CHAPTER 3

METHOD DESCRIPTION

There are two main issues to overcome in the state estimation algorithm; the numerical ill-conditioning problem and the convergence problem. When the system is ill-conditioned, it will manifest itself in the form of slow convergence or failure to converge. Thus, several methods have been proposed to circumvent this problem and they have been discussed in Chapter 1. This research proposes another approach in orthogonal decomposition method to solve the power system state estimation. Before going into the proposed method, let's go through the notion of numerical ill-conditioning in power system state estimation problem.

3.1 Notion of numerical ill-conditioning

The state estimation solution employs an iterative process. It generates a sequence of points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$…. At each iteration a subproblem is solved, i.e. the next point $\mathbf{x}_{k+1}$ is generated by using the current point $\mathbf{x}_k$ and the parameter value $\mathbf{p}$ (impedances, etc.). This procedure can be represented by a function $\mathbf{x}_1 = \varphi(\mathbf{x}_0, \mathbf{p})$, $\mathbf{x}_2 = \varphi(\mathbf{x}_1, \mathbf{p})$, ….The iterative process converges if $\mathbf{x}_n$ approaches the solution $\mathbf{x}$. Because of the finite precision representation a number $\mathbf{x}_1$ is actually stored as an approximation $\mathbf{x}_1^*$, the difference being the round-off error. The effect of the round-off error is that $\mathbf{x}_2^* = \varphi(\mathbf{x}_1^*, \mathbf{p})$ is computed rather than $\mathbf{x}_2 = \varphi(\mathbf{x}_1, \mathbf{p})$.

An algorithm is ill-conditioned if for a given $(\mathbf{x}_1, \mathbf{p})$ the difference between $\varphi(\mathbf{x}_1, \mathbf{p})$ and $\varphi(\mathbf{x}_1^*, \mathbf{p})$ or between $\varphi(\mathbf{x}_1, \mathbf{p})$ and $\varphi(\mathbf{x}_1, \mathbf{p}^*)$ is large for $\mathbf{x}_1$ and $\mathbf{x}_1^*$ very close and $\mathbf{p}$ and $\mathbf{p}^*$ very close. Therefore, for a normally fast convergent solution method, owing to ill-conditioning, the effect of round-off error may lead it to slow convergence or failure to converge at all.

One way to measure the degree of ill-conditioning of a problem formulation is by the condition number (Watkins, 2002) of the coefficient matrix. In solving the least squares equation, $\mathbf{Ax} = \mathbf{b}$, one is interested to know by how much does the result $(\mathbf{A} + \mathbf{E})^{-1}\mathbf{b}$ or $\mathbf{A}^{-1}(\mathbf{b} + \boldsymbol{\varepsilon})$, where matrix $\mathbf{E}$ and vector $\boldsymbol{\varepsilon}$ represent errors, differ from the true solution $\mathbf{A}^{-1}\mathbf{b}$ because of the error introduced in $\mathbf{A}$ and $\mathbf{b}$. If they differ greatly the matrix is said to be ill-conditioned. It is well known in numerical analysis that the relative error in $\mathbf{A}$ or $\mathbf{b}$ maybe be magnified by as much as the condition number of $\mathbf{A}$ in passing through the solution. (Wu, 1990)

In solving the state estimation equation, $[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]\Delta\mathbf{x} = [\mathbf{H}^T\mathbf{R}^{-1}][\mathbf{z} - \mathbf{f}(\mathbf{x})]$, the condition number of the coefficient matrix ($\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}$) is the square of the condition number of the Jacobian matrix $\mathbf{H}$. The prevalent approach, the normal equation method (refer to section 1.2; pp. 5) directly computes sparse matrix triangular factorization of the gain matrix. Therefore, a large condition number of the coefficient matrix may cause ill-conditioning when performing triangular factorization, which will lead to slow convergence or convergence failure.

Consequently, the basic idea in solving the state estimation problem is to avoid the formation of the gain matrix to alleviating the numerical ill-conditioning problem in state estimation. Many solving methods have been developed according to this basic idea, such as Hactel's augmented matrix method, normal equations with constraints method, Hybrid method and Peters Wilkinson method (refer to section 1.2).

Hactel's augmented matrix method (refer to section 1.2; pp. 9) is one of the methods that had successfully overcome the numerical ill-conditioning that occurs in the coefficient matrix. Hactel's augmented matrix method skillfully avoids the cross product of jacobian matrix by treating the residual vector as an unknown. The coefficient matrix of Hactel's is symmetric and indefinite. Moreover, the large dimension of the coefficient matrix will influence the factorization efficiency. Hence, the Hactel's augmented matrix method need more than the simple ordering and factorization. The implementation of this method does not seem to be extensive.

Despite of the numerical ill-conditioning that occurs due to the large condition number of the gain matrix, another major source of numerical ill-conditioning in solving the state estimation problem is the large weights of virtual measurements (Holten et al., 1988). Virtual measurements are the kind of information that does not require metering, for example, zeros injection at the switching station. Virtual measurements play an important role in power system state estimation. The accuracy of the estimation can be improved by the exact mathematical relationships of virtual measurements. Also, virtual measurements contribute to enhance the system observability. Hence, virtual measurements are included in the measurement matrix $\mathbf{z}$. The larger the weighting factor that is assigned, the more accurate the measurement. It has been observed that, assignment of large weighting factors to virtual measurements may cause numerical ill-conditioning of the system (Holten et al., 1988). Weighting factors are the error variance of the measurement device. Therefore, the large weights of virtual measurements are identified as another source of numerical ill-conditioning in power system state estimation.

The normal equations with constraints method (refer to section 1.2; pp. 8) has successfully overcome the numerical ill-conditioning problem due to the large weights of virtual measurements by separating the virtual measurements with zero injections from the telemetered measurements and treating them as equality constraints (refer to section 1.2; pp. 8). As a result, the coefficient matrix of this method is no longer positive definite. However, care must be exercised in the triangular factorization of the coefficient matrix, where ordering and factorization must be carried out simultaneously with special techniques like 1x1 and 2x2 pivots (Wu, 1987). Therefore, the normal equations with constraints method requires a more sophisticated method besides triangular factorization. The computational implementation is indeed complicated excessively.

Consequently, it is important to have a method with the property of numerical stability, computation efficiency and computation implementation simplicity in order to solve the power state estimation problem efficiently and accurately.

3.2 Orthogonal decomposition method

Our proposed method is called the orthogonal decomposition method using Householder transformation. Implementations have been found to be numerically stable as they use unitary transformations and handle the numerical ill-conditioning encountered in the power system state estimation problem satisfactory.

The orthogonal decomposition method triangularized the weighted jacobian directly as opposed to first forming the squared gain matrix as in normal equation method (refer to section 1.2; pp. 5) and alleviating the numerical ill-conditioning due to large condition number of coefficient matrix, $\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}$.

In addition, the numerical robustness of orthogonal decomposition approach allows for zero injection constraints to be modeled as heavily weighted measurements. The ability to handle very wide ranges of weights with an orthogonal decomposition method obviates the need for special treatment of zero injections equality constraints. This greatly simplifies the state estimator implementation. The orthogonal decomposition problem formulation discussed in section 3.2.2 is shown to be very simple and clean.

Besides eliminating sources of numerical ill-conditioning, the orthogonal decomposition method has also simplified the solution process. Furthermore, the computational efficiency of this approach is very competitive with other methods.

The orthogonal decomposition method has a good compromise of numerical stability, computational efficiency and implementation simplicity. The orthogonal decomposition method is competitive with the Hybrid method and Peters Wilkinson method. The comparisons with other solving methods are presented in Chapter 4.

3.3 Orthogonal decomposition method based on Householder transformation

QR factorization is the heart of the orthogonal decomposition method. Therefore, the efficiency of the whole power system state estimation depends on the efficiency of the QR factorization. Many different methods exist to perform QR factorization, e.g. Householder transformation, Givens rotation, and Gram-Schmidt decomposition. All three of them are known to be numerically robust, although several authors have claimed the superiority of the Householder method in limiting the accumulation of the round-off error (Ravishankar et al., 2005).

From the literature survey, the prevalent approach for orthogonal decomposition system state estimator is the Givens rotation method (Trefethen and Bau, 1997). Instead of the Givens rotation method, this research has proposed the Householder transformation as the ordering method in the QR factorization and applies in the orthogonal decomposition method to solve the power system state estimation problem.

The error analysis carried out by Wilkinson showed that the Householder transformation outperforms the Givens rotation method under finite precision computations (Wilkinson, 1965). Additionally, the Householder method is more numerically stable since it uses orthogonal similarity transform (Householder and Bauer, 1959). Straightforward implementation of Givens rotation method requires about 50% more work than Householder method, and also requires more storage. These disadvantages can be overcome, but requires more complicated implementation.

3.3.1 Householder transformation

The Householder transformation was introduced in 1958 by Alston Scott Householder. Householder transformation or Householder reflection is a linear transformation that describes a reflection about a plane or hyperplane containing the origin. Householder transformations are widely used in numerical linear algebra, to perform QR factorization and in the first step of the QR algorithm. This operation can be applied in the QR factorization of an m-by-n matrix with m ≥ n, by reflecting first one column of a matrix onto a multiple of a standard basis vector, calculating the transformation matrix, multiplying it with the original matrix and then recursing down the (*i*, *i*) minors of that product.

Let $\mathbf{x}$ be an arbitrary real m-dimensional column vector of a matrix $\mathbf{A}$ such that $\|\mathbf{x}\| = |a|$ for a scalar a. If the QR algorithm is implemented using floating-point arithmetic, then a should get the opposite sign as the first coordinate of $\mathbf{x}$ to avoid loss of significance. Then, set

$$\mathbf{u} = \mathbf{x} - a\,\mathbf{e}_1,$$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|},$$

$$\mathbf{Q} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^T,$$

where

$\mathbf{e}_1$ is the vector $(1, 0, ..., 0)^T$;

$\|\cdot\|$ is the Euclidean norm;

$\mathbf{Q}$ is a Householder matrix;

$\mathbf{I}$ is an m-by-m identity matrix.

This can be used to gradually transform an m-by-n matrix $\mathbf{A}$ to upper triangular form. First, $\mathbf{A}$ is multiplied with the Householder matrix $\mathbf{Q}_1$ that is obtained when we choose the first matrix column for **x**. This results in a matrix $\mathbf{Q}_1\mathbf{A}$ with zeros in the left column (except for the first row):

$$\mathbf{Q}_1\mathbf{A} = \begin{bmatrix} a_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & \mathbf{A}^1 & \\ 0 & & & \end{bmatrix}.$$

This can be repeated for $\mathbf{A}^1$ (obtained from $\mathbf{Q}_1\mathbf{A}$ by deleting the first row and the first column), resulting in another Householder matrix $\mathbf{Q}_2{}^1$. Note that $\mathbf{Q}_2{}^1$ is smaller than $\mathbf{Q}_1$. For $\mathbf{Q}_2{}^1$ to operate on $\mathbf{Q}_1\mathbf{A}$ instead of $\mathbf{A}^1$, $\mathbf{Q}_2{}^1$ needs to be expanded to the upper left, filling in a 1:

$$\mathbf{Q}_2 = \begin{pmatrix} \mathbf{I}_1 & 0 \\ 0 & \mathbf{Q}_2{}^1 \end{pmatrix},$$

or in general:

$$\mathbf{Q}_k = \begin{pmatrix} \mathbf{I}_{k-1} & 0 \\ 0 & \mathbf{Q}_k{}^1 \end{pmatrix}.$$

Then, $\mathbf{Q}_1\mathbf{A}$ is multiplied with the Householder matrix $\mathbf{Q}_2$ and results in a matrix $\mathbf{Q}_2\mathbf{Q}_1\mathbf{A}$. After t iterations of this process, t = min(m-1, n),

$$\mathbf{U} = \mathbf{Q}_t \cdots \mathbf{Q}_2\mathbf{Q}_1\mathbf{A}$$

is an upper triangular matrix. Hence, with

$$\mathbf{Q} = \mathbf{Q}_1{}^T\mathbf{Q}_2{}^T \cdots \mathbf{Q}_t{}^T$$

the QR factorization of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{QU}.$$

(Note that the standard designation of an upper triangular matrix is $\mathbf{R}$; however, we used $\mathbf{U}$ so as not to confuse the identity of the covariance matrix of the measurement error $\mathbf{R}$ in the earlier chapter.)

(http://en.wikipedia.org/wiki/QR_decomposition#Using_Householder_reflections)

The application of Householder methods can cause severe "intermediate" fill-ins (non-zero elements generated by transformation); these fill-ins will be annihilated eventually, but they can cause excessive storage and degrade the computation efficiency. In order to gain high efficiency, row and column ordering are adopted for Householder transformation during QR factorization to reduce intermediate fills.

3.3.2 Description of the orthogonal decomposition via Householder transformation algorithm

The idea of solving state estimator problem via orthogonal decomposition method started with elimination of matrix $\mathbf{R}^{-1}$ in the state estimation least-squares equation, eq. 2.45 as follow:

$$\mathbf{R}^{-1} = \mathbf{R}^{-1/2}\mathbf{R}^{-1/2} \tag{3.1}$$

where

$$\mathbf{R}^{-1/2} = \begin{bmatrix} \frac{1}{s_1} & & & \\ & \frac{1}{s_2} & & \\ & & \frac{1}{s_3} & \\ & & & \mathbf{O} \end{bmatrix}.$$

The gain matrix becomes

$$\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} = \mathbf{H}^T\mathbf{R}^{-1/2}\mathbf{R}^{-1/2}\mathbf{H}. \tag{3.2}$$

Thus, the state estimation least-squares equation becomes

$$\left[\mathbf{H}^T\mathbf{R}^{-1/2}\mathbf{R}^{-1/2}\mathbf{H}\right]\Delta\mathbf{x} = \left[\mathbf{H}^T\mathbf{R}^{-1/2}\mathbf{R}^{-1/2}\right]\left[\mathbf{z} - \mathbf{f}(\mathbf{x})\right]. \tag{3.3}$$

Eq. 3.3 can be written as a normal equation:

$$\left[\mathbf{A}^T\mathbf{A}\right]\Delta\mathbf{x} = \left[\mathbf{A}^T\mathbf{B}\right], \tag{3.4}$$

where

$$\mathbf{A} = \mathbf{R}^{-1/2}\mathbf{H} \quad \text{and} \quad \mathbf{B} = \mathbf{R}^{-1/2}\left[\mathbf{z} - \mathbf{f}(\mathbf{x})\right].$$

The idea of orthogonal decomposition algorithm is to perform the QR factorization on matrix $\mathbf{A}$. The QR Factorization block uses a sequence of successive Householder transformations to triangularize the input matrix **A**. The block factors a column permutation of the m-by-n input matrix **A** as

$$\mathbf{A} = \mathbf{QU}, \tag{3.5}$$

where the matrix $\mathbf{Q}$ is an orthogonal matrix and the matrix $\mathbf{U}$ is an unsquare upper triangular matrix since the Jacobian matrix $\mathbf{H}$ is not square. QR factorization is an important tool for solving nonlinear least-squares problem because of good error propagation properties and the invertability of unitary matrices:

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}, \tag{3.6}$$

where matrix $\mathbf{I}$ is the identity matrix, which to say that the transpose of $\mathbf{Q}$ is its inverse: .

$$\mathbf{Q}^T = \mathbf{Q}^{-1}. \tag{3.7}$$

Now, $\mathbf{A} = \mathbf{QU}$ is substituted into eq. 3.4:

$$\left[\mathbf{U}^T\mathbf{Q}^T\mathbf{QU}\right]\Delta\mathbf{x} = \left[\mathbf{U}^T\mathbf{Q}^T\right]\mathbf{B}. \tag{3.8}$$

Since $\mathbf{U}$ is nonsingular and $\mathbf{Q}$ is orthogonal, eq. 3.8 can be rewritten as follow:

$$\left[\mathbf{U}\right]\Delta\mathbf{x} = \left[\mathbf{Q}^T\right]\mathbf{B}. \tag{3.9}$$

Eq. 3.9 can be solved in two stages as follows:

$$\mathbf{y} = \mathbf{Q}^T\mathbf{B},$$

and $\quad \mathbf{U}\Delta\mathbf{x} = \mathbf{y}$.

3.3.3 The orthogonal decomposition algorithm

In summary, the application of the orthogonal decomposition method to power system state estimation results in the following algorithm:

Step 1: Read measurement and pick starting value for $\mathbf{x} = \mathbf{x}^0$.

Step 2: Form the residual matrix $\mathbf{z} - \mathbf{f}(\mathbf{x})$.

Step 3: Form the Jacobian matrix $\mathbf{H}$.

Step 4: Form $\mathbf{A} = \mathbf{R}^{-1/2}\mathbf{H}$ and $\mathbf{B} = \mathbf{R}^{-1/2}[\mathbf{z} - \mathbf{f}(\mathbf{x})]$.

Step 5: Factor $\mathbf{A}$ with the resulting QR factorization, $\mathbf{A} = \mathbf{QU}$.

Step 6: Compute the incremental change in $\mathbf{x}$ with one forward substitution and one backward substitution.

Continue this iterative process (Step2-6) until the absolute value of the difference between the consecutive increments of the state variables is less than a predetermined tolerance. Please refer to the Appendix C for the structure of the above matrices and vectors.

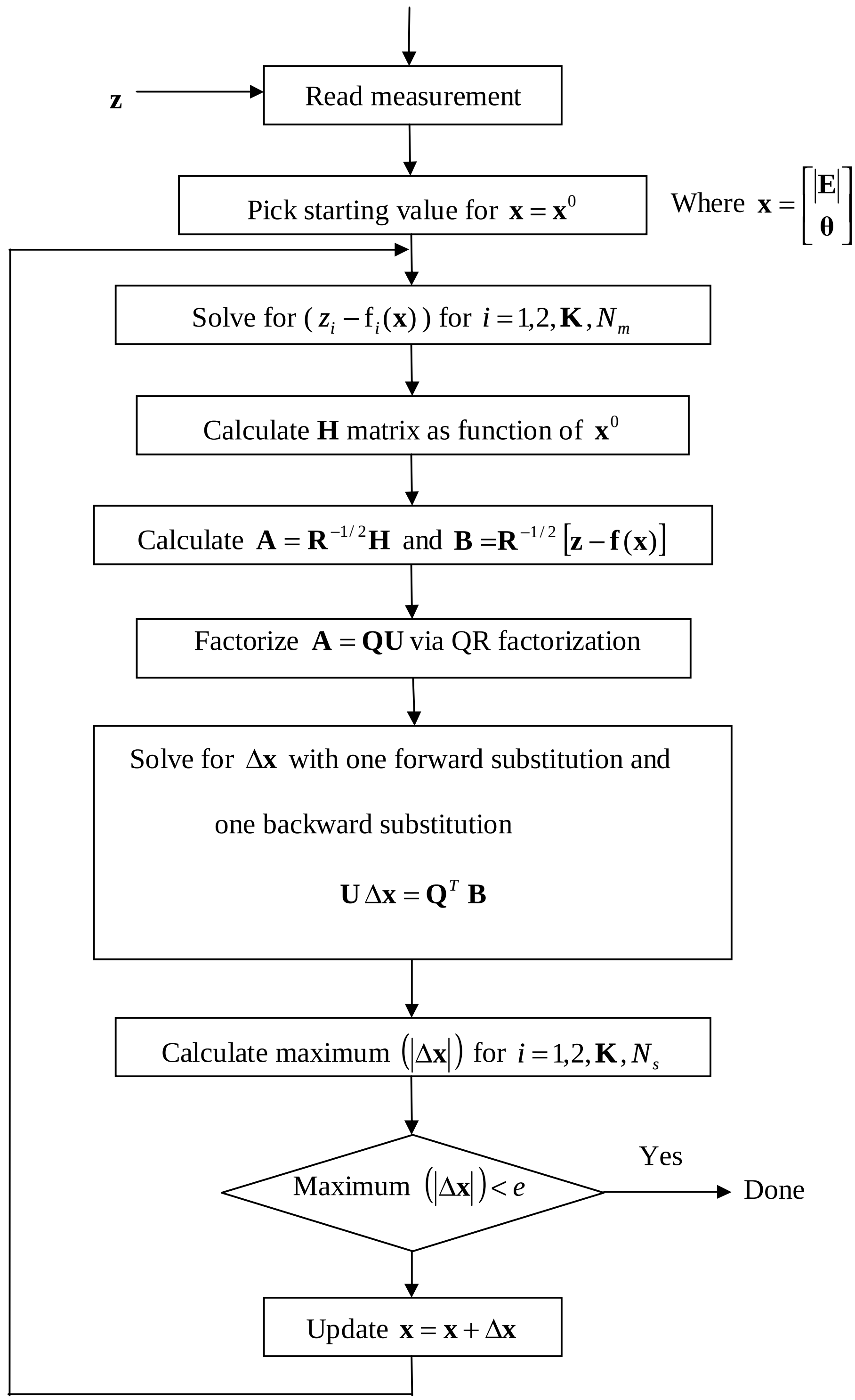

Figure 3.1: State estimation orthogonal decomposition algorithm