theta^* = \theta/(1/2)g$ where g is the ratio of the two angular momenta,

$$g = \frac{2j+1}{2l+1} \ . \tag{2.73}$$

Hence,

$$\theta^* = 2\theta \frac{2l+1}{2j+1} \tag{2.74}$$

The angular momentum gets into statistics (Shrivastava). It has been pointed out that shifts appearing in Haldane hierarchy, visualize the angular momentum of the quasiparticles [172]. The Z_2 topological invariant can be expressed as a Wilson loop of SU(2) Berry gauge field which is quantized due to time reversal symmetry [173]. The topological phase transitions are linked to the divergence of the fidelity metric [174]. The entanglement spectrum can be used to identify the topological order [175]. The fractional charges bound to topological defects obey fractional statistics [176]. In some cases, the spatial and temporal scaling is different which results into a gap in the topological phases [177]. In the case of a quarter filling, the system undergoes a quantum phase transition from metal to topological insulator [178]. It has been reported that a one-dimensional fermion excitation occurs when a solenoid carrying a flux,ϕ , is inserted in a topological insulator. In this case, the flux $\phi = hc/2e$ is spin filtered. It is not clear whether the charge of 2e implies any spin states. The solenoid inserted in the insulator creates a "wormhole" through which the Fermion mode can propagate with fixed spin [179]. In bilayers of two-dimensional electron gas with a layer of a superconductor, a fractional charge of e/2 is predicted [180]. The non-Abelian quasiholes can travel through the entanglement in the topological phases [181]. Several properties of topological insulators have been reviewed [182].

2.6.6 Tunneling

Consider the many body bound state at the impurity as consisting of one-particle orbits so that the orbit of an additional quasiparticle encloses the flux,

$$\phi = m\phi_0^* + N\phi_0 \tag{2.75}$$

where $\phi_0 = hc/e$ is the flux quantum for an electron and $\phi^* = q\phi_0$ is the flux quantum for a quasiparticle which is an anyon with charge $e^* = e/q$ but the electric field of the

electromagnetic field is ignored [183].In a strong magnetic field, the quasiparticles move along the lines of equal potential. Here N is the number of quasiparticles captured by the impurity and m is the angular momentum of the tunneling quasiparticle. The first term is required by gauge invariance while the second term shows that each quasiparticle is bound with one flux quantum so that N particles have $N\phi_0$. The energy of N-anyon states is characterized by two integer parameters. It is not necessary that m should be an integer. The interval in the magnetic field between two consequent bound states of quasiparticles is $\Delta B_q = \phi^*/A$ where A is the area. Due to the tunneling the number of quasiparticles N coupled with the impurity changes by one so that the flux ϕ changes by ϕ_0 instead of ϕ_0^* . The corresponding period is $\Delta B_1 = \phi_0/A$. Consider the situation when N quasiparticles are initially bound by the impurity and the tunneling quasiparticle arrives at an orbit enclosing all of them. The wave function of the quasiparticle gains a phase factor $z = exp(2\pi iN/q)$ after each complete revolution over the quantized orbit. Since γ is the probability of tunneling from the impurity to the left or right edge, the phase factor is $z(1-\gamma/2)$. The total tunneling amplitude contains a series,

$$t_{LR} = \sum_k z^k \, (1-\gamma/2)^k = \frac{1}{1-z(1-\gamma/2)} \tag{2.76}$$

Therefore, in the case of tunneling of anyons, the selection rules are modified by the impurity atoms[184]. Sasaki and Ezawa [185] suggest that Coulomb potential is modified by the impurities. In the case of bilayers, there is an incommensurate phase which has rippled dipole charge-density waves. The tunable dipole density wave instability can be detected [186]. Some nonlinearities are found in the tunneling conductance between edge states [187]. Some of the states found in the quantum Hall effect certainly correspond to S=0 singlets which have Meissner effect. However, a lot of fractions have some other value of the spin which is different from zero [188]. The magnetoexcitons are generated due to ionized impurity atoms with the velocity of the

excitons becoming critical so that there is a hydrodynamic breakdown in the current [189]. The tunneling of fractionally charged quasiparticles through a barrier is found to increase in going from mesoscopic quasi-particle dominated to electron dominated system [190].

2.6.7 Activation

The activation energy is measured by means of temperature dependence of the resistivity near a phase transition, $\rho \propto \exp(-\Delta/k_B T)$. Girvin et al. [191] have related the activation energy to the dynamical structure factor,

$$S(k,\omega) = \sum_n |< n|\rho_k|0 >|^2 \, \delta(E_n - E_0 - \omega) \tag{2.77}$$

where $\rho_k = \sum_j exp(ik.rj)$. The structure factor is of the form of,

$$S(k,w) = NS(k)\delta(\omega - \Delta_k) \tag{2.78}$$

In the single-mode approximation, the excitation energy is,

$$\Delta_k = f(k)/s(k) \tag{2.79}$$

where the oscillator strength is,

$$f(k) = \frac{1}{N}\int d\omega \, \omega S(k,\omega) \tag{2.80}$$

As in the case of Laughlin's wave function, the states, E_n lie within the lowest Landau level, (LLL), so that the structure factor as well as the oscillator strength has to be projected to the LLL. In the projected density operator, the coordinates and the momenta are complex, $z_j = x_j + iy_j$, $k = k_x + ik_y$ so that,

$$\overline{\rho}(k) = \sum_{j=1}^{N} exp\left(ik\partial/\partial z_j\right) exp\left(ik^* z_j/2\right) \tag{2.81}$$

and

$$\overline{s}(k) = s(k) - \left(1 - e^{-|k|^2/2}\right) \tag{2.82}$$

The excitation energy is then $\Delta(k) = \overline{f}(k)/\overline{s}\,(k)$. For the Laughlin ground state, $a_0 = (hc/eB)^{1/2} = 1$ so that $\overline{s}(k)$ vanishes as $|k|^4$,

$$\overline{s}(k) = |k|^4\,(1-\nu)/8\nu \tag{2.83}$$

where ν is the fractional filling factor such as $\nu = 1/3$. Usually, the $a_0 = 1$ condition is not satisfied because of the product of the area and the charge occurring in the flux quantization. The asymptotic excitons dispersion is $\Delta_\mu(k) = \Delta_\mu - \nu^3/k$ and the projected oscillator strength is,

$$f(k) = \frac{1}{N} < 0\left|\overline{\rho}_k^{+}\left[\overline{H}_{k,}\rho_k\right]\right|0 > \tag{2.84}$$

where the projected Hamiltonian is,

$$\overline{\mathcal{H}} = \frac{1}{2(2\pi)^2}\int \left[d^2q v(q)\overline{\rho}_q^{\dagger}\overline{\rho}_q - \rho e^{-q^2/2}\right]. \tag{2.85}$$

The Coulomb interaction is $v(q) = 2\pi e^2/(\epsilon_0 q)$. The calculated value of Δ as a function of k shows a minimum similar to that found in the dispersion of liquid helium. Hence, we predict "rotons" in the Laughlin state. In this theory the filling factor is $\nu < 1$ and the activation energy is very small compared with the Coulomb interaction. Zhang and Das Sarma [192] have calculated the excitation energy for small number of electrons on a sphere. Using finite width of the two dimensional layer, the activation energy is found to saturate at high magnetic fields. The excitation gap goes to zero at a critical field [193]. The impurity scattering and the surface scattering define the two critical fields. When the gap becomes zero, the Hall plateau also vanishes. On the basis of Coulomb interaction [194] we expect the quasiparticle energy to be $\Delta \simeq 0.1\, e^2/\epsilon a_0$. When we replace a_0 by $(hc/eB)^{1/2}$ we see that $\Delta \propto B^{1/2}$. The activation energies thus are a form of system's eigen values which are having solid state effects such as band gaps. It is also possible that there is a Goldstone mode of a finite or zero mass.

2.6.8 Thermodynamics

The Boltzmann equation approach shows that non-diagonal component of the conductivity acquires a factor of $(1 - e^{-E})$ where E is the single-particle energy. Usually $e^{-E} = \delta$ in which case the conductivity depends on δ^b where b depends on the effective mass of the quasiparticles, magnetic field and temperature [195]. The magnetization depends inversely on the cyclotron frequency [196] and the specific heat shows [197] a peak at the value of a fractional charge, such as 1/3.

2.6.9 Gauge

In the Laughlin's wave function two different values of m can be used in such a way that both are ground states. Hence, the wave function has degeneracies. This is not the text-book definition of the degeneracy but we use this definition here. The ground state ψ_0 of the two-dimensional electron gas on the surface is,

$$\mathcal{H}(p_1 - eA_1, \dots, p_N - eA_N)\psi_0 = E_0\psi_0. \tag{2.86}$$

Let us assume that ψ_0 is periodic in y direction with period L,

$$\psi_0(r_1, \dots, r_j + L\hat{y}, \dots.) = \psi_0(r_1, \dots, r_j, \dots)(j = 1,2, \dots, N). \tag{2.87}$$

We switch the field on to increase the flux from zero to ϕ. The system remains in the ground state which may be different from the original one. The new wave function has $A_1 + a$ instead of A_1 for the vector potential. We assume,

$$\psi(r_1, \dots, r_N) = \exp\left[i(ea/\hbar)\sum_{j=1}^{N} y_j\right] u(r_1, \dots, r_N). \tag{2.88}$$

Since ψ is periodic in y,

$$u(r_1, \dots, r_j + L\hat{y}, \dots) = \exp(-ie\phi/\hbar)\, u(r_1, \dots, r_j, \dots) \qquad (2.89)$$

where $\phi = aL$ is the flux of the solenoid. The system maps back to ϕ_0 at a finite value of $p\phi_0$ and there are p electrons to contribute to conductivity, $\sigma = (q/p)e^2/h$ is a fraction, the ground state must be degenerate for the gauge invariance [198]. Applying the field has no effect on the spin which is quite left out from the argument. Using the translational invariance should lead to the assumption of crystallization [199]. It has been argued that switching off the magnetic field destroys the quantization. In the flux quantization also when B = 0, there is no flux quantum [200]. The microscopic origin of the effective action is given by Pruisken et al. [201] and the general gauge invariance has been discussed by Shizuya [202].Consider the atomic wave function ψ_1 and the molecular wave function ψ_2 with chemical potential $\mu_2 = 2\mu_1 - \delta$. The Feshbach resonance occurs for $\delta = 0$, which is $\mu_2 = 2\mu_1$ where μ_1 and μ_2 are the chemical potentials corresponding to ψ_1 and ψ_2, respectively. The statistical gauge field $|\vec{\nabla} \times \vec{a}| = \phi_0(|\psi_1|^2 + 2|\psi_2|^2)$ is chosen such that the external magnetic field $\nabla \times (< a > + A) = 0$ for bosons. The action is invariant under the gauge transformation $\psi_\sigma \rightarrow \psi_\sigma \exp[i\sigma\theta(x,t)$ and $a_\mu \rightarrow a_\mu + \partial_\mu\theta(x,t)$. The system goes through a phase transition from $\nu = 1$ to $\nu = 1/4$. The quasiparticles interacting with the gauge fields show that charge fractionalization occurs. The gauge transformation thus plays an important role when the system goes from one filling factor to another [203]. The different plateaus are obviously associated with different charge per quasiparticle.

2.6.10 Optical spectra

The band gap optical recombination anomaly is found for $\nu < 1$. These anomalies areassociated with changes in population of the lowest spin-split Landau level [204]. An analysis of the

acceptor photoluminescence line shape can allow a measurement of the quasi-Fermi energy [205]. The polarized photoluminescence of charged excitons shows that spin-flip is important for the understanding of excitations [206]. The electron-electron interactions are shown to lead to splitting and oscillations of the recombination line [207]. The singlets as well as the triplets have been found in magnetoexcitons spectra [208]. The binding energy of the singlet decreases with increasing distance between the electron and the hole [209]. The circularly polarized photoluminescence lines show anomaly at a magnetic field associated with $\nu = 1/3$. The splitting of lines is observed at about 10.8 T which corresponds to singlet and triplet states [210]. A method to create fractionally charged states in optical lattices has been described by Sϕrensen et al. [211]. Some measurements of the two-photon photoemission are available [212]. Shining the material with microwaves also reveals the occurrence of fractionally charged states [213] and zero-resistance state has been detected [214]. The spin features may be linked to the incompressible states [215]. In the original Laughlin state spin is not involved. The optical response of ultrafast pulses has been calculated for the electrons and holes in a semiconducting lattice [216]. The photoluminescence emission from GaAs/AlGaAs near the filling factor 2/3 has been found to split but the components have not been interpreted [217]. The strong electronic correlations effect has been calculated [218].

2.6.11 Wigner crystallization

The electrons at low temperatures form a solid which is called the Wigner crystal. It is shown by Maki and Zotos [219] that the stability of the Wigner crystal depends on the filling factor. The electron lattice is stable for $0 < \nu < 1$ while the hole lattice is stable for $0.55 \leq \nu \leq 1$. In both the cases, the filling factor $\nu < 1$ so that all fractions are less than one. The excitation energy has

been calculated both for n = 0 and for n = 1 level of the Landau oscillator. The energy shows a shallow minimum at some characteristic wave vector [220]. The phonon modes of the Wigner crystal in the high magnetic field limit show that response is fully determined by the ground-state electron density [221].The periodic boundary conditions have been introduced at rational Landau-level filling which show [222] that there is a non-degenerate ground state at k=0. The broadening of the collective excitation spectrum reduces the minimum excitation energy and eventually the gap required for the occurrence of the fractions collapses [223]. In a model calculation, alternating layers have different electron densities with only one type of charge carriers. The interlayer correlations are important for the consideration of two types of charge carriers [224]. The translational symmetry predicts a degeneracy that varies periodically with system size and equals “one" for certain commensurate cases. In the incommensurate states, the Hall conductance is a fraction not equal to the filling factor [225]. The incommensurate ground state exists on the lattice at odd-filling fractions [226]. A pinning mode in a Wigner solid has been detected [227] at $\nu = 1/3$.

2.6.12 Phonons

The electron-phonon scattering rate via a deformation potential reduces the activation energy which is used in the thermal activation process. The current relaxation function is directly proportional to the compressibility, $g_L(q) = 1/(\upsilon^2 M_i)$ where v is the sound velocity and M_i is the mass of the host atom. The main features, such as location of plateaus in the quantum Hall resistivity are not seriously affected by the phonons [228]. The magnetic freezing of electrons which forms the Wigner crystal stimulates the phonons so that phonon replica occurs [229] in the magnetoluminescence at very small values such as $\nu = 0.09 - 0.05$. In some cases, it is possible that activation energy matches with the phonon energy and not with the electronic energy.

However, it is more likely that the activation energy is electronic energy level separation rather than phonon energy [230]. In some cases, the activation energy depends on the gap energy and it involves a two phonon process rather than the usual one phonon process.

2.6.13 Hierarchy

According to Shrivastava's theory the fractions are produced by (i) two principal series which differ in the sign of the spin which can be s=1/2. (ii) Some of the fractions are due to resonances such as $\nu_1 - \nu_2 = \nu$. (iii) Weak fractions occur due to two-particle states,$\nu_1 + \nu_2 = \nu$ and (iv) due to clustering property of the plateaus, the spin need not be 1/2, for example, it may be 3/2 or higher value. Let us see the results due to Haldane's hierarchy. The Haldane's hierarchy is an algebraic proposal which depends on the termination of a continued fraction [231]. One example of Haldane's hierarchy for $\nu = 1$, is shown in Fig. 2.1 and Shrivastava's hierarchy is shown in Fig. 2.2 for the principal series only. It should be noted that the quantum Hall effect dominant fractions are a property of the pure material and it does not require the existence of impurities. The effect of impurities is found to scale eventually terminating the hierarchy [232]. It is claimed by Mallett et al. [233] that 2/9 and 2/11 are “daughter" states of a 1/5 hierarchy parent state. Actually, one of the ways to find the origin of a fraction is to look at the denominator, such as 2/11. Equate 11 to 2*l* + 1 so that *l* = 5. According to Shrivastava this gives rise to two values *l*/(2l + 1) = 5/11 and *l* + 1/(2l + 1) = 6/11. The resonance produces 6/11-5/11=1/11. The eigen value formula is, $\frac{1}{2}g(n+\frac{1}{2})$ which gives,

$$\frac{1}{2}g\left(n_1+\frac{1}{2}\right)-\frac{1}{2}g\left(n_2+\frac{1}{2}\right)=\frac{1}{2}g(n_1-n_2). \qquad (2.90)$$

If $n_1 = 2$, $n_2 = 0$, $l = 5\frac{1}{2}g_- = 5/11$, $\frac{1}{2}g_+ = 6/11$, $\frac{1}{2}gn_1 = 10/11$ and 12/11 and then $\nu_2 - \nu_1 = 2/11$. The resonance condition is,

$$E_2 - E_1 = g\frac{e\hbar}{2mc}B \tag{2.91}$$

in which effective charge is introduced so that,

$$E_2 - E_1 = \hbar\frac{e^*B}{mc} \tag{2.92}$$

equivalent to,

$$E_2 - E_1 = \hbar\omega_c^*. \tag{2.93}$$

With effective charge,

$$e^* = \frac{1}{2}ge \tag{2.94}$$

which depends on L, S and J through the g value, $g = (2J + 1)/(2L + 1)$ and there are two values due to the $\pm$ sign in J = L$\pm$S. The energy of a state is given by $E = g\mu_B B$ with two values of g instead of the usual one. The plateau in the Hall resistivity is found at,

$$\rho_{xy} = \frac{1}{e^2\left[\frac{1}{2}g_1\left(n_1 + \frac{1}{2}\right) - \frac{1}{2}g_2\left(n_2 + \frac{1}{2}\right)\right]}. \tag{2.95}$$

There are degeneracies because there are other ways of obtaining the same number. d'Ambrumenil and Morf [234] have assumed the values of fractions first and then calculated the energies so that there is no way of deriving the fractions in their theory. In Laughlin's theory the fraction is $\nu = 1/m$ but *m* can not be calculated from the theory. Read [235] finds that hierarchy theory in some way should be equivalent to a Ginsburg-Landau model of phase transition. In the case of anyons, the statistical parameter $\alpha_s = \theta/\pi$ plays the role of the filling factor, ν. Gros and MacDonald [236] propose that for filling factors in the range $1/3 \leq \nu \leq 2/3$, the quantum Hall

fractions occur only when $\nu = \nu_n = n/(2n+1)$ and $\nu = 1 - \nu_n$ and at no other fractional filling factors with odd denominators. According to Shrivastava's theory the correct series is $l/(2l + 1)$ which when tabulated is the same as $n/(2n + 1)$. Similarly, the other correct series is $(l + 1)/(2l + 1)$. The interaction of quasiparticles obeying fractional statistics has been used to calculate the energies and gaps of the polarized and unpolarized states [237]. The spin model using single-ion anisotropy has been used by Hatsugai and Kohmoto [238] while Haldane's hierarchy is independent of spin. It has been pointed out by Yang and Su [239] that the Hilbert space is bosonic. However, it is important to consider the quantum Hall fractions as fermions. Some of the states may obey fractional statistics [240]. The Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms have denominators less than or equal to n arranged in order of increasing size. The Farey sequence starts with the value 0 denoted by 0/1 and ends with the value 1, denoted by 1/1. The terms (0/1,1/1), (0/1,1/2,1/1), (0/1, 1/3, 1/2, 2/3, 3/4, 1/1) etc do not have a sum like a geometric series. Hence, Farey numbers are of no significance to the physics of the problem [241]. The low energy behavior of a daughter state in the next level of the hierarchy is described by an interacting system of quasiparticles of the parent state. The quantitative consequences of this approach for electrons interacting via a pseudo potential interaction produces daughter states. The parent with 1/2 filling gives the daughter state at 2/3 filling [242]. The fractional charge of the edge states may also follow the hierarchy [243]. Some of the hierarchy states $\nu = n/[\,n(m_L + 1) + 1]$ are the same as non-hierarchy $1/m_L$states [244]. Since the Laughlin wave function does not depend on spin there must be a spin-charge decomposition picture [245]. Nakamura et al. [246] have found that the fraction 1/3 is linked to spin and lacks in parity whereas Laughlin's 1/3 fraction

does not have spin. Bonderson et al. [247] find that pairing of electrons is required to satisfy the non-Abelian Pfaffian determinant.

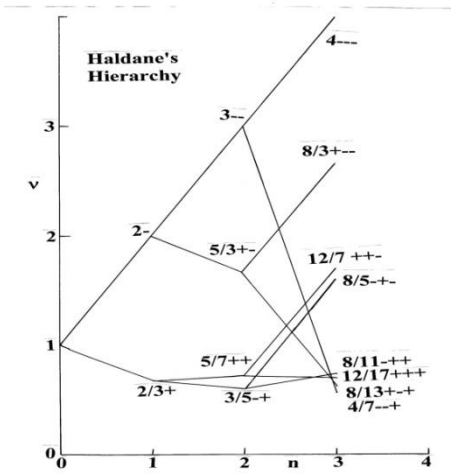

Figure 2.1: Haldane's hierarchy of fractions according to the continued fraction.

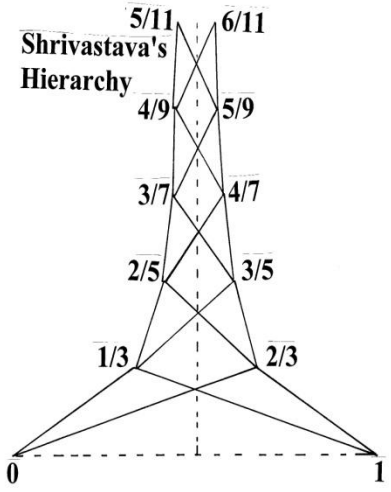

Figure 2.2: The hierarchy of fractions according to Shrivastava's formula of effective charge according to the principal series only.

2.6.14 Rotons

The quantum Hall effect states are regarded as a super fluid because of the flux quantization. Hence, its theory should be similar to that of Landau rotons. The dispersion relation was developed by Feynman [248] for liquid helium and hence similar dispersion should occur in the quantum Hall states. The predicted spectrum has a large gap at k=0 and a deep magneto-roton minimum at a finite wave vector. The magneto-roton minimum is a precursor to the gap collapse associated with Wigner crystallization [249]. The excitation energies of the no interacting two-dimensional electron gas are multiples of the cyclotron frequency [250]. The thermally excited magnetorotons are scattered by the phonons. The scattering of magnetorotons to the long wave length phonons is possible because of the shape of the magnetoroton dispersion relation [251]. The magnetorot on minima are observed [252] in the light scattering at 1/3 and at 2/5. There is a roton like structure [253] at $\nu = 1/2$. Suorsa et al. [254] have constructed the quasi-hole condensate at filling factor 2/3 and two three level states at 5/13 and at 5/7 which are built from combinations of quasi-electron and quasi-hole condensates but spin has been ignored.

2.6.15 Symmetries

The symmetry of the system can be broken by means of Goldstone theorem which creates a boson. When A(r), the electromagnetic vector potential or the magnetic induction $B(r) \rightarrow 0$, vortices are generated. This method of taking the vector potential to zero leads to symmetry breaking current carrying states which give vortices as the bosons [255].Ortiz et al. [256] have considered spinless fixed-phase complex wave function to map the bosons into fermions. Pryadko [257] has considered the possibility of adding current to the flux $\phi/2\pi \rightarrow \phi/2\pi + 2I$. However, the current represents both the electric as well as the magnetic vectors as in the

Maxwell equations so that detaching the current will produce both the electric as well as the magnetic fields. The problem of attaching flux quanta to the electrons without any care of the electric fields has not been completed. The effective charge in Laughlin's wave function, $1/m$, is not derivable. If it is treated as a new quantum number, p, the filling fraction $1/p$ is largely not explained [258].

2.6.16 Spin

Although there is no spin in the Laughlin's wave function several authors have discussed spin dependent properties [259]. Giuliani and Quinn [42] have considered two filled Landau levels of opposite spin in high field limit. The lowest lying excitations can be described in terms of singlet and triplet excitons. At a critical Zeeman energy, there is a first order transition to a fully spin polarized state in which two Landau levels of equal spin are filled. The possibility of off-diagonal long-range order in spin chains by using spinons (spin only without charge particles) is discussed by Talstra et al. [260]. The quantized spin currents which break the time reversal invariance are discussed by Haldane and Arovas [261]. Dorozhkin et al. [262] suggest that spin reverses in obtaining the plateau in the resistance at 2/3 but they do not give reference to the original work of Shrivastava [3]. Inelastic light scattering indicates that spin waves may occur in clusters of electrons [263] which means that some of the fractions are produced by electron clusters. The magnetic polarization decays with temperature and there are particles with spin up as well as with spin down [264]. The unpolarized electrons can tunnel between two samples which are 1/3 filled each [265]. The energy varies with the magnetic field as in a Landau fan diagram [266] and the spin is only partially polarized [267]. The real space spin-singlet pairing depends on the dimensions of the sample [268]. Some of the fractions correspond to partially

polarized states [269]. The Laughlin's wave function has been of considerable interest in condensed matter physics, particularly, in problems involving dipole-dipole interaction [270], group theory [271], entanglement [272], superconductors [273], liquid crystals [274], Kondo problem [275], solitons [276], stripes[277], Fermi liquids [278] life time [279] and density [280]. The energy gaps are found and can be measured by activation process. There is a possibility of a phase transition from incompressible to compressible states. The activation gap at 5/2 is found to vary as a function of magnetic field. It can be fitted [281] to an expression which has $B_{\perp}^{1/2}$ and $\left(B_{\perp}^{2}+B_{\parallel}^{2}\right)^{1/2}$. The leading quantum fluctuations lead to an effective spin-1 Hamiltonian [282]. In the case of two different spin species, a spin-drag Hall effect is possible due to dipole-dipole interaction [283]. It is found that the S=1/2 Heisenberg anti ferromagnet in one dimension has an entanglement spectrum similar to that of the fractional Hall state [284]. The magnetization of a soft ferromagnet is found to show vortices and anti vortices [285]. The frustrated ferromagnetic S=1/2 chain shows two-spinon and four-spinon continuum [286]. The spin Fermi surface is stable against weak interactions [287]. The quantum phases for strongly interacting one dimensional spin systems have been identified [288].

2.6.17 Currents

The persistent current as a function of temperature exhibits anomalous oscillations. The mesoscopic systems such as in quantum wires, there is a possibility of measuring fractional charge [289]. The spin polarization at the half-filled Landau level without the spin-orbit interaction does not affect the energy [290] but there is domain formation [291] and unusual potentials may give important information [292]. The oscillations in the amplitude of persistent current are predicted to arise due to the fractional charge [292]. There is a zero-field edge

singularity in the spectrum due to impurity atoms [293]. The quantum mechanics of coherence has been calculated in detail. There is a small energy spread but the states are localized [294].

2.6.18 Entanglement

We calculate the entanglement entropy and the fluctuations for a given model Hamiltonian in one or two dimensions [295].The system is partitioned and the particles can move from one subsystem to the other. The number of particles in a subsystem therefore fluctuates. In one dimension [296], the entropy of a subsystem A of size x embedded in a large system of size L is given by $(x, L) = -Tr\rho_A ln\rho_A$. The remainder of the system is called B. The central charge is c. The entanglement entropy at zero temperature for $L \to \infty$ has the universal behavior,

$$S(x) = \frac{c}{3}\, lnx + s_1\,. \tag{2.96}$$

The $Tr\rho_A^n$ transforms under conformal mapping with S(x, L) for finite L, for the number fluctuation in the subsystem A,

$$F_A = < (N_A - < (N_A) >)^2 > \tag{2.97}$$

where N_A is the number of particles in the subsystem A. For a pure state without vortices, $F_A = F_B$. The fluctuations occur in the boundary between two subsystems. For separable states, $F_A = 0$ and for valence-bond (VB) states F_A coincides with von Neumann and VB entropies. We consider the Luttinger liquids (LL) which describe the low energy properties of many one-dimensional systems. In the limit $L \to \infty$,

$$\pi^2 F_{LL} = < [\emptyset(x) - \emptyset(0)]^2 > \tag{2.98}$$

Where $\emptyset$ is the charge field. At T=0

$$\pi^2 F_{LL} = \mathrm{K} ln\frac{x}{a}\,. \tag{2.99}$$

Here K is the Luttinger parameter and a is a cutoff distance. The same entropy is obeyed when $\mathrm{K} \to \mathrm{K}/2$ and $x \to 2x$ with $\emptyset(0) = constant$. The fluctuation is related to the filling factor for $\mathrm{K} = \nu$. The charge fluctuation across a quantum point contact is given by,

$$\pi^2 F(t) = \nu ln \frac{t}{\delta} \tag{2.100}$$

where δ is a short time cutoff. We define the characteristic function,

$$M_A(\lambda) =< \exp[i\lambda(N_A - < N_A >)] \geq = \left(\frac{x}{a}\right)^{-g\lambda^2/2\pi^2} \tag{2.101}$$

which transforms under conformal mapping as,

$$\pi^2 F_A = -\pi^2 M_A^{"}(0) = gln\left(\frac{x}{a}\right). \tag{2.102}$$

The pre factor g can always be fixed by considering the physical meaning of the charge. At finite temperature $1/\beta$,the mapping $z \to z' = \frac{\beta}{2\pi} lnz$ in the above equation gives,

$$\pi^2 F(x, \beta) = gln\left(\frac{\beta}{\pi a} sinh \frac{\pi x}{\beta}\right) \tag{2.103}$$

For $x >> \beta(\hbar = k_B = \upsilon = 1)$ the interactions across the boundary can be neglected because correlations decay exponentially. Here $\upsilon$is the velocity which has been left out. We consider the subsystem A to be a grand canonical ensemble in equilibrium with a bath consisting of the remainder of the system. Total particle number is fixed, then,

$$F(x, \beta) \sim \kappa \frac{x}{\beta} \tag{2.104}$$

where $\kappa = \frac{\partial n}{\partial \mu}$ is the compressibility so that (2.103) for $x >> \beta$ gives,

$$g = \pi \upsilon \mathrm{K} \tag{2.105}$$

Where υ is the velocity of the particles and

$$\mathrm{K} = \pi \upsilon \kappa \tag{2.106}$$

for the Luttinger liquid. The entanglement and the fluctuations are related to the central charge c as,

$$\frac{S(x)}{\pi^2 F(x)} \simeq \frac{c}{3\pi\upsilon\kappa} \tag{2.107}$$

For $x >> a$, the detailed entropy becomes model dependent. Hence it depends on the Hamiltonian.

2.6.19 Mott insulator

We use the product of the Laughlin's wave function and that of a Mott insulator as a trial wave function in a Bose-Hubbard model so that the system phase separates. One phase is an insulator while the other phase is a Laughlin's hypothetical liquid. The bosons in a rotating optical lattice form analogs of Laughlin's state when tunneling is weak. The Bose-Hubbard model is written as,

$$\mathcal{H}_{\imath} = -t\sum_{i,j} a_i^{\dagger}\, a_j e^{iA_{i,j}} + \frac{1}{2}U\sum_i \hat{n}_i\,(\hat{n}_i - 1) \tag{2.108}$$

Where $a_i\left(a_j^{\dagger}\right)$ are the annihilation (creation) operators for bosons on sites, i or j, respectively. The boson number density is given by $\hat{n} = a_i^{\dagger}a_j$ and

$$A_{i,j} = exp\left(\frac{ie}{\hbar c}\int_{r_j}^{r_i}\vec{A}.d\vec{l}\right).$$

The phases acquired when hopping in the $\pm x$ direction are $\mp 2\pi\alpha i_y$ where i_y is the y coordinate scaled by the lattice constant a. According to the flux quantization $\alpha = B\,a^2/(hc/e) = p/q$ is the flux quantum per plaquette. The model is solved either on a sphere or on a torus. We consider the following variational ansatz,

$$|\psi\rangle = \sum_{z_1,\ldots,z_N} \psi_{La}\,(z_1,\ldots,z_N)a_{z1}^{\dagger}a_{z2}^{\dagger}\ldots a_{zN}^{\dagger}|\psi_{MI}\rangle \tag{2.109}$$

where $\psi_{La}$the Laughlin's wave function and $|\psi_{MI}\rangle$ is that of a Mott insulator,

$$|\psi_{MI}\rangle = \prod_j \left(a_j^\dagger\right)^{n_0} \left(\frac{1}{n_0!}\right)^{1/2} |vac\rangle. \quad (2.110)$$

Here $z_i = x_i + iy_i$ is the complex coordinate for i varying from 1 to N. The $\sum_{z1,\ldots,}$ is the sum over all lattice sites. We assume the $L \times L$ torus geometry with boundary conditions,

$$\psi(\ldots, z_k + L, \ldots) = \psi(\ldots, z_k, \ldots) \quad (2.111)$$

$$\psi(\ldots, z_k + iL, \ldots) = \exp[(-2i\pi mN/L)x_k]\, \psi(\ldots, z_k, \ldots) \quad (2.112)$$

leads to a phase separation [296].

A model has been made in such a way that electrons disappear from the symmetric positions along y axis and appear on the symmetric positions along x axis and vice versa. The coupling constant of such a model can be made to map with that for which the Laughlin's wave function is the ground state [297]. In a limiting case it provides a featureless Mott insulating state. The Laughlin's wave function is not the ground state of the Coulomb's Hamiltonian but it gives the zero-energy ground state of the Hamiltonian,

$$\mathcal{H} = \nabla^2 \delta(r - r'). \quad (2.113)$$

This Hamiltonian maps to a pair hopping model in two dimensional lattice, given by,

$$\mathcal{H} = \sum_{R,x,y} f^*(x) f(y) c_{R+x}^\dagger c_{R-y}^\dagger c_{R-y} c_{R+y} \quad (2.114)$$

with

$$f(x) = \kappa^{3/2} \sum_n (x - nL) \exp\{-\kappa^2 (x - nL)^2\} \quad (2.115)$$

where,

$$L = \frac{L_x L_y}{2\pi l_B^2}. \quad (2.116)$$

With a similar expression for $f(y)$ with $\kappa = 2\pi l_B / L_y$. Here L_x and $L_y$are the linear dimensions of the torus and $l_B = (\hbar c/eB)^{1/2}$ is the magnetic length. Note that this pair hopping model is

not the Coulomb interaction. This problem is quite different from that of the Coulomb interaction as far as distance dependence is concerned. It does not have a 1/r type potential and the coupling constant is proportional to the product *xy*. The operator part of (2.114) preserves the position of the centre-of-mass. Two electrons annihilated at $R \pm y$ hop to $R \pm x$ with the effective hopping range $1/\kappa$ and the centre of mass is preserved at *R*. he position of the centre of mass is indicated by q. The ground state of the 1/q fractional quantum Hall liquid with the q-fold degeneracy can be labeled by q different centre-of-mass positions. The ground state of (2.116) is a charge-density wave with amplitude ~ exp $(-c/\kappa^2)$, where c is a constant of the order O(1) and the energy gap is finite for any finite κ. For small κ the CDW amplitude is exponentially small and the ground state describes a featureless Mott insulator. In the presence of U(1) monopole flux, the single-particle wave function is described by the monopole vector spherical harmonics which can be denoted by the $SU(2)|l, m\rangle$ state where l can be integer or half-integer and m is the magnetic quantum number. The Landau level spectrum is given by,

$$E_k = \frac{\hbar^2}{2MR^2} l(l+1) \tag{2.117}$$

where M is the mass of the electrons and *R* is the radius of the sphere. If total magnetic flux is 2S, l = S + k, where k is the Landau level index. Because k is integer, S is either integer or half-integer. Each Landau level has 2(S + k) + 1 degeneracy [297]. The bosons in the lowest Landau level at the filling factor $\nu \to 1/2$ describe the ground state of a Mott insulator [298]. The density-matrix renormalization group (DMRG) method shows that there is a finite energy gap in the spectrum [299]. Paulin et al.[300] have discussed the time evolution of states. Some of the states are having short life time so that the full Hilbert space is not utilized whereas some of the states are long lived. Hence, there is a finite effect of life time on the properties of a quantum system. It is possible to encode the calculation of local operators in a Grassmann tensor network

[301]. Tang et al. [302] have defined a Chern number from which under suitable conditions they find flat bands which can make the quantized Hall states observable at high temperatures. The low energy part of the interacting particles in flatbands is described by a Jastrow correlated wave function which in spinless case shows a transition from a crystalline to liquid state [303]. Greiter [304] has shown that Landau levels on a sphere show oscillating as well as rotating energy levels. The highest weight MacDonald and Jack polynomials are described by Jolicoeur and Luque [305]. Kapit and Muller [306] find an exact Hamiltonian with the coupling constant of the type,

$$J(z_j, z_k) = W(z) \exp\left[\frac{\pi}{2}(z_j z^* - z_j^* z)\phi\right] \tag{2.118}$$

$$\mathcal{H} = \sum\nolimits_{j \neq k} J(z_j, z_k) a_j^\dagger a_k \tag{2.119}$$

where $z_j = x_i + iy_j$, $z = z_k - z_j$ for which Laughlin's wave function is a solution. Indeed, it is quite different from the Coulomb interaction. The flux is $\phi = n\phi_0, \phi_0 = hc/e$. For $W(z) = tG(z) \exp\left[-\frac{\pi}{2}(1-\phi)|z|^2\right], G(z) = (-1)^{x+y+xy}$ a single degenerate Landau level results. Similar results are found [307] for the Falicov-Kimball model in which Hofstadter buttery has two wings. The Hubbard model calculations for the kagome lattice are given by O'Brien et al [308]. Latorre et al. [309] construct a quantum algorithm that creates the Laughlin's state for arbitrary number of particles for $\nu = 1$. The circuit uses $n(n-1)/2$ qudits and hence it is efficient. Estienne et al. [310] find analytic expression for the generic wave function with one extra flux corresponding to the Jack polynomials. Deshpande et al. [311] have reported the liquids and solids in one dimension and Cane et al. [312] have discussed the asymptotes of Selbery-like integrals. Imada and Miyake [313] have also performed the first principle

calculations. The Laughlin correlations in cyclotron braids are discussed by Jacak and Jacak [314]. The interlayer coupling and the charge imbalance is discussed by Peterson et al. [315].

2.6.20 Experiments

There are a variety of experiments [316]. The plateaus in the Hall resistivity have been measured in detail. It is reported that the experimental data is well reproducible but the origin of some of the fractions such as 3/5, 2/5 and 2/7 is not clear in terms of a particle-hole symmetry [317]. There are measurements of noise at the plateau which measures the resistivity and hence the charge of the particles. The nuclear magnetic resonance did not yield much information about the origin of plateaus. The electron spin resonance is detected. Many different tricks are used to introduce new ideas to interpret the data. Oh and Chang [318] made an effort to calculate the Lande's g values which came out to be near the experimental value. Some authors tried to explain the fractional charge as a random number called the Chern-Simon's number. There is a lot of effort, theoretically as well as experimentally, to refer to Laughlin's paper which has wide applications (see for example [319]).

2.7 Data Interpretation

2.7.1 Graphite Hall effect

As we have seen above, the Laughlin's wave function has generated a lot of interest amongst theorists. It requires the knowledge of complex Riemann space, so that it is unlikely to be used for the interpretation of the plateaus measured in the Hall effect. Hence we give here the interpretation of the Hall effect data in graphite in terms of Shrivastava's theory [2,3] which treats the angular momentum with spin in a special way. The resistance of the spin parallel configuration is smaller than the resistance of the spin anti-parallel configuration. This effect is

called magnetoresistance because of the application of a magnetic field on the spins. In 1988, Albert Fert has performed the measurements in Fe/Cr films [320]. We find that this effect is the same as predicted by Shrivastava's theory [2,3] in 1985 where changing the sign of spin changes the resistivity. We make use of the data of Hall resistivity in graphite from which we prove that calculated values at the plateaus match with the experimental values. In particular, the simple g value defined as $(2j+1)/(2l+1)$ is used along with ideas of flux quantization. The idea of two particle states, resonances and principal $(1/2)g$ are used to explain the data of resistivity. Fert [320] wrote in his Nobel lecture "In 1988 the discovery of the magnetoresistance of the magnetic multilayer opened the way to an efficient control of the motion of the electrons by acting on their spin through the orientation of the magnetization". This means that spin has an effect on the resistance. The resistivity is changed by spin in a magnetic as well as in a nonmagnetic material [2,3]. In fact Shrivastava has found in 1985 that the resistivity changes by changing the sign of spin. We take the example of Hall Effect in graphite to explain the role of spin for the understanding of resistivity. According to Shrivastava calculation, the quantized resistivity may be written as, $\rho = h/e^2$ which may be corrected to, $\rho = h/[(1/2)\, ge^2]$ where,

$$\frac{1}{2}g = \frac{l + \frac{1}{2} \pm s}{2l + 1} \tag{2.120}$$

which makes the quantized value depend on the sign of the spin and the orbital angular momentum quantum number. The flux quantization is described by, $B.A = n'hc/e$ where $A^{1/2}$is the magnetic length and $n'$is an integer. The formulas given above are non-relativistic. The electrodynamics effect is included in the value of the charge of the electron. The value of $h/2e^2$ found by using the value of the Planck's constant, $h = 6.626068960 \times 10^{-34}$Js and that of the electron charge $e = 1.602176487 \times 10^{-19}$ Coulomb is 12.906 403 783 Ω. This value neither

requires two dimensionality nor does it require Landau levels. According to Shrivastava formula, the positive sign before s gives the resistivity,

$$\rho_+ = \frac{h}{e^2}\frac{2l+1}{l+\frac{1}{2}+s} \tag{2.121}$$

whereas for the negative sign the resistivity is,

$$\rho_- = \frac{h}{e^2}\frac{2l+1}{l+\frac{1}{2}-s}\ . \tag{2.122}$$

For $l = 0, s = 1/2\,, \rho_+ = h/e^2$ and $\rho_- = \infty$. Therefore, large changes in the resistivity are possible when the sign of the spin is changed. The above values suggest change in resistivity from $25.8k\Omega$ to infinity upon spin flip. If we substitute $i = 0, \rho = h/ie^2$, then the resistivity changes from $25.8k\Omega$ to infinity. Of course, there are other values which show the resistivity as a function of spin. We need not limit to s=1/2 only. Other values of the spin such as 3/2 or 5/2 are also possible. According to Shrivastava theory, there are quasiparticles of fractional as well as integer charge and the spin and charge are coupled. Hence, a modified Bohr magneton emerges and resistivity depends on the spin. There are fundamental charges given by $e^*/e = (1/2)g$, where $g = (2j+1)/(2l+1)$ so that the resistivity becomes $\rho = h/[(1/2)\,ge^2]$. In heterostructures, the spin need not be 1/2 because there is cluster formation. For example, the spin of a cluster may be3/2 or 5/2, etc. There are two-particle states so that $\omega_1 + w_2$ is possible. Similarly, there are resonances so that $\omega_1 - \omega_2$ is also allowed. Hence, quasi particle charge is determined from (i) spin-charge coupling, (ii) two-particle states, (iii) resonances and (iv) electron clustering. We explain the fractional charges found in graphite. The experimental measurements of the Hall effect in graphite have been performed by Kopelevich et al. [321] from which, we obtain the fractions, 2/7, 1/4, 2/9, 1/5, 2/11, 1/6, 2/15, 1/8,2/17 and 1/9. The

interpretation of the data, shown in Fig.2.3 is as follows. The energy of a state is given by $(1/2)g\left(n+\frac{1}{2}\right)$ so that we consider two oscillators with energies,$E_1=\frac{1}{2}g_1(n_1+\frac{1}{2})$and $E_2=\frac{1}{2}g_2\left(n_2+\frac{1}{2}\right)$. The energy difference between these states is $\frac{1}{2}g(n_1-n_2)$. For $l=2, 2l+1=7$ and for positive sign in $\frac{1}{2}g=[l+\frac{1}{2}+s]/(2l+1)=4/7$,$\frac{1}{2}g_1n_1-\frac{1}{2}g_2n_2+(1/2)g_1(1/2)-(1/2)g_2(1/2)=(1/2)(1/2)g_1$ for $n_1=n_2=0$ forthe second oscillator which has*l* = 0, negative sign and s=1/2 so that$(1/2)g_2=0$. Hence, $(1/2)(1/2)g_1=2/7$. The ingredients we put are two oscillators with different values of *l* and s which are the orbital and spin angular momenta quantum numbers, in different parts of the sample. The effective charge which depends on spin also determines the resistivity. Hence the resistivity depends on spin. In the electron clusters, spin can become zero, so that we put S=0 to obtain $(1/2)g=[l+(1/2)]/(2l+1)=1/2\ \ or\ \ g=1$. In the formula $(1/2)[(1/2)g-0]\ for\ g=1$ we obtain 1/4. For *l* = 4, 2*l* +1 = 9 and for s = 1/2 for negative sign, we obtain $(1/2)g=4/9$. In the expression, $(1/2)[(1/2)g_1-(1/2)g_2]$ we have $g_2=0$ and $(1/2)g_1=4/9$ so that the effective charge becomes $(1/2)(1/2)g_1=2/9$. For *l* = 2, we have 2*l* +1 = 5 and *l*/[2l +1] = 2/5 and (*l* +1)/(2*l* +1) = 3/5. We calculate the resonance state at $(1/2)g[n_1+(1/2)]-(1/2)g[n_2+(1/2)]$ at$(1/2)g(n_1-n_2)$ which comes at 3/5 – 2/5 = 1/5. For *l* = 5 we have *l*/2*l* + 1 = 5/11 and (*l* + 1)/(2*l* + 1) = 6/11. The value of (1/2)(6/11 - 0) = 3/11. The resonance state now occurs at 5/11 – 3/11 = 2/11. Let us look at the flux quantization at $n'h\,c/e$ so that for $n'=2$, the charge is e/2. Hence for $n'=2$, the effective value of 1/3 changes to 1/6. The original value for *l* = 1, 2*l* + 1 = 3 for negative sign is 1/3. For *l* = 7, the two series, *l*/2*l* + 1 = 7/15 and (*l* + 1)/(2*l* + 1) = 8/15,$(1/2)(1/2)g=4/15$ and for $n'=2$, 4/15 becomes 2/15. We have already obtained 1/4 which for $n'=2$ becomes 1/8. For *l* = 8, 2*l* + 1 = 17 and the principal fractions are *l*/2*l* + 1 =

8/17 and 9/17. We have $(1/2)(1/2)g = 4/17$ which for $n' = 2$ gives 2/17. For l = 4, $l/(2l + 1)$ = 4/9 and $(l + 1)/(2l + 1)$ = 5/9. The resonance state of these two occurs at 5/9 – 4/9 = 1/9. This explains all of the fractions observed in the fractional quantum Hall effect of graphite experimentally measured by Kopelevich et al. [321]. Fert has not explained the microscopic origin of the change in resistivity by spin. We find that the spin-charge effect in which charge becomes related to spin as found by Shrivastava's theory a few years earlier gives the spin dependent resistivity. All of the observed fractional charges agree with the scheme of a spin-charge relationship. The relationship of resistivity with charge is, $\rho = h/[(1/2)g\,e^2]$ which is amply demonstrated. The equation (2.120) has a zero energy state when l=0 and s=1/2 with negative sign in the expression.

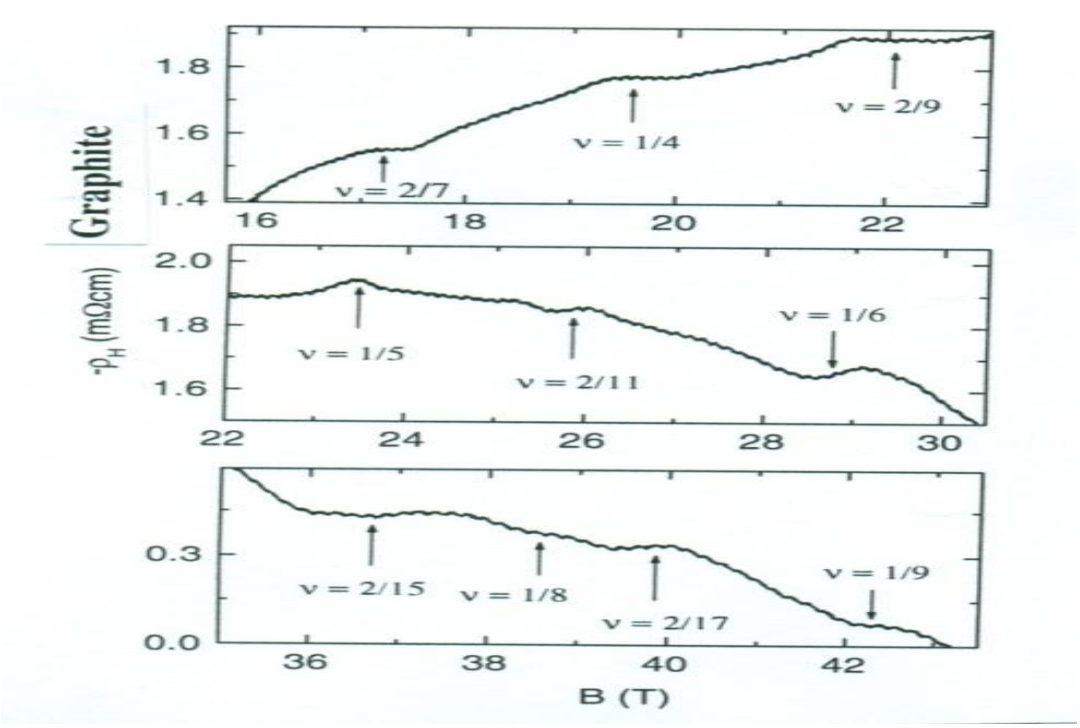

Figure 2.3: The quantum Hall effect of graphite showing several fractions.

2.7.2 Electron clusters

All of the experimental data can be explained on the basis of the formulae (2.120-2.122). The interpretation of the Stormer's data is given in the reference [322] and the Pan's data is explained in the reference [323]. The fractions observed in graphene are also explained by this theory [2]. Recently, Kumar et al. [324] have reexamined the fractions in the quantum Hall effect and found that some of the fractions depend on the sample preparation which is linked with the formation of many domains and clusters of electrons. Since electrons have spin, these clusters have a finite spin which may be NS where N is the number of ferromagnetically aligned electrons in the cluster. The fractions observed by Kumar et al. [324] are the same as in Pan's data except for one value which can be understood on the basis of electron clusters of finite spin. The fractions of charge which are measured from the plateaus in the Hall resistivity, ρ_{xy}, are usually the same as those measured from the minima in ρ_{xx}. The fractions occur in four categories, (i) principal fractions, (ii) resonances, (iii) sum processes and the (iv) electron clusters. In the case of electron clusters, the fractions derived from ρ_{xx} are slightly different from those found from ρ_{xy} . The effective charge of an electron becomes anisotropic due to the spin wave propagation in the micro-clusters of electrons. The wave vector of the spin waves appears in the effective value of the spin. There is an explicit dependence of the charge on the spin which modifies the condition of the flux quantization and leads to an anisotropic charge. In the Hall Effect, the resistivity is a linear function of the magnetic field. When field is quantized there occur plateaus in the ρ_{xy} and minima in ρ_{xy}. The plateaus in the $\rho_{xy} = h/(ie^2)$ occur at the integer values of i. When temperature is reduced, it is found that i need not be an integer and it can become a fraction such as 1/3 or 2/3. These fractions occur in pairs showing particle-hole symmetry. At this stage,

Laughlin [1] wrote a wave function by using the first principles and suggested that the charge may become fractional such as e/3 or e/5 due to electron correlations. At low temperatures, the number of excitations greatly reduces so that it is likely that correlations are reducing data low temperatures and the experimentally observed phenomenon may be explained by spin. Instead of starting from a wave function, we looked at the factors which multiply the Bohr magneton and hence change the effective charge of the electron. We found that there is a possibility of a spin-charge coupling which changes the charge of a particle. The factor of $g = (2j + 1)/(2l + 1)$ multiplies the Bohr magneton and hence changes the charge of the electron to $e^* = \frac{1}{2} ge$ which is e for $g = 2$. For $j = l \pm s$, we obtain the particle-hole symmetry because of the sign of spin which is related to helicity,$\vec{p}.\vec{s}$. Here $\vec{p}$ is the linear momentum of the electron and $\vec{s}$ is the spin. The expression $g = (2j + 1)/(2l + 1)$ explainsmany fractions which are found in the experimental data . By changing the value of *l* we develop a series of fractions and another series is obtained by using the negative sign in $\pm s$. This way we obtained two series, one for + sign and the other for - sign. These series are correct and give the experimental values correctly and hence called “principal fractions". By going to lower temperatures more fractions are obtained. To understand these fractions, we invent the “resonances", $\nu_1 - \nu_2$, which explain several experimentally observed fractions. Here, $\nu_i = \frac{1}{2} g_i$ in which the values of g_i are obtained by changing the values of l . It is also important to invent the "sum process" which yields the fractional charges at $\nu_i + \nu_j$. So far we have used only spin 1/2 particles. In actual samples which have a layer of GaAs over Al doped GaAs, electron clusters occur. In the clusters the spin need not be 1/2. For example it can be 3/2 if three electrons are aligned parallel to each other as in a ferromagnetic. Hence the spin becomes large, such as NS. It is also possible that the electrons are aligned in such a way that there are $N_\uparrow$ electrons with spin upand there are $N_\downarrow$

electrons with spin down so that the spin of the electrons in the cluster becomes NS with $N = N_{\uparrow} - N_{\downarrow}$. Hence, the idea of "clusters" of the electrons explains the data. In this way, all of the 101 fractions can be explained. Instead of detecting one fraction at a time, sometimes a full series of fractions can be written down. Such series can be generated by using higher than s=1/2 and by varying the orbital angular momentum quantum number, *l*. These series of fractions are observed in graphene and correctly predicted [2]. Recently, Kumar et al. [324] have found a fraction which shows one value when measured from the plateau in ρ_{xy} and a slightly different value when measured from the minimum in ρ_{xx}. Hence, there is a possibility that in some cases, the fractions obtained from ρ_{xy} are not equal to those found from ρ_{xx}. We show that there are spin waves in small clusters of electrons which create a finite "spin deviation" giving rise to anisotropy in the fractional charge. The spin deviation depends on the spin wave vector and makes the spin value anisotropic which due to spin-charge coupling makes the charge anisotropic and hence the effective charge along *xy* plane is not equal to that in the xx direction. We find that a small difference occurs in the value of the fraction in going from xx to xy value.

The Hamiltonian of the ferromagnetic spin waves with exchange interaction is given by,

$$\mathcal{H} = -J\sum_{j\delta} S_j . S_{j+\delta} - g\mu_B H \sum_j S_{jz} \qquad (2.123)$$

where *J* is the exchange interaction and *H* is the magnetic field. S_j are the spin operators at the *j*th site and the summation can be carried out to nearest neighbors,$S_{j+\delta}$. The number of nearest neighbors is z so that the Fourier transforms of spin operators requires,

$$\gamma_k = \frac{1}{z}\sum_{\delta} \exp\left(ik.\delta\right) . \qquad (2.124)$$

Leaving out the magnetic field dependent term, the unperturbed frequency of a magnon is,

$$\hbar\omega_k = 2JzS(1-\gamma_k). \tag{2.125}$$

Hence, the spin has been changed from S to $S(1-\gamma_k)$ in going from the site variables to spin wave variables. This is called the spin deviation and it amounts to a few percent inreal materials. For$k.\delta << 1$, exp $(ik.\delta)$ has only sine term and the cosine term is zero sothat for small wave vectors,

$$z(1-\gamma_k) \simeq \frac{1}{2}\sum_k (k.\delta)^2 \tag{2.126}$$

So that

$$\hbar\omega_k \simeq JS\sum_k (k.\delta)^2 \tag{2.127}$$

which for cubic lattices is $2JS(\vec{k},\vec{a})^2$where a is the lattice constant. Hence, the spin ischanged to$S(\vec{k},\vec{a})^2$. In the case of two sublattice antiferromagnets, $\hbar\omega_k \simeq 4\sqrt{3}\,JSka$.In a cluster, in the a-b plane, the frequency of a magnon becomes $2JS[\left(\vec{k}.\vec{a}\right)^2 + \left(\vec{k}\vec{b}\right)^2]$ whereas along the *x* direction it will appear as $2JS(\vec{k},\vec{a})^2$. Hence the spin value becomesanisotropic.

The Hall effect resistivity is,

$$\rho_{xy} = \frac{h}{\frac{1}{2}ge^2} = \frac{h}{\frac{l+\frac{1}{2}\pm s}{(2l+1)}e^2} \tag{2.128}$$

In the case of an electron cluster, the value of S along xx direction is$NS = (N_\uparrow - N_{\downarrow)S}$ which due to spin waves becomes $S(\vec{k},\vec{a})^2 N$ whereas along the xy direction it will be $\frac{1}{2}S[\left(\vec{k}.\vec{a}\right)^2 + \left(\vec{k}.\vec{b}\right)^2]N$. Hence the value of $\frac{1}{2}g$ depends on the direction. The value of the filling factor,

$$\nu = \frac{1}{2}g \tag{2.129}$$

also depends on the direction. Hence ν_{xx} need not be equal to $\nu_{xy}$where ν_{xx} is deduced from ρ_{xx} and ν_{xy} from ρ_{xy}.Kumar et al. [324] find all the same fractions as in the Pan's data [323] except one value which depends on the sample preparation. It was reported that in the xx direction the filling factor for one of the plateaus is $\nu = 2.463 \pm 0.002$ whereas in the *xy* direction$\nu = 2.461$, Hence, the value deduced from the ρ_{xy} is slightly different from that deduced from ρ_{xx}, which we assign to the value of the spin due to spin waves. For *l* =0, the filling fraction is,

$$\frac{1}{2} \pm s = \frac{2463}{1000} \ . \tag{2.130}$$

Hence $\pm s = 1.963$ which shows 1.8 percent spin deviation compared with S = 2. The Hall effect experiment is performed on a polycrystalline heterostructures film so that the crystallographic directions are not the same as those used for the Hall effect. The difference between the two values along the xx direction and the *xy* direction is a result of anisotropy of the spin waves. Gallis et al. [325] also find that near $\nu = 1$, ferromagnetic spinwave collapses and low energy spin wave emerges below the Zeeman energy.

2.7.3 Graphene

The resistance diverges as the charge approaches zero because of the infinity in $\rho = h/e^2$at e = 0 which is predicted at $g = 0$. The zero charge arises due to a zero in the g valueas shown by Shrivastava [2]. For $l = 0, (1/2)g = \frac{1}{2} \pm s$, which goes to zero with the negative sign. Hou et al. [326] also find that the strong magnetic field does not remove the zero-energy modes and there is a binding of a fraction of charge in the hexagonal structure of graphene. Abanin et al. [327] find that the measured gap energy is much larger than expected from a two level theory. This is because of the multilevel nature of the levels in graphene. Papic et al. [328] have calculated a four-component trial wave function which is adiabatically connected to the

Laughlin's wave function in the upper spin branch and hence may provide a representation for graphene. It is connected to valley-isospin ferromagnetic ordering and a completely filled lower spin branch which is similar to that of splitting in the Zeeman effect. However, graphene is not linked to incompressible states. It is possible to go from the incompressible state to the compressible state but in that case the charge will have to be adjusted [329]. The idea of isospin has been applied to graphene [330]. A trilateral system has also been considered [331]. For the simple layer of carbon atoms ordinary non-relativistic quantum mechanics should be sufficient. Accordingly Zabidi et al.[332] have done the most reasonable and correct calculation of the band structure of graphene including that of a wire pulled from graphene which shows that graphene is a non relativistic material and the Dirac super symmetry is not immediately found [332,333]. Some of the fractions observed in the quantum Hall effect of graphene are due to clusters [334]. The flux quantization condition becomes spin dependent [335]. There is a zero-energy state as g becomes zero [336,337] but the Laughlin's wave function does not apply to graphene [338]. The transition from a plateau to another plateau has been described in terms of a critical exponent which shows the change in the area in which flux is quantized [339]. The correct interpretation of the quantum Hall effect data of graphene is given in ref. [2]. Indeed it modifies the flux quantization and makes a fundamental contribution to the understanding of the angular momentum [340,341].The vibrational frequencies of graphene are calculated by Rosli et al. [342] from the correct and proper quantum mechanics. Some authors are making efforts to change the kinetic energy variation from $(1/2)mv^2 = \hbar^2k^2/2m$ to those proportional to k or k^3 where k is the wave vector of the electron. Hence instead of energy proportional to k^2 as in a parabola, it will become linear, a hyperbola or an ellipse, etc. These dispersion relations should not be a serious matter for the physical properties of graphene.

2.7.4 AlGaAs/GaAs

Let us see as to how the interpretation of the data in AlGaAs leads to a completeness of the interpretation of the quantum Hall effect. The data of Kumar et al. [324] shows that there are clusters which are explained by a spin deviation. We see that it leads to the completeness of the solution. The cyclotron resonance occurs at,

$$g\mu_B B = \hbar\omega. \tag{2.131}$$

The Bohr magneton is,

$$\mu_B = \frac{e\hbar}{2mc}\,. \tag{2.132}$$

Hence,

$$\frac{g}{2}\frac{e\hbar}{mc}B = \hbar\omega. \tag{2.133}$$

Since, for L=0, $g = 2$, it is a common practice to define the cyclotron frequency as,

$$\omega_c = eB/mc\ \ . \tag{2.134}$$

The electrons in a magnetic field behave like harmonic oscillators. Hence, the energy of astate is given by,

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) = \hbar\frac{eB}{mc}\left(n + \frac{1}{2}\right). \tag{2.135}$$

It is proper to include the factor of $\frac{1}{2}g$ so that the above energy is modified to,

$$E_n = \frac{1}{2}g\hbar\frac{eB}{mc}\left(n + \frac{1}{2}\right). \tag{2.136}$$

The same effect can be obtained by replacing e by$e^* = \frac{1}{2}ge$, so that the energy can be written as,

$$E_n = \hbar\frac{Be^*}{mc}\left(n + \frac{1}{2}\right). \tag{2.137}$$

The transition can occur from$E_n\ to\ E_{n'}$. The Hall effect resistivity is,

$$\rho_{xy} = \frac{B}{nec} = \frac{B}{ne^*c} \quad . \tag{2.138}$$

Hence, effective charge can be measured. The flux quantization is given by,

$$B.A = n' \frac{hc}{e} \quad . \tag{2.139}$$

Substituting this field in the Hall effect formula gives,

$$\rho_{xy} = \frac{n'hc}{ne^*ceA} = \frac{n'}{nA}\frac{h}{ee^*} = n'' \frac{h}{\frac{1}{2}ge^2} \tag{2.140}$$

which gives the quantum Hall effect. We do not use the Lande's formula because it does not have the particle-hole symmetry but suggest a formula linear in the angular momenta,

$$g = \frac{2j+1}{2l+1} \tag{2.141}$$

where $j = l \pm s$ is the total angular momentum quantum number and due to the $\pm$ sign it has the particle-hole symmetry. The effective charge is thus,

$$\nu_{\pm} e = \frac{1}{2} g_{\pm} e = \frac{l + \frac{1}{2} \pm s}{2l+1} e \ . \tag{2.142}$$

For positive sign and s=1/2,

$$\nu_+ = \frac{l+1}{2l+1} \tag{2.143}$$

and for the negative sign and s=1/2

$$\nu_- = \frac{l}{2l+1} \quad . \tag{2.144}$$

Here $\nu_{\pm} = \frac{1}{2} g_{\pm}$ gives the correct fractional charges, such as for $l = 1, \nu_- = 1/3$ and $\nu_+ = 2/3$, etc. These values are the same as tabulated in 1985 [2]. These values also agree with the

experimental data. We call these “principal fractions". In addition to the “principal fractions", we can also generate resonances, $\nu_1 - \nu_2$. This process produces some more fractions not already present in the “principal fractions", in addition, we also consider the energy level difference, $E_1 - E_2$ with $E_2 = 0$. Indeed, there is a zero energy state for $l = 0$ with negative sign, s = 1/2, $\nu_- = 0$. Many resonances are in fact present in the experimental data and predicted from the linear theory. At low temperatures, the excitation populations are small so that interactions are minimized. Hence, we predict the “resonances". We are thus able to produce a large number of principal fractions and resonances so that “two-particle states" occur. For these particles we have $\nu_1 + \nu_2$so that this process produces more fractions than are found in resonances. Hence we have (i) principal fractions, (ii) resonances and (iii) two particle states. The real material is often having electron clusters so that the spin need not be 1/2. The spin becomes NS with $N = N_\uparrow - N_\downarrow$. Hence the spin becomes 3/2, 5/2, etc. For example, for *l*=0, the fractional charge can occur at,

$$e^*/e = \frac{1}{2} \pm s \ . \tag{2.145}$$

For finite *l*,

$$e^*/e = \frac{l + \frac{1}{2} \pm s}{2l + 1} \quad . \tag{2.146}$$

For S=0 which can arise in pairs for which $N_\uparrow - N_\downarrow = 0$, the effective charge becomes,

$$e^* = \frac{1}{2}\frac{2l+1}{2l+1} = \frac{1}{2} \tag{2.147}$$

which has even denominator. For S=1,

$$e^* = \frac{l + \frac{1}{2} \pm s}{2l + 1} = \frac{l + \frac{3}{2}}{2l + 1}; \frac{l - \frac{1}{2}}{2l + 1} \tag{2.148}$$

which have even denominators and occur in pairs. For a large cluster, there is a small spin deviation due to “spin waves" so that the spin ceases to be an integer. For example, S becomes $S - \delta S = 2 - 0.02 = 1.98$ which has 1 per cent spin deviation. All of the data is correctly explained on the basis of this theory and we have written down the interpretation of 101 fractions satisfactorily [323,343]. Recently, a lot of the latest fractions have been examined and found to be satisfactorily explained [344-346]. The clustering is also seen in eq. (2.130) which explains the data. We first select a few of the Landau level filling factors which are observed in the experiments of Li et al. [347]. These fractions are,

1/5, 2/9, 2/3, 3/5, 2/5, 1/3, 5/3, 4/3, 4/7, 7/11, 3/7 and 63/100, (2.149)

Laughlin's theory suggests a wave function which can produce a charge at 1/*m*. If we assume that *m* = 3, then effective charge becomes 1/3, which will explain the fraction 1/3 observed in the Hall resistivity. The Laughlin's wave function,

$$|m\rangle = (2^{m+1}\pi m!)^{-1/2} z^m \exp\left(-\frac{1}{4}|z|^2\right) \qquad (2.150)$$

is not a ground state of the Coulomb interaction but it proposes its own new Hamiltonian which is a function of $\nabla^2\delta(r_i - r_j)$ in the complex space. Usually, the coordinates, (x,y,z), of the electrons are represented by real numbers but in the complex Riemann space, complex coordinates are used which is unlikely to be found in the physics of the problem. This wave function is also very difficult to use for the interpretation of the experimental data. Its exponential with a minus sign makes the value very small and the square on $|z|^2$ makes it very sharp and unrealistic. Theoretically it was done intentionally to make a two dimensional probability of finding the electrons. The fractions observed are explained in terms of (i) principal fractions, (ii) two particle states, (iii) resonance states and (iv) clusters. The spin is usually 1/2 but in clusters it can be some other value such as 3/2, 5/2... N/2, etc.

The principal fractions are the same as those obtained from $l/(2l+1)$ and $(l+1)/(2l+1)$. For s = 1/2,$(1/2)g = [l + (1/2) \pm s]/(2l + 1)$ becomes $l/(2l + 1)$ for negative sign and $(l + 1)/(2l + 1)$ for the positive sign. For various values of l we predict,

$$l = 0, \nu_{\pm} = 0,1$$

$$l = 1, \nu_{\pm} = 1/3\,, 2/3$$

$$l = 2\,, \nu_{\pm} = 2/5\,, 3/5$$

$$l = 3\,, \nu_{\pm} = 3/7\,, 4/7$$

$$l = 4\,, \nu_{\pm} = 4/9\,, 5/9$$

$$l = 5\,, \nu_{\pm} = 5/11\,, 6/11$$

$$l = 6\,, \nu_{\pm} = 6/13\,, 7/13$$

$$l = 7\,, \nu_{\pm} = 7/15\,, 8/15 \qquad (2.151)$$

This explains 2/3, 3/5, 2/5, 1/3, 4/7, and 3/7 found in the experimental data of Liet al. [347]. That leaves out 1/5, 2/9, 5/3, 4/3, 7/11 and 63/100. Let us now look at 1/5. The denominator 5 comes from $2l + 1 = 5$ so that $l = 2$. The principal fractions are 2/5 and 3/5. The resonance state occurs at 3/5 – 2/5 = 1/5. Hence 1/5 is explained as a resonance state. Let us look at 2/9. The denominator 9 comes from $2l + 1 = 9$ so that $l = 4$. The principal fractions are 4/9 and 5/9. The resonance state is at 5/9-4/9 =1/9.The energy levels occur at $(1/2)g(n + 1/2)$ so that for two oscillators with,

$$E_{2,1} = \frac{1}{2} g_2 \left(n_2 + \frac{1}{2}\right) - \frac{1}{2} g_1 \left(n_1 + \frac{1}{2}\right). \qquad (2.152)$$

For $g_1 = 0(l = 0), n_2 = 0$

$$E_{2,1} = \frac{1}{4} g \quad . \qquad (2.153)$$

For $(1/2)g_2 = 4/9$ from eq. (1.151), $(1/2)g = 2/9$ from (1.153). This explains 2/9. The value of 5/3 is more than one. Hence this can not occur in *n* = 0 state which is the lowest value. To understand the values higher than one, we need to excite a few Landau levels. Now consider *l* = 1 so that 2*l* + 1 = 3. For this value, we have $(1/2)g_- = 1/3$ and $(1/2)g_+ = 2/3$. The energy level difference, for $g_1 = g_2$, can be written as,

$$\frac{1}{2}g_2\left(n_2+\frac{1}{2}\right)-\frac{1}{2}g_1\left(n_1+\frac{1}{2}\right)=\frac{1}{2}g(n_2-n_1). \tag{2.154}$$

For $(1/2)g = 1/3$, we obtain $(1/3)(n_2 - n_1)$ which is 5/3 for $n_2 = 5, n_1 = 0$. For $n_2 = 4, n_1 = 0, (1/3)n_2 = 4/3$. Let us look at the denominator 2*l* + 1 = 11 so that *l* = 5. Then *l*/2*l* + 1 = 5/11 and *(l* + 1)/(2*l* + 1) = 6/11. The resonance state occurs at $6/11 - 5/11 = 1/11$ and the two-particle state occurs at $6/11 + 1/11 = 7/11$. The sample of $Al_xGa_{1-x}As$ may have clusters which tend to diffuse by heating. Let us look at 63/25. The value of 2*l* + 1 = 25 so that *l* = 12. Now the g value is given by,

$$\frac{1}{2}g = \frac{12+\frac{1}{2}\pm s}{25}\ . \tag{2.155}$$

We want to get $12 + (1/2) \pm s = 63$. Therefore, $\pm s = 63 - 12 - (1/2) = 101/2$ so that,

$$\frac{1}{2}g = \frac{12+\frac{1}{2}\pm s}{25} = \frac{\frac{25}{2}\pm\frac{101}{2}}{25} = \frac{126}{50}, -\frac{76}{50} \tag{2.156}$$

so that for the positive sign $(1/2)g = 63/25$. From the previous calculation (1.153), $(1/4)g = 63/50$. Usually the flux is quantized at *hc*/e so that these are the strongest plateaus in the Hall resistivity. In order to understand weak effects in the data, we consider $n'hc/e$ with $n' = 2$ so that $hc/(e/2)$ quantizes the flux which is at half the charge. Hence the charge of 63/50 occurs as

a weak plateau at 63/100. This is because of a cluster of spin equal to 101/2 which is perfectly reasonable when Al doping is being donein GaAs. Note that,

$$\frac{1}{2}g_{+} = \frac{l+\frac{1}{2}+s}{2l+1} \tag{2.157}$$

and

$$\frac{1}{2}g_{-} = \frac{l+\frac{1}{2}-s}{2l+1} \tag{2.158}$$

add up to

$$\frac{1}{2}g_{+} + \frac{1}{2}g_{-} = 1 \tag{2.159}$$

and the particle-hole symmetry is established by two signs of spin in the helicity which is $\vec{p}.\vec{s}$ where $\vec{p}$ is the linear momentum. Only the sign of the helicity is needed and it changes by changing the sign of s. When $n' = 1$ in the flux quantization, $B.A = hc/e$ and the particle-hole symmetry uses the flux as *hc*/e. When we consider $n' = 2$, the flux quantum is 2*hc*/e and the effective charge is reduced to 1/2 the value so that the flux is quantized with *hc*/(e/2). Now, the effective particle-hole symmetric charges need not add to one but they add up to 1/2, so the charge can be both positive as well as negative. When $n' = 2$, the fraction 63/50 becomes 63/100 which is just the experimental value. There is a cluster of several atoms and many electrons so that the spin is NS = 101/2. Upon doping x increases from 0.21 to 0.85. The extra atoms help diffuse the cluster so that single electron again contributes to plateaus in the resistivity. Hence the plateau at 63/100 changes to plateau at 1. Li et al.[344] show the Hall resistivity as a function of filling fraction, ν , for two concentrations x=0.21 and x=0.85. In between 0.6 and 0.7 at 0.63 the x=0.21 graph shows a minimum the bottom of which is the value 0.63 for which the resistivity is 1.58. When the concentration is changed to x=0.85 this minimum at 0.63 disappears

and now the minimum is at the resistivity of 1. The directions, z for the magnetic field, x for the electric field and y for the Hall voltage are fixed by the electromagnetic field equations. Hence, as long as the coordinate system is orthogonal, there is no way of mixing xx with xy in the Maxwell equations. The impurity doping makes the dielectric constant and the permeability anisotropic and non-uniform which reflects in the non-uniformity of the magnetic field and hence in the resistivity. The data at certain fields correspond to single-particle plateaus which upon doping changes to a cluster state and upon further doping or cooling can change to the single- particle state. As the magnetic field is changed, the clusters of electrons are formed which break when another plateau approaches. The clusters also explain the short-range disorder which depends on the size of the clusters which gives small contribution to resistivity near the bending points. The Landau levels require two-dimensionality which is broken when clusters of atoms are made. The harmonic oscillator type levels $\hbar\omega_c(n+1/2)$ occur for electrons in two dimensions. Here, at very low temperatures such as a few mK, the population in the excited states is very low so that n = 0 in n + 1/2 = 1/2 is more appropriate than full n+1/2. However, as the temperature is increased, finite value of n has to be taken. The samples are non uniform and hence the value of n is not the same in the entire sample. Similarly, there are many clusters of electrons and the number of electrons in each cluster is not equal. The cluster formation thus depends on the method of preparation of the sample. We have examined the experimental data of the measurement of Hall resistivity as a function of Al doping of GaAs. Pan et al.[348] have obtained some more results about electron clusters in GaAs. We briefly discuss these results. In a single particle picture, the energy gap at a filling factor is due to the Zeeman splitting equal to $|g|\mu_B B$ where $g = -0.44$ in GaAs. The modified Zeeman splitting is equal to $g_{\pm}\mu_B B$ because at high magnetic fields the g value must have particle-hole symmetry. It is found that Lande's

formula is a single value without the particle-hole symmetry. The particle-hole symmetry is described by $g = (2j + 1)/(1l + 1)$ with $j = l \pm s$. The energy gap is not fully due to Zeeman effect. In part, the energy gaps can arise from the periodic potential of the lattice. The experimentally measured single-particle splitting are much larger than given by the old Zeeman effect due to the field-independent splitting and the Coulomb interaction. Due to the thermodynamics not all electrons are aligned along the field and then there is a difference between S and S_z. While S_z deals with alignment along the z direction, S need not align. At low temperatures a spin aligned ferromagnetic cluster of electrons is formed but due to lack of unfilled shells, the clusters are not stable. The zero-point modes are important and lead to making and breaking of the clusters. The Jahn-Teller effect which breaks the symmetry of a degenerate state interacting with bosons of symmetry with respect to which the degenerate state is unstable, destroys the same symmetries of clusters. The size of the antidotes becomes equal to the wave length of the electron so that the energy corresponding to this wave length contributes to the energy gap. The magnetization varies with field as, $M \sim H^{1/\delta}$.Hence, the energy appears to vary as $\sim H^{(1/\delta)+1}$ in the scaling region. The energy gap varies with the magnetic field as$\Delta \sim (B - B_0)^{(1/\delta)-1}$. In the sample of ref. [347] there is an electroncluster so that for $g = -0.44, 3g = 1.32 = 132/100 = 33/25$. For l = 12 the denominator is 2l + 1 = 25 and 12 + (1/2)$\pm$ s = 33 which suggests $\pm s = 41/2$ which is possible in a spin cluster. Indeed, clusters are formed and contribute to order-disorder phenomenon and unstable ferromagnetism is generated in the clusters. We have seen that the filling factors become anisotropic due to spin wave formation in the clusters consistent with the experimental observations [349].There are three different theories, (i) Laughlin's wave function theory, (ii) Wilczek's composite Fermion theory [349] and the (iii) Shrivastava's angular momentum theory. We find that Laughlin's wave

function is not convenient for the calculation of resistivity. In the original work of Laughlin, it was not the intention to derive many experimental properties but only to find the ground state. Similarly, in Wilczek's composite Fermion theory [350], the feasibility of theoretical consistency is extended to explore new possibilities but the electromagnetic field Maxwell equations have been violated. We are left with Shrivastava's angular momentum theory which indeed explains the experimental data correctly.

2.8 de Haas-van Alphen effect

The de Haas van Alphen effect occurs when the field dependent linear energy becomes equal to the Fermi energy. The physical properties oscillate when electrons move from the field dependent energy to the constant Fermi energy, ε_F. This phenomenon keeps coming many times for different values of n in the usual harmonic oscillator type of levels which have (n+1/2).

$$\hbar\omega_c(n+1/2)=\varepsilon_F\,[\mathrm{deHaas-vanAlphen}]. \tag{2.160}$$

On the other hand, the present effect in the quantum Hall effect, there is no need of Fermi energy and transitions occur within the energy levels of the type,

$$E_n=\frac{1}{2}g_{\pm}\mu_B H\left(n+\frac{1}{2}\right)[Shrivastava] \tag{2.161}$$

in which the field H is flux quantized and the Lande's factor is replaced by particle-hole symmetric $j=l\pm s$ which leads to the effective charge of $e^*=\frac{1}{2}ge$ which determines the quantized resistivity,

$$\rho_{xy}=\frac{h}{e^*e} \tag{2.162}$$

Further details of this effective charge can be found in ref. [323]. This effect does not require Laughlin's wave function.

2.9 Conclusions

There is a big leap between the Laughlin's wave function and the experiments on the quantum Hall effect. On one hand we have the Riemann space and on the other hand we have AlGaAs, graphene or graphite. They are unlikely to meet. Laughlin's wave function is solved on the basis of quantum mechanics and uses “incompressibility". It has a very short range and it is a ground state of new δ function type Hamiltonian. The experimental data is explained on the basis of Shrivastava's theory which is also based on quantum mechanics. The natural question is that whether the two theories, Laughlin's and Shrivastava's are equivalent? If Laughlin's charge is changed to $1/[2l + 1]$ then it is possible to see some remote equivalence. In Shrivastava’s theory, spin is included by using $j = l \pm s$ in which negative sign is non-relativistic whereas in Laughlin's theory there is no spin at all. It is possible to form monopoles consisting of composites of electrons and vortices in which case there is no contact between experiments and “attaching flux quanta". The usual experiments by using GaAs are unlikely to attach flux quanta to electrons. The data on the quantum Hall effect is thus explained by the angular momenta series, $l/(2l + 1)$ and $(l + 1)/(2l + 1)$, resonances, sum processes and electron clusters