CHAPTER 4

CLUSTERING OF ELECTRONS AND THE FILLING FACTOR IN QUANTUM HALL EFFECT

4.1 Introduction

The relative angular momentum of two particles, L_2 is used to define a projection operator P_2^p which corresponds to the relative angular momentum $L_2 + p$ the filling factor is defined by $\nu = 1/(L_2^{min} + p)$. The number of particles is g. If one more particle comes near it for the purpose of forming a cluster, the filling factor becomes $\nu = [\frac{1}{2}g(g+1) + p]^{-1}$. The g=2, p=4 is called Haffnian and g=2, p=3 is called Gaffnian. For g=2, p=3, v=1/6. This state has a ground state of a special pseudopotential type Hamiltonian. Given a fraction, we find the ground state energy of a special Hamiltonian. The state wave function, the ground state energy and the special Hamiltonian are linked together. In this methodology, the flux quanta cannot be attached to the electrons and the projection operator is not linked to the Coulomb Hamiltonian. Therefore, composite fermions (CF) with the flux quanta attached to the electrons, suggested by Jain cannot satisfy the wave functions, the Hamiltonian and the ground state requirements. The calculated ground state does not attach flux quanta.

In a simple paper with Coulomb interactions between electrons and attractive Coulomb interactions between electrons and nuclei, Laughlin [1] found that there is a simple wave function, the ground state of which is lower than Wang et al.[32] calculation of the ground state of the charge density waves. It looks quite impressive that this ground state has fractionally charged excitations. It turns out that the ground state of the charge density wave is not very accurate and hence, the lower energy argument of Laughlin requires quite a lot of work and then also we may not be able to prove that Laughlin found the ground state. Let us relax the requirement of the Hamiltonian and pretend that the Hamiltonian is satisfactory and we will work on this problem at a later time. In that case, we have to find a renormalization group transformation which transforms $\mathcal{H}_{CF}$ into $\mathcal{H}_{C}$. Such a transformation is also not found, or we can say that it does not exist. Besides, even if the transformation is found, it turns out that the Wilson renormalization group is not a true group in the meaning used in mathematics. The insufficient overlap of the CF wave function points out the need to better understand the FQH states from a theoretical stand point. It is known that at the plateau, the number of electrons is one or a small number. As the field is increased, there is a critical point characterized by a Goldstone mode. This mode can have zero energy and hence should be predicted by the theory. In the transformed state, there are lots of electrons which form a “cluster “. The zero energy mode, usually, has a gap parameter which is introduced in the proper Coulomb Hamiltonian so that the transformed Hamiltonian may be compared with a pseudo potential. The zero energy mode property is insufficient to completely determine the structure of CF wave functions. It is also possible to introduce the Goldstone mode in the Coulomb Hamiltonian in which case CF will not find zero energy Goldstone modes. One can try "squeezing representation" of the operators [354]. It changes the shape of absorption and emission lines and can change the uncertainty relation from ≥ to =. Overall it has only a small effect on the regular quantum mechanics and it will not help in attaching flux quanta which are algebraically incomplete. The wave function Ψ_α or its eigen value a_α in $a_\alpha|\Psi_\alpha\rangle$ is a vector in an infinite dimensional Hilbert space. The component of the state vector along some axis of the Hilbert space is numerically equal to the corresponding element of the matrix. The "norm" is a special case of inner product of two state vectors Ψ_α and Ψ_β which is defined as

$$\Psi_\alpha \Psi_\beta = \Psi_\alpha^+ \Psi_\beta = \int \Psi_\alpha^+ (r) \Psi_\beta(r) d^3 r = 1. \tag{4.1}$$

This is a number called norm when $\alpha = \beta$. In quantum mechanics the normalization integral must be unity. We obtain a transformed wave function by operating it with another operator so that the transformed wave function looks like $t\Psi_\beta$. We refer to this transformation as a "generalized transformation". This includes rotation of the state vector in the Hilbert space. The generalized rotation usually does not conserve the norm of the state vector. Even then the normalization integral will still be unity:

$$\int t\, \Psi_\alpha^*(r) t' \Psi_\beta(r) d^3 r = 1\,. \tag{4.2}$$

The transformation t reduces the wave function Ψ_β to $t\Psi_\beta$ but has no effect on the Hilbert space. The electric and magnetic fields are defined by the equations $\vec{E} = -4\pi c\vec{P}$, $\vec{H} = \vec{\nabla} \times \vec{A}$. Both $\vec{E}$ and $\vec{H}$ are described by the plane waves which are vector functions of $\vec{r}$ that are polarized perpendicular to the propagation vector so that $\nabla \cdot A = \nabla \cdot P = 0$. Both A and P are expanded in terms of $u_{\kappa\lambda}$

$$A(\vec{r}, t) = \sum_{\kappa\lambda}' [q_{\kappa\lambda}(t) u_{\kappa\lambda}(r) + q'_{\kappa\lambda}(t) u^*_{\kappa\lambda}(r)]$$
$$P(\vec{r}, t) = \sum_{\kappa\lambda}' [p_{\kappa\lambda}(t) u_{\kappa\lambda}(r) + p'_{\kappa\lambda}(t) u^*_{\kappa\lambda}(r)] \tag{4.3}$$

Both $p_{\kappa\lambda}$ and $q_{\kappa\lambda}$ are coupled together in the definitions of the creation and annihilation perators

$$a_{\kappa\lambda} = \frac{1}{2}\left(q_{\kappa\lambda} + \frac{4\pi i c}{k} p_{\kappa\lambda}\right) e^{i\kappa c t} \ ,$$

$$a^{+}_{\kappa\lambda} = \frac{1}{2}\left(q_{\kappa\lambda} - \frac{4\pi i c}{k} p_{\kappa\lambda}\right) e^{-i\kappa c t} \qquad (4.4)$$

Hence, in this theory there is no way to separate $\vec{E}$ and $\vec{H}$. Hence, we cannot take only the $\vec{H}$ and leave out $\vec{E}$. Separating out the $p_{k\lambda}$ and $q_{k\lambda}$ is impossible when photons have to be described as creation and annihilation operators. In the case of Laguerre's polynomials, the recursion relation is known to be,

$$a_{\nu+1} = \frac{\nu + \ell + 1 - \lambda}{(\nu + 1)(\nu + 2\ell + 2)} a_{\nu} \quad . \qquad (4.5)$$

This can be useful for the production of fractional numbers. There is a term containing $\ell(\ell+1)$ in the differential equation, and the potential can be taken to be a Coulomb potential in which case the Schrödinger equation for the hydrogen atom separates out and we get the eigen values in terms of eigen functions which are exact solutions of a given Hamiltonian. In such solutions, the Hamiltonian has a simple interpretation in terms of kinetic energy and the potential energy and this may be called a "physical Hamiltonian". When the Hamiltonian cannot be written in terms of the kinetic energy and potential energy we may called that Hamiltonian "non-physical". The transformed wave function $t\Psi$ can be a "squeezed wave function". Similarly, we have the "squeezed Laguerre's polynomial" to represent a transformed Laguerre's polynomial. Let us say that there was a polynomial which managed to attach flux quanta to the electron. Then "squeezed polynomial" will have effect on the line shape. Apparently, at this time such a polynomial which can attach flux quanta automatically to the electron is not found. Some authors have assumed a series quite different from (4.5) in such a way that it will give the experimental fractions correctly but they found non-physical Hamiltonians [355]. Instead of the angular momentum, the minimum angular momentum is defined as $\mathrm{g}(\mathrm{g}+1)/2$ so that the relative angular momentum is $L_{\mathrm{g}+1} < L_{\mathrm{g}+1}^{min} + p$. This will make the filling factor $[\frac{1}{2}\mathrm{g}(\mathrm{g}+1)+\mathrm{p}]^{-1}$. For g=2, p=4, $\nu = \nu(Haffnian)$ and g=2, p=3 $(Gaffnian)$ as if there is a need for names for every different value of ν . These series do belong to certain differential equations but then the Hamiltonian is not formed from the Coulomb interactions and hence they are called non-physical [4]. Usually, the Coulomb potential is of the form of $1/r$ and this is the correct form of the interaction in physical systems. However, if there is an additional interaction of the type of $1/r^2$, then it is possible to construct the Hamiltonian such that the eigen function are Jack's polynomials. This can also be classified as almost non-physical. The ground state energy of the Gaffnian is very near that of CF so both of them are non-physical. The Slater determinant for N electrons has the electrons Ψ_{11} as the first electron at r_1, Ψ_{12} first electron at r_2, etc. so that the first row looks like $\Psi_1(r_1), \Psi_2(r_1)$,…., $\Psi_N(r_1)$ for N electrons. This kind of determinant for N electrons becomes,

$$\Psi(1,2,\ldots,N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \Psi_1(\mathrm{r}_1) & \Psi_2(\mathrm{r}_1) & \cdots & \Psi_\mathrm{N}(\mathrm{r}_1) \\ \Psi_1(\mathrm{r}_2) & \Psi_2(\mathrm{r}_2) & \cdots & \Psi_\mathrm{N}(\mathrm{r}_2) \\ \Psi_1(\mathrm{r}_\mathrm{N}) & \Psi_2(\mathrm{r}_\mathrm{N}) & \cdots & \Psi_\mathrm{N}(\mathrm{r}_\mathrm{N}) \end{vmatrix} . \qquad (4.6)$$

We can multiply it by a polynomial. It is also possible to use hydrogen type wave function $e^{-\zeta r} r^{-\alpha x}$, etc. The wave function used by Laughlin is of the form

$\prod (z_i - z_j)^n \prod \Psi_i(z_e)$.

The plane waves are like $\Psi_1(r_1) \cong e^{ikr_1} \cong kr_1$. Since, $k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$ we can use $\Psi_1(r_1) \approx \sin\left(\frac{2\pi\nu}{c} r_1\right)$. There are many such possible trial wave functions. By using $k = \frac{2\pi}{h} p$,$k\, r_1 \cong \frac{1}{\hbar} p_1 r_1$ and $p_x = -i\hbar \frac{\partial}{\partial x_1}$. We can write $k_1 r_1 \cong \frac{1}{\hbar}(-i\hbar)\frac{\partial}{\partial x_1} r_1$. If we use the wave vector as a wave function, $\Psi_1(\mathrm{r}_1) \cong \frac{\partial}{\partial \mathrm{x}_1}$. This type of trials are nice to do but will not lead to a minimum in the energy. These will also be called non-physical trial wave functions. There can be two states at each angular momentum in the determinant because both z^m and $z^m \frac{\partial}{\partial z}$ operators can be used in the determinant which have the same angular momentum m. We can have $\Psi_1(r_1) \cong z_1^{t-1} \frac{\partial}{\partial x_1}$. There is nothing against $z^{m+2} \frac{\partial^2}{\partial z^2}orz^{m+3} \frac{\partial^3}{\partial z^3}$, etc. We can use the product of a determinant and a polynomial, for example,

$$\begin{vmatrix} \partial_1^2 & \partial_2^2 & \cdots & \cdots & \partial_N^2 \\ z_1^{t-1}\partial_1^2 & \cdots & \cdots & \cdots & z_N^{t-2}\partial_N^2 \\ 1 & 1 & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} . \tag{4.7}$$

The basic problem is to find the ground state energy for $\nu = 1/m$ (Laughlin), $\nu = \left[g(g+1)/2 + p\right]^{-1}$ (Haffnian,Gaffnian), $\nu = k/k+1$(Read – Rezai) and $\nu = k/2k+1$(CF). The Laughlin ground state has been calculated by using Coulomb Hamiltonian and there is no problem in obtaining the ground state. However, it does not have a minimum and stability is not found. When the other formulas are used Gaffnian and CF are very near to each other but the calculation depends on the energy gap. The number of energy levels in each case is not equal which makes it difficult to compare them. The "flux – attachment" in the CF is also not found because of the incompleteness of the problem. There are more variables than the number of equations and “detachment” of the flux quanta has not been done.

4.2 Conclusions

We find that the Laughlin’s calculation of the ground state energy is a good calculation using Coulomb Hamiltonian but it compares with the charge-density waves. In other calculations such as those of Haldane [356], the Coulomb Hamiltonian has not been restored. In particular where Jack’s polynomials are used the potentials are not near Coulomb interactions. Several series have been attempted which do not match with Coulomb interactions. All of the trial wave functions, which we have calculated, are based on Laughlin’s wave functions.