LINEAR SPACES AND PRESERVERS OF PERSYMMETRIC TRIANGULAR MATRICES OF BOUNDED RANK-TWO

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ABSTRACT

Let $\mathbb{F}$ be a field and n an integer $\geqslant 2$. We say that a square matrix A is persymmetric if A is symmetric in the second diagonal. Let $\mathcal{ST}_n(\mathbb{F})$ denote the linear space of all $n \times n$ persymmetric upper triangular matrices over $\mathbb{F}$. A subspace S of $\mathcal{ST}_n(\mathbb{F})$ is said to be a space of bounded rank-two matrices if each matrix in S has rank bounded above by two, and a rank-two space if each nonzero element in it has rank two. In this dissertation, we classify subspaces of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$ over a field $\mathbb{F}$ with at least three elements. As a corollary, a complete description of rank-two subspaces of $\mathcal{ST}_n(\mathbb{F})$ is obtained. We next deduce from the structural results of subspaces of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$, a characterization of linear maps $\phi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$, $m \geqslant n \geqslant 2$, that send nonzero matrices with rank at most two to nonzero matrices with rank at most two.

ABSTRAK

Katakan $\mathbb{F}$ adalah medan dan n adalah integer $\geqslant 2$. Suatu matriks segiempat sama A dikatakan persimetri jika A adalah simetri pada pepenjuru yang kedua. Biar $\mathcal{ST}_n(\mathbb{F})$ menandakan ruang linear yang terdiri daripada semua matriks persimetri segitiga atas jenis $n \times n$ terhadap $\mathbb{F}$. Suatu subruang S bagi $\mathcal{ST}_n(\mathbb{F})$ dikenali sebagai ruang matriks disempadani pangkat-dua jika setiap matriks dalam S mempunyai pangkat yang disempadani atas oleh dua, dan dikenali sebagai ruang pangkat-dua jika setiap unsur bukan sifar mempunyai pangkat dua. Dalam disertasi ini, kami mencirikan subruang matriks disempadani pangkat-dua bagi $\mathcal{ST}_n(\mathbb{F})$ terhadap medan $\mathbb{F}$ yang mempunyai sekurang-kurangnya tiga unsur. Sebagai korolari, pencirian lengkap tentang subruang pangkat-dua bagi $\mathcal{ST}_n(\mathbb{F})$ telah diperolehi. Seterusnya, daripada hasil struktur subruang matriks disempadani pangkat-dua bagi $\mathcal{ST}_n(\mathbb{F})$, kami dapat mendeduksikan suatu pencirian pemetaan linear $\phi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$, $m \geqslant n \geqslant 2$, yang menghantar matriks bukan sifar dengan pangkat selebih-lebihnya dua kepada matriks bukan sifar dengan pangkat selebih-lebihnya dua.

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INTRODUCTION

One of the most active research topics in matrix theory and operator theory in the last century is the linear preserver problem. Linear preserver problems concern the characterization of linear maps between matrix spaces leaving invariant certain functions, subsets, or relations. A solution of a linear preserver problem consists of a structural description of the preserver. One well-known result, dating back over one hundred years, is the Frobenius's classical theorem see [5], concerning determinant preservers on $\mathcal{M}_n(\mathbb{C})$ (the algebra of $n \times n$ matrices over the complex field $\mathbb{C}$). He proved that every linear map ϕ on $\mathcal{M}_n(\mathbb{C})$ satisfying

$$det(\phi(A)) = det(A) \quad \text{for all } A \in \mathcal{M}_n(\mathbb{C}),$$

is either of the form

$$\phi(A) = MAN \quad \text{for all } A \in \mathcal{M}_n(\mathbb{C}) \tag{1}$$

or

$$\phi(A) = MA^tN \quad \text{for all } A \in \mathcal{M}_n(\mathbb{C}) \tag{2}$$

for some invertible matrices $M, N \in \mathcal{M}_n(\mathbb{C})$ with $det(MN) = 1$. Here and throughout the dissertation, A^t denotes the transpose of A. The mappings ϕ above are examples of function preservers.

In general, let $\mathcal{M}$ be a matrix space or a tensor space. Typically, there are four types of linear preserver problem:

(I) (Function-preserving) Let F be a (scalar-valued, vector-valued, or set-valued) given function on $\mathcal{M}$. Characterize those linear maps ϕ on $\mathcal{M}$ which satisfy

$$F(\phi(A)) = F(A) \quad \text{for all } A \in \mathcal{M}.$$

(II) (Subset-preserving) Let $\mathcal{S}$ be a given subset of $\mathcal{M}$. Characterize those linear maps ϕ on $\mathcal{M}$ which satisfy

$$A \in \mathcal{S} \ \Rightarrow \ \phi(A) \in \mathcal{S} \quad \text{for all } A \in \mathcal{M}$$

or satisfy

$$A \in \mathcal{S} \ \Leftrightarrow \ \phi(A) \in \mathcal{S} \quad \text{for all } A \in \mathcal{M}.$$

(III) (Relation-preserving) Let $\sim$ be a relation on $\mathcal{M}$. Characterize those linear maps ϕ on $\mathcal{M}$ which satisfy

$$A \sim B \Rightarrow \phi(A) \sim \phi(B) \quad \text{for all } A, B \in \mathcal{M}$$

or satisfy

$$A \sim B \Leftrightarrow \phi(A) \sim \phi(B) \quad \text{for all } A, B \in \mathcal{M}.$$

(IV) (Function-commuting) Let $F : \mathcal{M} \to \mathcal{M}$ be a function. Characterize those linear maps ϕ on $\mathcal{M}$ which satisfy

$$\phi \circ F = F \circ \phi$$

that is, $F(\phi(A)) = \phi(F(A))$ for all $A \in \mathcal{M}$.

One may see that the formulation of linear preserver problems is simple and natural. The answers are often very elegant. In the last few decades, thousands of papers have been published on linear preservers and many interesting results have been obtained. We note that not all linear maps with a special preserving property have the standard forms (1) or (2). For example, Hiai [6] proved that a linear map ϕ on $\mathcal{M}_n(\mathbb{C})$ preserves the relation of similarity if and only if there exist $\alpha, \beta \in \mathbb{C}$ and an invertible $S \in \mathcal{M}_n(\mathbb{C})$ such that ϕ has the form

$$A \mapsto S^{-1}AS + \alpha(trA)I_n \quad \text{or} \quad A \mapsto S^{-1}A^tS + \beta(trA)I_n,$$

or there exists a fixed $B \in \mathcal{M}_n(\mathbb{C})$ such that ϕ has the form

$$A \mapsto (trA)B.$$

Here, A^t denotes the transpose of A. Furthermore, in the 1990s, Pierce and other researchers wrote a monograph [12] which can be viewed as a summary of the results on linear preserver problems ranging from 1897 to 1991.

Without restricting themselves to linear maps acting on the same space, some researchers started to consider linear maps between different matrix spaces, i.e. $\phi : \mathcal{M} \to \mathcal{M}'$ for some matrix spaces $\mathcal{M}, \mathcal{M}'$. Such problems are more challenging and their study might lead to the discovery of hidden structures due to the differences between the two spaces, and hence to a generalization of the results for the case $\mathcal{M} = \mathcal{M}'$. However, it is usually hard to obtain new structures. For example, the characterization of additive rank-one preservers from the space of triangular matrices to the space of rectangular matrices was obtained in [2] while the characterization of additive rank-one preservers between the spaces of rectangular matrices of different sizes was obtained in [14]. It turns out that the structure for additive rank one preservers on spaces of triangular matrices is much more complicated than the one on spaces of rectangular matrices.

Linear preserver problems can be divided into many subcategories. It is impossible to cover all of them in this section. In the following, we give only a brief survey of those results related to the title of this dissertation.

Research on rank linear preservers was carried out by Marcus and Moyls [9]. They described the structure of rank-one linear preservers on $\mathcal{M}_{m,n}(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic 0. The preservers have the standard form (1) or, when $m = n$, possibly the standard form (2) with invertible matrices M, N of suitable sizes. The authors proved the above result in the setting of tensor spaces whereas Minc [10] gave an alternative proof using only elementary matrix theory fifteen years later.

Back in the 1970s, L. J. Cummings [3] characterized the maximal decomposable subspaces of the k^{th} symmetric product space $\bigvee_k V$, where V is a finite-dimensional vector space over an algebraically closed field of characteristic 0. In particular, when $k = 2$, the result is closely related to subspaces of symmetric matrix spaces of bounded rank-two only if the field has characteristic not equal to two. Later, he used the structure obtained for decomposable subspaces and some lemmas from [3] to investigate linear transformations on the k-fold symmetric product of an n-dimensional vector space V, $n > k + 1$, which carry nonzero decomposable tensors to nonzero decomposable tensors, see [4].

Meanwhile, M. H. Lim [8] studied the structure of linear maps on the vector space of all $n \times n$ symmetric matrices preserving matrices of rank one, two or n by using the tools of second symmetric product spaces. The paper [8] also contains a characterization of subspaces of symmetric tensors of order two consisting of elements of rank less than or equal to two.

Rank non-increasing linear maps first appeared in [7]. The author employed the language of tensors to obtain a result which was then used to obtain the structure of those rank-one non-increasing linear maps on $\mathcal{M}_{m,n}(\mathbb{F})$. Then in the late 90's, research on certain preserver problems on spaces of upper triangular matrices was carried out by W.L. Chooi, M.H. Lim [1] and L. Molnár, P. Šemrl [11]. They first classify rank-one linear preserver structures and then make use of these structures to classify other preserver problems on triangular spaces, specifically the adjugate commuting problem and the rank-one idempotent preserver problem.

Motivated by all these results, we carry out, in this dissertation, a study on bounded rank-two linear preservers on persymmetric upper triangular spaces. We now give the basic notations and definitions needed to describe our work.

Let $\mathbb{F}$ be a field and let m, n be integers $\geqslant 2$. Let $\mathcal{M}_{m,n}(\mathbb{F})$ be the linear space of all $m \times n$ matrices over $\mathbb{F}$. We abbreviate $\mathcal{M}_{n,n}(\mathbb{F})$ to $\mathcal{M}_n(\mathbb{F})$. Let $A = (a_{ij}) \in \mathcal{M}_{m,n}(\mathbb{F})$. We denote by A^+ the matrix $(b_{ij}) \in \mathcal{M}_{n,m}(\mathbb{F})$ such that

$$b_{ij} = a_{n+1-j,m+1-i} \quad \text{for every} \quad 1 \leqslant i \leqslant n \quad \text{and} \quad 1 \leqslant j \leqslant m.$$

We see that $A^+ = J_n A^t J_m$ where J_n is the $n \times n$ matrix with $1's$ on the second diagonal and $0's$ elsewhere, and A^t stands for the transpose of A. We say that a square matrix $A \in \mathcal{M}_n(\mathbb{F})$ is *persymmetric* if it is symmetric in the second diagonal, that is, $A^+ = A$. Let $\mathcal{T}_n(\mathbb{F})$ denote the subspace of $\mathcal{M}_n(\mathbb{F})$ consisting of all $n \times n$ upper triangular matrices. We denote by $\mathcal{ST}_n(\mathbb{F})$ the set of all $n \times n$ persymmetric upper triangular matrices over $\mathbb{F}$. Symbolically,

$$\mathcal{ST}_n(\mathbb{F}) := \{A \in \mathcal{T}_n(\mathbb{F}) \mid A^+ = A\}.$$

Clearly, $\mathcal{ST}_n(\mathbb{F})$ forms a subspace of $\mathcal{T}_n(\mathbb{F})$; we shall call it the *persymmetric triangular matrix space* over $\mathbb{F}$.

Now, let U and V be subspaces of $\mathcal{M}_n(\mathbb{F})$ and $\mathcal{M}_m(\mathbb{F})$, respectively, and let k be a positive integer with $k \leqslant \min\{m, n\}$. A mapping $\phi : U \to V$ is said to be a *bounded rank-k linear preserver* if ϕ is linear and satisfies

$$1 \leqslant \operatorname{rank} \phi(A) \leqslant k \quad \text{whenever} \quad 1 \leqslant \operatorname{rank} A \leqslant k$$

where $\operatorname{rank} A$ denotes the rank of the matrix A. Let S be a subspace of U. We say that S is a *subspace of bounded rank-k matrices* if each matrix in S has rank bounded above by k, and that S is a *rank-k subspace* if each nonzero element in it has rank k.

We assume throughout this dissertation, unless otherwise stated, that $\mathbb{F}$ is an arbitrary field and $n \geqslant 2$. We use $\langle u_1, \dots, u_r \rangle$ to denote the linear space spanned by the vectors $u_1, \dots, u_r$. As usual, we denote the vectors of the standard basis of $\mathcal{M}_{n,1}(\mathbb{F})$ by $\{e_1, \dots, e_n\}$ and employ the notation $E_{ij} := e_i \cdot e_j^t$ to denote the matrix in $\mathcal{M}_n(\mathbb{F})$ having the (i, j)-th entry equal to one and all others equal to zero.

This dissertation is divided into three chapters. We now give a brief description of each chapter.

Chapter 1 provides all the notations needed in this dissertation. We then describe the rank canonical form of a persymmetric upper triangular matrix. The result enables us to define a tensor for symmetric matrices and this definition holds even for a field with characteristic two. Some properties of the new tensor can also be found in this chapter.

In Chapter 2, we characterize subspaces of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$. Consequently, a characterization of rank-two subspaces of $\mathcal{ST}_n(\mathbb{F})$ is obtained. As shown in [4, 8], the characterization of subspaces of bounded rank-two matrices is essential to a study of bounded rank-two linear preservers.

Chapter 3 carries out a study of bounded rank-two linear preservers between persymmetric triangular matrix spaces. We see from the results that if the dimension of the domain is very small, then we obtain some surprisingly odd structures. Besides the persymmetric and the upper triangular properties, the structures appear mainly due to the difference of dimensions between two linear spaces and there is one special form when the underlying field has four elements.

Chapter 1

PRELIMINARIES

We begin this chapter by introducing the rank canonical form of an arbitrary matrix in $\mathcal{ST}_n(\mathbb{F})$. If we consider only those matrices with rank at most two, then the rank canonical forms enable us to define a new tensor product as shown in Lemma 1.2. This allows us to establish a link between a matrix with rank at most two in $\mathcal{ST}_n(\mathbb{F})$ and a tensor product of two vectors from $\mathcal{M}_{n,1}(\mathbb{F})$. Although the representation of the matrix under this new tensor is not unique, it proves to be very useful in the construction of our subspaces of bounded rank-two matrices, as shown in the next chapter.

Let $\mathbb{F}$ be a field and let n be an integer $\geqslant 2$. Let $\alpha \in \mathbb{F}$, and let $1 \leqslant i, j \leqslant n$ be integers such that $1 \leqslant i \leqslant j < n+1-i$. We define

$$Z_{ij}^{\alpha} := E_{ij} + E_{ij}^{+} + \alpha E_{i,n+1-i} \tag{1.1}$$

and $Z_{ij} := Z_{ij}^{0}$. For example, if $n = 4$, then

$$Z_{11}^{\alpha} = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Z_{13}^{\alpha} = \begin{pmatrix} 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Z_{22}^{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $Z_{ij}^{\alpha} \in \mathcal{ST}_n(\mathbb{F})$ for every $\alpha \in \mathbb{F}$, $1 \leqslant i \leqslant j < n+1-i \leqslant n$. We begin our work by finding the rank canonical forms of the vectors in $\mathcal{ST}_n(\mathbb{F})$.

Lemma 1.1. *Let $\mathbb{F}$ be an arbitrary field, and let n, k be integers such that $n \geqslant 2$*

and $0 \leqslant k \leqslant n$. *Then* $A \in \mathcal{ST}_n(\mathbb{F})$ *is of rank* k *if and only if there exist an integer* $0 \leqslant h \leqslant \frac{k}{2}$, *scalars* $\alpha_1, \ldots, \alpha_h \in \mathbb{F}$, *nonzero scalars* $\beta_{2h+1}, \ldots, \beta_k \in \mathbb{F}$, *and an invertible matrix* $P \in \mathcal{T}_n(\mathbb{F})$ *such that*

$$A = P\left(\sum_{i=1}^{h} Z_{s_i t_i}^{\alpha_i} + \sum_{i=2h+1}^{k} \beta_i E_{p_i, n+1-p_i} \right) P^+$$

where $\{s_1, \ldots, s_h, p_{2h+1}, \ldots, p_k\}$ *and* $\{t_1, \ldots, t_h, n+1-p_{2h+1}, \ldots, n+1-p_k\}$ *are two sets of distinct positive integers such that* $1 \leqslant s_i \leqslant t_i < n+1-s_i$ *for* $i = 1, \ldots, h$, *and* $1 \leqslant p_i \leqslant \frac{n+1}{2}$ *for* $i = 2h+1, \ldots, k$. *Moreover, if there exists an integer* $1 \leqslant j \leqslant h$ *such that* $\alpha_j \neq 0$, *then* $\mathbb{F}$ *has characteristic* 2.

Proof. The sufficiency part is clear. We now prove the necessity part. Suppose that $A = (a_{ij})$ is nonzero. Let R_i and C_j denote the i-th row and the j-th column of A, respectively. Since $0 \neq A \in \mathcal{ST}_n(\mathbb{F})$, there exists a pair of integers (i_0, j_0) with $1 \leqslant i_0 \leqslant j_0 \leqslant n+1-i_0$ such that

$$a_{i_0,j_0} \neq 0$$

and $a_{i,j_0} = 0$ for all $i_0 < i \leqslant n$, and $a_{ij} = 0$ for all $1 \leqslant j < j_0$ and $1 \leqslant i \leqslant n$. We divide our proof into the following two cases:

Case I: $j_0 = n+1-i_0$. For each $1 \leqslant s < i_0$, we apply the following elementary row and column operations:

$$R_s \to R_s - a_{s,j_0} a_{i_0,j_0}^{-1} R_{i_0} \quad \text{and} \quad C_{n+1-s} \to C_{n+1-s} - a_{i_0,n+1-s} a_{i_0,j_0}^{-1} C_{j_0}$$

on A. We note that, for each $1 \leqslant s < i_0$, there exists the elementary matrix $I_n - c_s E_{s,i_0} \in \mathcal{T}_n(\mathbb{F})$ corresponding to the the row operation $R_s \to R_s - c_s R_{i_0}$, where $c_s = a_{s,j_0} a_{i_0,j_0}^{-1} \in \mathbb{F}$. Since $A^+ = A$, it follows that $a_{i_0,n+1-s} = a_{s,j_0}$ for every $1 \leqslant s < i_0$, and so, there exists an invertible matrix $P_1 \in \mathcal{T}_n(\mathbb{F})$ such that

$$P_1 A P_1^+ = a_{i_0,j_0} E_{i_0,j_0} + A_1 \tag{1.2}$$

for some matrix $A_1 = (b_{ij}) \in \mathcal{ST}_n(\mathbb{F})$ such that $b_{i,j_0} = 0$ for all $1 \leqslant i \leqslant n$, $b_{i_0,j} = 0$ for all $1 \leqslant j \leqslant n$, and $b_{ij} = 0$ for all $1 \leqslant j < j_0$ and $1 \leqslant i \leqslant n$.

Case II: $j_0 \neq n+1-i_0$. Then $a_{n+1-j_0,n+1-i_0} = a_{i_0,j_0} \neq 0$. Without loss of generality, we may assume $a_{i_0,j_0} = 1 = a_{n+1-j_0,n+1-i_0}$. For each $1 \leqslant s < i_0$, we apply the elementary row and column operations:

$$R_s \to R_s - a_{s,j_0} R_{i_0} \quad \text{and} \quad C_{n+1-s} \to C_{n+1-s} - a_{n+1-j_0,n+1-s} C_{n+1-i_0}$$

on A, and this is followed by the following elementary row and column operations:

$$R_t \to R_t - a_{t,n+1-i_0} R_{n+1-j_0} \quad \text{and} \quad C_{n+1-t} \to C_{n+1-t} - a_{i_0,n+1-t} C_{j_0}$$

for every $1 \leqslant t < n+1-j_0$. We note that, for each $1 \leqslant s < i_0$ (respectively, for each $1 \leqslant t < n+1-j_0$), there exists the elementary matrix $I_n - a_{s,j_0} E_{s,i_0}$ (respectively, $I_n - a_{t,n+1-i_0} E_{t,n+1-j_0}$) in $\mathcal{T}_n(\mathbb{F})$ corresponding to the row operation $R_s \to R_s - a_{s,j_0} R_{i_0}$ (respectively, $R_t \to R_t - a_{t,n+1-i_0} R_{n+1-j_0}$). Since $a_{n+1-j_0,n+1-s} = a_{s,j_0}$ for every $1 \leqslant s < i_0$, and $a_{i_0,n+1-t} = a_{t,n+1-i_0}$ for every $1 \leqslant t < n+1-j_0$, there exists an invertible matrix $P_2 \in \mathcal{T}_n(\mathbb{F})$ such that

$$P_2 A P_2^+ = Z_{i_0 j_0}^{\alpha_1} + A_1 \tag{1.3}$$

for some scalar $\alpha_1 \in \mathbb{F}$ and some matrix $A_1 = (b_{ij}) \in \mathcal{ST}_n(\mathbb{F})$ such that $b_{i,j_0} = 0$ for all $1 \leqslant i \leqslant n$, $b_{i_0,j} = 0$ for $1 \leqslant j \leqslant n$, and $b_{ij} = 0$ for all $1 \leqslant j < j_0$ and $1 \leqslant i \leqslant n$.

In view of (1.2) and (1.3), if $A_1 = 0$, then we are done. If $A_1 \neq 0$, then, by repeating a similar argument on A_1, since A is of rank k, there exist an integer $0 \leqslant h \leqslant \frac{k}{2}$, scalars $\alpha_1, \ldots, \alpha_h, \beta_{2h+1}, \ldots, \beta_k \in \mathbb{F}$, and an invertible matrix $Q \in \mathcal{T}_n(\mathbb{F})$ such that

$$QAQ^+ = \sum_{i=1}^{h} Z_{s_i t_i}^{\alpha_i} + \sum_{i=2h+1}^{k} \beta_i E_{p_i, n+1-p_i} \tag{1.4}$$

where $\{s_1, \ldots, s_h, p_{2h+1}, \ldots, p_k\}$ and $\{t_1, \ldots, t_h, n+1-p_{2h+1}, \ldots, n+1-p_k\}$ are two sets of distinct positive integers such that $1 \leqslant s_i \leqslant t_i < n+1-s_i$ for $i = 1, \ldots, h$, and $1 \leqslant p_i \leqslant \frac{n+1}{2}$ for $i = 2h+1, \ldots, k$. If $\mathbb{F}$ has characteristic 2, then the proof is complete. If $\mathbb{F}$ has characteristic not 2, then, for each $1 \leqslant i \leqslant h$, we further perform the elementary row and column operations:

$$R_{s_i} \to R_{s_i} - 2^{-1}\alpha_i R_{n+1-t_i} \quad \text{and} \quad C_{n+1-s_i} \to C_{n+1-s_i} - 2^{-1}\alpha_i C_{t_i}$$

on QAQ^+ in (1.4) to annihilate α_i in $Z_{s_i t_i}^{\alpha_i}$. Since $s_i < n+1-t_i$ for all $1 \leqslant i \leqslant h$, there exists an invertible $P \in \mathcal{T}_n(\mathbb{F})$ such that

$$PAP^+ = \sum_{i=1}^{h} Z_{s_i t_i} + \sum_{i=2h+1}^{k} \beta_i E_{p_i, n+1-p_i}.$$

The proof is complete. □

Let $\mathbb{F}$ be a field and let n be an integer $\geqslant 2$. Let $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$. Define

$$u \oslash v := u \cdot v^+ + v \cdot u^+ \qquad \text{and} \qquad u^2 := u \cdot u^+ \tag{1.5}$$

where $u \cdot v^+ \in \mathcal{M}_n(\mathbb{F})$ is the usual matrix product of the vectors $u \in \mathcal{M}_{n,1}(\mathbb{F})$ and

$v^+ \in \mathbb{F}^n$. We verify easily from (1.5) that $(u,v) \mapsto u \oslash v$ is a symmetric bilinear map from $\mathcal{M}_{n,1}(\mathbb{F}) \times \mathcal{M}_{n,1}(\mathbb{F})$ into $\mathcal{M}_n(\mathbb{F})$. Also, for each $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$, we have

$$(u \oslash v)^+ = u \oslash v \quad \text{and} \quad (u^2)^+ = u^2$$

and

$$P(u \oslash v)P^+ = (Pu) \oslash (Pv) \quad \text{and} \quad P(u^2)P^+ = (Pu)^2 \tag{1.6}$$

for every $P \in \mathcal{M}_n(\mathbb{F})$. It is easy to see that $e_i \oslash e_j = E_{i,n+1-j} + E_{j,n+1-i}$ and $e_i^2 = E_{i,n+1-i}$ for every pair of integers $1 \leqslant i, j \leqslant n$. It can immediately be seen from (1.5) that $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ are linearly independent vectors if and only if each of the matrices $u \oslash v$, $u \oslash v + u^2$ and $u^2 + v^2$ is of rank two.

Now, let $1 \leqslant i \leqslant n$. We write

$$\mathcal{U}_i := \left\{ (u_1, \ldots, u_i, 0, \ldots, 0)^T \,\middle|\, u_1, \ldots, u_i \in \mathbb{F} \right\}.$$

Let $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ and $w \in \mathcal{U}_r$ for some positive integers $1 \leqslant p, q, r \leqslant n$. It is immediate from the definitions in (1.5) that $u^2 \in \mathcal{ST}_n(\mathbb{F})$ if and only if $1 \leqslant p \leqslant \frac{n+1}{2}$, and $v \oslash w \in \mathcal{ST}_n(\mathbb{F})$ when $1 \leqslant q \leqslant n+1-r \leqslant n$.

The following lemma allows us to express the matrices in $\mathcal{ST}_n(\mathbb{F})$ in tensor language.

Lemma 1.2. *Let $\mathbb{F}$ be a field and let n be an integer $\geqslant 2$. The following statements hold.*

(a) *Then $A \in \mathcal{ST}_n(\mathbb{F})$ with $0 \leqslant \operatorname{rank} A \leqslant 2$ if and only if A is of one of the following forms:*

$$A = \alpha u^2 + \beta v^2 \tag{1.7}$$

for some linearly independent vectors $u, v \in \mathcal{U}_p$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and scalars* $\alpha, \beta \in \mathbb{F}$*; or*

$$A = u \oslash v + \lambda u^2 \tag{1.8}$$

for some linearly independent vectors $u \in \mathcal{U}_p$ *and* $v \in \mathcal{U}_q$ *with* $1 \leqslant p \leqslant n+1-q < n+1-p$ *and scalar* $\lambda \in \mathbb{F}$. *Further, if* A *is of form* (1.8) *with* $\lambda \neq 0$, *then* $\mathbb{F}$ *has characteristic two.*

(b) *Let* $A = u \oslash v + \lambda u^2$ *for some linearly independent vectors* $u \in \mathcal{U}_p$ *and* $v \in \mathcal{U}_q$ *with* $1 \leqslant p < q \leqslant \frac{n+1}{2}$ *and scalar* $\lambda \in \mathbb{F}$. *If* $\operatorname{char}\mathbb{F} \neq 2$ *or* $\operatorname{char}\mathbb{F} = 2$ *with* $\lambda \neq 0$, *then* A *can be rewritten in form* (1.7), *i.e.,*

$$A = \mu w^2 + \eta z^2$$

for some linearly independent vectors $w, z \in \mathcal{U}_q$ *and nonzero scalars* $\mu, \eta \in \mathbb{F}$.

Proof. (a) The sufficiency part is clear. We now consider the necessity part. Let $A \in \mathcal{ST}_n(\mathbb{F})$ be a matrix of bounded rank-two. In view of Lemma 1.1, there exists an invertible matrix $P \in \mathcal{T}_n(\mathbb{F})$ such that either

$$A = P(\alpha E_{p,n+1-p} + \beta E_{q,n+1-q})P^+ \tag{1.9}$$

for some scalars $\alpha, \beta \in \mathbb{F}$ and distinct integers $1 \leqslant q < p \leqslant \frac{n+1}{2}$; or

$$A = PZ_{pq}^{\alpha}P^+ = P(E_{pq} + E_{pq}^+ + \lambda E_{p,n+1-p})P^+ \tag{1.10}$$

for some integers $1 \leqslant p \leqslant q < n+1-p$ and some scalar $\lambda \in \mathbb{F}$ such that $\lambda \neq 0$ implies $\operatorname{char}\mathbb{F} = 2$. If A is of form (1.9), then $0 \leqslant \operatorname{rank} A \leqslant 2$, and $A = \alpha P(e_p^2)P^+ + \beta P(e_q^2)P^+ = \alpha u^2 + \beta v^2$, where $u = Pe_p$ and $v = Pe_q$ are linearly independent vectors in $\mathcal{U}_p$. If A is of form (1.10), then $\operatorname{rank} A = 2$, and $A = P(e_p \oslash e_{n+1-q})P^+ + \lambda P(e_p^2)P^+ = u \oslash v + \lambda u^2$ by (1.6), where $u = Pe_p \in \mathcal{U}_p$ and $v = Pe_{n+1-q} \in \mathcal{U}_{n+1-q}$ are linearly independent vectors. We are done.

(b) We divide into two cases:

Case I: $\lambda \neq 0$. Then $A = u \oslash v + \lambda u^2 = \lambda(u + \lambda^{-1}v)^2 + (-\lambda^{-1})v^2 = \lambda w^2 + (-\lambda^{-1})z^2$, where $w = u + \lambda^{-1}v$ and $z = v$ are linearly independent vectors in $\mathcal{U}_q$.

Case II: $\lambda = 0$. Then $A = u \oslash v = P_1(e_p \oslash e_q)P_1^+$ for some invertible matrix $P_1 \in \mathcal{T}_n(\mathbb{F})$. If $\operatorname{char} \mathbb{F} \neq 2$, then we perform the following elementary row and column operations on $e_p \oslash e_q$:

$$R_p \to R_p + R_q \quad \text{and} \quad C_{n+1-p} \to C_{n+1-p} + C_{n+1-q}.$$

Then there is an invertible matrix $P_2 \in \mathcal{T}_n(\mathbb{F})$ such that

$$A = (P_1P_2)(e_p \oslash e_q + 2e_p^2)(P_1P_2)^+ = (P_1P_2e_p) \oslash (P_1P_2e_q) + 2(P_1P_2e_p)^2 = x \oslash y + 2x^2$$

by (1.6), where $x = P_1P_2e_p$ and $y = P_1P_2e_q$ are linearly independent vectors in $\mathcal{U}_q$. By a similar argument as in Case I, we are done. $\square$

The following two lemmas illustrate some properties of the new tensor product.

Lemma 1.3. *Let $\mathbb{F}$ be a field and let n be an integer $\geqslant 2$. Let $u, v, x, y \in \mathcal{M}_{n,1}(\mathbb{F})$ and let $\alpha, \beta, \gamma, \mu, \eta \in \mathbb{F}$ be scalars such that $\mu, \eta \neq 0$. Then the following assertions hold.*

(a) *If $u \oslash v + \alpha u^2 = x \oslash y + \beta x^2 \neq 0$, then $\langle u, v\rangle = \langle x, y\rangle$.*

(b) *If $u \oslash v + \alpha u^2 = \mu x^2 + \eta y^2 \neq 0$, then $\langle u, v\rangle = \langle x, y\rangle$.*

(c) $0 \neq u \oslash v \in \mathcal{ST}_n(\mathbb{F})$ *if and only if either*

(i) *$u \in \mathcal{U}_p$ and $v \in \mathcal{U}_q$ for some integers $1 \leqslant p \leqslant n+1-q \leqslant n$, or*

(ii) *$u \in \mathcal{U}_q$ and $v = \alpha u + w \in \mathcal{U}_q$ for some nonzero scalar $\alpha \in \mathbb{F}$ and some vector $w \in \mathcal{U}_p$ with $1 \leqslant p \leqslant n+1-q < \frac{n+1}{2}$ such that u, w are linearly independent.*

Furthermore, if (ii) *holds, then $\mathbb{F}$ has characteristic* 2.

(d) *Suppose that $\mathbb{F}$ has characteristic 2, and that u, v, y are linearly independent vectors. Let $A = \alpha(u \oslash v) + \beta(u \oslash y) + \mu(v \oslash y)$ and $B = u \oslash x + \gamma u^2$. Then $\gamma = 0$ and $x \in \langle u, v, y\rangle$ if and only if $\operatorname{rank}(A + \lambda B) \leqslant 2$ for every nonzero scalar $\lambda \in \mathbb{F}$.*

Proof. (a) We divide our proof into two cases.

Case I: u, v are linearly dependent. Then $\operatorname{rank}(u \oslash v + \alpha u^2) = 1$, and so x, y are linearly dependent. Evidently, u, x are nonzero. It follows that $v = \lambda_1 u$ and $y = \lambda_2 x$ for some scalars $\lambda_1, \lambda_2 \in \mathbb{F}$, and hence $(2\lambda_2 + \beta)x^2 = (2\lambda_1 + \alpha)u^2 \neq 0$. Therefore, $\langle x\rangle = \langle u\rangle$, and the result holds.

Case II: u, v are linearly independent. Then x, y are linearly independent. We claim that $\langle u, v\rangle = \langle x, y\rangle$. Suppose to the contrary that $\langle u, v\rangle \neq \langle x, y\rangle$. Then $x \notin \langle u, v\rangle$ or $y \notin \langle u, v\rangle$.

Case II-1: $x \notin \langle u, v\rangle$. Then $y \in \langle u, v, y\rangle$. Let $y = a'u + b'v + c'y$ for some $a', b', c' \in \mathbb{F}$. So,

$$0 = u \oslash v + \alpha u^2 - x \oslash y - \beta x^2 = u \oslash v + \alpha u^2 - a'x \oslash u - b'x \oslash v - (\beta + 2c')x^2. \quad (1.11)$$

We extend $\{x, u, v\}$ to an ordered basis $\mathscr{B}$ of $\mathcal{M}_{n,1}(\mathbb{F})$. Let $P \in \mathcal{M}_n(\mathbb{F})$ be the transition matrix obtained from $\mathscr{B}$ to the standard ordered basis of $\mathcal{M}_{n,1}(\mathbb{F})$. From (1.11), we see that $P(u \oslash v + \alpha u^2 - a'x \oslash u - b'x \oslash v - (\beta + 2c')x^2)P^{+} = 0$, and by

(1.6), we obtain

$$\begin{pmatrix} -b' & -a' & -(\beta+2c') \\ 1 & \alpha & -a' \\ 0 & 1 & -b' \end{pmatrix} = 0$$

an impossibility.

Case II-2: $y \notin \langle u, v\rangle$. Then $x \in \langle u, v, y\rangle$. We write $x = au + bv + cy$ for some scalars $a, b, c \in \mathbb{F}$. So, $-u \oslash v - \alpha u^2 + x \oslash y + \beta x^2 = 0$ implies that

$$(\beta ab - 1)u \oslash v - (\alpha - a^2\beta)u^2 + b^2\beta v^2 + a(1+\beta c)u \oslash y + b(1+\beta c)v \oslash y + c^2\beta y^2 = 0.$$

We extend $\{y, u, v\}$ to an ordered basis $\mathscr{C}$ of $\mathcal{M}_{n,1}(\mathbb{F})$. Let $Q \in \mathcal{M}_n(\mathbb{F})$ be the transition matrix obtained from $\mathscr{C}$ to the standard ordered basis of $\mathcal{M}_{n,1}(\mathbb{F})$. Then $Q((\beta ab-1)u\oslash v-(\alpha-a^2\beta)u^2+b^2\beta v^2+a(1+\beta c)u\oslash y+b(1+\beta c)v\oslash y+c^2\beta y^2)Q^+ = 0$, and by (1.6), we have

$$\begin{pmatrix} b(1+\beta c) & a(1+\beta c) & c^2\beta \\ \beta ab - 1 & a^2\beta - \alpha & a(1+\beta c) \\ b^2\beta & \beta ab - 1 & b(1+\beta c) \end{pmatrix} = 0.$$

Note that $\beta ab - 1 = 0$. If $\beta = 0$, then $1 = 0$, a contradiction. If $\beta \neq 0$, then $b^2\beta = 0$ yields $b = 0$. Therefore, $\beta ab - 1 = 0$ implies that $1 = 0$, an impossibility. In both Case II-1 and Case II-2 , we conclude that $x, y \in \langle u, v\rangle$. By the linear independence of x, y, we have $\langle u, v\rangle = \langle x, y\rangle$.

(b) If u, v are linearly dependent, then the result holds by arguments similar to Case I of (a). Suppose that u, v are linearly independent. Then x, y are linearly independent. If $x \notin \langle u, v\rangle$, then $\text{rank}\,(\eta y^2) = \text{rank}\,(u \oslash v + \alpha u^2 - \mu x^2) = 3$, a contradiction. So, $x \in \langle u, v\rangle$. Similarly, we obtain $y \in \langle u, v\rangle$. Hence, by the linear independence of x and y, $\langle u, v\rangle = \langle x, y\rangle$, as desired.

(c) The sufficiency part is clear. We now consider the necessity part. Since $u \oslash v \neq 0$, we have the following two cases:

Case A: $\operatorname{rank}(u \oslash v) = 1$. By Lemma 1.2, there exist a nonzero vector $x \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and a nonzero scalar $\alpha \in \mathbb{F}$ such that $u \oslash v = \alpha x^2$. Then $\operatorname{char}\mathbb{F} \neq 2$ and u, v are nonzero linearly dependent vectors such that $\langle u\rangle = \langle x\rangle = \langle v\rangle$. Thus, $u, v \in \mathcal{U}_p$, and so (i) holds true.

Case B: $\operatorname{rank}(u \oslash v) = 2$. By Lemma 1.2, we have the following two cases:

Case B-1: $u \oslash v = \alpha x^2 + \beta y^2$ for some nonzero scalars $\alpha, \beta \in \mathbb{F}$ and linearly independent vectors $x, y \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$. By (b), we have $\langle u, v\rangle = \langle x, y\rangle$. So, $u, v \in \mathcal{U}_p$ and (i) holds.

Case B-2: $u \oslash v = x \oslash y + \lambda x^2$ for some linearly independent vectors $x \in \mathcal{U}_p$ and $y \in \mathcal{U}_q$ with $1 \leqslant p \leqslant n + 1 - q < n + 1 - p \leqslant n$, and some scalar $\lambda \in \mathbb{F}$. By (a), we have $\langle u, v\rangle = \langle x, y\rangle$. Then $u = ax + by$ and $v = cx + dy$ for some scalars $a, b, c, d \in \mathbb{F}$, and so

$$u \oslash v = (2ac)\, x^2 + (2bd)\, y^2 + (ad + bc)\, x \oslash y. \tag{1.12}$$

Consider $\operatorname{char}\mathbb{F} \neq 2$. If $bd \neq 0$, then $1 \leqslant q \leqslant \frac{n+1}{2}$. So, $u, v \in \mathcal{U}_\ell$ with $\ell = \max\{p, q\} \leqslant \frac{n+1}{2}$. Therefore, (i) holds. If $bd = 0$, then either $b = 0$ or $d = 0$. It follows that either $u \in \mathcal{U}_p$ or $v \in \mathcal{U}_p$ with $p \leqslant \frac{n+1}{2}$, and thus (i) holds true. We now consider $\operatorname{char}\mathbb{F} = 2$. By (1.12), we have $u \oslash v = (ad + bc)\, x \oslash y$ with $ad + bc \neq 0$. If $q \leqslant \frac{n+1}{2}$, $b = 0$ or $d = 0$, then (i) holds. If $q > \frac{n+1}{2}$ and $b, d \neq 0$, then $1 \leqslant p \leqslant n + 1 - q < \frac{n+1}{2}$ and $v = cx + dy = \alpha u + w$ where $\alpha = b^{-1}d \in \mathbb{F}$ and $w = b^{-1}(ad + bc)x \in \mathcal{U}_p$. It is clear that $\alpha \neq 0$ and u, w are linearly independent. Hence, (ii) holds true.

(d) We first prove the necessity part. Let $x = au + bv + cy$ for some scalars $a, b, c \in \mathbb{F}$. Then $u \oslash x = b(u \oslash v) + c(u \oslash y)$, and thus $\operatorname{rank}(A + \lambda B) \leqslant 2$ for every nonzero scalar $\lambda \in \mathbb{F}$ because

$$\det \begin{pmatrix} \beta + \lambda c & \alpha + \lambda b & 0 \\ \mu & 0 & \alpha + \lambda b \\ 0 & \mu & \beta + \lambda c \end{pmatrix} = -2\mu(\beta + \lambda c)(\alpha + \lambda b) = 0.$$

We now prove the sufficiency part. If $x \notin \langle u, v, y \rangle$, then $\operatorname{rank}(A + \lambda B) = 4 > 2$ for every nonzero scalar $\lambda \in \mathbb{F}$ since

$$\det \begin{pmatrix} \beta & \lambda & \alpha & \lambda\gamma \\ \mu & 0 & 0 & \alpha \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \mu & \beta \end{pmatrix} = \mu^2\lambda^2 \neq 0$$

for every nonzero scalar $\lambda \in \mathbb{F}$. Suppose that $\gamma \neq 0$. The result holds if $x \notin \langle u, v, y \rangle$. We now consider $x \in \langle u, v, y \rangle$. Let $x = au + bv + cy$ for some scalars $a, b, c \in \mathbb{F}$. Then $A + \lambda B = (\alpha + \lambda b)u \oslash v + (\beta + \lambda c)u \oslash y + \mu(v \oslash y) + \lambda\gamma u^2$ is of rank 3 for every nonzero scalar $\lambda \in \mathbb{F}$ since

$$\det \begin{pmatrix} \beta + \lambda c & \alpha + \lambda b & \lambda\gamma \\ \mu & 0 & \alpha + \lambda b \\ 0 & \mu & \beta + \lambda c \end{pmatrix} = \lambda\gamma\mu^2 \neq 0.$$

We are done. $\square$

Chapter 2

SPACES OF BOUNDED RANK-TWO MATRICES AND RANK-TWO SPACES

We shall use this chapter to determine subspaces of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$ and eventually give a classification of these subspaces in Theorem 2.6. The classification of subspaces is very important and should be viewed as groundwork for the study of bounded rank-two linear preservers in Chapter 3. In the process of proving our main theorem, we develop a few lemmas which, at the same time, illustrate the behaviour of subspaces of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$.

In [8], M. H. Lim proved Theorem 2.1 below. We need the following preamble: Let $\mathbb{F}$ denote an infinite field of characteristic not equal to two. Let U be a finite dimensional vector space of dimension n over $\mathbb{F}$ and $U^{(2)}$ be the second symmetric product space over U. Let J_k denote the set of all vectors in $U^{(2)}$ of the form $\sum_{i=1}^{k} \lambda_i x_i^2$, where $x_1, \ldots, x_k$ are linearly independent vectors and $\lambda_1, \ldots, \lambda_k \in \mathbb{F}\backslash\{0\}$.

Theorem 2.1. *Let M be a subspace of $U^{(2)}$ such that $M \subseteq \{0\} \cup J_1 \cup J_2$. Then either*

(I) *$M \subseteq W^{(2)}$ for some 2-dimensional subspace W of U or*

(II) *$M \subseteq u \cdot U$ for some nonzero vector $u \in U$.*

We strengthen Theorem 2.1 to Theorem 2.6 by replacing the underlying field $\mathbb{F}$ with any field containing at least three elements. To begin our investigation, we first give the following definitions concerning the subspaces of bounded rank-two matrices.

Let $x \in \mathcal{M}_{n,1}(\mathbb{F})$ and let U be a subspace of $\mathcal{M}_{n,1}(\mathbb{F})$. We denote

$$x \oslash U := \{\, x \oslash u \mid u \in U \,\} \tag{2.1}$$

and

$$U^2 := \left\langle\, u^2 \,\middle|\, u \in U \,\right\rangle . \tag{2.2}$$

Lemma 2.2. *Let $\mathbb{F}$ be a field and let n be an integer $\geqslant 2$. Let $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ and let U be a subspace of $\mathcal{M}_{n,1}(\mathbb{F})$. Then the following assertions hold.*

(a) *If $A \in u \oslash U$, then $\operatorname{rank} A \leqslant 2$. Moreover, if $\operatorname{char} \mathbb{F} = 2$, then $\operatorname{rank} A = 2$ for every nonzero $A \in u \oslash U$.*

(b) *If u, v are linearly independent, then*

(i) *$\langle u, v\rangle^2 = \langle u^2, v^2, u \oslash v\rangle$ is a 3-dimensional subspace of $\mathcal{M}_n(\mathbb{F})$.*

(ii) *If $A \in \langle u, v\rangle^2$, then rank $A \leqslant 2$.*

Proof. (a) Let $A \in u \oslash U$. By (2.1), we have $A = u \oslash y$ for some $y \in U$. Since $\operatorname{rank}(u \cdot y^+) \leqslant 1$, it follows that $\operatorname{rank} A = \operatorname{rank}(u \cdot y^+ + y \cdot u^+) \leqslant \operatorname{rank}(u \cdot y^+) + \operatorname{rank}(y \cdot u^+) \leqslant 2$. Consider now $\operatorname{char} \mathbb{F} = 2$. If $y \in U$ is a vector such that $u \oslash y \neq 0$, then $\langle y\rangle \neq \langle u\rangle$. For, if not, then $y = \alpha u$ for some $\alpha \in \mathbb{F}$, and so $u \oslash y = 2\alpha u^2 = 0$. Hence, $\operatorname{rank}(u \oslash y) = \operatorname{rank}(u \cdot y^+ + y \cdot u^+) = 2$.

(b) (i) By (2.2), we see that

$$\langle u, v\rangle^2 = \langle w^2 \mid w \in \langle u, v\rangle\rangle = \langle(\alpha u + \beta v)^2 \mid \alpha, \beta \in \mathbb{F}\rangle. \tag{2.3}$$

Note that $(\alpha u+\beta v)^2 = \alpha^2 u^2 + \beta^2 v^2 + (\alpha\beta) u \oslash v \in \langle u^2, v^2, u \oslash v\rangle$ for $\alpha, \beta \in \mathbb{F}$. By (2.3), we get $\langle u, v\rangle^2 \subseteq \langle u^2, v^2, u \oslash v\rangle$. On the other hand, we see that $u^2 = ((1)u + (0)v)^2$

and $v^2 = ((0)u + (1)v)^2$ with $0, 1 \in \mathbb{F}$, and $u \oslash v = (u+v)^2 - u^2 - v^2$. It follows from (2.3) that $u^2, v^2, u \oslash v \in \langle u, v\rangle^2$. Hence, we have $\langle u, v\rangle^2 = \langle u^2, v^2, u \oslash v\rangle$. Note that u^2, v^2, $u \oslash v$ are linearly independent whenever u, v are linearly independent. It follows that $\langle u, v\rangle^2$ is a 3-dimensional subspace of $\mathcal{M}_n(\mathbb{F})$.

(ii) Let $A \in \langle u, v\rangle^2$ be nonzero. By (i), we have $A = \alpha u^2 + \beta v^2 + \gamma u \oslash v$ for some $\alpha, \beta, \gamma \in \mathbb{F}$. If $\alpha = \beta = 0$, then $A = \gamma(u \cdot v^+ + v \cdot u^+)$ is of rank 2. Suppose that $\alpha \neq 0$ or $\beta \neq 0$. Without loss of generality, we may assume $\alpha \neq 0$. Then

$$A = \alpha(u + \gamma(\alpha^{-1})v)^2 + (\beta - \alpha^{-1}\gamma^2)v^2.$$

Since $\operatorname{rank} w^2 \leqslant 1$ for every $w \in \mathcal{M}_{n,1}(\mathbb{F})$, it follows that $\operatorname{rank} A \leqslant 2$. This completes the proof. $\square$

The next three lemmas are essential for us to prove the main theorem.

Lemma 2.3. *Let $\mathbb{F}$ be a field and let n be an integer $\geqslant 2$. Then the following assertions hold.*

(a) *Let $u \in \mathcal{U}_p$ and let U be a subspace of $\mathcal{U}_q$ with $1 \leqslant p \leqslant n+1-q \leqslant n$. Then $u \oslash U$ is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$.*

(b) *Let $u \in \mathcal{U}_p$ and let U be a subspace of $\mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant p \leqslant n+1-q \leqslant n$. Then $u \oslash U + \langle u^2\rangle$ is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$.*

(c) *Let U be a subspace of $\mathcal{U}_q$ with $1 \leqslant q \leqslant \frac{n+1}{2}$. Then U^2 is a subspace of $\mathcal{ST}_n(\mathbb{F})$. Moreover, if U is a 2-dimensional subspace of $\mathcal{U}_q$, then U^2 is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$.*

(d) *Let $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ and $w \in \mathcal{U}_r$ be linearly independent vectors such that $1 \leqslant p, q \leqslant n+1-r \leqslant n$ and either $p \leqslant n+1-q$, or $p = q > \frac{n+1}{2}$*

and $v = \alpha u + z$ *for some nonzero scalar* $\alpha \in \mathbb{F}$ *and vector* $z \in \mathcal{U}_\ell$ *with* $1 \leqslant \ell \leqslant n+1-p < \frac{n+1}{2}$ *such that* u, z *are linearly independent. If* $\operatorname{char}\mathbb{F} = 2$*, then* $\langle u \oslash v,\, u \oslash w,\, v \oslash w \rangle$ *is a rank-two subspace of* $\mathcal{ST}_n(\mathbb{F})$*.*

Proof. (a) Let $v \in U$. Since $1 \leqslant p \leqslant n+1-q$, we have $u \cdot v^+ \in \mathcal{T}_n(\mathbb{F})$, and so $v \cdot u^+ = (u \cdot v^+)^+ \in \mathcal{T}_n(\mathbb{F})$. Further, since $(u \oslash v)^+ = u \oslash v$, we conclude that $u \oslash v \in \mathcal{ST}_n(\mathbb{F})$ for every $v \in U$. Next, we claim that $u \oslash U$ is a subspace of $\mathcal{ST}_n(\mathbb{F})$. Clearly, $0 \in u \oslash U$, and so $u \oslash U \neq \varnothing$. Let $A_1, A_2 \in u \oslash U$ and $\alpha \in \mathbb{F}$. Then there exist $v_1, v_2 \in U$ such that $A_i = u \oslash v_i$ for $i = 1, 2$. So

$$A_1 + \alpha A_2 = u \oslash v_1 + \alpha(u \oslash v_2) = u \oslash v_1 + u \oslash \alpha v_2 = u \oslash (v_1 + \alpha v_2).$$

Since $v_1 + \alpha v_2 \in U$, we get $A_1 + \alpha A_2 \in u \oslash U$, and hence $u \oslash U$ is a subspace of $\mathcal{ST}_n(\mathbb{F})$. Together with Lemma 2.2 (a), we have $u \oslash U$ is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$.

(b) Since $1 \leqslant p \leqslant \frac{n+1}{2}$ and $(u^2)^+ = u^2$, we have $u^2 \in \mathcal{ST}_n(\mathbb{F})$, and so $\langle u^2 \rangle$ is a subspace of $\mathcal{ST}_n(\mathbb{F})$. Let $A \in u \oslash U + \langle u^2 \rangle$. Then $A = u \oslash y + \alpha u^2$ for some $y \in U$ and $\alpha \in \mathbb{F}$. So, $\operatorname{rank} A = \operatorname{rank}(u \cdot (y^+ + \alpha u^+) + y \cdot u^+) \leqslant 2$. Therefore, $u \oslash U + \langle u^2 \rangle$ is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$.

(c) Since $1 \leqslant q \leqslant \frac{n+1}{2}$, it follows that $y^2 \in \mathcal{ST}_n(\mathbb{F})$ for every $y \in U$. So, U^2 is a subspace of $\mathcal{ST}_n(\mathbb{F})$. Further, if U is 2-dimensional, then, by Lemma 2.2 (b)(ii), U^2 is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$.

(d) Let $A \in \langle u \oslash v,\, u \oslash w,\, v \oslash w \rangle$ be nonzero. Then $A = a(u \oslash v) + b(u \oslash w) + c(v \oslash w)$ for some scalars $a, b, c \in \mathbb{F}$ with $(a, b, c) \neq 0$. It is clear that $A \in \mathcal{ST}_n(\mathbb{F})$ by Lemma 1.3 (c). Let

$$X = \begin{pmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{pmatrix}.$$

Since $\operatorname{rank} X \neq 1$ and $\det X = -2abc = 0$ since $\operatorname{char} \mathbb{F} = 2$. It follows that $\operatorname{rank} X = 2$, and so $\operatorname{rank} A = 2$. Therefore, $\langle u \oslash v,\, u \oslash w,\, v \oslash w \rangle$ is a rank-two subspace of $\mathcal{ST}_n(\mathbb{F})$. $\square$

Lemma 2.4. *Let $\mathbb{F}$ be a field of characteristic two and let n be an integer $\geqslant 2$. Let $u \in \mathcal{U}_p$ and $v_1, \ldots, v_k \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $1 \leqslant q \leqslant n + 1 - p \leqslant n$, and let $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$ such that $u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2$ are matrices in $\mathcal{ST}_n(\mathbb{F})$.*

(a) *If $u, v_1, \ldots, v_k$ are linearly independent, then $u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2$ are linearly independent, and each nonzero element of $\langle u \oslash v_1 + \lambda_1 u^2, \ldots,\, u \oslash v_k + \lambda_k u^2 \rangle$ is of rank two.*

(b) *Suppose that $u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2$ are linearly independent. If $u, v_1, \ldots, v_k$ are linearly dependent, then there exists an integer $1 \leqslant i_0 \leqslant k$ such that*

$$\left\langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \right\rangle = u \oslash U + \left\langle u^2 \right\rangle$$

for some $(k-1)$-dimensional subspace $U = \langle v_1, \ldots, v_{i_0 - 1}, v_{i_0 + 1}, \ldots, v_k \rangle$ of $\mathcal{U}_q$.

Proof. We denote $\mathcal{G} := \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \rangle$.

(a) Suppose that $\mu_1(u \oslash v_1 + \lambda_1 u^2) + \cdots + \mu_k(u \oslash v_k + \lambda_k u^2) = 0$ for some scalars $\mu_1, \ldots, \mu_k \in \mathbb{F}$. Then $u \oslash (\mu_1 v_1 + \cdots + \mu_k v_k) + (\mu_1 \lambda_1 + \cdots + \mu_k \lambda_k) u^2 = 0$. Since $u, v_1, \ldots, v_k$ are linearly independent, it follows that $u, \mu_1 v_1 + \cdots + \mu_k v_k$ are linearly independent if $(\mu_1, \ldots, \mu_k) \neq 0$. Hence $\mu_1 = \cdots = \mu_k = 0$, and thus $u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2$ are linearly independent.

Let $A \in \mathcal{G}$ be a nonzero matrix. Then there exist scalars $\beta_1, \ldots, \beta_k \in \mathbb{F}$ not all of which are zero such that

$$\begin{aligned} A &= \beta_1(u \oslash v_1 + \lambda_1 u^2) + \cdots + \beta_k(u \oslash v_k + \lambda_k u^2) \\ &= u \oslash (\beta_1 v_1 + \cdots + \beta_k v_k) + (\beta_1\lambda_1 + \cdots + \beta_k\lambda_k)u^2. \end{aligned}$$

Since $u, v_1, \ldots, v_k$ are linearly independent and $(\beta_1, \ldots, \beta_k) \neq 0$, it follows that $\beta_1 v_1 + \cdots + \beta_k v_k, u$ are linearly independent, and so $\operatorname{rank} A = 2$, as desired.

(b) If $u, v_1, \ldots, v_k$ are linearly dependent, then there exist scalars $\alpha, \alpha_1, \ldots, \alpha_k \in \mathbb{F}$ not all of which are zero such that $\alpha u + \alpha_1 u_1 + \cdots + \alpha_k u_k = 0$. Since $\operatorname{char} \mathbb{F} = 2$, we have

$$\begin{aligned} &\alpha_1(u \oslash v_1 + \lambda_1 u^2) + \cdots + \alpha_k(u \oslash v_k + \lambda_k u^2) \\ &\qquad = u \oslash (\alpha u + \alpha_1 u_1 + \cdots + \alpha_k u_k) + (\alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k)u^2 \\ &\qquad = (\alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k)u^2. \end{aligned}$$

If $\alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k = 0$, then $\alpha_1 = \cdots = \alpha_k = 0$ as $u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2$ are linearly independent. Therefore $\alpha u = 0$, and hence $\alpha = 0$ since $u \neq 0$. This leads to a contradiction because $(\alpha, \alpha_1, \ldots, \alpha_k) \neq 0$. Hence $\alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k \neq 0$. It follows that

$$u^2 = (\alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k)^{-1}(\alpha_1(u \oslash v_1 + \lambda_1 u^2) + \cdots + \alpha_k(u \oslash v_k + \lambda_k u^2)) \in \mathcal{G},$$

and hence $\langle u^2 \rangle \subseteq \mathcal{G}$. Therefore $\mathcal{G} + \langle u^2 \rangle = \mathcal{G}$. On the other hand, it is easily verified that $u \oslash \langle v_1, \ldots, v_k \rangle + \langle u^2 \rangle = \mathcal{G} + \langle u^2 \rangle$, and so

$$\mathcal{G} = u \oslash \langle v_1, \ldots, v_k \rangle + \left\langle u^2 \right\rangle . \tag{2.4}$$

By Lemma 2.2 (a), we see that each nonzero element of $u \oslash \langle v_1, \ldots, v_k \rangle$ is of rank two. It follows that $u \oslash \langle v_1, \ldots, v_k \rangle \cap \langle u^2 \rangle = \{0\}$. Since $\mathcal{G}$ is k-dimensional, it follows from (2.4) that $u \oslash \langle v_1, \ldots, v_k \rangle$ has dimension $k-1$. Thus $v_1, \ldots, v_k$ are linearly dependent, and so there exists $i_0 \in \{1, \ldots, k\}$ such that $v_1, \ldots, v_{i_0-1}, v_{i_0+1}, \ldots, v_k$ are linearly independent. Hence $\mathcal{G} = u \oslash \langle v_1, \ldots, v_{i_0-1}, v_{i_0+1}, \ldots, v_k \rangle + \langle u^2 \rangle$. This completes our proof. $\square$

Lemma 2.5. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n be an integer $\geqslant 2$. Let $\mathcal{S}$ be a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$. Let $A_1, A_2 \in \mathcal{S}$ be rank two matrices such that*

$$A_1 = \alpha u^2 + \beta v^2 \tag{2.5}$$

for some linearly independent vectors $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and nonzero scalars $\alpha, \beta \in \mathbb{F}$, or

$$A_1 = u \oslash v + \gamma u^2 \tag{2.6}$$

for some linearly independent vectors $u \in \mathcal{U}_p$ and $v \in \mathcal{U}_q$ with $1 \leqslant p \leqslant n+1-q < n+1-p$ and scalar $\gamma \in \mathbb{F}$; and

$$A_2 = \alpha_1 u_1^2 + \beta_1 v_1^2 \tag{2.7}$$

for some linearly independent vectors $u_1, v_1 \in \mathcal{U}_s$ with $1 \leqslant s \leqslant \frac{n+1}{2}$ and nonzero scalars $\alpha_1, \beta_1 \in \mathbb{F}$, or

$$A_2 = u_1 \oslash v_1 + \gamma_1 u_1^2 \tag{2.8}$$

for some linearly independent vectors $u_1 \in \mathcal{U}_s$ *and* $v_1 \in \mathcal{U}_t$*, with* $1 \leqslant s \leqslant n+1-t < n+1-s$ *and scalar* $\gamma_1 \in \mathbb{F}$*. If*

$$u_1 \notin \langle u, v\rangle \quad \text{or} \quad v_1 \notin \langle u, v\rangle\,, \tag{2.9}$$

then the following assertions hold.

(a) *If* $\operatorname{char}\mathbb{F} \neq 2$*, then there exist linearly independent vectors* $w \in \mathcal{U}_\ell$ *and* $y, z \in \mathcal{U}_h$ *such that* A_1 *and* A_2 *can be represented as*

$$A_1 = w \oslash y \qquad \text{and} \qquad A_2 = w \oslash z \tag{2.10}$$

for some integers $1 \leqslant \ell \leqslant n+1-h \leqslant n$*.*

(b) *If* $\operatorname{char}\mathbb{F} = 2$*, then there exist linearly independent vectors* $w \in \mathcal{U}_\ell$ *and* $y, z \in \mathcal{U}_h$ *and scalars* $\alpha_1, \alpha_2 \in \mathbb{F}$ *such that* A_1 *and* A_2 *can be represented as*

$$A_1 = w \oslash y + \alpha_1 w^2 \qquad \text{and} \qquad A_2 = w \oslash z + \alpha_2 w^2 \tag{2.11}$$

for some integers $1 \leqslant \ell \leqslant n+1-h \leqslant n$*, and* $1 \leqslant \ell \leqslant \frac{n+1}{2}$ *when* $(\alpha_1, \alpha_2) \neq 0$*.*

Proof. (a) We divide our proof into the following four cases:

Case A-(i): A_1 is of form (2.5) and A_2 is of form (2.7). In view of (2.9), we consider only $u_1 \notin \langle u, v\rangle$ as the second case can be verified similarly. Since $\operatorname{rank}(A_1 + A_2) \leqslant 2$, it follows that $v_1 \in \langle u, v, u_1\rangle$. Let $v_1 = c_1 u + d_1 v + g_1 u_1$ for some scalars $c_1, d_1, g_1 \in \mathbb{F}$. Let $\lambda \in \mathbb{F}$. Note that

$$\lambda(\beta^{-1}A_1) + \beta_1^{-1}A_2 = \lambda a u^2 + \lambda v^2 + b u_1^2 + (c_1 u + d_1 v + g_1 u_1)^2 \in \mathcal{S}$$

where $a = \alpha\beta^{-1}$ and $b = \alpha_1\beta_1^{-1}$ are nonzero scalars in $\mathbb{F}$, and so

$$\lambda(\beta^{-1}A_1) + \beta_1^{-1}A_2 = (\lambda a + c_1^2)u^2 + (\lambda + d_1^2)v^2 + (b + g_1^2)u_1^2 + \\ c_1d_1(u \oslash v) + c_1g_1(u \oslash u_1) + d_1g_1(v \oslash u_1)$$

is of rank at most 2. Therefore, we have

$$\det\begin{pmatrix} c_1g_1 & c_1d_1 & \lambda a + c_1^2 \\ d_1g_1 & \lambda + d_1^2 & c_1d_1 \\ b + g_1^2 & d_1g_1 & c_1g_1 \end{pmatrix} = 0,$$

and hence $a(b + g_1^2)\lambda^2 + b(ad_1^2 + c_1^2)\lambda = 0$ for every $\lambda \in \mathbb{F}$. Since $|F| \geqslant 3$, it follows that

$$ad_1^2 + c_1^2 = 0, \tag{2.12}$$

$$b + g_1^2 = 0. \tag{2.13}$$

Since u_1, v_1 are linearly independent, by (2.12), we get $c_1, d_1 \neq 0$. By (2.12) and (2.13),

$$\beta^{-1}A_1 = au^2 + v^2 = 2^{-1}(c_1u + d_1v) \oslash (ac_1^{-1}u + d_1^{-1}v), \tag{2.14}$$

$$\beta_1^{-1}A_2 = bu_1^2 + (c_1u + d_1v + g_1u_1)^2 = 2^{-1}(c_1u + d_1v) \oslash (c_1u + d_1v + 2g_1u_1). \tag{2.15}$$

From (2.14) and (2.15), we have $A_1 = w_1 \oslash y_1$ and $A_2 = w_1 \oslash z_1$, where $w_1 = 2^{-1}(c_1u + d_1v) \in \mathcal{U}_p$, $y_1 = \beta(ac_1^{-1}u + d_1^{-1}v) \in \mathcal{U}_{h_1}$ and $z_1 = \beta(c_1u + d_1v + 2g_1u_1) \in \mathcal{U}_{h_1}$, with $h_1 = \max\{p, s\}$, are linearly independent vectors. So, (2.10) is proved.

Case A-(ii): A_1 is of form (2.5) and A_2 is of form (2.8). Then $\gamma_1 = 0$, and by Lemma 1.2 (b), we may assume without loss of generality that $u_1 \in \mathcal{U}_s$ and $v_1 \in \mathcal{U}_t$ for some integers $s \leqslant n + 1 - t < n + 1 - s$ and $t > \frac{n+1}{2}$. Since $p, s < t$, we have $v_1 \notin \langle u, v\rangle$. Since $\operatorname{rank}(A_1 + A_2) \leqslant 2$, it follows that $\{u, v, u_1, v_1\}$ is linearly dependent. Then $u_1 \in \langle u, v\rangle$, and so $u_1 = c_2u + d_2v$ for some scalars $c_2, d_2 \in \mathbb{F}$. Let $\lambda \in \mathbb{F}$. Then

$$\lambda(\beta^{-1}A_1) + A_2 = \lambda a u^2 + \lambda v^2 + u_1 \oslash v_1 = \lambda a u^2 + \lambda v^2 + c_2(u \oslash v_1) + d_2(v \oslash v_1)$$

where $a = \alpha\beta^{-1} \in \mathbb{F}$. Since $\rho(\lambda(\beta^{-1}A_1) + A_2) \leqslant 2$, it follows that

$$0 = \det\begin{pmatrix} c_2 & 0 & \lambda a \\ d_2 & \lambda & 0 \\ 0 & d_2 & c_2 \end{pmatrix} = (ad_2^2 + c_2^2)\lambda$$

for every $\lambda \in \mathbb{F}$. Then $ad_2^2 + c_2^2 = 0$. Since $u_1 \neq 0$ and $a \neq 0$, we have $c_2, d_2 \neq 0$. So

$$\beta^{-1}A_1 = au^2 + v^2 = 2^{-1}u_1 \oslash (ac_2^{-1}u + d_2^{-1}v). \tag{2.16}$$

$$A_2 = u_1 \oslash v_1 = 2^{-1}u_1 \oslash 2v_1. \tag{2.17}$$

From (2.16) and (2.17), we see that $A_1 = w_2 \oslash y_2$ and $A_2 = w_2 \oslash z_2$, where $w_2 = 2^{-1}u_1 \in \mathcal{U}_s$, $y_2 = \beta(ac_2^{-1}u + d_2^{-1}v) \in \mathcal{U}_{h_2}$ and $z_2 = 2v_1 \in \mathcal{U}_{h_2}$, with $h_2 = \max\{s, t\}$, are linearly independent vectors. So, (2.10) is proved.

Case A-(iii) : A_1 is of form (2.6) and A_2 is of form (2.7). By (2.9), we obtain $u \notin \langle u_1, v_1\rangle$ or $v \notin \langle u_1, v_1\rangle$. By a similar argument as in Case A-(ii), (2.10) holds.

Case A-(iv) : A_1 is of form (2.6) and A_2 is of form (2.8). Then $\gamma = \gamma_1 = 0$. By Lemma 1.2 (b), we assume without loss of generality that $u \in \mathcal{U}_p$ and $v \in \mathcal{U}_q$ for some integers $p \leqslant n+1-q < n+1-p$ and $q > \frac{n+1}{2}$, and $u_1 \in \mathcal{U}_s$ and $v_1 \in \mathcal{U}_t$ for some integers $s \leqslant n+1-t < n+1-s$ and $t > \frac{n+1}{2}$. By (2.9), we argue in the following two cases:

Sub-case A-(iv)-1: $u_1 \notin \langle u, v\rangle$. Then $v_1 \in \langle u, v, u_1\rangle$. Let $v_1 = c_3u + d_3v + g_3u_1$

for some scalars $c_3, d_3, g_3 \in \mathbb{F}$. Since $t > s, p$, we get $d_3 \neq 0$. Let $\lambda \in \mathbb{F}$. Then

$$\lambda A_1 + A_2 = \lambda u \oslash v + u_1 \oslash v_1 = 2g_3 u_1^2 + \lambda u \oslash v + c_3 u \oslash u_1 + d_3 v \oslash u_1.$$

Since rank $(\lambda A_1 + A_2) \leqslant 2$, we obtain

$$0 = \det \begin{pmatrix} c_3 & \lambda & 0 \\ d_3 & 0 & \lambda \\ 2g_3 & d_3 & c_3 \end{pmatrix} = 2g_3\lambda^2 - 2c_3 d_3 \lambda$$

for every $\lambda \in \mathbb{F}$. Since $|F| \geqslant 3$ and $d_3 \neq 0$, it follows that $g_3 = c_3 = 0$. Thus, $v_1 = d_3 v$, and so

$$A_1 = v \oslash u, \tag{2.18}$$

$$A_2 = v \oslash d_3 u_1. \tag{2.19}$$

From (2.18) and (2.19), we have $A_1 = w_3 \oslash y_3$ and $A_2 = w_3 \oslash z_3$, where $w_3 = v \in \mathcal{U}_q$, $y_3 = u \in \mathcal{U}_{h_1}$ and $z_3 = d_3 u_1 \in \mathcal{U}_{h_1}$, with $h_1 = \max\{p, s\}$, are linearly independent vectors, and so (2.10) holds.

Sub-case A-(iv)-2: $v_1 \notin \langle u, v\rangle$. Then $u_1 \in \langle u, v, v_1\rangle$. Let $u_1 = c_4 u + d_4 v + g_4 v_1$ for some scalars $c_4, d_4, g_4 \in \mathbb{F}$. Since $u_1 \in \mathcal{U}_s$, we have $d_4 = 0 \Leftrightarrow g_4 = 0$. By a similar argument as in Sub-case A-(iv)-1, it can be shown that $d_4 = 0$, and so $g_4 = 0$. Thus, $u_1 = c_4 u$, and so $A_1 = w_4 \oslash y_4$ and $A_2 = w_4 \oslash z_4$, where $w_4 = u \in \mathcal{U}_p$, $y_4 = v \in \mathcal{U}_{h_3}$ and $z_4 = c_4 v_1 \in \mathcal{U}_{h_3}$, with $h_3 = \max\{q, t\}$, are linearly independent vectors. Consequently, (2.10) is proved.

(b) We divide our proof into the following six cases:

Case B-(i): A_1 is of form (2.5) and A_2 is of form (2.7). We consider only $u_1 \notin \langle u, v\rangle$ as the second case $v_1 \notin \langle u, v\rangle$ can be verified similarly. Then $v_1 =$

$c_5u + d_5v + g_5u_1$ for some scalars $c_5, d_5, g_5 \in \mathbb{F}$. Let $\lambda \in \mathbb{F}$. So, we have

$$\lambda(\beta^{-1}A_1) + \beta_1^{-1}A_2 = (\lambda a + c_5^2)u^2 + (\lambda + d_5^2)v^2 + (b + g_5^2)u_1^2 + c_5d_5(u \oslash v) + c_5g_5(u \oslash u_1) + d_5g_5(v \oslash u_1) \in \mathcal{S}$$

where $a = \alpha\beta^{-1}$ and $b = \alpha_1\beta_1^{-1}$ are nonzero scalars in $\mathbb{F}$. Therefore

$$\det\begin{pmatrix} c_5g_5 & c_5d_5 & \lambda a + c_5^2 \\ d_5g_5 & \lambda + d_5^2 & c_5d_5 \\ b + g_5^2 & d_5g_1 & c_5g_5 \end{pmatrix} = 0.$$

Since $|F| \geqslant 3$, we get $ad_5^2 + c_5^2 = 0$ and $b + g_5^2 = 0$, and so $c_5, d_5 \neq 0$. Then

$$\beta^{-1}A_1 = (c_5u + d_5v) \oslash (d_5^{-1}v) + d_5^{-2}(c_5u + d_5v)^2,$$

$$\beta_1^{-1}A_2 = (c_5u + d_5v) \oslash (g_5u_1) + (c_5u + d_5v)^2.$$

So, we have $A_1 = w_5 \oslash y_5 + (\beta d_5^{-2})w_5^2$ and $A_2 = w_5 \oslash z_5 + \beta_1 w_5^2$, where $w_5 = c_5u + d_5v \in \mathcal{U}_p$, $y_5 = \beta d_5^{-1}v \in \mathcal{U}_{h_1}$ and $z_5 = \beta_1 g_5 u_1 \in \mathcal{U}_{h_1}$, with $h_1 = \max\{p, s\}$, are linearly independent vectors. Hence, (2.11) holds.

Case B-(ii) : A_1 is of form (2.5) and A_2 is of form (2.8) with $\gamma_1 \neq 0$. By Lemma 1.2 (b) and (2.8), we may assume without loss of generality that $u_1 \in \mathcal{U}_s$ and $v_1 \in \mathcal{U}_t$ for some integers $s \leqslant n + 1 - t < n + 1 - s$ and $t > \frac{n+1}{2}$. Since $p, s < t$, $v_1 \notin \langle u, v\rangle$ and $\{u, v, u_1, v_1\}$ is linearly dependent, we have $u_1 \in \langle u, v\rangle$, and so $u_1 = c_6u + d_6v$ for some $c_6, d_6 \in \mathbb{F}$. Let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned}\lambda(\beta^{-1}A_1) + A_2 &= \lambda a u^2 + \lambda v^2 + u_1 \oslash v_1 + \gamma_1 u_1^2 \\ &= (\lambda a + \gamma_1 c_6^2)u^2 + (\lambda + \gamma_1 d_6^2)v^2 + \gamma_1 c_6 d_6 (u \oslash v) \\ &\quad + c_6(u \oslash v_1) + d_6(v \oslash v_1)\end{aligned}$$

where $a = \alpha\beta^{-1} \in \mathbb{F}$. Since $\rho(\lambda(\beta^{-1}A_1) + A_2) \leqslant 2$, it follows that

$$0 = \det\begin{pmatrix} c_6 & \gamma_1 c_6 d_6 & \lambda a + \gamma_1 c_6^2 \\ d_6 & \lambda + \gamma_1 d_6^2 & \gamma_1 c_6 d_6 \\ 0 & d_6 & c_6 \end{pmatrix} = (a d_6^2 + c_6^2)\lambda$$

for every $\lambda \in \mathbb{F}$. Then $a d_6^2 + c_6^2 = 0$. Since $u_1 \neq 0$, it follows that $c_6, d_6 \neq 0$, and so $a = c_6^2 (d_6^{-1})^2$. Therefore,

$$\beta^{-1}A_1 = u_1 \oslash (d_6^{-1}v) + d_6^{-2}u_1^2 \qquad \text{and} \qquad A_2 = u_1 \oslash v_1 + \gamma_1 u_1^2.$$

Thus, $A_1 = w_6 \oslash y_6 + (\beta d_6^{-2}) w_6^2$ and $A_2 = w_6 \oslash z_6 + \gamma_1 w_6^2$, where $w_6 = u_1 \in \mathcal{U}_s$, $y_6 = \beta d_6^{-1} v \in \mathcal{U}_{h_4}$ and $z_6 = v_1 \in \mathcal{U}_{h_4}$, with $h_4 = \max\{p, t\}$, are linearly independent vectors. So, (2.11) holds.

Case B-(iii) : A_1 is of form (2.6) with $\gamma \neq 0$, and A_2 is of form (2.7). By a similar argument as in the proof of Case B-(ii), (2.11) holds true.

Case B-(iv) : A_1 is of form (2.6) with $\gamma \neq 0$, and A_2 is of form (2.8) with $\gamma_1 \neq 0$. In view of Lemma 1.2 (b), we may assume without loss of generality that $u \in \mathcal{U}_p$ and $v \in \mathcal{U}_q$ for some integers $p \leqslant n+1-q < n+1-p$ and $q > \frac{n+1}{2}$, and $u_1 \in \mathcal{U}_s$ and $v_1 \in \mathcal{U}_t$ for some integers $s \leqslant n+1-t < n+1-s$ and $t > \frac{n+1}{2}$. Suppose that $u_1 \notin \langle u, v\rangle$. Then $v_1 \in \langle u, v, u_1\rangle$, and so $v_1 = c_7 u + d_7 v + g_7 u_1$ for some $c_7, d_7, g_7 \in \mathbb{F}$. Since $p, s < q, t$, we have $d_7 \neq 0$. Let $\lambda \in \mathbb{F}$. Then $\lambda A_1 + A_2 = \lambda\gamma u^2 + \gamma_1 u_1^2 + \lambda(u \oslash v) + c_7(u \oslash u_1) + d_7(v \oslash u_1) \in \mathcal{S}$. So

$$0 = \det\begin{pmatrix} c_7 & \lambda & \lambda\gamma \\ d_7 & 0 & \lambda \\ \gamma_1 & d_7 & c_7 \end{pmatrix} = \gamma_1\lambda^2 + \gamma d_7^2 \lambda$$

for every $\lambda \in \mathbb{F}$. Since $|F| \geqslant 3$ and $d_7 \neq 0$, we conclude that $\gamma_1 = 0$ and $\gamma = 0$, a contradiction. Then $v_1 \notin \langle u, v\rangle$ by (2.9). Thus, $u_1 \in \langle u, v, v_1\rangle$, and so $u_1 =$

$c_8u + d_8v + g_8v_1$ for some scalars $c_8, d_8, g_8 \in \mathbb{F}$. Note that $g_8 = 0$ for if not, then $v_1 = g_8^{-1}u_1 - g_8^{-1}c_8u - g_8^{-1}d_8v$ and an argument analogous to the above yields a contradiction. Hence, $g_8 = 0$, and so $d_8 = 0$ since $u_1 \in \mathcal{U}_s$. Then

$$A_1 = u \oslash v + \gamma u^2 \quad \text{and} \quad A_2 = u \oslash (c_8v_1) + \gamma_1 c_8^2 u^2.$$

Therefore, we have $A_1 = w_8 \oslash y_8 + \gamma w_8^2$ and $A_2 = w_8 \oslash z_8 + (\gamma_1 c_8^2)w_8^2$, where $w_8 = u \in \mathcal{U}_p$, $y_8 = v \in \mathcal{U}_{h_3}$ and $z_8 = c_8v_1 \in \mathcal{U}_{h_3}$, with $h_3 = \max\{q, t\}$, are linearly independent vectors. Hence, (2.11) is proved.

Case B-(v): A_1 is of form (2.6) with $\gamma = 0$, and A_2 is of form (2.7). In view of (2.9), we have $\langle u_1, v_1\rangle \neq \langle u, v\rangle$, and so $u \notin \langle u_1, v_1\rangle$ or $v \notin \langle u_1, v_1\rangle$. We consider only the case $u \notin \langle u_1, v_1\rangle$ as the second case can be verified similarly. So, $v = c_9u_1 + d_9v_1 + g_9u$ for some $c_9, d_9, g_9 \in \mathbb{F}$. Let $\lambda \in \mathbb{F}$. Then $\lambda A_1 + \beta_1^{-1}A_2 = au_1^2 + v_1^2 + \lambda c_9 u \oslash u_1 + \lambda d_9 u \oslash v_1 \in \mathcal{S}$ with $a = \alpha_1\beta_1^{-1} \in \mathbb{F}$. So

$$0 = \det\begin{pmatrix} \lambda d_9 & \lambda c_9 & 0 \\ 0 & a & \lambda c_9 \\ 1 & 0 & \lambda d_9 \end{pmatrix} = (ad_9^2 + c_9^2)\lambda^2$$

for every $\lambda \in \mathbb{F}$. Since $|F| \geqslant 3$, we have $ad_9^2 + c_9^2 = 0$, and so $c_9, d_9 \neq 0$ and $a = c_9^2(d^{-1})^2$. Then

$$A_1 = v \oslash u = (c_9d_9^{-1}u_1 + v_1) \oslash d_9u,$$

$$\beta_1^{-1}A_2 = au_1^2 + v_1^2 = (c_9d_9^{-1}u_1 + v_1) \oslash v_1 + (c_9d_9^{-1}u_1 + v_1)^2.$$

Thus, $A_1 = w_9 \oslash y_9$ and $A_2 = w_9 \oslash z_9 + \beta_1 w_9^2$, where $w_9 = c_9d_9^{-1}u_1 + v_1 \in \mathcal{U}_s$, $y_9 = d_9u \in \mathcal{U}_{h_1}$ and $z_9 = \beta_1 v_1 \in \mathcal{U}_{h_1}$, with $h_1 = \max\{p, s\}$, are linearly independent vectors. So, (2.11) is proved.

Case B-(vi): A_1 is of form (2.6) with $\gamma = 0$, and A_2 is of form (2.8).

We first consider $\gamma_1 \neq 0$. If $u_1 \notin \langle u, v\rangle$, then $\operatorname{rank}(A_1 + A_2) = \operatorname{rank}(u \oslash v + u_1 \oslash v_1 + \gamma_1 u_1^2) > 2$, a contradiction. By (2.9), we have $v_1 \notin \langle u, v\rangle$, and so $u_1 \in \langle u, v, v_1\rangle$. Let $u_1 = c_{10}u + d_{10}v + g_{10}v_1$ for some scalars $c_{10}, d_{10}, g_{10} \in \mathbb{F}$. Let $\lambda \in \mathbb{F}$. Then

$$\lambda A_1 + A_2 = \gamma_1 c_{10}^2 u + \gamma_1 d_{10}^2 v^2 + \gamma_1 g_{10}^2 v_1^2 + (\lambda + \gamma_1 c_{10} d_{10})(u \oslash v) + (c_{10} + \gamma_1 c_{10} g_{10})(u \oslash v_1) + (d_{10} + \gamma_1 d_{10} g_{10})(v \oslash v_1).$$

Since $\operatorname{rank}(\lambda A_1 + A_2) \leqslant 2$, we have

$$0 = \det\begin{pmatrix} c_{10} + \gamma_1 c_{10} g_{10} & \lambda + \gamma_1 c_{10} d_{10} & \gamma_1 c_{10}^2 \\ d_{10} + \gamma_1 d_{10} g_{10} & \gamma_1 d_{10}^2 & \lambda + \gamma_1 c_{10} d_{10} \\ \gamma_1 g_{10}^2 & d_{10} + \gamma_1 d_{10} g_{10} & c_{10} + \gamma_1 c_{10} g_{10} \end{pmatrix} = \gamma_1 g_{10}^2 \lambda^2$$

for every $\lambda \in \mathbb{F}$. Since $|F| \geqslant 3$ and $\gamma_1 \neq 0$, we get $g_{10} = 0$, and so $u_1 = c_{10}u + d_{10}v$. If $c_{10} = 0$, then $d_{10} \neq 0$, and so $v = d_{10}^{-1}u_1 \in \mathcal{U}_s$. Thus, $A_1 = v \oslash u = u_1 \oslash d_{10}^{-1}u$. Hence, we have $A_1 = w_{10} \oslash y_{10}$ and $A_2 = w_{10} \oslash z_{10} + \gamma_1 w_{10}^2$, where $w_{10} = u_1 \in \mathcal{U}_s$, $y_{10} = d_{10}^{-1}u \in \mathcal{U}_{h_4}$ and $z_{10} = v_1 \in \mathcal{U}_{h_4}$, with $h_4 = \max\{p, t\}$, are linearly independent vectors. Thus, (2.11) holds. If $c_{10} \neq 0$, then $A_1 = u \oslash v = u_1 \oslash c_{10}^{-1}v$. So, we have $A_1 = w_{11} \oslash y_{11}$ and $A_2 = w_{11} \oslash z_{11} + \gamma_1 w_{11}^2$, where $w_{11} = u_1 \in \mathcal{U}_s$, $y_{11} = c_{10}^{-1}v \in \mathcal{U}_{h_3}$ and $z_{11} = v_1 \in \mathcal{U}_{h_3}$, with $h_3 = \max\{q, t\}$, are linearly independent vectors. So, (2.11) holds true.

We now consider $\gamma_1 = 0$. We consider only $u_1 \notin \langle u, v\rangle$ as the second case $v_1 \notin \langle u, v\rangle$ can be verified similarly. Then $v_1 \in \langle u, v, u_1\rangle$. Let $v_1 = c_{11}u + d_{11}v + g_{11}u_1$ for some $c_{11}, d_{11}, g_{11} \in \mathbb{F}$. If $c_{11} = 0$, then $A_2 = u_1 \oslash v_1 = u_1 \oslash (d_{11}v + g_{11}u_1) = v \oslash d_{11}u_1$. So, $A_1 = w_{12} \oslash y_{12}$ and $A_2 = w_{12} \oslash z_{12}$, where $w_{12} = v \in \mathcal{U}_q$, $y_{12} = u \in \mathcal{U}_{h_1}$ and $z_{12} = d_{11}u_1 \in \mathcal{U}_{h_1}$, with $h_1 = \max\{p, s\}$, are linearly independent vectors. Therefore, (2.11) is proved. If $c_{11} \neq 0$, then $A_1 = u \oslash v = (c_{11}u + d_{11}v) \oslash c_{11}^{-1}v$ and

$A_2 = u_1 \oslash v_1 = u_1 \oslash (c_{11}u + d_{11}v + g_{11}u_1) = (c_{11}u + d_{11}v) \oslash u_1$. Therefore,

$$A_1 = w_{13} \oslash y_{13} \qquad \text{and} \qquad A_2 = w_{13} \oslash z_{13}$$

where $y_{13} = c_{11}^{-1}v \in \mathcal{U}_q$, $z_{13} = u_1 \in \mathcal{U}_s$ and $w_{13} = c_{11}u + d_{11}v \in \mathcal{U}_{h_5}$, with $h_5 = \max\{p, q\}$, are independent vectors. We see that if $q \leqslant \frac{n+1}{2}$, then (2.11) holds. If $q > \frac{n+1}{2}$, then $h_5 = q$. Since $w_{13} \oslash y_{13} \in \mathcal{ST}_n(\mathbb{F})$, it follows from Lemma 1.3 (c)(ii) that nonzero scalar $\alpha \in \mathbb{F}$ and some vector $y'_{13} \in \mathcal{U}_\ell$ with $1 \leqslant \ell \leqslant n + 1 - h_5 < \frac{n+1}{2}$ such that w_{13}, y'_{13} are linearly independent. Then

$$A_1 = w_{13} \oslash (\alpha w_{13} + y'_{13}) = w_{13} \oslash y'_{13}.$$

Note that if y'_{13}, z_{13} are linearly dependent, then $\langle w_{13}, y'_{13}\rangle = \langle w_{13}, z_{13}\rangle$, and so $\langle u_1, v_1\rangle = \langle w_{13}, z_{13}\rangle = \langle w_{13}, y'_{13}\rangle = \langle u, v\rangle$, a contradiction. Further, since $h_5 > \frac{n+1}{2} > \ell, s$, it follows that w_{13}, y'_{13}, z_{13} are linearly independent. Consequently, (2.11) holds. The proof is complete. $\square$

We are now in a position to give a classification of spaces of bounded rank-two persymmetric triangular matrices over a field with at least three elements.

Theorem 2.6. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n be an integer $\geqslant 2$. Then $\mathcal{S}$ is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$ if and only if $\mathcal{S}$ is of one of the following forms:*

(a) *$\mathcal{S} \subseteq \langle u^2, v^2, u \oslash v\rangle$ for some linearly independent vectors $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$.*

(b) *$\mathcal{S} = u \oslash U$ for some nonzero vector $u \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant n + 1 - q \leqslant n$.*

(c) $\mathcal{S} = u \oslash U + \langle u^2 \rangle$ *for some nonzero vector* $u \in \mathcal{U}_p$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and some subspace* U *of* $\mathcal{U}_q$ *with* $1 \leqslant q \leqslant n+1-p \leqslant n$.

(d) $\mathcal{S} = \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \rangle$ *for some linearly independent vectors* $u, v_1, \ldots, v_k$ *such that* $u \in \mathcal{U}_p$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and* $v_1, \ldots, v_k \in \mathcal{U}_q$ *with* $1 \leqslant q \leqslant n+1-p \leqslant n$, *and scalars* $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ *with* $(\lambda_1, \ldots, \lambda_k) \neq 0$.

(e) $\mathcal{S} = \langle u \oslash v,\ u \oslash w,\ v \oslash w \rangle$ *for some linearly independent vectors* $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ *and* $w \in \mathcal{U}_r$ *such that* $1 \leqslant p, q \leqslant n+1-r \leqslant n$ *and either* $p \leqslant n+1-q$, *or* $p = q > \frac{n+1}{2}$ *and* $v = \alpha u + z$ *for some nonzero scalar* $\alpha \in \mathbb{F}$ *and vector* $z \in \mathcal{U}_k$ *with* $1 \leqslant k \leqslant n+1-p < \frac{n+1}{2}$ *such that* u, z *are linearly independent.*

Moreover, if $\mathcal{S}$ *takes any of the forms* (**c**), (**d**) *or* (**e**), *then* $\mathbb{F}$ *has characteristic two.*

Proof. The sufficiency part follows immediately from Lemmas 2.3 and 2.4.

We now consider the necessity part. Let $\mathcal{S} \neq \{0\}$. Suppose that $\mathcal{S}$ has no rank two matrices. Let $A, B \in \mathcal{S}$ be nonzero matrices. Then A and B are of rank one, and by Lemma 1.2 (a), there exist nonzero vectors $x \in \mathcal{U}_p$ and $u \in \mathcal{U}_q$, with $1 \leqslant p, q \leqslant \frac{n+1}{2}$, such that $A = \alpha x^2$ and $B = \beta u^2$ for some nonzero scalars $\alpha, \beta \in \mathbb{F}$. If $\langle x \rangle \neq \langle u \rangle$, then $A + B = \alpha x^2 + \beta u^2 \in \mathcal{S}$ is of rank two, a contradiction. Hence, $\langle x \rangle = \langle u \rangle$, and so $\mathcal{S} = \langle u^2 \rangle$ for some vector $u \in \mathcal{U}_p$ such that $1 \leqslant p \leqslant \frac{n+1}{2}$.

Suppose that $\mathcal{S}$ has a rank two matrix, say A_1. In view of Lemma 1.2 (a), we see that either

$$A_1 = \alpha u^2 + \beta v^2 \tag{2.20}$$

for some linearly independent vectors $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$, and nonzero scalars $\alpha, \beta \in \mathbb{F}$; or

$$A_1 = u \oslash v + \gamma u^2 \tag{2.21}$$

for some linearly independent vectors $u \in \mathcal{U}_p$ and $v \in \mathcal{U}_q$ with $1 \leqslant p \leqslant n+1-q <$

$n+1-p$, and some scalar $\gamma \in \mathbb{F}$. We distinguish our proof into the following two main cases:

Case I: $\mathcal{S} \subseteq \langle u, v\rangle^2$. In view of Lemma 2.2 (b)(i), we notice that $\langle u, v\rangle^2 = \langle u^2, v^2, u \oslash v\rangle$ and it is a 3-dimensional subspace of $\mathcal{M}_n(\mathbb{F})$ since $\{u, v\}$ is linearly independent. If $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$, then we have

$$\mathcal{S} \subseteq \langle u^2, v^2, u \oslash v\rangle.$$

However, if $u \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $v \in \mathcal{U}_q$ with $\frac{n+1}{2} < q \leqslant n$, then $\mathcal{S} \subseteq \langle u^2, u \oslash v\rangle$. Since $\mathcal{S}$ contains a rank two matrix, in this case, we conclude that $\mathcal{S} = u \oslash \langle v\rangle$, or

$$\mathcal{S} = u \oslash \langle v\rangle + \langle u^2\rangle \quad \text{or} \quad \mathcal{S} = \langle u \oslash v + \gamma u^2\rangle \tag{2.22}$$

for some nonzero scalar $\gamma \in \mathbb{F}$. Note that if $\mathcal{S}$ is of a form in (2.22), then $\mathbb{F}$ has characteristic two by Lemma 1.2 (a). We are done.

Case II: $\mathcal{S} \nsubseteq \langle u, v\rangle^2$. Let $A_2 \in \mathcal{S}$ be a matrix such that $A_2 \notin \langle u, v\rangle^2$. By Lemma 1.2 (a), we have either $A_2 = \alpha_1 u_1^2 + \beta_1 v_1^2$ for some linearly independent vectors $u_1, v_1 \in \mathcal{U}_s$ with $1 \leqslant s \leqslant \frac{n+1}{2}$, and scalars $\alpha_1, \beta_1 \in \mathbb{F}$; or $A_2 = u_1 \oslash v_1 + \gamma_1 u_1^2$ for some linearly independent vectors $u_1 \in \mathcal{U}_s$ and $v_1 \in \mathcal{U}_t$, with $1 \leqslant s \leqslant n+1-t < n+1-s$, and scalar $\gamma_1 \in \mathbb{F}$. Since $A_2 \notin \langle u, v\rangle^2$, it follows that $u_1 \notin \langle u, v\rangle$ or $v_1 \notin \langle u, v\rangle$. We first show that $\operatorname{rank} A_2 = 2$. Suppose to the contrary that $\operatorname{rank} A_2 = 1$. Then $A_2 = \alpha_1 u_1^2$ for some nonzero scalar $\alpha_1 \in \mathbb{F}$ with $u_1 \notin \langle u, v\rangle$. If A_1 is of form (2.20), then $A_1 + A_2 = \alpha u^2 + \beta v^2 + \alpha_1 u_1^2$ is of rank 3, a contradiction. Therefore, A_1 is of form (2.21), and so $A_1 + A_2 = u \oslash v + \gamma u^2 + \alpha_1 u_1^2$ is of rank 3 because

$$\det\begin{pmatrix} 0 & 0 & \alpha_1 \\ 1 & \gamma & 0 \\ 0 & 1 & 0 \end{pmatrix} = \alpha_1 \neq 0,$$

a contradiction. So, $\operatorname{rank} A_2 = 2$. We now divide our proof into the following two cases:

Case II-1: $\mathbb{F}$ has characteristic two. By Lemma 2.5 (b), A_1 and A_2 can be represented as

$$A_1 = w \oslash y + \varsigma_1 w^2 \qquad \text{and} \qquad A_2 = w \oslash z + \varsigma_2 w^2 \tag{2.23}$$

for some scalars $\varsigma_1, \varsigma_2 \in \mathbb{F}$ and linearly independent vectors $w \in \mathcal{U}_\ell$ and $y, z \in \mathcal{U}_h$ with $1 \leqslant \ell \leqslant n+1-h \leqslant n$, and $1 \leqslant \ell \leqslant \frac{n+1}{2}$ when $(\varsigma_1, \varsigma_2) \neq 0$. Let $A \in \mathcal{S}$ be a nonzero element. If A is of rank one, then, in view of Lemma 1.2 (a), A is of the form (1.7), i.e., $A = \lambda x^2$ for some nonzero scalar $\lambda \in \mathbb{F}$ and nonzero vector $x \in \mathcal{U}_{\ell_1}$ with $1 \leqslant \ell_1 \leqslant \frac{n+1}{2}$. Since $\operatorname{rank}(A + A_i) \leqslant 2$ for $i = 1, 2$, it follows from (2.23) that $x \in \langle w, y\rangle$ and $x \in \langle w, z\rangle$. Since w, y, z are linearly independent, we have $x \in \langle w, y\rangle \cap \langle w, z\rangle = \langle w\rangle$. Then $\ell = \ell_1 \leqslant \frac{n+1}{2}$ and

$$A = \lambda_A w^2 \qquad \text{for some scalar } \lambda_A \in \mathbb{F}. \tag{2.24}$$

We now consider A is of rank two. By Lemma 1.2 (a), we have either

$$A = \alpha_2 x_1^2 + \beta_2 x_2^2 \tag{2.25}$$

for some linearly independent vectors $x_1, x_2 \in \mathcal{U}_{h_1}$ with $1 \leqslant h_1 \leqslant \frac{n+1}{2}$, and scalars $\alpha_2, \beta_2 \in \mathbb{F}$; or

$$A = x_1 \oslash x_2 + \gamma_2 x_1^2 \tag{2.26}$$

for some linearly independent vectors $x_1 \in \mathcal{U}_{\ell_2}$ and $x_2 \in \mathcal{U}_{h_2}$ with $1 \leqslant \ell_2 \leqslant n+1-h_2 < n+1-\ell_2$, and scalar $\gamma_2 \in \mathbb{F}$. We divide our argument into the following two cases:

Case II-1-A: $x_i \notin \langle w, y, z\rangle$ for some $1 \leqslant i \leqslant 2$. For each $\zeta \in \mathbb{F}$, we denote

$$z_\zeta := y + \zeta z \qquad \text{and} \qquad B_\zeta := w \oslash z_\zeta + (\varsigma_1 + \zeta\varsigma_2)w^2.$$

Since $B_\zeta = A_1 + \zeta A_2 \in \mathcal{S}$ and $\langle x_1, x_2\rangle \neq \langle w, z_\zeta\rangle$ for every $\zeta \in \mathbb{F}$, it follows from Lemma 2.5 (b) that A and B_ζ can be rewritten in the form as in (2.11), i.e., there exist linearly independent vectors $w_\zeta \in \mathcal{U}_{\ell'}$ and $y_\zeta, v_\zeta \in \mathcal{U}_{h'}$ and scalars $\theta_\zeta, \vartheta_\zeta \in \mathbb{F}$ such that

$$A = w_\zeta \oslash y_\zeta + \theta_\zeta w_\zeta^2 \qquad \text{and} \qquad B_\zeta = w_\zeta \oslash v_\zeta + \vartheta_\zeta w_\zeta^2. \tag{2.27}$$

for some integers $1 \leqslant \ell' \leqslant n+1-h' \leqslant n$, and $1 \leqslant \ell' \leqslant \frac{n+1}{2}$ when $(\theta_\zeta, \vartheta_\zeta) \neq 0$. Since $w \oslash z_\zeta + (\varsigma_1 + \zeta\varsigma_2)w^2 = B_\zeta = w_\zeta \oslash v_\zeta + \vartheta_\zeta w_\zeta^2$, it follows from Lemma 1.3 (a) that $\langle w_\zeta, v_\zeta\rangle = \langle w, z_\zeta\rangle$. Therefore, for each $\zeta \in \mathbb{F}$, there exist scalars $a_\zeta, b_\zeta \in \mathbb{F}$ such that

$$w_\zeta = a_\zeta w + b_\zeta z_\zeta. \tag{2.28}$$

Let ζ_1 and ζ_2 be a pair of distinct scalars in $\mathbb{F}$. By an argument analogous to (2.27), we obtain $w_{\zeta_1} \oslash y_{\zeta_1} + \theta_{\zeta_1} w_{\zeta_1}^2 = A = w_{\zeta_2} \oslash y_{\zeta_2} + \theta_{\zeta_2} w_{\zeta_2}^2$, and so $\langle w_{\zeta_1}, y_{\zeta_1}\rangle = \langle w_{\zeta_2}, y_{\zeta_2}\rangle$ by Lemma 1.3 (a). Therefore

$$w_{\zeta_1} = c w_{\zeta_2} + d y_{\zeta_2} \tag{2.29}$$

for some $c, d \in \mathbb{F}$. On the other hand, in view of (2.25) and (2.26), and by Lemma 1.3 (a), we get $\langle w_{\zeta_2}, y_{\zeta_2}\rangle = \langle x_1, x_2\rangle$. By (2.28), we note that $w_{\zeta_2} \in \langle w, y, z\rangle$. It follows that $y_{\zeta_2} \notin \langle w, y, z \rangle$ since $x_i \notin \langle w, y, z \rangle$ for some $1 \leqslant i \leqslant 2$. Together with (2.28) and (2.29), we have $d = 0$, and so

$$w_{\zeta_1} = c\, w_{\zeta_2} \quad \text{with } c \neq 0. \tag{2.30}$$

Since w, y, z are linearly independent, we have $\langle z_{\zeta_1} \rangle \neq \langle z_{\zeta_2} \rangle$ for every pair of distinct elements $\zeta_1, \zeta_2 \in \mathbb{F}$. Thus, $w, z_{\zeta_1}, z_{\zeta_2}$ are linearly independent. By (2.28) and (2.30), we obtain

$$a_{\zeta_1} w + b_{\zeta_1} z_{\zeta_1} = c(a_{\zeta_2} w + b_{\zeta_2} z_{\zeta_2}) \quad \Rightarrow \quad (a_{\zeta_1} - c a_{\zeta_2}) w + b_{\zeta_1} z_{\zeta_1} + (-c b_{\zeta_2}) z_{\zeta_2} = 0. \tag{2.31}$$

Hence, $b_{\zeta_1} = b_{\zeta_2} = 0$. Since the result holds true for any two distinct scalars $\zeta_1, \zeta_2 \in \mathbb{F}$, it follows that $b_\zeta = 0$ for every $\zeta \in \mathbb{F}$. By (2.28), we have $\langle w_\zeta \rangle = \langle w \rangle$ for every $\zeta \in \mathbb{F}$. It follows from (2.27) that

$$A = w \oslash y_A + \alpha_A w^2 \tag{2.32}$$

for some scalar $\alpha_A \in \mathbb{F}$ and some nonzero vector $y_A \in \mathcal{U}_{h_3}$ such that either $1 \leqslant h_3 \leqslant n + 1 - \ell \leqslant n$, and $1 \leqslant \ell \leqslant \frac{n+1}{2}$ when $\alpha_A \neq 0$.

Case II-1-B: $x_i \in \langle w, y, z \rangle$ for $i = 1, 2$. Let $x_1 = a_1 w + b_1 y + c_1 z$ and $x_2 = a_2 w + b_2 y + c_2 z$ for some scalars $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{F}$. We first consider A is of form (2.25). Let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} A + \lambda A_1 = {} & (\alpha_2 a_1^2 + \beta_2 a_2^2 + \lambda \varsigma_1) w^2 + (\alpha_2 b_1^2 + \beta_2 b_2^2) y^2 + (\alpha_2 c_1^2 + \beta_2 c_2^2) z^2 \\ & + w \oslash [(\alpha_2 a_1 b_1 + \beta_2 a_2 b_2 + \lambda) y + (\alpha_2 a_1 c_1 + \beta_2 a_2 c_2) z] + (\alpha_2 b_1 c_1 + \beta_2 b_2 c_2) y \oslash z \end{aligned}$$

by (2.23) and (2.25). Since rank $(A+\lambda A_1) \leqslant 2$, we have

$$\det\begin{pmatrix} \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 + \lambda & \alpha_2 a_1^2 + \beta_2 a_2^2 + \lambda\varsigma_1 \\ \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 b_1^2 + \beta_2 b_2^2 & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 + \lambda \\ \alpha_2 c_1^2 + \beta_2 c_2^2 & \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 \end{pmatrix} = 0.$$

Then $(\alpha_2 c_1^2 + \beta_2 c_2^2)\lambda^2 + \varsigma_1 \alpha_2 \beta_2 (b_1 c_2 + b_2 c_1)^2 \lambda = 0$ for every $\lambda \in \mathbb{F}$. Since $|\mathbb{F}| \geqslant 3$, we get

$$\alpha_2 c_1^2 + \beta_2 c_2^2 = 0 = \varsigma_1 \alpha_2 \beta_2 (b_1 c_2 + b_2 c_1)^2. \tag{2.33}$$

Likewise, since rank $(A+\lambda A_2) \leqslant 2$, we get

$$\det\begin{pmatrix} \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 + \lambda & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 & \alpha_2 a_1^2 + \beta_2 a_2^2 + \lambda\varsigma_2 \\ \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 b_1^2 + \beta_2 b_2^2 & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 \\ \alpha_2 c_1^2 + \beta_2 c_2^2 & \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 + \lambda \end{pmatrix} = 0,$$

and so $(\alpha_2 b_1^2 + \beta_2 b_2^2)\lambda^2 + \varsigma_2 \alpha_2 \beta_2 (b_1 c_2 + b_2 c_1)^2 \lambda = 0$ for every $\lambda \in \mathbb{F}$. Therefore, we obtain

$$\alpha_2 b_1^2 + \beta_2 b_2^2 = 0 = \varsigma_2 \alpha_2 \beta_2 (b_1 c_2 + b_2 c_1)^2. \tag{2.34}$$

If $(\varsigma_1, \varsigma_2) \neq 0$, then, by (2.33) and (2.34), we have $\alpha_2 \beta_2 (b_1 c_2 + b_2 c_1)^2 = 0$. Notice that

$$(\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)^2 = (\alpha_2 b_1^2 + \beta_2 b_2^2)(\alpha_2 c_1^2 + \beta_2 c_2^2) + \alpha_2 \beta_2 (b_1 c_2 + b_2 c_1)^2 = 0.$$

Thus, $\alpha_2 b_1 c_1 + \beta_2 b_2 c_2 = 0$. Together with (2.33) and (2.34), we have

$$A = w \oslash ((\alpha_2 a_1 b_1 + \beta_2 a_2 b_2)y + (\alpha_2 a_1 c_1 + \beta_2 a_2 c_2)z) + (\alpha_2 a_1^2 + \beta_2 a_2^2)w^2. \tag{2.35}$$

Suppose that $(\varsigma_1, \varsigma_2) = 0$, and that $\alpha_2 b_1 c_1 + \beta_2 b_2 c_2 \neq 0$. Then, by (2.33) and (2.34), we have

$$A = (\alpha_2 a_1^2 + \beta_2 a_2^2)w^2 + (\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)\, y \oslash z$$

$$+\ (\alpha_2 a_1 b_1 + \beta_2 a_2 b_2)\, w \oslash y + (\alpha_2 a_1 c_1 + \beta_2 a_2 c_2)\, w \oslash z.$$

Since $\operatorname{rank} A \leqslant 2$, it follows that $\alpha_2 a_1^2 + \beta_2 a_2^2 = 0$. Hence, we have

$$A = (\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)\, y \oslash z + (\alpha_2 a_1 b_1 + \beta_2 a_2 b_2)\, w \oslash y + (\alpha_2 a_1 c_1 + \beta_2 a_2 c_2)\, w \oslash z. \quad (2.36)$$

We now consider A is of form (2.26). Let $\lambda \in \mathbb{F}$. Then

$$A + \lambda A_1 = (\gamma_2 a_1^2 + \lambda \varsigma_1)w^2 + \gamma_2 b_1^2 y^2 + \gamma_2 c_1^2 z^2 + (\gamma_2 b_1 c_1 + b_1 c_2 + b_2 c_1) y \oslash z$$

$$+\ w \oslash [(\gamma_2 a_1 b_1 + a_1 b_2 + a_2 b_1 + \lambda) y + (\gamma_2 a_1 c_1 + a_1 c_2 + a_2 c_1) z]$$

by (2.23) and (2.26). Since $\operatorname{rank}(A + \lambda A_1) \leqslant 2$, it follows that

$$\det\begin{pmatrix} \gamma_2 a_1 c_1 + a_1 c_2 + a_2 c_1 & \gamma_2 a_1 b_1 + a_1 b_2 + a_2 b_1 + \lambda & \gamma_2 a_1^2 + \lambda \varsigma_1 \\ \gamma_2 b_1 c_1 + b_1 c_2 + b_2 c_1 & \gamma_2 b_1^2 & \gamma_2 a_1 b_1 + a_1 b_2 + a_2 b_1 + \lambda \\ \gamma_2 c_1^2 & \gamma_2 b_1 c_1 + b_1 c_2 + b_2 c_1 & \gamma_2 a_1 c_1 + a_1 c_2 + a_2 c_1 \end{pmatrix} = 0.$$

Then $\gamma_2 c_1^2 \lambda^2 + \varsigma_1 (b_1 c_2 + b_2 c_1)^2 \lambda = 0$ for every $\lambda \in \mathbb{F}$. Since $|\mathbb{F}| \geqslant 3$, we obtain

$$\gamma_2 c_1^2 = 0 = \varsigma_1 (b_1 c_2 + b_2 c_1)^2. \quad (2.37)$$

Similarly, since $\operatorname{rank}(A + \lambda A_2) \leqslant 2$, we get

$$\det\begin{pmatrix} \gamma_2 a_1 c_1 + a_1 c_2 + a_2 c_1 + \lambda & \gamma_2 a_1 b_1 + a_1 b_2 + a_2 b_1 & \gamma_2 a_1^2 + \lambda \varsigma_2 \\ \gamma_2 b_1 c_1 + b_1 c_2 + b_2 c_1 & \gamma_2 b_1^2 & \gamma_2 a_1 b_1 + a_1 b_2 + a_2 b_1 \\ \gamma_2 c_1^2 & \gamma_2 b_1 c_1 + b_1 c_2 + b_2 c_1 & \gamma_2 a_1 c_1 + a_1 c_2 + a_2 c_1 + \lambda \end{pmatrix} = 0,$$

and so $\gamma_2 b_1^2 \lambda^2 + \varsigma_2 (b_1 c_2 + b_2 c_1)^2 \lambda = 0$ for every $\lambda \in \mathbb{F}$. Thus

$$\gamma_2 b_1^2 = 0 = \varsigma_2 (b_1 c_2 + b_2 c_1)^2. \tag{2.38}$$

If $(\varsigma_1, \varsigma_2) \neq 0$, then, in view of (2.37) and (2.38), we have $(b_1 c_2 + b_2 c_1)^2 = 0$, and so $b_1 c_2 + b_2 c_1 = 0$. Furthermore, $(\gamma_2 b_1 c_1)^2 = (\gamma_2 b_1^2)(\gamma_2 c_1^2) = 0$ implies that $\gamma_2 b_1 c_1 = 0$. Hence, we obtain

$$A = w \oslash ((\gamma_2 a_1 b_1 + a_1 b_2 + a_2 b_1) y + (\gamma_2 a_1 c_1 + a_1 c_2 + a_2 c_1) z) + (\gamma_2 a_1^2) w^2. \tag{2.39}$$

Suppose now that $(\varsigma_1, \varsigma_2) = 0$ and that $b_1 c_2 + b_2 c_1 \neq 0$. Then $(b_1, c_1) \neq 0$. By (2.37) and (2.38), we conclude that $\gamma_2 = 0$, and so

$$A = (b_1 c_2 + b_2 c_1)\, y \oslash z + (a_1 b_2 + a_2 b_1)\, w \oslash y + (a_1 c_2 + a_2 c_1)\, w \oslash z. \tag{2.40}$$

Since $y \oslash z \in \mathcal{ST}_n(\mathbb{F})$, it follows that if $h > \frac{n+1}{2}$, then, by Lemma 1.3 (c) (ii), we have $z = \alpha y + y'$ for some nonzero scalar $\alpha \in \mathbb{F}$ and some vector $y' \in \mathcal{U}_{\ell'}$ with $1 \leqslant \ell' \leqslant n + 1 - h < \frac{n+1}{2}$ such that y, y' are linearly independent. Consequently, since $A_1, A_2 \in \mathcal{S}$ and $w \in \mathcal{U}_\ell$, $y, z \in \mathcal{U}_h$ are linearly independent vectors, it follows from (2.24), (2.32), (2.35), (2.36), (2.39) and (2.40) that if $\mathcal{S}$ contains no rank one matrices, then we have either

$$\mathcal{S} = w \oslash V_1$$

for some subspace V_1 of $\mathcal{U}_{k_1}$ with $1 \leqslant \ell \leqslant n + 1 - k_1 \leqslant n$, and $y, z \in V_1$ and $(\varsigma_1, \varsigma_2) = 0$, or by Lemma 2.4 (a), we get

$$\mathcal{S} = \left\langle w \oslash y_1 + \lambda_1 w^2, \ldots, w \oslash y_k + \lambda_k w^2 \right\rangle$$

for some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$, and some vectors $y_1, \ldots, y_k \in \mathcal{U}_{k_2}$ with $1 \leqslant k_2 \leqslant n+1-\ell$ and $1 \leqslant \ell \leqslant \frac{n+1}{2}$ such that $w, y_1, \ldots, y_k$ are linearly independent, or by Lemma 1.3 (d), we have

$$\mathcal{S} = \langle w \oslash y,\, w \oslash z,\, y \oslash z \rangle\,;$$

or if $\mathcal{S}$ contains rank one matrices, then, by Lemma 2.4 (b), we get

$$\mathcal{S} = w \oslash V_2 + \langle w^2 \rangle$$

for some subspace V_2 of $\mathcal{U}_{k_3}$ with $1 \leqslant k_3 \leqslant n+1-\ell \leqslant n$ and $1 \leqslant \ell \leqslant \frac{n+1}{2}$, and $y, z \in V_2$. We are done.

Case II-2: $\mathbb{F}$ has characteristic not two. In view of Lemma 2.5 (a), we see that A_1 and A_2 can be rewritten as

$$A_1 = w \oslash y \qquad \text{and} \qquad A_2 = w \oslash z \tag{2.41}$$

for some linearly independent vectors $w \in \mathcal{U}_p$ and $y, z \in \mathcal{U}_q$ with $1 \leqslant p \leqslant n+1-q \leqslant n$. Let A be an arbitrary nonzero element of $\mathcal{S}$. If A is of rank one, then, by an argument analogous to (2.24), we can show that $p \leqslant \frac{n+1}{2}$ and

$$A = \lambda_A w^2 \quad \text{ for some scalar } \lambda_A \in \mathbb{F}. \tag{2.42}$$

We now consider A is of rank two. By Lemma 1.2 (a), we have

$$A = \alpha_2 x_1^2 + \beta_2 x_2^2 \tag{2.43}$$

for some linearly independent vectors $x_1, x_2 \in \mathcal{U}_{q_1}$ with $1 \leqslant q_1 \leqslant \frac{n+1}{2}$, and scalars $\alpha_2, \beta_2 \in \mathbb{F}$; or

$$A = x_1 \oslash x_2 + \gamma_2 x_1^2 \tag{2.44}$$

for some linearly independent vectors $x_1 \in \mathcal{U}_{p_2}$ and $x_2 \in \mathcal{U}_{q_2}$, with $1 \leqslant p_2 \leqslant n+1-q_2 < n+1-p_2$, and scalar $\gamma_2 \in \mathbb{F}$. We divide our argument into the following two cases:

Case II-2-A: $x_i \notin \langle w, y, z \rangle$ for some $1 \leqslant i \leqslant 2$. For each $\zeta \in \mathbb{F}$, we denote

$$z_\zeta = y + \zeta z \quad \text{ and } \quad B_\zeta = w \oslash z_\zeta.$$

By the hypothesis of Case II-1-A, we see that $\langle x_1, x_2 \rangle \neq \langle w, z_\zeta \rangle$ for every $\zeta \in \mathbb{F}$. Since $B_\zeta = A_1 + \zeta A_2 \in \mathcal{S}$, it follows from Lemma 2.5 (a) that there exist linearly independent vectors $w_\zeta \in \mathcal{U}_{p_3}$ and $y_\zeta, v_\zeta \in \mathcal{U}_{q_3}$ with $1 \leqslant p_3 \leqslant n+1-q_3 \leqslant n$ such that A and B_ζ can be rewritten as

$$A = w_\zeta \oslash y_\zeta \quad \text{ and } \quad B_\zeta = w_\zeta \oslash v_\zeta. \tag{2.45}$$

By an argument analogous to Case II-1-A, we can show that $\langle w_\zeta \rangle = \langle w \rangle$ for every $\zeta \in \mathbb{F}$. In view of (2.45), we have

$$A = w \oslash y_A \tag{2.46}$$

for some nonzero vector $y_A \in \mathcal{U}_{p_4}$ with $1 \leqslant p \leqslant n+1-p_4 \leqslant n$.

Case II-2-B: $x_i \in \langle w, y, z \rangle$ for $i = 1, 2$. Let $x_1 = a_1 w + b_1 y + c_1 z$ and $x_2 = a_2 w + b_2 y + c_2 z$ for some scalars $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{F}$. We first consider A is of form (2.43). Let $\lambda \in \mathbb{F}$. Then

$$A + \lambda A_1 = \alpha_2 x_1^2 + \beta_2 x_2^2 + \lambda\, w \oslash y$$

$$= (\alpha_2 a_1^2 + \beta_2 a_2^2) w^2 + (\alpha_2 b_1^2 + \beta_2 b_2^2) y^2 + (\alpha_2 c_1^2 + \beta_2 c_2^2) z^2$$

$$+ w \oslash [(\alpha_2 a_1 b_1 + \beta_2 a_2 b_2 + \lambda) y + (\alpha_2 a_1 c_1 + \beta_2 a_2 c_2) z] + (\alpha_2 b_1 c_1 + \beta_2 b_2 c_2) y \oslash z.$$

Since rank $(A + \lambda A_1) \leqslant 2$, we have

$$\det \begin{pmatrix} \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 + \lambda & \alpha_2 a_1^2 + \beta_2 a_2^2 \\ \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 b_1^2 + \beta_2 b_2^2 & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 + \lambda \\ \alpha_2 c_1^2 + \beta_2 c_2^2 & \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 \end{pmatrix} = 0,$$

and so

$$(\alpha_2 c_1^2 + \beta_2 c_2^2)\lambda^2 + 2[(\alpha_2 c_1^2 + \beta_2 c_2^2)(\alpha_2 a_1 b_1 + \beta_2 a_2 b_2) - (\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)(\alpha_2 a_1 c_1 + \beta_2 a_2 c_2)]\lambda = 0$$

for every $\lambda \in \mathbb{F}$. Since $|\mathbb{F}| \geqslant 3$, it follows that

$$\alpha_2 c_1^2 + \beta_2 c_2^2 = 0, \tag{2.47}$$

$$(\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)(\alpha_2 a_1 c_1 + \beta_2 a_2 c_2) = 0. \tag{2.48}$$

Similarly, since rank $(A + \lambda A_2) \leqslant 2$, we get

$$\det \begin{pmatrix} \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 + \lambda & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 & \alpha_2 a_1^2 + \beta_2 a_2^2 \\ \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 b_1^2 + \beta_2 b_2^2 & \alpha_2 a_1 b_1 + \beta_2 a_2 b_2 \\ \alpha_2 c_1^2 + \beta_2 c_2^2 & \alpha_2 b_1 c_1 + \beta_2 b_2 c_2 & \alpha_2 a_1 c_1 + \beta_2 a_2 c_2 + \lambda \end{pmatrix} = 0,$$

and so

$$(\alpha_2 b_1^2 + \beta_2 b_2^2)\lambda^2 + 2[(\alpha_2 b_1^2 + \beta_2 b_2^2)(\alpha_2 a_1 c_1 + \beta_2 a_2 c_2)$$
$$-(\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)(\alpha_2 a_1 b_1 + \beta_2 a_2 b_2)]\lambda = 0$$

for every $\lambda \in \mathbb{F}$. Then

$$\alpha_2 b_1^2 + \beta_2 b_2^2 = 0, \tag{2.49}$$

$$(\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)(\alpha_2 a_1 b_1 + \beta_2 a_2 b_2) = 0. \tag{2.50}$$

We now claim that $\alpha_2 b_1 c_1 + \beta_2 b_2 c_2 = 0$. Suppose that $\alpha_2 b_1 c_1 + \beta_2 b_2 c_2 \neq 0$. By (2.48) and (2.50), $\alpha_2 a_1 c_1 + \beta_2 a_2 c_2 = 0 = \alpha_2 a_1 b_1 + \beta_2 a_2 b_2$. Together with (2.41), (2.43), (2.47) and (2.49), we have

$$A + \lambda A_1 + A_2 = (\alpha_2 a_1^2 + \beta_2 a_2^2)w^2 + (\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)y \oslash z + w \oslash (\lambda y + z)$$

is of rank at most two. Then $2(\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)\lambda - (\alpha_2 a_1^2 + \beta_2 a_2^2)(\alpha_2 b_1 c_1 + \beta_2 b_2 c_2)^2 = 0$ for all $\lambda \in \mathbb{F}$. Since $|\mathbb{F}| \geqslant 3$, we conclude that $\alpha_2 b_1 c_1 + \beta_2 b_2 c_2 = 0$, a contradiction. Therefore, we have $\alpha_2 b_1 c_1 + \beta_2 b_2 c_2 = 0$, as claimed. So, we have

$$A = w \oslash ((\alpha_2 a_1 b_1 + \beta_2 a_2 b_2)y + (\alpha_2 a_1 c_1 + \beta_2 a_2 c_2)z + 2^{-1}(\alpha_2 a_1^2 + \beta_2 a_2^2)w). \tag{2.51}$$

Next, we consider A is of form (2.44). Let $\lambda \in \mathbb{F}$. By (2.41) and (2.44), we have

$$A + \lambda A_1 = 2a_1 a_2 w^2 + 2b_1 b_2 y^2 + 2c_1 c_2 z^2$$
$$+ w \oslash [(a_1 b_2 + a_2 b_1 + \lambda)y + (a_1 c_2 + a_2 c_1)z] + (b_1 c_2 + b_2 c_1)y \oslash z.$$

Since $\operatorname{rank}(A+\lambda A_1) \leqslant 2$, it follows that

$$\det\begin{pmatrix} a_1c_2+a_2c_1 & a_1b_2+a_2b_1+\lambda & 2a_1a_2 \\ b_1c_2+b_2c_1 & 2b_1b_2 & a_1b_2+a_2b_1+\lambda \\ 2c_1c_2 & b_1c_2+b_2c_1 & a_1c_2+a_2c_1 \end{pmatrix} = 0.$$

So, $2c_1c_2\lambda^2 + 2[(2c_1c_2)(a_1b_2+a_2b_1) - (b_1c_2+b_2c_1)(a_1c_2+a_2c_1)]\lambda = 0$ for all $\lambda \in \mathbb{F}$. Since $|\mathbb{F}| \geqslant 3$, we have

$$2c_1c_2 = 0 = (b_1c_2+b_2c_1)(a_1c_2+a_2c_1). \tag{2.52}$$

Similarly, since $\operatorname{rank}(A+\lambda A_2) \leqslant 2$, we have

$$\det\begin{pmatrix} a_1c_2+a_2c_1+\lambda & a_1b_2+a_2b_1 & 2a_1a_2 \\ b_1c_2+b_2c_1 & 2b_1b_2 & a_1b_2+a_2b_1 \\ 2c_1c_2 & b_1c_2+b_2c_1 & a_1c_2+a_2c_1+\lambda \end{pmatrix} = 0,$$

and so $2b_1b_2\lambda^2 + 2[(2b_1b_2)(a_1c_2+a_2c_1) - (b_1c_2+b_2c_1)(a_1b_2+a_2b_1)]\lambda = 0$ for all $\lambda \in \mathbb{F}$. Then

$$2b_1b_2 = 0 = (b_1c_2+b_2c_1)(a_1b_2+a_2b_1). \tag{2.53}$$

Suppose that $b_1c_2+b_2c_1 \neq 0$. By (2.52) and (2.53), we have $a_1c_2+a_2c_1 = 0 = a_1b_2+a_2b_1$. Therefore, we obtain

$$A+\lambda A_1+A_2 = 2a_1a_2w^2 + (b_1c_2+b_2c_1)y \oslash z + w \oslash (\lambda y + z)$$

is of rank at most 2 for all $\lambda \in \mathbb{F}$. So, $(b_1c_2+b_2c_1)\lambda - a_1a_2(b_1c_2+b_2c_1)^2 = 0$ for all $\lambda \in \mathbb{F}$. Since $|\mathbb{F}| \geqslant 3$, we obtain $b_1c_2+b_2c_1 = 0$, a contradiction. So, we have

$b_1c_2 + b_2c_1 = 0$, and hence

$$A = w \oslash (a_1a_2w + (a_1b_2 + a_2b_1)y + (a_1c_2 + a_2c_1)z). \tag{2.54}$$

Since $A_1, A_2 \in \mathcal{S}$ and $w \in \mathcal{U}_\ell$, $y, z \in \mathcal{U}_h$ are linearly independent vectors, together with (2.42), (2.46), (2.51) and (2.54), we conclude that

$$\mathcal{S} = w \oslash U$$

for some nonzero vector $w \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_r$ with $1 \leqslant p \leqslant n+1-r \leqslant n$. The proof is complete. $\square$

As an immediate consequence of Lemmas 2.3 (d) and 2.4 (a) and Theorem 2.6, we give a complete description of rank-two subspaces of $\mathcal{ST}_n(\mathbb{F})$ over a field $\mathbb{F}$ with at least three elements.

Theorem 2.7. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n be an integer $\geqslant 2$. Then $\mathcal{S}$ is a rank-two subspace of $\mathcal{ST}_n(\mathbb{F})$ if and only if $\mathcal{S}$ is of one of the following forms:*

(a) $\mathcal{S} = \langle \alpha_1 u \oslash v + \alpha_2 u^2 + \alpha_3 v^2,\ \beta_1 u \oslash v + \beta_2 u^2 + \beta_3 v^2 \rangle$ *for some linearly independent vectors $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$, and some fixed scalars $\alpha_1, \alpha_2, \alpha_3,\ \beta_1, \beta_2, \beta_3 \in \mathbb{F}$ such that*

$$\det\begin{pmatrix} a\alpha_1 + b\beta_1 & a\alpha_2 + b\beta_2 \\ a\alpha_3 + b\beta_3 & a\alpha_1 + b\beta_1 \end{pmatrix} \neq 0$$

for every scalar $a, b \in \mathbb{F}$ with $(a, b) \neq 0$.

(b) $\mathcal{S} = u \oslash U$ *for some nonzero vector $u \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant n + 1 - q \leqslant n$ when $\operatorname{char}\mathbb{F} = 2$, and $U \cap \langle u \rangle = \{0\}$ when $\operatorname{char}\mathbb{F} \neq 2$.*

(c) $\mathcal{S} = \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \rangle$ *for some linearly independent vectors*

$u, v_1, \ldots, v_k$ such that $u \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $v_1, \ldots, v_k \in \mathcal{U}_q$ with $1 \leqslant q \leqslant n+1-p$, and some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$.

(d) *$\mathcal{S} = \langle u \oslash v,\ u \oslash w,\ v \oslash w\rangle$ for some linearly independent vectors $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ and $w \in \mathcal{U}_r$ such that $1 \leqslant p, q \leqslant n+1-r \leqslant n$ and either $p \leqslant n+1-q$, or $p = q > \frac{n+1}{2}$ and $v = \alpha u + z$ for some nonzero scalar $\alpha \in \mathbb{F}$ and vector $z \in \mathcal{U}_k$ with $1 \leqslant k \leqslant n+1-p < \frac{n+1}{2}$ such that u, z are linearly independent.*

Moreover, if $\mathcal{S}$ takes the form (**c**) *or* (**d**)*, then $\mathbb{F}$ has characteristic two.*

Proof. The sufficiency part of the theorem is clear. We now prove the necessity. Since $\mathcal{S}$ is a rank-two subspace of $\mathcal{ST}_n(\mathbb{F})$, it follows from Lemmas 2.3 (d) and 2.4 (a) and Theorem 2.6 that $\mathcal{S}$ takes one of the following forms:

(A) $\mathcal{S} \subseteq \langle u^2, v^2, u \oslash v\rangle$ for some linearly independent vectors $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$.

(B) $\mathcal{S} = u \oslash U$ for some nonzero vector $u \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant n+1-q \leqslant n$ when $\operatorname{char}\mathbb{F} = 2$, and $U \cap \langle u\rangle = \{0\}$ when $\operatorname{char}\mathbb{F} \neq 2$.

(C) $\mathcal{S} = \langle u \oslash v_1 + \lambda_1 u^2,\ \ldots,\ u \oslash v_k + \lambda_k u^2\rangle$ for some linearly independent vectors $u, v_1, \ldots, v_k$ such that $u \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $v_1, \ldots, v_k \in \mathcal{U}_q$ with $1 \leqslant q \leqslant n+1-p$, and some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$.

(D) $\mathcal{S} = \langle u \oslash v,\ u \oslash w,\ v \oslash w\rangle$ for some linearly independent vectors $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ and $w \in \mathcal{U}_r$ such that $1 \leqslant p, q \leqslant n+1-r \leqslant n$ and either $p \leqslant n+1-q$, or $p = q > \frac{n+1}{2}$ and $v = \alpha u + z$ for some nonzero scalar $\alpha \in \mathbb{F}$ and vector $z \in \mathcal{U}_k$ with $1 \leqslant k \leqslant n+1-p < \frac{n+1}{2}$ such that u, z are linearly independent.

Moreover, if $\mathcal{S}$ is of Form (**C**) or Form (**D**), then $\mathbb{F}$ has characteristic two. It is clear that Forms (**B**), (**C**) and (**D**) are rank-two subspaces of $\mathcal{ST}_n(\mathbb{F})$. We now consider

$\mathcal{S}$ of Form (**A**). If $\dim \mathcal{S} = 3$, then $\mathcal{S} = \langle u^2, v^2, u \oslash v \rangle$, and so $\mathcal{S}$ consists of rank one matrices, a contradiction. Thus $\dim \mathcal{S} \leqslant 2$ and we have

$$\mathcal{S} = \left\langle \alpha_1 u \oslash v + \alpha_2 u^2 + \alpha_3 v^2,\ \beta_1 u \oslash v + \beta_2 u^2 + \beta_3 v^2 \right\rangle$$

for some fixed scalars $\alpha_1, \alpha_2, \alpha_3,\ \beta_1, \beta_2, \beta_3 \in \mathbb{F}$. If all scalars α_i and β_i are zero, then $\mathcal{S} = \{0\}$ and it is of Form (**B**). Suppose that $\mathcal{S} \neq \{0\}$. Let $A \in \mathcal{S}$ be nonzero. Then there exist scalars $a, b \in \mathbb{F}$ with $(a, b) \neq 0$ such that $A = a(\alpha_1 u \oslash v + \alpha_2 u^2 + \alpha_3 v^2) + b(\beta_1 u \oslash v + \beta_2 u^2 + \beta_3 v^2)$, and so $A = (a\alpha_1 + b\beta_1) u \oslash v + (a\alpha_2 + b\beta_2) u^2 + (a\alpha_3 + b\beta_3) v^2$ is of rank two. Then we have

$$\det \begin{pmatrix} a\alpha_1 + b\beta_1 & a\alpha_2 + b\beta_2 \\ a\alpha_3 + b\beta_3 & a\alpha_1 + b\beta_1 \end{pmatrix} \neq 0.$$

This completes our proof. $\square$

We give a few examples of rank-two subspaces of $\mathcal{ST}_n(\mathbb{F})$ to illustrate the form of type (a) in Theorem 2.7.

Example 2.8. Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n be an integer $\geqslant 3$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathcal{M}_{n,1}(\mathbb{F})$.

(a) Let $\mathcal{S}_1 = \langle e_1^2 + e_2^2 \rangle$. Clearly, each nonzero element in $\mathcal{S}_1$ is of the form $a(E_{1n} + E_{2,n-1})$ with $a \in \mathbb{F}$ nonzero which is certainly of rank two. Therefore, $\mathcal{S}_1$ is a 1-dimensional rank-two subspace of $\mathcal{ST}_n(\mathbb{F})$.

(b) Suppose that $\mathbb{F} = \mathbb{R}$.

(i) Let $\mathcal{S}_2 = \langle e_1 \oslash e_2 + e_1^2,\ e_1 \oslash e_2 + 2e_2^2 \rangle$. Then $\mathcal{S}_2$ is a 2-dimensional subspace of $\mathcal{ST}_n(\mathbb{R})$. Let $A \in \mathcal{S}_2$ be nonzero. Then there are scalars $a, b \in \mathbb{R}$ not all zero such that

$$\begin{aligned} A &= a(e_1 \oslash e_2 + e_1^2) + b(e_1 \oslash e_2 + 2e_2^2) \\ &= (a+b)E_{1,n-1} + aE_{1n} + 2bE_{2,n-1} + (a+b)E_{2n}. \end{aligned}$$

Note that $(a,b) \neq 0$ implies

$$\det \begin{pmatrix} a+b & a \\ 2b & a+b \end{pmatrix} = (a+b)^2 - 2ab = a^2 + b^2 \neq 0.$$

Thus, A is of rank two. Then $\mathcal{S}_2$ is a rank-two subspace of $\mathcal{ST}_n(\mathbb{R})$.

(ii) Let $\mathcal{S}_3 = \langle e_1 \oslash e_2,\, e_1 \oslash e_2 + e_1^2 - e_2^2 \rangle$. Let A be a nonzero element in $\mathcal{S}_3$. Then $A = (a+b)E_{1,n-1} + bE_{1n} - bE_{2,n-1} + (a+b)E_{2n}$ for some scalars $a, b \in \mathbb{R}$ with $(a,b) \neq 0$. Since

$$\det \begin{pmatrix} a+b & b \\ -b & a+b \end{pmatrix} = (a+b)^2 + b^2 \neq 0,$$

it follows that $\mathcal{S}_3$ is a 2-dimensional rank-two subspace of $\mathcal{ST}_n(\mathbb{R})$.

(c) Let $\mathbb{F} = \{0, 1, \alpha, \beta\}$ be a finite field with four elements with the addition and multiplication tables

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

Clearly $\mathbb{F}$ has characteristic two. Let $\mathcal{S}_4 = \langle e_1 \oslash e_2 + e_1^2, e_1 \oslash e_2 + \alpha e_2^2 \rangle \subseteq \mathcal{ST}_n(\mathbb{F})$. We wish to show that every nonzero element in $\mathcal{S}_4$ is of rank two, that is, $\lambda_1(e_1 \oslash e_2 + e_1^2) + \lambda_2(e_1 \oslash e_2 + \alpha e_2^2)$ is of rank two for any nonzero $\lambda_1, \lambda_2 \in \mathbb{F}$. Let $A = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 \\ \lambda_2 \alpha & \lambda_1 + \lambda_2 \end{pmatrix}$. Then we have $\det A = \lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2 \alpha$. The following tables

λ_1	λ_2	$\det A$	λ_1	λ_2	$\det A$	λ_1	λ_2	$\det A$
1	1	α	α	1	1	β	1	α
1	α	1	α	α	1	β	α	β
1	β	α	α	β	β	β	β	β

show that $\det A \neq 0$ for any nonzero $\lambda_1, \lambda_2 \in \mathbb{F}$. Hence $\mathcal{S}_4$ is a rank-two subspace of $\mathcal{ST}_n(\mathbb{F})$.

Chapter 3

BOUNDED RANK-TWO LINEAR PRESERVERS

As mentioned in the introduction, from the structural result in Theorem 2.6, we shall proceed to give a characterization of bounded rank-two linear preservers between persymmetric triangular matrix spaces.

Let $\mathbb{F}$ be a field with at least three elements and $m \geqslant n \geqslant 2$ be integers. Let $T : \mathcal{ST}_n(\mathbb{F}) \longrightarrow \mathcal{ST}_m(\mathbb{F})$ be a bounded rank-two linear preserver. Suppose that S is a subspace of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$. Then from our definition of bounded rank-two linear preservers, it is immediate that $T(S)$ is again a subspace of bounded rank-two matrices of $\mathcal{ST}_m(\mathbb{F})$. Consequently, Theorem 2.6 gives us the structure of $T(S)$.

The characterization of rank linear preservers can be a great help in the study of other types of linear preserver problems. For example, in [13], Watkins used a result of Marcus and Moyls [9] on rank-one linear preservers to characterize those nonsingular linear transformations on the space of $n \times n$ matrices ($n > 4$) over an algebraically closed field of characteristic 0 that preserve commuting pairs of matrices.

Here, we adapt the technique of constructing rank linear preservers from [1] to study bounded rank-two linear preservers between persymmetric triangular matrix spaces. The lemmas below play an important role in proving all the theorems in this chapter.

Lemma 3.1. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n be an integer $\geqslant 2$. Let $w \in \mathcal{U}_p$ and $u, v \in \mathcal{U}_q$ be vectors and let $a, b \in \mathbb{F}$ be scalars such that $w \oslash u + aw^2, w \oslash v + bw^2$*

are linearly independent. If

$$w \oslash u + aw^2, w \oslash v + bw^2 \in x \oslash X + \left\langle x^2 \right\rangle .$$

for some vector $x \in \mathcal{M}_{n,1}(\mathbb{F})$ and subspace X of $\mathcal{M}_{n,1}(\mathbb{F})$, then x, w are linearly dependent.

Proof. Suppose that $\langle u \rangle = \langle w \rangle$. Then $a \neq 0$. Since $w \oslash u + aw^2 \in x \oslash X + \langle x^2 \rangle$, then by Lemma 1.3(a) we have $x \in \langle w, u \rangle = \langle w \rangle$. The case $\langle v \rangle = \langle w \rangle$ can be verified similarly. Suppose that u, w are linearly independent vectors. Then consider the case $v \in \langle u, w \rangle$. Let $v = cu + dw$ for some scalars $c, d \in \mathbb{F}$ with $c \neq 0$. Then $w \oslash v + bw^2 = cw \oslash u + (2d + b)w^2$. Hence $w^2 \in x \oslash X + \langle x^2 \rangle$ implying $w \in \langle x \rangle$. Consider the case w, u, v are linearly independent. Since $w \oslash u + aw^2, w \oslash v + bw^2 \in x \oslash X + \langle x^2 \rangle$, then by Lemma 1.3(a) we have $x \in \langle w, u \rangle \cap \langle w, u \rangle = \langle w \rangle$. We are done. □

Lemma 3.2. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n be an integer $\geqslant 2$. Let $u, v, x, y \in \mathcal{M}_{n,1}(\mathbb{F})$ and let $\alpha, \beta \in \mathbb{F}$ such that $u \oslash v + \alpha u^2 = x \oslash y + \beta x^2$. If u, x are linearly independent, then $u \oslash v + \alpha u^2$ is of rank two, and the following assertions hold.*

(a) *If $\mathbb{F}$ has characteristic two, then $\alpha = \beta = 0$ and $x = au + bv$ and $y = b^{-1}(1 - ac)u + cv$ for some $a, b, c \in \mathbb{F}$ with $b \neq 0$.*

(b) *If $\mathbb{F}$ has characteristic not two, then either*

- **(i)** *$x = dv$ and $y = d^{-1}u - 2^{-1}\beta dv$ for some nonzero scalar $d \in \mathbb{F}$, or*
- **(ii)** *$x = gu + (\beta g + (2g)^{-1}\alpha)^{-1}v$ and $y = ((2g)^{-1}(\alpha - \beta g^2))u + (-2^{-1}\beta(\beta g + (2g)^{-1}\alpha)^{-1})v$ for some nonzero scalar $g \in \mathbb{F}$.*

Further if $\alpha = \beta = 0$, then **(i)** *holds.*

Proof. By Lemma 1.3(a), we have $\langle u, v\rangle = \langle x, y\rangle$. If $u \oslash v + \alpha u^2$ is of rank one then $\langle u\rangle = \langle v\rangle$ and so $x \in \langle u, v\rangle = \langle u\rangle$, a contradiction. Hence $u \oslash v + \alpha u^2$ is of rank two. This implies that u, v are linearly independent. Let $x = au + bv$ and $y = cu + dv$ for some scalars $a, b, c, d \in \mathbb{F}$ with $b \neq 0$. Then

$$u \oslash v + \alpha u^2 = (2ac + \beta a^2)u^2 + (ad + bc + 2\beta ab)u \oslash v + (2bd + \beta b^2)v^2.$$

Hence

$$2ac + \beta a^2 = \alpha, \tag{3.1}$$

$$ad + bc + 2\beta ab = 1, \tag{3.2}$$

$$2bd + \beta b^2 = 0. \tag{3.3}$$

We argue in the following two cases.

Case I: $\mathbb{F}$ has characteristic two. From (3.3), $\beta b^2 = 0$. This implies that $\beta = 0$ since $b \neq 0$. So by (3.1), $\alpha = 0$. Then (3.2) implies that $ad+bc = 1$ or $c = b^{-1}(1-ad)$.

Case II: $\mathbb{F}$ has characteristic not two. By (3.3), $d = -2^{-1}\beta b$. If $a = 0$, then it follows from (3.1), $\alpha = 0$. Then (3.2) implies that $bc = 1$ or $c = b^{-1}$ yielding $x = bv$ and $y = b^{-1}u - 2^{-1}\beta bv$. Suppose that $a \neq 0$. Then by (3.1), $c = (2a)^{-1}(\alpha - \beta a^2)$. It follows from (3.2) that $b = (\beta a + \frac{1}{2}a^{-1}\alpha)^{-1}$. Note that if $\alpha = \beta = 0$, then from (3.2) and (3.3), we see that $d = 0$ and $bc = 1$ implying $c \neq 0$. Then by (3.1) we have $a = 0$. The proof is complete. $\square$

Lemma 3.3. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n be an integer $\geqslant 2$. Let $u, v, w, z \in \mathcal{M}_{n,1}(\mathbb{F})$ be vectors and let $\alpha, \beta \in \mathbb{F}$ be scalars such that $u \oslash v + \alpha u^2, w \oslash z + \beta w^2$ are linearly independent. If w, u, v are linearly independent vectors such* *that* $u \oslash v + \alpha u^2 + \lambda(w \oslash z + \beta w^2)$ *has rank bounded by two for every* $\lambda \in \mathbb{F}$*, then the following assertions hold.*

(a) *If* $\mathbb{F}$ *has characteristic two, then either* $\beta = 0$ *and* $z \in \langle u, w\rangle \setminus \langle w\rangle$*, or* $\alpha = \beta = 0$ *and* $z \in \langle u, v, w\rangle \setminus \langle u, w\rangle$*.*

(b) *If* $\mathbb{F}$ *has characteristic not two, then either* $z = au - 2^{-1}\beta w$ *for some nonzero scalar* $a \in \mathbb{F}$ *or* $z = b(2^{-1}\alpha u + v) - 2^{-1}\beta w$ *for some nonzero scalar* $b \in \mathbb{F}$*.*

Proof. It is clear that $z \in \langle u, v, w\rangle$, otherwise $u \oslash v + \alpha u^2 + w \oslash z + \beta w^2$ is of rank four. Let $z = au + bv + cw$ for some $a, b, c \in \mathbb{F}$. We check that

$$u \oslash v + \alpha u^2 + \lambda(w \oslash z + \beta w^2) = u \oslash (\lambda a w + v) + \alpha u^2 + \lambda b w \oslash v + \lambda(\beta + 2c)w^2$$

is of rank bounded above by two for any $\lambda \in \mathbb{F}$, yielding

$$det \begin{pmatrix} 1 & \lambda a & \alpha \\ \lambda b & \lambda(\beta + 2c) & \lambda a \\ 0 & \lambda b & 1 \end{pmatrix} = \lambda^2(\alpha b^2 - 2ab) + \lambda(\beta + 2c) = 0.$$

Since $|\mathbb{F}| \geqslant 3$, we have $(\alpha b - 2a)b = \beta + 2c = 0$.

Case I: $\mathbb{F}$ has characteristic two. Then $\beta = 0$. If $b = 0$, then we have $z \in \langle u, w\rangle$ and we are done. Suppose that $b \neq 0$. Then $\alpha = 0$ and the result follows.

Case II: $\mathbb{F}$ has characteristic not two. Then $c = -2^{-1}\beta$. If $b = 0$, then $z = au - 2^{-1}\beta w$. If $b \neq 0$, then $a = 2^{-1}\alpha b$ and so $z = b(2^{-1}\alpha u + v) - 2^{-1}\beta w$ and we are done. □

Let $\{f_1, \ldots, f_m\}$ denote the standard ordered basis of $\mathcal{M}_{m,1}(\mathbb{F})$. Recall that

$$\mathcal{U}_i := \left\{ (u_1, \ldots, u_i, 0, \ldots, 0)^T \,\middle|\, u_1, \ldots, u_i \in \mathbb{F} \right\}.$$

Lemma 3.4. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$ and let n, m be integers such that $m \geqslant n \geqslant 2$. Let $P \in \mathcal{M}_{m,n}(\mathbb{F})$ be a matrix of rank n and let $T : \mathcal{ST}_n(\mathbb{F}) \longrightarrow \mathcal{ST}_m(\mathbb{F})$ be the linear map defined by*

$$T(A) = PAP^+$$

for all $A \in \mathcal{ST}_n(\mathbb{F})$. Then there exists an invertible matrix $Q \in \mathcal{M}_m(\mathbb{F})$ such that

$$T(A) = Q \left(\begin{array}{c|c} 0_{n,m-n} & A \\ \hline 0 & 0_{m-n,n} \end{array} \right) Q^+ \tag{3.4}$$

for all $A \in \mathcal{ST}_n(\mathbb{F})$, where $Qf_i = Pe_i \in \mathcal{U}_{p_i}$ and $Qf_j = Pe_j \in \mathcal{U}_{q_j}$ with $1 \leqslant p_i \leqslant \frac{m+1}{2}$ for every $1 \leqslant i \leqslant \frac{n+1}{2}$, and $q_j = max\{m + 1 - p_k \,|\, 1 \leqslant k \leqslant n + 1 - j\}$ for every $\frac{n+1}{2} < j \leqslant n + 1 - i$. In particular, $Q \in \mathcal{T}_n(\mathbb{F})$ when $m = n$.

Proof. We note that PAP^+ is upper triangular, it follows that $f^+_{m+1-s}(PAP^+)f_t = 0$ for all $s > t$. By letting $A = e^2_k$ for some $1 \leqslant k \leqslant \frac{n+1}{2}$, we see that $f^+_{m+1-s}Pe_k(f^+_t Pe_k)^+ = 0$. Since P is of rank n, then for each k there exists a corresponding t_k such that $f^+_{t_k}Pe_k \neq 0$ and $f^+_{m+1-s}Pe_k = 0$ for all $s > t_k$. Further, $1 \leqslant m + 1 - t_k \leqslant \frac{m+1}{2}$; otherwise if $t_k < \frac{m+1}{2}$ or $t_k < m + 1 - t_k$, then we have

$$(f^+_{t_k}Pe_k)^2 = f^+_{t_k}Pe^2_kP^+f_{t_k} = 0.$$

Hence $f^+_{t_k}Pe_k = 0$ which contradicts our earlier assumption. Therefore $Pe_k = z_k \in \mathcal{U}_{p_k}$ for some $1 \leqslant p_k \leqslant \frac{m+1}{2}$, for all $1 \leqslant k \leqslant \frac{n+1}{2}$. Next, consider z_j where $\frac{n+1}{2} < j \leqslant n$. Note that for a fixed j, we have $z_i \oslash z_j \in \mathcal{ST}_m(\mathbb{F})$ for all $1 \leqslant i < j \leqslant n + 1 - i$. Hence this implies that $z_j \in \mathcal{U}_{q_j}$ for some integer q_j where $q_j \leqslant m + 1 - p_i$ for all $1 \leqslant i < j \leqslant n + 1 - i$.

Let $Q \in \mathcal{M}_m(\mathbb{F})$ be the transition matrix from the ordered basis $\{f_1, \ldots, f_m\}$ to the ordered basis $\{z_1, \ldots, z_m\}$ such that $Qf_k = z_k$ for all $1 \leqslant k \leqslant n$, where $\{z_1, \ldots, z_m\}$ is an ordered basis obtained by extending $\{z_1, \ldots, z_n\}$. Then we have

$$T(A) = \lambda Q \left(\begin{array}{c|c} 0_{n,m-n} & A \\ \hline 0 & 0_{m-n,n} \end{array} \right) Q^+.$$

In particular, when $m = n$, we wish to show that $Q \in \mathcal{T}_n(\mathbb{F})$. Notice that $Qe_1 = u_1, \ldots, Qe_n = u_n$ are linearly independent vectors, then we have $u_i \in \mathcal{U}_n \backslash \mathcal{U}_{n-1}$ for some $1 \leqslant i \leqslant n$ and it follows that $u_1 \in \mathcal{U}_1$ for $u_1^2, u_1 \oslash u_2, \ldots, u_1 \oslash u_n \in \mathcal{ST}_n(\mathbb{F})$. Further, being linearly independent, we have $u_i \notin \mathcal{U}_1$ for all $2 \leqslant i \leqslant n$. Suppose that $u_j \in \mathcal{U}_j \backslash \mathcal{U}_{j-1}$ for all $1 \leqslant j \leqslant k$. Then we claim that $u_{k+1} \in \mathcal{U}_{k+1} \backslash \mathcal{U}_k$. Since $u_1, \ldots, u_n$ are linearly independent, so we have $u_{k+1} \notin \mathcal{U}_k$, in other words, $u_{k+1} \in \mathcal{U}_l \backslash \mathcal{U}_k$ for some $k+1 \leqslant l \leqslant n$. Further, $u_1 u_{k+1}^+, \ldots, u_{n-k} u_{k+1}^+ \in \mathcal{T}_n(\mathbb{F})$ would imply that there exists some i_{k+1} where $1 \leqslant i_{k+1} \leqslant n-k$ such that $u_{i_{k+1}} \in \mathcal{U}_{n-k}$ and it follows that $u_{k+1} \in \mathcal{U}_{n+1-(n-k)} = \mathcal{U}_{k+1}$, the claim then holds. Hence we conclude that $u_i \in \mathcal{U}_i \backslash \mathcal{U}_{i-1}$ where we define $\mathcal{U}_0 = \{0\}$, for all $1 \leqslant i \leqslant n$ and hence $Q \in \mathcal{T}_n(\mathbb{F})$. $\square$

We now prove our main theorem.

Theorem 3.5. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let n, m be integers such that $m \geqslant n \geqslant 5$. If $T : \mathcal{ST}_n(\mathbb{F}) \longrightarrow \mathcal{ST}_m(\mathbb{F})$ is a bounded rank-two linear preserver, then T is one of the following forms:*

(a) $\operatorname{Im} T = u \oslash U$ *for some nonzero vector $u \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant m+1-q \leqslant m$.*

(b) *$\mathbb{F}$ has characteristic two and* $\operatorname{Im} T = u \oslash U + \langle u^2 \rangle$ *for some nonzero vector $u \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant q \leqslant m+1-p$.*

(c) *$\mathbb{F}$ has characteristic two and* $\operatorname{Im} T = \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \rangle$ *for* *some linearly independent vectors* $u \in \mathcal{U}_p$ *and* $v_1, \ldots, v_k \in \mathcal{U}_q$ *with* $1 \leqslant p \leqslant \frac{m+1}{2}$ *and* $1 \leqslant q \leqslant m+1-p$*, and some scalars* $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ *with* $(\lambda_1, \ldots, \lambda_k) \neq 0$*.*

(d) *there exist an invertible matrix* $P \in \mathcal{M}_m(\mathbb{F})$ *and some scalars* $\lambda, \lambda' \in \mathbb{F}$ *with* $\lambda \neq 0$ *such that*

$$T(A) = \lambda P \left(\begin{array}{c|c} 0_{n,m-n} & A \\ \hline 0 & 0_{m-n,n} \end{array} \right) P^{+} + \lambda' a_{11} (Pe_1)^2$$

for all $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$*, where* $Pe_i \in \mathcal{U}_{p_i}$ *and* $Pe_j \in \mathcal{U}_{q_j}$ *with* $1 \leqslant p_i \leqslant \frac{m+1}{2}$ *for every* $1 \leqslant i \leqslant \frac{n+1}{2}$*, and* $q_j = max\{m+1-p_k \,|\, 1 \leqslant k \leqslant n+1-j\}$ *for every* $\frac{n+1}{2} < j \leqslant n+1-i$*, and* $\lambda' \neq 0$ *only if* $\mathbb{F}$ *has characteristic two. In particular,* $P \in \mathcal{T}_n(\mathbb{F})$ *when* $m = n$*.*

Proof. We distinguish our proof into two parts:

Case I: $\mathbb{F}$ has characteristic two. Since T is a bounded rank-two linear preserver, then we have $T(e_1^2), T(e_1 \oslash e_2), \ldots, T(e_1 \oslash e_n)$ are linearly independent. Further, since $n \geqslant 5$, then by Theorem 2.6 we see that

$$T(e_1^2) = u_1 \oslash u_{n+1} + \alpha_{n+1} u_1^2, \quad T(e_1 \oslash e_i) = u_1 \oslash u_i + \alpha_i u_1^2 \tag{3.5}$$

for some $u_1 \in \mathcal{U}_p$, $u_{n+1}, u_i \in \mathcal{U}_q$ satisfying $1 \leqslant p \leqslant m+1-q \leqslant m$ and some $\alpha_{n+1}, \alpha_i \in \mathbb{F}$ such that $1 \leqslant p \leqslant \frac{m+1}{2}$ whenever $(\alpha_2, \ldots, \alpha_{n+1}) \neq 0$, for all $2 \leqslant i \leqslant n$. Using a similar argument, we have $T(e_2 \oslash e_1), T(e_2^2), T(e_2 \oslash e_3), \ldots, T(e_2 \oslash e_{n-1})$ are linearly independent and

$$T(e_2^2) = v_2 \oslash v_n + \beta_n v_2^2, \quad T(e_2 \oslash e_j) = v_2 \oslash v_j + \beta_j v_2^2 \tag{3.6}$$

for some $v_2 \in \mathcal{U}_s$, $v_n, v_j \in \mathcal{U}_t$ satisfying $1 \leqslant s \leqslant m+1-t \leqslant m$ and some $\beta_n, \beta_j \in \mathbb{F}$

such that $1 \leqslant s \leqslant \frac{m+1}{2}$ whenever $(\beta_1, \beta_3, \dots, \beta_n) \neq 0$, for all $j = 1$ and $3 \leqslant j \leqslant n-1$.

Case I-A: $v_2 = \gamma u_1$ for some $\gamma \in \mathbb{F}$. We first claim that for any $3 \leqslant k \leqslant \frac{n+1}{2}$

$$T(e_k^2) \in u_1 \oslash \mathcal{U}_{q_1} + \langle u_1^2 \rangle \tag{3.7}$$

for some subspace $\mathcal{U}_{q_1}$ satisfying $1 \leqslant p \leqslant m + 1 - q_1 \leqslant m$. Note that $\langle T(e_1 \oslash e_k), T(e_2 \oslash e_k), T(e_k^2) \rangle$ is a 3-dimensional subspace of bounded rank-two matrices and so by Theorem 2.6 and Lemma 3.1, we have either $T(e_k^2) \in u_1 \oslash \mathcal{U}_{q_1} + \langle u_1^2 \rangle$, as claimed, or $\alpha_k = \beta_k = 0$ such that u_1, u_k, v_k are linearly independent and

$$T(e_k^2) = a_1 u_k \oslash v_k + b_1 u_1 \oslash u_k + c_1 u_1 \oslash v_k$$

for some scalars $a_1, b_1, c_1 \in \mathbb{F}$ with $a_1 \neq 0$. If $\langle u_{n+1} \rangle = \langle u_1 \rangle$, then $T(e_1^2 + e_k^2)$ is of rank three, a contradiction. A similar argument can be applied to the case $\langle v_n \rangle = \langle v_2 \rangle$. Hence $T(e_1^2), T(e_2^2)$ are of rank two. For arbitrary $\lambda_1, \lambda_2 \in \mathbb{F}$, we have $T(\lambda_1 e_1^2 + e_k^2)$ and $T(\lambda_2 e_2^2 + e_k^2)$ both have ranks bounded by two. Hence by Lemma 3.3, we obtain $\alpha_{n+1} = \beta_n = 0$ and

$$T(e_1^2) = u_1 \oslash (a_2 u_k + b_2 v_k), \quad T(e_2^2) = \gamma u_1 \oslash (a_3 u_k + b_3 v_k)$$

for some scalars $a_2, a_3, b_2, b_3 \in \mathbb{F}$ such that $a_2 b_3 - a_3 b_2 \neq 0$. Consider $T(e_1 \oslash e_2)$. Since $\langle T(e_1 \oslash e_2), T(e_1^2), T(e_2^2) \rangle$ is a subspace of bounded rank-two matrices and from (3.5), we have $u_2 \notin \langle u_{n+1}, v_n \rangle = \langle u_k, v_k \rangle$. This implies that

$$T((e_1 + e_2)^2 + e_k^2) = a_1 u_k \oslash v_k + u_1 \oslash (u_2 + (b_1 + a_2 + \gamma a_3) u_k + (c_1 + b_2 + \gamma b_3) v_k) + \alpha_2 u_1^2$$

has rank > 2, a contradiction. Hence claim (3.7) is proved. Using the fact that

$\langle T(e_s \oslash e_t), T(e_1 \oslash e_s), T(e_2 \oslash e_s), T(e_s^2)\rangle$ is a subspace of bounded rank-two matrices, it follows from Theorem 2.6 and Lemma 3.1 that

$$T(e_s \oslash e_t) \in u_1 \oslash \mathcal{U}_{q_2} + \langle u_1^2 \rangle \tag{3.8}$$

for some subspace $\mathcal{U}_{q_2}$ satisfying $1 \leqslant p \leqslant m+1-q_2 \leqslant m$, for all $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$. Conclusively, by $(3.5)-(3.8)$, $\operatorname{Im} T$ is one of the following forms:

(a) $\operatorname{Im} T = u_1 \oslash U$ for some nonzero vector $u_1 \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant m+1-q \leqslant m$.

(b) $\operatorname{Im} T = u_1 \oslash U + \langle u_1^2 \rangle$ for some nonzero vector $u_1 \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant q \leqslant m+1-p$.

(c) $\operatorname{Im} T = \langle u_1 \oslash w_1 + \lambda_1 u_1^2, \ldots, u_1 \oslash w_k + \lambda_k u_1^2 \rangle$ for some linearly independent vectors $u_1 \in \mathcal{U}_p$ and $w_1, \ldots, w_k \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m+1-p$, and some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$.

Case I-B: u_1, v_2 are linearly independent. Note that $T(e_1 \oslash e_2) = u_1 \oslash u_2 + \alpha_2 u_1^2 = v_2 \oslash v_1 + \beta_1 v_2^2$. Then by Lemma 3.2, we have $\alpha_2 = \beta_1 = 0$ and $T(e_1 \oslash e_2)$ is of rank two such that $v_1 = \gamma_1 u_1 + \gamma_2 u_2$, $v_2 = \gamma_3 u_1 + \gamma_4 u_2$ for some $\gamma_1, \ldots, \gamma_4 \in \mathbb{F}$ with $\gamma_4 \neq 0$. We also note that

$$T(e_1 \oslash e_2) = u_1 \oslash u_2 = u_1 \oslash \gamma_4^{-1} v_2. \tag{3.9}$$

We claim that $\{u_1, \ldots, u_{n-1}\}$ is linearly independent. Suppose to the contrary that $\{u_1, \ldots, u_{n-1}\}$ is linearly dependent. Then $T(e_1 \oslash f) = \alpha_f u_1^2$ for some $f \in \langle e_1, \ldots, e_{n-1}\rangle \setminus \langle e_2 \rangle$ where $\alpha_f \in \mathbb{F}$. Hence $\alpha_f \neq 0$ for T is a bounded rank-two linear preserver. On the other hand, in view of (3.6), we have $T(e_2 \oslash f) = v_2 \oslash v_f + \beta_f v_2^2$

for some $v_f \in \langle v_1, \ldots, v_{n-1}\rangle$ where $\beta_f \in \mathbb{F}$. Since $T((e_1+e_2) \oslash f)$ has rank bounded by two, then $v_f \in \langle u_1, v_2\rangle$. Hence

$$T(e_2 \oslash f) = a_4 v_2 \oslash u_1 + \beta_f v_2^2$$

for some scalar $a_4 \in \mathbb{F}$. Note that $T(e_2 \oslash (f + a_4\gamma_4 e_1)) = \beta_f v_2^2$, and so $\beta_f \neq 0$. Consider $T(e_1 \oslash e_l)$ for some $3 \leqslant l \leqslant n-1$ such that e_l, f are linearly independent. If $u_l \in \langle u_1, v_2\rangle$, then we see that

$$T(e_1 \oslash e_l) \in \langle T(e_1 \oslash e_2), T(e_1 \oslash f)\rangle$$

a contradiction, and so $u_l \notin \langle u_1, v_2\rangle$. Since $T(e_1 \oslash e_l + \lambda e_2 \oslash e_l)$ has rank bounded by two for any $\lambda \in \mathbb{F}$, applying Lemma 3.3, then we have $\beta_l = 0$ and either $v_l \in \langle u_1, v_2\rangle \setminus \langle v_2\rangle$ or $\alpha_l = 0$ such that $v_l \in \langle u_1, v_2, u_l\rangle \setminus \langle u_1, v_2\rangle$. The first case will give us $T(e_2 \oslash e_l) \in \langle T(e_1 \oslash e_2), T(e_2 \oslash f)\rangle$ leading to a contradiction. Hence

$$T(e_1 \oslash e_l) = b_5^{-1} u_1 \oslash u_l', \quad T(e_2 \oslash e_l) = v_2 \oslash u_l'$$

where $u_l' = a_5 u_1 + b_5 u_l$ for some $a_5, b_5 \in \mathbb{F}$ with $b_5 \neq 0$ and so u_1, v_2, u_l' are linearly independent. Now for any $\lambda \in \mathbb{F}$,

$$T((\lambda e_1 + e_2) \oslash (f + e_l + a_4\gamma_4 e_2)) = \lambda\alpha_f u_1^2 + \beta_f v_2^2 + (\lambda b_5^{-1} u_1 + v_2) \oslash u_l'$$

has rank bounded by two, giving us

$$det \begin{pmatrix} \lambda b_5^{-1} & 0 & \lambda\alpha_f \\ 1 & \beta_f & 0 \\ 0 & 1 & \lambda b_5^{-1} \end{pmatrix} = \lambda\alpha_f + \lambda^2 \beta_f b_5^{-2} = 0.$$

Since $|\mathbb{F}| \geqslant 3$, we have $\alpha_f = 0$, a contradiction. Hence $\{u_1, \ldots, u_{n-1}\}$ is linearly independent, as claimed. In a similar manner, we may show that $\{v_1, \ldots, v_{n-1}\}$ is linearly independent. Next we see that for any $3 \leqslant k \leqslant n-1$, $T((e_1 + \lambda e_2) \oslash e_k)$ has rank bounded by two for any $\lambda \in \mathbb{F}$. Thus applying Lemma 3.3, we have $\beta_k = 0$, and either $T(e_2 \oslash e_k) \in \langle T(e_1 \oslash e_2) \rangle$, or $\alpha_k = 0$ with $v_k \in \langle u_1, v_2, u_k \rangle \setminus \langle u_1, v_2 \rangle$. The first case obviously is a contradiction and so we conclude that

$$T(e_1 \oslash e_k) = b_{0k}^{-1} u_1 \oslash v_k, \quad T(e_2 \oslash e_k) = v_2 \oslash v_k \tag{3.10}$$

such that $v_k = a_{0k} u_1 + b_{0k} u_k$ for some $a_{0k}, b_{0k} \in \mathbb{F}$ with $b_{0k} \neq 0$, for all $3 \leqslant k \leqslant n-1$. Here we wish to show that $b_{0k} = b_0 \in \mathbb{F}$ for all $3 \leqslant k \leqslant n-1$. So, by considering any two distinct k_1, k_2 where $3 \leqslant k_1, k_2 \leqslant n-1$, we have $T((e_1 + e_2) \oslash (e_{k_1} + e_{k_2}))$ is of rank bounded by two implying that $b_{0k_1} = b_{0k_2}$ and we are done. Next, we have two sub-cases to be considered:

Case I-B-(i): $n \geqslant 6$. For any pair of integers (s, t) satisfying $3 \leqslant s \leqslant \frac{n+1}{2}$ and $3 \leqslant s < t \leqslant n+1-s$,

$$\left\langle T(e_1 \oslash e_s), T(e_2 \oslash e_s), T(e_s^2), T(e_s \oslash e_t) \right\rangle$$

forms a 4-dimensional subspace of bounded rank-two matrices. Hence by Theorem 2.6 and Lemma 3.1, we obtain

$$T(e_s^2),\ T(e_s \oslash e_t) \in v_s \oslash \mathcal{U}_{q_3} + \left\langle v_s^2 \right\rangle \tag{3.11}$$

for some subspace $\mathcal{U}_{q_3}$ satisfying $1 \leqslant p \leqslant m+1-q_3 \leqslant m$, for all (s, t). On the other hand,

$$\langle T(e_1 \oslash e_t), T(e_2 \oslash e_t), T(e_s \oslash e_t)\rangle$$

is a 3-dimensional subspace of bounded rank-two matrices and so by Theorem 2.6 and Lemma 3.1, we obtain either $T(e_s \oslash e_t) = a_6 u_1 \oslash u_2 + b_6 u_1 \oslash v_t + c_6 u_2 \oslash v_t$ for some scalars $a_6, b_6, c_6 \in \mathbb{F}$ with $a_6 \neq 0$, or

$$T(e_s \oslash e_t) \in v_t \oslash \mathcal{U}_{q_4} + \left\langle v_t^2 \right\rangle$$

for some subspace $\mathcal{U}_{q_4}$ satisfying $1 \leqslant p \leqslant m+1-q_4 \leqslant m$, for all $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$. The first case is not possible by (3.11) and so

$$T(e_s \oslash e_t) \in (v_s \oslash \mathcal{U}_{q_3} + \left\langle v_s^2 \right\rangle) \cap (v_t \oslash \mathcal{U}_{q_4} + \left\langle v_t^2 \right\rangle) = \langle v_s \oslash v_t \rangle$$

and so we have

$$T(e_s \oslash e_t) = \gamma_{st} v_s \oslash v_t \tag{3.12}$$

for some nonzero scalar $\gamma_{st} \in \mathbb{F}$, for all $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$.

We wish to show that T sends rank one matrices to rank one matrices, or in other words T preserves rank one matrices. We start off by considering $T(e_1^2)$. Suppose to the contrary that $\alpha_{n+1} = 0$. Then $u_{n+1} \notin \langle u_1, \ldots, u_{n-1}\rangle$. Note that $T(e_1^2 + \lambda e_2^2)$ has rank bounded by two for any $\lambda \in \mathbb{F}$. Applying Lemma 3.3, we have $\beta_n = 0$ and either $T(e_2^2) \in \langle T(e_1 \oslash e_2)\rangle$, or $v_n \in \langle u_1, v_2, u_{n+1}\rangle \setminus \langle u_1, v_2\rangle$. The first case is a contradiction. Hence $T(e_2^2) = v_2 \oslash (a_7 u_1 + b_7 u_{n+1})$ for some $a_7, b_7 \in \mathbb{F}$ with $b_7 \neq 0$.

Then for any $3 \leqslant s \leqslant \frac{n+1}{2}$, by (3.11), we get

$$T(e_s^2) = v_s \oslash v_s' + \gamma_5 v_s^2$$

for some vector v_s' and some scalar $\gamma_5 \in \mathbb{F}$ with v_s, v_s' linearly independent, if not then $T(e_s^2) = \gamma_6 v_s^2$ for some $\gamma_6 \in \mathbb{F}$ would imply that $T(e_2^2 + e_s^2)$ is of rank three. We first note that $T(e_1^2), T(e_2^2), T(e_s^2)$ are of rank two. Since $T(e_1^2 + \lambda_1 e_s^2), T(e_2^2 + \lambda_2 e_s^2)$ both have rank bounded by two for any $\lambda_1, \lambda_2 \in \mathbb{F}$, so by Lemma 3.3 we conclude that $\gamma_5 = 0$ and $v_s' \in \langle v_s, u_{n+1} \rangle \setminus \langle v_s, u_{n+1} \rangle$. Hence $T(e_s^2) = a_8 u_s \oslash u_{n+1}$ for some nonzero $a_8 \in \mathbb{F}$. But this gives us that, for any $\lambda_0 \in \mathbb{F} \setminus \{\gamma_4 a_7, 0\}$,

$$T(e_s^2 + (\lambda_0 e_1 + e_2)^2) = u_{n+1} \oslash (a_8 u_s + b_7 v_2 + \lambda_0^2 u_1) + (\lambda_0 \gamma_4^{-1} + a_7) u_1 \oslash v_2$$

is of rank four, a contradiction. Hence $\alpha_{n+1} \neq 0$. Further, if u_{n+1}, u_1, v_2 are linearly independent and $T(e_1^2 + \lambda e_2^2)$ has rank bounded by two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we have $v_n \in \langle u_1, v_2 \rangle$ as $\alpha_{n+1} \neq 0$. But this implies that $T(e_2^2), T(e_1 \oslash e_2)$ are linearly dependent, a contradiction. Hence we have $u_{n+1} \in \langle u_1, v_2 \rangle$ and it follows that $v_n \in \langle u_1, v_2 \rangle$ as $\alpha_{n+1}, \beta_n \neq 0$. Moreover, for $3 \leqslant s \leqslant \frac{n+1}{2}$, we see that $T(e_1^2 + e_s^2), T(e_2^2 + e_s^2)$ both are of rank bounded by two. So by (3.11), it follows that

$$T(e_1^2) = \alpha_{n+1} u_1^2, \quad T(e_2^2) = \beta_n v_2^2, \quad T(e_s^2) = \gamma_{ss} v_s^2 \tag{3.13}$$

for some nonzero scalar $\gamma_{ss} \in \mathbb{F}$. Hence T preserves rank one matrices. Without loss of generality, we take $\alpha_{n+1} = 1$, that is, $T(e_1^2) = u_1^2$. Since

$$T((e_1 + e_2 + e_s)^2 - e_s^2) = (b_0^{-1} u_1 + v_2) \oslash v_s + \gamma_4^{-1} u_1 \oslash v_2 + u_1^2 + \beta_n v_2^2$$

has rank bounded above by two, we have

$$det\begin{pmatrix} b_0^{-1} & \gamma_4^{-1} & 1 \\ 1 & \beta_n & \gamma_4^{-1} \\ 0 & 1 & b_0^{-1} \end{pmatrix} = 1 + \beta_n b_0^{-2} = 0.$$

This implies that $\beta_n = b_0^2$ or $T(e_2^2) = b_0^2 v_2^2$. On the other hand, for any $\lambda \in \mathbb{F}$, by (3.10) and (3.13), we have

$$T((e_1+e_2+e_s)^2+(\lambda-1)e_2^2) = (b_0^{-1}u_1+v_2)\oslash v_s + \gamma_4^{-1}u_1\oslash v_2 + u_1^2 + \lambda b_0^{-2}v_2^2 + \gamma_{ss}v_s^2$$

is of rank bounded by two. This gives rise to

$$det\begin{pmatrix} b_0^{-1} & \gamma_4^{-1} & 1 \\ 1 & \lambda b_0^2 & \gamma_4^{-1} \\ \gamma_{ss} & 1 & b_0^{-1} \end{pmatrix} = \lambda(\gamma_{ss}b_0^2+1)+1+\gamma_{ss}\gamma_4^{-2} = 0.$$

Since $|\mathbb{F}| \geqslant 3$, we obtain $\gamma_{ss} = \gamma_4^2 = b_0^{-2}$. Hence from (3.10) and (3.13),

$$T(e_1\oslash e_k) = \gamma_4 u_1 \oslash v_k, \quad T(e_{s_1}^2) = (\gamma_4 v_{s_1})^2$$

for all $3 \leqslant k \leqslant n-1$ and $2 \leqslant s_1 \leqslant \frac{n+1}{2}$. For arbitrary pair of (s,t) satisfying $3 \leqslant s \leqslant \frac{n+1}{2}$ and $3 \leqslant s < t \leqslant n+1-s$, by (3.12), we have

$$T((e_1+e_s+e_t)^2-e_t^2) = (\gamma_4 u_1+\gamma_{st}v_s)\oslash v_t + \gamma_4 u_1\oslash v_s + u_1^2 + \gamma_4^2 v_s^2$$

has rank bounded above by two implying

$$det\begin{pmatrix} \gamma_4 & \gamma_4 & 1 \\ \gamma_{st} & \gamma_4^2 & \gamma_4 \\ 0 & \gamma_{st} & \gamma_4 \end{pmatrix} = (\gamma_{st}+\gamma_4^2)^2 = 0.$$

Thus, $\gamma_{st} = \gamma_4^2$ yielding $T(e_s\oslash e_t) = \gamma_4 v_s \oslash \gamma_4 v_t$ for all (s,t). Finally, we claim that

$u_1, \ldots, u_n$ are linearly independent. Recall that we have shown that $u_1, \ldots, u_{n-1}$ are linearly independent (just after equation (3.9)), so it suffices to show that $u_n \notin \langle u_1, \ldots, u_{n-1} \rangle$. Suppose to the contrary that $u_n \in \langle u_1, \ldots, u_{n-1} \rangle$. Then $T(e_1 \oslash f') \in \langle T(e_1^2) \rangle$ for some nonzero vector $f' \in \langle u_1, \ldots, u_n \rangle$, a contradiction.

Let $z_1 = u_1$, $z_2 = \gamma_4^{-1} v_2$, $z_n = u_n$ and $z_j = \gamma_4 v_j$ for all $3 \leqslant j \leqslant n-1$. Define $Pe_i = z_i$ for all $1 \leqslant i \leqslant n$. Then $P \in \mathcal{M}_{m,n}(\mathbb{F})$ is of full column rank and since $m \geqslant n$, we say P is of rank n. For any $1 \leqslant i \leqslant \frac{n+1}{2}$ and $i < j \leqslant n+1-i$ with $j \neq n$, we have

$$T(E_{1,1} + E_{n,n}) = T(e_1 \oslash e_n) = z_1 \oslash z_n + \alpha_n z_1^2 = P[(e_1 \oslash e_n) + \alpha_n e_1^2]P^+$$

$$T(E_{i,n+1-i}) = T(e_i^2) = z_i^2 = (Pe_i)^2 = P(e_i^2)P^+$$

$$T(E_{i,n+1-j} + E_{j,n+1-i}) = T(e_i \oslash e_j) = z_i \oslash z_j = (Pe_i) \oslash (Pe_j) = P(e_i \oslash e_j)P^+.$$

Thus this proves that

$$T(A) = \lambda PAP^+ + \alpha_n a_{11}(Pe_1)^2$$

for all $A \in \mathcal{ST}_n(\mathbb{F})$ and some nonzero $\lambda \in \mathbb{F}$ where a_{11} denotes the $(1,1)$-th entry of matrix A. Then by Lemma 3.4, we are done.

Case I-B-(ii): $n = 5$. Note that

$$\left\langle T(e_1 \oslash e_3), T(e_2 \oslash e_3), T(e_3^2) \right\rangle$$

forms a 3-dimensional subspace of bounded rank-two matrices. By Theorem 2.6 and

Lemma 3.1, we have either (3.11) with $s = 3$, or

$$T(e_3^2) = a_9 u_1 \oslash v_2 + v_3 \oslash (b_9 u_1 + c_9 v_2)$$

for some scalars $a_9, b_9, c_9 \in \mathbb{F}$ with $a_9 \neq 0$. If the first case happens, then a similar argument from Case I-B-(i) can be applied here and the result follows easily.

Suppose the later holds. From (3.5) with $n = 5$, we first claim that $\alpha_6 \neq 0$. Suppose to the contrary that $\alpha_6 = 0$. Then by (3.10), $u_6 \notin \langle u_1, v_2, v_3 \rangle$. Since $T(e_1^2 + e_3^2)$ has rank bounded above by two, then we have $c_9 = 0$ and so

$$T((e_1 + e_2 + e_3)^2 - e_2^2) = v_2 \oslash v_3 + u_1 \oslash (u_6 + (a_9 + \gamma_4^{-1})v_2 + (b_9 + b_0^{-1})v_3)$$

is of rank four, a contradiction. Hence $\alpha_6 \neq 0$, as claimed. Notice that if u_6, u_1, v_2 are linearly independent and $T(e_1^2 + \lambda e_2^2)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we conclude that $T(e_2^2) \in \langle T(e_1 \oslash e_2) \rangle$ as $\alpha_6 \neq 0$, which is a contradiction. Thus $u_6 \in \langle u_1, v_2 \rangle$. Using a similar argument, we may also show that $\beta_5 \neq 0$ and $v_5 \in \langle u_1, v_2 \rangle$. Hence

$$T(e_1^2) = d_9 u_1 \oslash v_2 + \alpha_6 u_1^2, \quad T(e_2^2) = h_9 v_2 \oslash u_1 + \beta_5 v_2^2$$

for some scalars $d_9, h_9 \in \mathbb{F}$ and $\alpha_6, \beta_5 \neq 0$. Further, since $T(e_1^2 + e_3^2)$ and $T(e_2^2 + e_3^2)$ are of rank bounded by two, it follows that $b_9 = c_9 = 0$. But we see that

$$T((e_1 + e_2 + e_3)^2 - e_2^2) = \alpha_6 u_1^2 + v_2 \oslash v_3 + u_1 \oslash ((a_9 + d_9 + \gamma_4^{-1})v_2 + b_0^{-1} v_3)$$

is of rank three leading to a contradiction. Hence we conclude that for $n = 5$, $T(e_3^2)$ is of the form (3.11) with $s = 3$. We are done.

Case II: $\mathbb{F}$ has characteristic not two. Since T is a bounded rank-two linear preserver, then we have $T(e_1^2), T(e_1 \oslash e_2), \ldots, T(e_1 \oslash e_n)$ are linearly independent. Further, since $n \geqslant 5$, then by Theorem 2.6, we see that

$$T(e_1^2) = u_1 \oslash u_{n+1}, \quad T(e_1 \oslash e_i) = u_1 \oslash u_i \tag{3.14}$$

for some $u_1 \in \mathcal{U}_p$, $u_{n+1}, u_i \in \mathcal{U}_q$ satisfying $1 \leqslant p \leqslant m+1-q \leqslant m$ for all $2 \leqslant i \leqslant n$. Using a similar argument, we have $T(e_2 \oslash e_1), T(e_2^2), T(e_2 \oslash e_3), \ldots, T(e_2 \oslash e_{n-1})$ are linearly independent and

$$T(e_2^2) = v_2 \oslash v_n, \quad T(e_2 \oslash e_j) = v_2 \oslash v_j \tag{3.15}$$

for some $v_2 \in \mathcal{U}_s$, $v_n, v_j \in \mathcal{U}_t$ satisfying $1 \leqslant s \leqslant m+1-t \leqslant m$ for all $j = 1$ and $3 \leqslant j \leqslant n-1$.

Case II-A: $v_2 = \gamma u_1$ for some $\gamma \in \mathbb{F}$. Note that for any $3 \leqslant k \leqslant \frac{n+1}{2}$, $\langle T(e_1 \oslash e_k), T(e_2 \oslash e_k), T(e_k^2)\rangle$ is a 3-dimensional subspace of bounded rank-two matrices. Then by Theorem 2.6 and Lemma 3.1, we have

$$T(e_k^2) \in u_1 \oslash \mathcal{U}_{q_1} \tag{3.16}$$

for some subspace $\mathcal{U}_{q_1}$ satisfying $1 \leqslant p \leqslant m+1-q_1 \leqslant m$. Similarly for any pair of (s,t) satisfying $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$, we have

$$\left\langle T(e_1 \oslash e_s), T(e_2 \oslash e_s), T(e_s^2), T(e_s \oslash e_t)\right\rangle$$

is a 4-dimensional subspace of bounded rank-two matrices. Applying Theorem 2.6

and Lemma 3.1, we obtain

$$T(e_s \oslash e_t) \in u_1 \oslash \mathcal{U}_{q_2} \tag{3.17}$$

for some subspace $\mathcal{U}_{q_2}$ satisfying $1 \leqslant p \leqslant m+1-q_2 \leqslant m$. Conclusively, by (3.14) – (3.17) with $v_2 \in \langle u_1 \rangle$, we conclude that

$$\operatorname{Im} T = u_1 \oslash U$$

for some nonzero vector $u_1 \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ such that $1 \leqslant p \leqslant m+1-q \leqslant m$.

Case II-B: u_1, v_2 are linearly independent. Note that $T(e_1 \oslash e_2) = u_1 \oslash u_2 = v_2 \oslash v_1$ and so by Lemma 3.2, we have $T(e_1 \oslash e_2)$ is of rank two such that $v_1 = \gamma u_1$, $v_2 = \gamma^{-1} u_2$ for some nonzero scalar $\gamma \in \mathbb{F}$. Hence

$$T(e_1 \oslash e_2) = u_1 \oslash u_2 = u_1 \oslash \gamma v_2.$$

Next we claim that $u_{n+1} \in \langle u_1, v_2 \rangle$. Suppose to the contrary that u_1, v_2, u_{n+1} are linearly independent. Then we see that $T(e_1^2 + \lambda e_2^2)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$. Hence by Lemma 3.3, we conclude that $v_n = b_4 u_{n+1}$ for some nonzero scalar $b_4 \in \mathbb{F}$. But this implies that

$$T((e_1+e_2)^2) = u_1 \oslash u_{n+1} + \gamma u_1 \oslash v_2 + b_4 v_2 \oslash u_{n+1}$$

is of rank three, a contradiction. Thus the claim is proved. It follows that

$$T(e_1^2) = a_5 u_1 \oslash v_2 + \alpha_{n+1} u_1^2, \quad T(e_2^2) = b_5 u_1 \oslash v_2 + \beta_n v_2^2 \tag{3.18}$$

for some scalars $a_5, b_5, \alpha_{n+1}, \beta_n \in \mathbb{F}$ with $\alpha_{n+1}, \beta_n \neq 0$. Further with $\alpha_{n+1}, \beta_n \neq 0$, it is immediate that $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_{n-1}\}$ are linearly independent. Next we see that for any $3 \leqslant k \leqslant n-1$, $T((e_1 + \lambda e_2) \oslash e_k)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$. Thus applying Lemma 3.3, we have $v_k \in \langle u_1 \rangle \cup \langle u_k \rangle$. Clearly $v_k \notin \langle u_1 \rangle$ and so we conclude that

$$T(e_1 \oslash e_k) = u_1 \oslash u_k, \quad T(e_2 \oslash e_k) = b_{0k} v_2 \oslash u_k \tag{3.19}$$

for some nonzero scalar $b_{0k} \in \mathbb{F}$, for all $3 \leqslant k \leqslant n-1$. Here we wish to show that $b_{0k} = b_0 \in \mathbb{F}$ for all $3 \leqslant k \leqslant n-1$. So, by considering any two distinct k_1, k_2 where $3 \leqslant k_1, k_2 \leqslant n-1$, we have $T((e_1 + e_2) \oslash (e_{k_1} + e_{k_2}))$ is of rank bounded by two implying that $b_{0k_1} = b_{0k_2}$ and we are done.

For any s, t satisfying $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$, we have

$$\left\langle T(e_1 \oslash e_s), T(e_2 \oslash e_s), T(e_s^2) \right\rangle$$

is a 3-dimensional subspace of bounded rank-two matrices and

$$\left\langle T(e_1 \oslash e_s), T(e_2 \oslash e_s), T(e_s^2), T(e_s \oslash e_t) \right\rangle$$

is a 4-dimensional subspace of bounded rank-two matrices. Thus applying Theorem 2.6 and Lemma 3.1, we obtain

$$T(e_s^2), T(e_s \oslash e_t) \in u_s \oslash \mathcal{U}_{q_3} \tag{3.20}$$

for some subspace $\mathcal{U}_{q_3}$ satisfying $1 \leqslant p \leqslant m+1-q_3 \leqslant m$, for all $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant$

$n+1-s$. On the other hand,

$$\langle T(e_1 \oslash e_t), T(e_2 \oslash e_t), T(e_s \oslash e_t)\rangle$$

is a 3-dimensional subspace of bounded rank-two matrices. Then by Theorem 2.6 and Lemma 3.1, we obtain

$$T(e_s \oslash e_t) \in u_t \oslash \mathcal{U}_{q_4}$$

for some subspace $\mathcal{U}_{q_4}$ satisfying $1 \leqslant p \leqslant m+1-q_4 \leqslant m$, for all $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$. Hence we see that

$$T(e_s \oslash e_t) \in (u_s \oslash \mathcal{U}_{q_3}) \cap (u_t \oslash \mathcal{U}_{q_4}) = \langle u_s \oslash u_t\rangle$$

and so we have

$$T(e_s \oslash e_t) = \gamma_{st} u_s \oslash u_t$$

for some nonzero scalar $\gamma_{st} \in \mathbb{F}$, for all $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$. On the other hand, since $T(e_1^2 + e_s^2), T(e_2^2 + e_s^2)$ both have rank bounded above by two, so from (3.20) and (3.18), we obtain

$$T(e_1^2) = \alpha_{n+1} u_1^2, \quad T(e_2^2) = \beta_n v_2^2, \quad T(e_s^2) = \gamma_{ss} v_s^2 \tag{3.21}$$

for some nonzero scalar $\gamma_{ss} \in \mathbb{F}$. Without loss of generality, we let $\alpha_{n+1} = 1$, that is

$T(e_1^2) = u_1^2$. Then

$$T((e_1+e_2)^2 + e_s^2) = u_1^2 + \gamma u_1 \oslash v_2 + \beta_n v_2^2 + \gamma_{ss} v_s^2$$

has rank bounded above by two yielding

$$det \begin{pmatrix} 0 & \gamma & 1 \\ 0 & \beta_n & \gamma \\ \gamma_{ss} & 0 & 0 \end{pmatrix} = (\gamma^2 - \beta_n)\gamma_{ss} = 0$$

and so $\beta_n = \gamma^2$ or $T(e_2^2) = \gamma^2 v_2^2$. On the other hand, for any $\lambda \in \mathbb{F}$, we see that

$$T((e_1+e_2+e_s)^2 + (\lambda - 1)e_2^2) = (u_1 + b_0 v_2) \oslash v_s + \gamma u_1 \oslash v_2 + u_1^2 + \lambda\gamma^2 v_2^2 + \gamma_{ss} u_s^2$$

has rank bounded above by two giving rise to

$$det \begin{pmatrix} 1 & \gamma & 1 \\ b_0 & \lambda\gamma^2 & \gamma \\ \gamma_{ss} & b_0 & 1 \end{pmatrix} = \lambda\gamma^2(1-\gamma_{ss}) + b_0^2 + \gamma^2\gamma_{ss} - 2\gamma b_0 = 0.$$

Since $|\mathbb{F}| \geqslant 3$, we obtain $1 - \gamma_{ss} = b_0^2 + \gamma^2\gamma_{ss} - 2\gamma b_0 = 0$. Hence $\gamma_{ss} = 1$ and $b_0 = \gamma$. It follows from (3.19) and (3.21) that

$$T(e_s^2) = u_s^2, \quad T(e_2 \oslash e_k) = \gamma v_2 \oslash u_k$$

for all $3 \leqslant s \leqslant \frac{n+1}{2}$ and $3 \leqslant k \leqslant n-1$. Now for any pair of (s,t) satisfying $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$, we have

$$T((e_1+e_s+e_t)^2 - e_t^2) = (u_1 + \gamma_{st} u_s) \oslash u_t + u_1 \oslash u_s + u_1^2 + u_s^2$$

has rank bounded above by two implying

$$det\begin{pmatrix} 1 & 1 & 1 \\ \gamma_{st} & 1 & 1 \\ 0 & \gamma_{st} & 1 \end{pmatrix} = (\gamma_{st}-1)^2 = 0$$

and thus we have $\gamma_{st} = 1$ yielding $T(e_s \oslash e_t) = u_s \oslash u_t$ for all $3 \leqslant s \leqslant \frac{n+1}{2} < t \leqslant n+1-s$. Let $z_1 = u_1$, $z_2 = \gamma v_2$ and $z_j = u_j$ for all $3 \leqslant j \leqslant n$. Define $Pe_i = z_i$ for all $1 \leqslant i \leqslant n$. Then $P \in \mathcal{M}_{m,n}(\mathbb{F})$ is of rank n and for any $1 \leqslant i \leqslant \frac{n+1}{2}$ and $i < j \leqslant n+1-i$, we have

$$T(E_{i,n+1-i}) = T(e_i^2) = z_i^2 = (Pe_i)^2 = P(e_i^2)P^+$$

$$T(E_{i,n+1-j} + E_{j,n+1-i}) = T(e_i \oslash e_j) = z_i \oslash z_j = (Pe_i) \oslash (Pe_j) = P(e_i \oslash e_j)P^+.$$

Thus this proves that

$$T(A) = \lambda PAP^+$$

for all $A \in \mathcal{ST}_n(\mathbb{F})$ and some nonzero $\lambda \in \mathbb{F}$. Then by Lemma 3.4, we are done. The proof is complete. $\square$

We give a few examples of bounded rank-two linear preservers $\mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$, $m > n \geqslant 5$, to illustrate the forms **(a)**, **(b)** and **(c)** listed in Theorem 3.5.

Example 3.6. Let $\mathbb{F}$ be a field with at least three elements and of characteristic two. Let m, n be integers such that $m > n \geqslant 5$. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathcal{M}_{m,1}(\mathbb{F})$.

(a) Let $T_1 : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$ be the linear map defined by

$$T_1(A) = a_{1,n} e_1 \oslash e_{n+1} + \sum_{k=1}^{n-1} \left(\sum_{i=1}^{\lfloor \frac{n-k}{2} \rfloor + 1} a_{i,i+k-1} \right) e_1 \oslash e_{n+1-k}$$

for all $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$. Then T_1 is a bounded rank-two linear preserver with

$$\operatorname{Im} T_1 = e_1 \oslash \langle e_2, \ldots, e_{n+1} \rangle .$$

(b) Let $T_2 : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$ be the linear map defined by

$$T_2(A) = a_{1,n} e_1^2 + \sum_{k=1}^{n-1} \left(\sum_{i=1}^{\lfloor \frac{n-k}{2} \rfloor + 1} a_{i,i+k-1} \right) e_1 \oslash e_{n+1-k}$$

for all $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$. Then T_2 is a bounded rank-two linear preserver with

$$\operatorname{Im} T_2 = e_1 \oslash \langle e_2, \ldots, e_n \rangle + \langle e_1^2 \rangle .$$

(c) Let $T_3 : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$ be the linear map defined by

$$T_3(A) = a_{1,n} \left(e_1 \oslash e_{n+1} + e_1^2 \right) + \sum_{k=1}^{n-1} \left(\sum_{i=1}^{\lfloor \frac{n-k}{2} \rfloor + 1} a_{i,i+k-1} \right) e_1 \oslash e_{n+1-k}$$

for all $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$. Then T_3 is a bounded rank-two linear preserver with

$$\operatorname{Im} T_3 = \left\langle e_1 \oslash e_{n+1} + e_1^2, e_1 \oslash e_2, \ldots, e_1 \oslash e_n \right\rangle .$$

We now consider the bounded rank-two linear preservers $T : \mathcal{ST}_4(\mathbb{F}) \longrightarrow \mathcal{ST}_m(\mathbb{F})$ for some integer m. It is clear that $m \geqslant 4$ since there always exists some subspace U of bounded rank-two matrices of $\mathcal{ST}_4(\mathbb{F})$ such that $\dim T(U) = 4$.

Theorem 3.7. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let m be an integer with $m \geqslant 4$. If $T : \mathcal{ST}_4(\mathbb{F}) \longrightarrow \mathcal{ST}_m(\mathbb{F})$ is a bounded rank-two linear preserver, then T is one of the following forms:*

(a) $\operatorname{Im} T = u \oslash U$ *for some nonzero vector $u \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with*

$1 \leqslant p \leqslant m+1-q \leqslant m.$

(b) *$\mathbb{F}$ has characteristic two and* $\operatorname{Im} T = u \oslash U + \langle u^2 \rangle$ *for some nonzero vector $u \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant q \leqslant m+1-p$.*

(c) *$\mathbb{F}$ has characteristic two and* $\operatorname{Im} T = \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \rangle$ *for some linearly independent vectors $u \in \mathcal{U}_p$ and $v_1, \ldots, v_k \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m+1-p$, and some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$.*

(d) *there exist an invertible matrix $P \in \mathcal{M}_m(\mathbb{F})$ and some scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{F}$ with $\lambda_1, \lambda_4 \neq 0$ such that*

$$T(A) = P \left(\begin{array}{c|cccc} & a_{11} & a_{1s_1} & \lambda_1 a_{13} + \lambda_2 a_{1s_2} + \lambda_3 a_{2t_2} & \lambda_4 a_{1s_2} + \lambda_5 a_{11} \\ & 0 & a_{2t_1} & \lambda_4 a_{2t_2} & \lambda_1 a_{13} + \lambda_2 a_{1s_2} + \lambda_3 a_{2t_2} \\ 0_{4,m-4} & 0 & 0 & a_{2t_1} & a_{1s_1} \\ & 0 & 0 & 0 & a_{11} \\ \hline 0 & & & 0_{m-4,4} & \end{array} \right) P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$, where $Pe_i \in \mathcal{U}_{p_i}$ and $Pe_j \in \mathcal{U}_{q_j}$ with $1 \leqslant p_i \leqslant \frac{m+1}{2}$ for $i = 1, 2$, and $q_j = max\{m+1-p_k \,|\, 1 \leqslant k \leqslant 5-j\}$ for $j = 3, \ldots, 5-i$, $\{s_1, s_2\} = \{2, 4\}$, $\{t_1, t_2\} = \{2, 3\}$, and $(\lambda_2, \lambda_3, \lambda_5) \neq 0$ only if $\mathbb{F}$ has characteristic two and $(s_1, s_2) = (4, 2)$ only if $|\mathbb{F}| = 4$. In particular, $P \in \mathcal{T}_4(\mathbb{F})$ when $m = 4$.

Proof. We distinguish our proof into two parts:

Case I: $\mathbb{F}$ has characteristic two. Since T is a bounded rank-two linear preserver, then we have $T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3), T(e_1 \oslash e_4)$ are linearly independent. Further, by Theorem 2.6, we see that

$$T(e_1^2) = u_1 \oslash u_5 + \alpha_5 u_1^2, \quad T(e_1 \oslash e_i) = u_1 \oslash u_i + \alpha_i u_1^2 \tag{3.22}$$

for some $u_1 \in \mathcal{U}_p$, $u_5, u_i \in \mathcal{U}_q$ satisfying $1 \leqslant p \leqslant m+1-q \leqslant m$ and some $\alpha_i \in \mathbb{F}$ such that $1 \leqslant p \leqslant \frac{m+1}{2}$ whenever $(\alpha_2, \ldots, \alpha_5) \neq 0$, for $i = 2, 3, 4$. On the other hand, we have $T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3)$ are linearly independent. Then by Theorem 2.6, there are three possible forms:

$$T(e_2^2) = v_2 \oslash v_4 + \beta_4 v_2^2, \quad T(e_2 \oslash e_j) = v_2 \oslash v_j + \beta_j v_2^2 \tag{3.23}$$

for some $v_2 \in \mathcal{U}_s$, $v_4, v_j \in \mathcal{U}_t$ satisfying $1 \leqslant s \leqslant m+1-t \leqslant m$ and some $\beta_j \in \mathbb{F}$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$ whenever $(\beta_1, \beta_3, \beta_4) \neq 0$, for $j = 1, 3$.

$$\big\langle T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3)\big\rangle = \langle w_1 \oslash w_2, w_1 \oslash w_3, w_2 \oslash w_3 \rangle \tag{3.24}$$

for some linearly independent vectors $w_1 \in \mathcal{U}_p$, $w_2 \in \mathcal{U}_q$ and $w_3 \in \mathcal{U}_r$ such that $p, q \leqslant m+1-r$ and either $p \leqslant m+1-q$, or $p = q > \frac{m+1}{2}$ and $w_2 = \alpha w_1 + z$ for some nonzero scalar $\alpha \in \mathbb{F}$ and some vector $z \in \mathcal{U}_k$ with $1 \leqslant k \leqslant m+1-p < \frac{m+1}{2}$ such that w_2, z are linearly independent.

$$\big\langle T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3)\big\rangle = \big\langle x^2, y^2, x \oslash y\big\rangle \tag{3.25}$$

for some linearly independent vectors $x, y \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$.

Case I-A: (3.23) holds. If $v_2 \in \langle u_1 \rangle$, then $\operatorname{Im} T$ is one of the following forms:

(a) $\operatorname{Im} T = u_1 \oslash U$ for some nonzero vector $u_1 \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant m+1-q \leqslant m$.

(b) $\operatorname{Im} T = u_1 \oslash U + \langle u_1^2 \rangle$ for some nonzero vector $u_1 \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant q \leqslant m+1-p$.

(c) $\operatorname{Im} T = \langle u_1 \oslash w_1 + \lambda_1 u_1^2, \ldots, u_1 \oslash w_k + \lambda_k u_1^2 \rangle$ for some linearly independent vectors $u_1 \in \mathcal{U}_p$ and $w_1, \ldots, w_k \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m+1-p$, and some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$.

Suppose now that u_1, v_2 are linearly independent. Then we see that $T(e_1 \oslash e_2) = u_1 \oslash u_2 + \alpha_2 u_1^2 = v_2 \oslash v_1 + \beta_1 v_2^2$. Hence by Lemma 3.2, we have $\alpha_2 = \beta_1 = 0$ and that $\langle v_1, v_2 \rangle = \langle u_1, u_2 \rangle$. It follows that

$$T(e_1 \oslash e_2) = u_1 \oslash u_2 = u_1 \oslash \gamma_1 v_2 = v_1 \oslash v_2$$

for some nonzero $\gamma_1 \in \mathbb{F}$. Suppose that $u_3 \in \langle u_1, u_2 \rangle$. Then we have $\zeta_1 u_1 + \zeta_2 u_2 + u_3 = 0$ for some $\zeta_1, \zeta_2 \in \mathbb{F}$ such that $(\zeta_1, \zeta_2) \neq 0$. This implies that $T(e_1 \oslash (\zeta_1 e_1 + \zeta_2 e_2 + e_3)) = \alpha_3 u_1^2 \neq 0$ and hence

$$T(e_1 \oslash e_3) = \alpha_3 u_1^2 - \zeta_2 \gamma_1 u_1 \oslash v_2.$$

On the other hand, we see that $T(e_2 \oslash (\zeta_1 e_1 + \zeta_2 e_2 + e_3)) = v_2 \oslash (\zeta_1 v_1 + v_3) + (\zeta_1 \beta_1 + \beta_3) v_2^2$. Hence we have $\zeta_1 v_1 + v_3 \in \langle u_1, v_2 \rangle$ yielding $T(e_2 \oslash (\zeta_1 e_1 + \zeta_2 e_2 + e_3)) = a_1 v_2 \oslash u_1 + (\zeta_1 \beta_1 + \beta_3) v_2^2$ for some scalar $a_1 \in \mathbb{F}$ such that $\zeta_1 \beta_1 + \beta_3 \neq 0$. Thus

$$T(e_2 \oslash e_3) = (\zeta_1 \beta_1 + \beta_3) v_2^2 + (a_1 - \zeta_1 \gamma_1) v_2 \oslash u_1.$$

Next by (3.22) and (3.23), it is easy to verify that $\{u_1, u_2, u_5\}$ and $\{v_1, v_2, v_4\}$ are linearly independent. Since $T(e_1^2 + \lambda e_2^2)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we have $\beta_4 = 0$ and either $v_4 \in \langle u_1, v_2 \rangle \setminus \langle v_2 \rangle$ or $\alpha_5 = 0$ with $v_4 \in \langle u_1, v_2, u_5 \rangle \setminus \langle u_1, v_2 \rangle$. The first case would imply that

$$T(e_2^2) \in \langle T(e_1 \oslash e_2), T(e_1 \oslash (\zeta_1 e_1 + \zeta_2 e_2 + e_3)) \rangle$$

a contradiction. Thus we obtain

$$T(e_1^2) = u_1 \oslash u_5 = b_2^{-1} u_1 \oslash v_4, \quad T(e_2^2) = v_2 \oslash v_4$$

such that $v_4 = a_2 u_1 + b_2 u_5$ for some $a_2, b_2 \in \mathbb{F}$ with $b_2 \neq 0$. Here we note that u_1, u_2, v_4 are linearly independent. Hence for any $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} T((\lambda e_1 + e_2 + e_3)^2 - e_3^2) &= (\lambda\gamma_1(1-\zeta_2) + (a_1 - \zeta_1\gamma_1))u_1 \oslash v_2 \\ &+ \lambda\alpha_3 u_1^2 + (\zeta_1\beta_1 + \beta_3)v_2^2 + \lambda^2 b_2^{-1} u_1 \oslash v_4 + v_2 \oslash v_4 \end{aligned}$$

has rank bounded above by two. Let $P(\lambda) = \lambda\gamma_1 + \lambda\gamma_1(1-\zeta_2) + (a_1 - \zeta_1\gamma_1)$. Then this implies that

$$det \begin{pmatrix} \lambda^2 b_2^{-1} & P(\lambda) & \lambda\alpha_3 \\ 1 & (\zeta_1\beta_1 + \beta_3) & P(\lambda) \\ 0 & 1 & \lambda^2 b_2^{-1} \end{pmatrix} = \lambda(\alpha_3 + \lambda^3(\zeta_1\beta_1 + \beta_3)b_2^{-2}) = 0. \qquad (3.26)$$

Case I-A-(i): $|\mathbb{F}| \geqslant 5$. Then it suffices to conclude that $\alpha_3 = 0$. But this contradicts the fact that $\alpha_3 \neq 0$. Therefore we have $\{u_1, u_2, u_3\}$ is linearly independent and a similar argument holds to show $\{v_1, v_2, v_3\}$ is linearly independent. Since $T((e_1 + \lambda e_2) \oslash e_3)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we have $\beta_3 = 0$ and either $v_3 \in \langle u_1, v_2\rangle \setminus \langle v_2\rangle$ or $\alpha_3 = 0$ with $v_3 \in \langle u_1, v_2, u_3\rangle \setminus \langle u_1, v_2\rangle$. The first case is not possible since $T(e_1 \oslash e_2), T(e_2 \oslash e_3)$ are linearly independent, so the second case holds. Hence we have

$$T(e_1 \oslash e_3) = b_4^{-1} u_1 \oslash v_3, \quad T(e_2 \oslash e_3) = v_2 \oslash v_3$$

such that $v_3 = a_4 u_1 + b_4 u_3$ for some scalars $a_4, b_4 \in \mathbb{F}$ with $b_4 \neq 0$. Next, we claim that $\alpha_5 \neq 0$. Suppose to the contrary that $\alpha_5 = 0$. Then u_1, v_2, u_3, u_5 are linearly independent. Since $T(e_1^2 + \lambda e_2^2)$ is of rank two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we have $\beta_4 = 0$ and either $v_4 \in \langle u_1, v_2 \rangle \setminus \langle v_2 \rangle$ or $v_4 \in \langle u_1, v_2, u_5 \rangle \setminus \langle u_1, v_2 \rangle$. The first case is not possible and so the second case holds. Therefore we obtain

$$T(e_1^2) = b_5^{-1} u_1 \oslash v_4, \quad T(e_2^2) = v_2 \oslash v_4$$

such that $v_4 = a_5 u_1 + b_5 u_5$ for some scalars $a_5, b_5 \in \mathbb{F}$ with $b_5 \neq 0$. Then we see that

$$T((\lambda e_1 + e_2 + e_3)^2 - e_3^2) = u_1 \oslash (\lambda \gamma_1 v_2 + \lambda b_4^{-1} v_3 + \lambda^2 b_5^{-1} v_4) + v_2 \oslash (v_3 + v_4)$$

has rank bounded above by two for any $\lambda \in \mathbb{F}$, yielding

$$det \begin{pmatrix} \lambda^2 b_5^{-1} & \lambda b_4^{-1} & \lambda \gamma_1 & 0 \\ 1 & 1 & 0 & \lambda \gamma_1 \\ 0 & 0 & 1 & \lambda b_4^{-1} \\ 0 & 0 & 1 & \lambda^2 b_5^{-1} \end{pmatrix} = \lambda^2 (\lambda b_5^{-1} - b_4^{-1})^2 = 0.$$

But if we take any nonzero scalar $\lambda \neq b_5 b_4^{-1}$, then the determinant is nonzero and so a contradiction. Hence $\alpha_5 \neq 0$. Suppose that $u_5 \notin \langle u_1, u_2 \rangle$. Then, since $T(e_1^2 + \lambda e_2^2)$ is of rank bounded by two for any $\lambda \in \mathbb{F}$, by Lemma 3.3, we conclude that $v_4 \in \langle u_1, v_2 \rangle$ since $\alpha_5 \neq 0$, but this give rise to $T(e_2^2) \in \langle T(e_1 \oslash e_2) \rangle$, a contradiction. Hence $u_5 \in \langle u_1, u_2 \rangle$ and we obtain

$$T(e_1^2) = a_6 u_1 \oslash v_2 + \alpha_5 u_1^2, \quad T(e_2^2) = b_6 u_1 \oslash v_2 + \beta_4 v_2^2$$

for some $a_6, b_6 \in \mathbb{F}$ and that $\alpha_5, \beta_4 \neq 0$. Then we see that

$$T((\lambda e_1+e_2+e_3)^2-e_3^2) = \lambda^2\alpha_5 u_1^2+\beta_4 v_2^2+(\lambda(\lambda a_6+\gamma)+b_6)u_1\oslash v_2+(\lambda b_4^{-1}u_1+v_2)\oslash v_3$$

has rank bounded above by two for any $\lambda \in \mathbb{F}$, thus it yields that

$$det\begin{pmatrix} \lambda b_4^{-1} & \lambda(\lambda a_6+\gamma)+b_6 & \lambda^2\alpha_5 \\ 1 & \beta_4 & \lambda(\lambda a_6+\gamma)+b_6 \\ 0 & 1 & \lambda b_4^{-1} \end{pmatrix} = \lambda^2(\alpha_5+\beta_4 b_4^{-2}) = 0.$$

Hence we obtain $\beta_4 = \alpha_5 b_4^2$. Conclusively we have

$$T(e_1^2) = a_6 b_4^{-1}u_1 \oslash (b_4 v_2) + \alpha_5 u_1^2, \quad T(e_2^2) = b_6 b_4^{-1}u_1 \oslash (b_4 v_2) + \alpha_5 (b_4 v_2)^2$$

$$T(e_1 \oslash e_2) = \gamma b_4^{-1}u_1 \oslash (b_4 v_2), \quad T(e_1 \oslash e_4) = u_1 \oslash u_4 + \alpha_4 u_1^2$$

$$T(e_1 \oslash e_3) = u_1 \oslash (b_4^{-1}v_3), \quad T(e_2 \oslash e_3) = (b_4 v_2) \oslash (b_4^{-1}v_3)$$

such that u_1, v_2, v_3, u_4 are linearly independent and $\alpha_5 \neq 0$. Let $z_1 = u_1$, $z_2 = b_4 v_2$, $z_3 = b_4^{-1}v_3$ and $z_4 = u_4$ and we define $Pe_i = z_i$ for all $1 \leqslant i \leqslant 4$. Then by Lemma 3.4, we get the required result.

Case I-A-(ii): $|\mathbb{F}| = 3, 4$. Since $\mathbb{F}$ has characteristic two, then $|\mathbb{F}| \neq 3$ and so we consider $|\mathbb{F}| = 4$ such that $\mathbb{F} = \{0, 1, \lambda_1, \lambda_2\}$. By (3.26), we obtain $\lambda_1^3 = \lambda_2^3 = 1$ which yields the relations $\lambda_1 + \lambda_2 + 1 = 0$ and $\lambda_1\lambda_2 = 1$. Hence we obtain $\zeta_1\beta_1 + \beta_3 = \alpha_3 b_2^2$ and so

$$T(e_1^2) = u_1 \oslash (b_2^{-1}v_4), \quad T(e_2^2) = b_2 v_2 \oslash (b_2^{-1}v_4)$$

$$T(e_1 \oslash e_2) = \gamma b_2^{-1}u_1 \oslash (b_2 v_2), \quad T(e_1 \oslash e_4) = u_1 \oslash u_4 + \alpha_4 u_1^2$$

$$T(e_1 \oslash e_3) = \alpha_3 u_1^2 - \zeta_2\gamma_1 u_1 \oslash v_2,$$

$$T(e_2 \oslash e_3) = \alpha_3 (b_2 v_2)^2 + (a_1 - \zeta_1\gamma_1) v_2 \oslash u_1.$$

For the case $v_3 \in \langle v_1, v_2 \rangle$, we may argue in a similar way as in Case-I-A to show that the result of Case I-A-(ii) still holds. On the other hand, if $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are linearly independent, then the argument and result follow from Case-I-A-(i).

Case I-B: (3.24) holds. We first note that $T(e_1 \oslash e_2) = u_1 \oslash u_2 + \alpha_2 u_1^2$. Then by the form of (3.24), we have $\alpha_2 = 0$. Next, we see that $T(e_2 \oslash e_3) = u_{23} \oslash u'_{23}$ for some $u_{23} \in \langle w_1, w_2 \rangle$ and $u'_{23} \in \langle w_1, w_2, w_3 \rangle$. Without loss of generality, say u_1, u_2, u_{23} are linearly independent as the case for u_1, u_2, u'_{23} can be verified similarly. Since $T((e_1 + \lambda e_2) \oslash e_3)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we conclude that

$$T(e_2 \oslash e_3) = u_{23} \oslash (a_7 u_1 + b_7 u_2) \neq 0 \tag{3.27}$$

for some scalars $a_7, b_7 \in \mathbb{F}$. By (3.24), we may write

$$\left\langle T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3) \right\rangle = \langle u_1 \oslash u_2, u_1 \oslash u_{23}, u_2 \oslash u_{23} \rangle$$

implying, in view of (3.27), $T(e_2^2) = u_{23} \oslash (a_8 u_1 + b_8 u_2) + c_8 u_{23} \oslash (a_7 u_1 + b_7 u_2) + d_8 u_1 \oslash u_2$ for some $a_8, b_8, c_8, d_8 \in \mathbb{F}$ such that $a_7 b_8 + a_8 b_7 \neq 0$. Further, we see that

$$T(e_1 \oslash e_2) = u_1 \oslash u_2 = (a_7 b_8 + a_8 b_7)^{-1} (a_7 u_1 + b_7 u_2) \oslash (a_8 u_1 + b_8 u_2).$$

Let $v_1 = (a_7 b_8 + a_8 b_7)^{-1} (a_8 u_1 + b_8 u_2)$, $v_2 = a_7 u_1 + b_7 u_2$ and $v_3 = u_{23}$. Then

$$T(e_1 \oslash e_2) = v_1 \oslash v_2,\ T(e_2 \oslash e_3) = v_2 \oslash v_3,$$

$$T(e_2^2) = (a_9 v_1 + b_9 v_2) \oslash v_3 + c_9 v_1 \oslash v_2 \tag{3.28}$$

for some scalars $a_9, b_9, c_9 \in \mathbb{F}$ with $a_9 \neq 0$ such that v_1, v_2, v_3 are linearly independent. On the other hand, we have

$$\langle T(e_1 \oslash e_3), T(e_1 \oslash e_2), T(e_2 \oslash e_3)\rangle = \langle T(e_1 \oslash e_3), v_1 \oslash v_2, v_2 \oslash v_3\rangle$$

is a 3-dimensional subspace of bounded rank-two matrices. Hence by Theorem 2.6 and Lemma 3.1, we have either

$$T(e_1 \oslash e_3) = v_2 \oslash v_{13} + \beta_{13} v_2^2 \tag{3.29}$$

for some vector v_{13} and some scalar $\beta_{13} \in \mathbb{F}$, or

$$T(e_1 \oslash e_3) = a_{10} v_1 \oslash v_3 + b_{10} v_2 \oslash v_3 + c_{10} v_1 \oslash v_2 \tag{3.30}$$

for some scalars $a_{10}, b_{10}, c_{10} \in \mathbb{F}$ with $a_{10} \neq 0$. Since $\langle u_1, u_2\rangle = \langle v_1, v_2\rangle$, then we consider the following two possible cases:

Case I-B-(i): $v_2 = \gamma u_1$ for some nonzero scalar $\gamma \in \mathbb{F}$. Then u_1, v_1, v_3 are linearly independent. By (3.22), it is not possible to have (3.30) and so (3.29) holds. Now since $T(\lambda e_1^2 + (c_9 e_1 \oslash e_2 + e_2^2))$ has rank bounded above by two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we have $\alpha_5 = 0$ and $u_5 \in \langle u_1, a_9 v_1 + b_9 \gamma u_1, v_3\rangle$. If $u_5 \in \langle u_1, a_9 v_1 + b_9 \gamma u_1\rangle$, then we have $T(e_1^2) \in \langle T(e_1 \oslash e_2)\rangle$, a contradiction. Hence we obtain $T(e_1^2) = v_2 \oslash (a_{11} v_3 + b_{11} v_1)$ for some scalars $a_{11}, b_{11} \in \mathbb{F}$ with $a_{11} \neq 0$. Then we see that

$$\begin{aligned} T((e_1 + e_2 + e_3)^2 - e_3^2) &= a_9 v_1 \oslash v_3 + \beta_{13} v_2^2 \\ &\quad + v_2 \oslash (v_{13} + (a_{11} + b_9 + 1) v_3 + (b_{11} + c_9 + 1) v_1) \end{aligned}$$

has rank bounded above by two and thus $\beta_{13} = 0$ and $v_{13} \in \langle v_1, v_2, v_3\rangle$. Hence we obtain $T(e_1 \oslash e_3) = v_2 \oslash (a_{12}v_3 + b_{12}v_1)$ such that $a_{12}b_{11} + a_{11}b_{12} \neq 0$ for $T(e_1^2), T(e_1 \oslash e_3)$ are linearly independent. But we see that $T(e_1 \oslash e_2) = v_1 \oslash v_2$, $T(b_{11}e_1 \oslash e_2 + e_1^2) = a_{11}v_2 \oslash v_3$ implying $T(e_1 \oslash e_3) \in \langle T(e_1^2), T(e_1 \oslash e_2)\rangle$, a contradiction. Hence Case I-B-(i) is not possible.

Case I-B-(ii): u_1, v_2 are linearly independent. Suppose that (3.29) holds. Then by (3.22), we have $v_2 \oslash v_{13} + \beta_{13}v_2^2 = u_1 \oslash u_3 + \alpha_3 u_1^2$ and so by Lemma 3.2, we obtain $T(e_1 \oslash e_3) = v_2 \oslash (a_{13}v_1 + b_{13}u_3)$ for some $a_{13}, b_{13} \in \mathbb{F}$. Further we may show that $u_3 \notin \langle v_1, v_2, v_3\rangle$ and that $b_{13} \neq 0$. On the other hand, $T(e_1 \oslash (a_{13}e_2 + e_3) + \lambda e_1^2)$ is of rank bounded by two for any $\lambda \in \mathbb{F}$. By Lemma 3.3, we have $\alpha_5 = 0$ and hence $T(e_1^2) = u_1 \oslash (a_{14}v_2 + b_{14}u_3)$. Since $\langle u_1, u_2\rangle = \langle v_1, v_2\rangle$, then we obtain

$$T(e_1^2) = (a_{15}v_1 + b_{15}v_2) \oslash u_3 + c_{15}v_1 \oslash v_2$$

for some $a_{15}, b_{15}, c_{15} \in \mathbb{F}$ with $a_{15} \neq 0$. Next, we see that

$$\begin{aligned} T((\lambda e_1 + e_2 + e_3)^2 - e_3^2) &= (\lambda^2 a_{15}v_1 + (\lambda b_{13} + \lambda^2 b_{15})v_2) \oslash u_3 \\ &\quad + (\lambda^2 c_{15} + \lambda a_{13} + \lambda + c_9)v_1 \oslash v_2 + (a_9 v_1 + (b_9 + 1)v_2) \oslash v_3 \end{aligned}$$

is of rank bounded by two for any $\lambda \in \mathbb{F}$, it yields that

$$\begin{aligned} &det \begin{pmatrix} \lambda^2 a_{15} & a_9 & \lambda^2 c_{15} + \lambda a_{13} + \lambda + c_9 & 0 \\ \lambda b_{13} + \lambda^2 b_{15} & b_9 + 1 & 0 & \lambda^2 c_{15} + \lambda a_{13} + \lambda + c_9 \\ 0 & 0 & b_9 + 1 & a_9 \\ 0 & 0 & \lambda b_{13} + \lambda^2 b_{15} & \lambda^2 a_{15} \end{pmatrix} \\ &\qquad = \lambda^2(\lambda(a_{15}(b_9 + 1) + a_9 b_{15}) + a_9 b_{13})^2 = 0. \end{aligned}$$

If $|\mathbb{F}| \geqslant 5$, then it is immediate that $a_9 b_{13} = 0$. On the other hand, if $|\mathbb{F}| = 4$, then

$\lambda^4 = \lambda \in \mathbb{F}$, since every element in the field is of order 3. Hence by substituting in any two nonzero distinct scalars, we have $a_9 b_{13} = 0$. But this is not true since a_9, b_{13} are nonzero. Hence (3.30) holds. We first note that $v_3 \notin \langle v_1, v_2 \rangle = \langle u_1, u_2 \rangle$. Then for $T(e_1 \oslash (e_3 + c_{10} e_2))$, we get $v_3 \oslash (a_{10} v_1 + b_{10} v_2) = u_1 \oslash (c_{10} u_2 + u_3) + \alpha_3 u_1^2$. Hence by Lemma 1.3(a), we obtain

$$u_1 \in \langle v_3, a_{10} v_1 + b_{10} v_2 \rangle \cap \langle v_1, v_2 \rangle = \langle a_{10} v_1 + b_{10} v_2 \rangle$$

and for the sake of convenience, we let $u_1 = a_{10} v_1 + b_{10} v_2$. Hence $v_1 = a_{10}^{-1}(u_1 + b_{10} v_2)$ which implies that $v_1 \oslash v_2 = a_{10}^{-1} u_1 \oslash v_2$. Then it follows from (3.28) that

$$T(e_1 \oslash e_3) = u_1 \oslash v_3 + a_{10}^{-1} c_{10} v_1 \oslash v_2.$$

We now assume that α_5 in (3.22) is nonzero. Since $T(e_1^2 + e_2^2)$ has rank bounded above by two, then we have $a_9 v_1 + b_9 v_2 = a_{16} u_1$ for some nonzero scalar $a_{16} \in \mathbb{F}$. It follows from (3.28) that

$$T(e_2^2) = a_{16} u_1 \oslash v_3 + a_{10}^{-1} c_9 u_1 \oslash v_2.$$

Note that u_1, v_2, v_3 are linearly independent and so

$$\begin{aligned} T((e_1 + e_2 + e_3)^2 - e_3^2) = {} & \alpha_5 u_1^2 + v_2 \oslash v_3 \\ & + u_1 \oslash (u_5 + (a_{16} + 1) v_3 + a_{10}^{-1}(c_9 + c_{10} + 1) v_2) \end{aligned}$$

has rank > 2, a contradiction. Thus $\alpha_5 = 0$ and it follows that $u_5 \notin \langle v_1, v_2, v_3 \rangle$. Further since $T(e_1^2 + e_2^2 + c_9 e_1 \oslash e_2)$ has rank bounded above by two, we conclude that $a_9v_1 + b_9v_2 = a_{17}u_1$ for some nonzero scalar $a_{17} \in \mathbb{F}$. But this implies that

$$T((e_1 + e_2 + e_3)^2 - e_3^2) = v_2 \oslash v_3 + u_1 \oslash (u_5 + (a_{17} + 1)v_3 + a_{10}^{-1}(c_9 + c_{10} + 1)v_2)$$

is of rank four, a contradiction. Hence Case I-B-(ii) will not occur.

Case I-C: Suppose that (3.25) holds. Then

$$u_1 \oslash u_2 + \alpha_2 u_1^2 = T(e_1 \oslash e_2) \in \left\langle x^2, y^2, x \oslash y \right\rangle . \tag{3.31}$$

Case I-C-(i): u_1, u_2 are linearly dependent. Then $T(e_1 \oslash e_2) = \alpha_2 u_1^2$ and u_1, u_3, u_5 are linearly independent. Further, without loss of generality, in view of (3.31), we see that $T(e_1 \oslash e_2) \in \langle x^2 \rangle$. Thus u_1, x are linearly dependent, and so

$$\left\langle x^2, y^2, x \oslash y \right\rangle = \left\langle u_1^2, y^2, u_1 \oslash y \right\rangle .$$

Let $T(e_2^2) = a_{11}u_1^2 + b_{11}u_1 \oslash y + c_{11}y^2$ for some $a_{11}, b_{11}, c_{11} \in \mathbb{F}$. If $c_{11} \neq 0$, then

$$T(e_1^2 + e_2^2) = u_1 \oslash u_5 + (\alpha_5 + a_{11})u_1^2 + b_{11}u_1 \oslash y + c_{11}y^2$$

has rank bounded above by two. Thus, $y \in \langle u_1, u_5 \rangle$. It follows that

$$\left\langle T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3) \right\rangle = \left\langle x^2, y^2, x \oslash y \right\rangle = \left\langle u_1^2, u_5^2, u_1 \oslash u_5 \right\rangle .$$

Since $T(e_1 \oslash e_3) \notin \langle u_1^2, u_5^2, u_1 \oslash u_5 \rangle$, then we have $T((e_1 + \lambda_1 e_2 + \lambda_2 e_3)^2 + \lambda_2^2 e_3^2)$ is of rank three for some $\lambda_1, \lambda_2 \in \mathbb{F}$ with $\lambda_2 \neq 0$, a contradiction.

Suppose that $c_{11} = 0$. Then $T(e_2 \oslash e_3) = a'_{11}u_1^2 + b'_{11}u_1 \oslash y + c'_{11}y^2$ for some

$a'_{11}, b'_{11}, c'_{11} \in \mathbb{F}$ with $c'_{11} \neq 0$. It follows that

$$T((e_1 + e_2) \oslash e_3) = u_1 \oslash u_3 + (\alpha_3 + a'_{11})u_1^2 + b'_{11}u_1 \oslash y + c'_{11}y^2$$

has rank bounded above by two. Thus, $y \in \langle u_1, u_3 \rangle$ and this implies that

$$\left\langle T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3) \right\rangle = \left\langle x^2, y^2, x \oslash y \right\rangle = \left\langle u_1^2, u_3^2, u_1 \oslash u_3 \right\rangle.$$

Since $T(e_1^2) \notin \langle u_1^2, u_3^2, u_1 \oslash u_3 \rangle$, then we have $T((e_1 + \lambda'_1 e_2 + \lambda'_2 e_3)^2 + (\lambda'_2)^2 e_3^2)$ is of rank three for some $\lambda'_1, \lambda'_2 \in \mathbb{F}$, a contradiction. Hence Case I-C-(i) is not possible.

Case I-C-(ii): u_1, u_2 are linearly independent. By Lemma 2.2 and Lemma 1.3(a) and (b), we obtain that $\langle u_1, u_2 \rangle = \langle x, y \rangle$. Thus

$$\left\langle T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3) \right\rangle = \left\langle x^2, y^2, x \oslash y \right\rangle = \left\langle u_1^2, u_2^2, u_1 \oslash u_2 \right\rangle.$$

Note that either $T(e_2^2)$ or $T(e_2 \oslash e_3)$ has nonzero u_2^2 term. Hence we first consider $T(e_2^2)$. Suppose that $T(e_2^2) = a_{12}u_1^2 + b_{12}u_1 \oslash u_2 + c_{12}u_2^2$ for some $a_{12}, b_{12}, c_{12} \in \mathbb{F}$ with $c_{12} \neq 0$. Then, since $T(e_1^2 + e_2^2)$ has rank bounded above by two, it follows that $\langle T(e_1^2), T(e_1 \oslash e_2) \rangle = \langle u_1^2, u_1 \oslash u_2 \rangle$. Hence $T(e_1 \oslash e_3) \notin \langle u_1^2, u_2^2, u_1 \oslash u_2 \rangle$. But this implies that $T((e_1 + \lambda_1 e_2 + \lambda_2 e_3)^2 + \lambda_2^2 e_3^2)$ is of rank three for some $\lambda_1, \lambda_2 \in \mathbb{F}$ where $\lambda_2 \neq 0$, a contradiction.

Now, suppose that $T(e_2 \oslash e_3) = a'_{12}u_1^2 + b'_{12}u_1 \oslash u_2 + c'_{12}u_2^2$ for some $a'_{12}, b'_{12}, c'_{12} \in \mathbb{F}$ with $c'_{12} \neq 0$. Since $T((e_1 + e_2) \oslash e_3)$ has rank bounded above by two, it follows that $\langle T(e_1^2), T(e_1 \oslash e_2) \rangle = \langle u_1^2, u_1 \oslash u_2 \rangle$. Hence $T(e_1 \oslash e_3) \notin \langle u_1^2, u_2^2, u_1 \oslash u_2 \rangle$. But this implies that $T((e_1 + \lambda'_1 e_2 + \lambda'_2 e_3)^2 + (\lambda'_2)^2 e_3^2)$ is of rank three for some $\lambda'_1, \lambda'_2 \in \mathbb{F}$ where $\lambda'_2 \neq 0$, a contradiction. Hence Case-I-C-(ii) is also not possible.

Case II: $\mathbb{F}$ has characteristic not two. Since T is a bounded rank-two linear preserver, then we have $T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3), T(e_1 \oslash e_4)$ are linearly independent. Further, by Theorem 2.6, we see that

$$T(e_1^2) = u_1 \oslash u_5, \quad T(e_1 \oslash e_i) = u_1 \oslash u_i$$

for some $u_1 \in \mathcal{U}_p$, $u_5, u_i \in \mathcal{U}_q$ satisfying $1 \leqslant p \leqslant m+1-q \leqslant m$ and some $\alpha_i \in \mathbb{F}$, for $i = 2, 3, 4$. On the other hand, we have $T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3)$ are linearly independent. Then by Theorem 2.6, we have either

$$T(e_2^2) = v_2 \oslash v_4, \quad T(e_2 \oslash e_j) = v_2 \oslash v_j \tag{3.32}$$

for some $v_2 \in \mathcal{U}_s$, $v_4, v_j \in \mathcal{U}_t$ satisfying $1 \leqslant s \leqslant m+1-t \leqslant m$ such that $\{T(e_2 \oslash e_1), T(e_2^2), T(e_2 \oslash e_3)\}$ is linearly independent, for $j = 1, 3$; or

$$\big\langle T(e_2^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3)\big\rangle = \big\langle x^2, y^2, x \oslash y\big\rangle \tag{3.33}$$

for some linearly independent vectors $x, y \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$.

Case II-A: Suppose that (3.32) holds. Notice that if $v_2 \in \langle u_1 \rangle$, then we have

$$\operatorname{Im} T = u_1 \oslash U$$

for some nonzero vector $u_1 \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant m+1-q \leqslant m$. Suppose now that u_1, v_2 are linearly independent. Then $T(e_1 \oslash e_2) = u_1 \oslash u_2 = v_2 \oslash v_1$, by Lemma 3.2, implies that $u_2 = \gamma v_2$ for some nonzero $\gamma \in \mathbb{F}$ and so $u_1 = \gamma^{-1} v_1$. We now claim that $u_5, v_4 \in \langle u_1, v_2 \rangle$. Suppose to the contrary that u_5, u_1, v_2 are linearly independent. Then we see that $T(e_1^2 + \lambda e_2^2)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$, yielding $v_4 = \lambda_0 u_5$ for some nonzero scalar $\lambda_0 \in \mathbb{F}$. But this implies that

$$T((e_1 + e_2)^2) = (u_1 + \lambda_0 v_2) \oslash u_5 + u_1 \oslash \gamma v_2$$

is of rank three, a contradiction and so the claim is proved. So we obtain

$$T(e_1^2) = a_6 u_1 \oslash v_2 + \alpha_5 u_1^2, \quad T(e_2^2) = b_6 u_1 \oslash v_2 + \beta_4 v_2^2$$

for some scalars $a_6, b_6, \alpha_5, \beta_4 \in \mathbb{F}$ with $\alpha_5, \beta_4 \neq 0$. Thus, this implies that $\{u_1, u_2, u_3, u_4\}$ and $\{v_1, v_2, v_3\}$ are linearly independent. Now, since $T((e_1 + \lambda e_2) \oslash e_3)$ has rank bounded above by two for any $\lambda \in \mathbb{F}$, then by Lemma 3.3 we conclude that $v_3 = b_4 u_3$ for some nonzero scalar $b_4 \in \mathbb{F}$. Hence we have

$$T(e_1 \oslash e_3) = b_4^{-1} u_1 \oslash v_3, \quad T(e_2 \oslash e_3) = v_2 \oslash v_3.$$

Then we see that

$$T((\lambda e_1 + e_2 + e_3)^2 - e_3^2) = \lambda^2 \alpha_5 u_1^2 + \beta_4 v_2^2 + (\lambda(\lambda a_6 + \gamma) + b_6) u_1 \oslash v_2 + (\lambda b_4^{-1} u_1 + v_2) \oslash v_3$$

has rank bounded above by two for any $\lambda \in \mathbb{F}$, thus it yields that

$$det \begin{pmatrix} \lambda b_4^{-1} & \lambda(\lambda a_6 + \gamma) + b_6 & \lambda^2 \alpha_5 \\ 1 & \beta_4 & \lambda(\lambda a_6 + \gamma) + b_6 \\ 0 & 1 & \lambda b_4^{-1} \end{pmatrix}$$
$$= \lambda^2(\alpha_5 + \beta_4 b_4^{-2} - 2b_4^{-1}\gamma) - 2\lambda^3 b_4^{-1} a_6 - 2\lambda b_4^{-1} b_6 = 0.$$

By substituting $\lambda = -1, 1$ we can easily conclude that $\beta_4 = 2b_4\gamma - \alpha_5 b_4^2$ and so

$b_6 = -a_6$. Conclusively we have

$$T(e_1^2) = a_6 b_4^{-1} u_1 \oslash (b_4 v_2) + \alpha_5 u_1^2, \quad T(e_2^2) = (\gamma v_2 - a_6 b_4^{-1} u_1) \oslash (b_4 v_2) - \alpha_5 (b_4 v_2)^2$$

$$T(e_1 \oslash e_2) = \gamma b_4^{-1} u_1 \oslash (b_4 v_2), \quad T(e_1 \oslash e_4) = u_1 \oslash u_4$$

$$T(e_1 \oslash e_3) = u_1 \oslash (b_4^{-1} v_3), \quad T(e_2 \oslash e_3) = (b_4 v_2) \oslash (b_4^{-1} v_3)$$

such that u_1, v_2, v_3, u_4 are linearly independent and $\alpha_5 \neq 0$. Let $z_1 = u_1$, $z_2 = b_4 v_2$, $z_3 = b_4^{-1} v_3$ and $z_4 = u_4$ and we define $Pe_i = z_i$ for all $1 \leqslant i \leqslant 4$. Then by Lemma 3.4, we get the required result. Here we remark that if $|\mathbb{F}| \geqslant 4$, then $a_6 = 0$.

Case II-B: Suppose that (3.33) holds. Then we may show in a similar way as in Case-I-C above that Case-II-B is not possible. We are done. $\square$

We give the following example to illustrate the special form of the linear preserver T when the underlying field $\mathbb{F}$ has exactly four elements.

Example 3.8. Let $\mathbb{F}$ be a field with four elements. Let $P \in \mathcal{T}_4(\mathbb{F})$ be an invertible matrix and let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{F}$ with $\lambda_1, \lambda_4 \neq 0$. Let T be the linear map on $\mathcal{ST}_4(\mathbb{F})$ defined by,

$$T(A) = P \begin{pmatrix} a_{11} & a_{14} & \lambda_1 a_{13} + \lambda_2 a_{12} + \lambda_3 a_{22} & \lambda_4 a_{12} + \lambda_5 a_{11} \\ 0 & a_{23} & \lambda_4 a_{22} & \lambda_1 a_{13} + \lambda_2 a_{12} + \lambda_3 a_{22} \\ 0 & 0 & a_{23} & a_{14} \\ 0 & 0 & 0 & a_{11} \end{pmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$. To show that T is a bounded rank-two linear preserver, it suffices to consider the following two matrices:

$$B = \begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{13} \\ 0 & a_{22} & a_{12} \end{pmatrix}, C = \begin{pmatrix} a_{14} & \lambda_1 a_{13} + \lambda_2 a_{12} + \lambda_3 a_{22} & \lambda_4 a_{12} \\ a_{23} & \lambda_4 a_{22} & \lambda_1 a_{13} + \lambda_2 a_{12} + \lambda_3 a_{22} \\ 0 & a_{23} & a_{14} \end{pmatrix}.$$

Note that

$$det(B) = a_{14}a_{22}^2 + a_{33}a_{12}^2 = 0 \Leftrightarrow det(C) = \lambda_4(a_{12}a_{33}^2 + a_{22}a_{14}^2) = 0$$

since every nonzero element in $\mathbb{F}$ is of order 3.

Theorem 3.9. *Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$, and let m be an integer such that $m \geqslant 3$. If $T : \mathcal{ST}_3(\mathbb{F}) \longrightarrow \mathcal{ST}_m(\mathbb{F})$ is a bounded rank-two linear preserver, then T is one of the following forms:*

(a) $\operatorname{Im} T = u \oslash U$ *for some nonzero vector $u \in \mathcal{U}_p$ and some nonzero subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant m+1-q \leqslant m$.*

(b) $\operatorname{Im} T = \langle x^2, y^2, x \oslash y\rangle$ *for some linearly independent vectors $x, y \in \mathcal{U}_s$ with $1 \leqslant s \leqslant \frac{m+1}{2}$.*

(c) *$\mathbb{F}$ has characteristic two, and $\operatorname{Im} T = u \oslash U + \langle u^2\rangle$ for some nonzero vector $u \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and some nonzero subspace U of $\mathcal{U}_q$ with $1 \leqslant q \leqslant m+1-p$.*

(d) *$\mathbb{F}$ has characteristic two, and $\operatorname{Im} T = \langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2\rangle$ for some linearly independent vectors $u \in \mathcal{U}_p$ and $v_1, \ldots, v_k \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m+1-p$, and some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$.*

(e) *there exist an invertible matrix $P \in \mathcal{M}_m(\mathbb{F})$ and some scalars $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{F}$ with $\lambda_4, \lambda_5 \neq 0$, such that*

$$T(A) = P \left(\begin{array}{c|ccc} & a_{s_1s_1} & a_{1s_2} + \lambda_0 a_{t_1t_1} & \lambda_1 a_{s_1s_1} + \lambda_2 a_{1s_2} + \lambda_3 a_{t_1t_1} + \lambda_4 a_{1t_2} \\ 0_{3,m-3} & 0 & \lambda_5 a_{t_1t_1} & a_{1s_2} + \lambda_0 a_{t_1t_1} \\ & 0 & 0 & a_{s_1s_1} \\ \hline 0 & & 0_{m-3,3} & \end{array} \right) P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where $Pe_i \in \mathcal{U}_{p_i}$ and $Pe_3 \in \mathcal{U}_q$ with $1 \leqslant p_i \leqslant \frac{m+1}{2}$

and $q \leqslant m+1-p_1$, *for* $i=1,2$, $\{s_1,t_1\}=\{1,2\}$, $\{s_2,t_2\}=\{2,3\}$, *and* $(\lambda_1,\lambda_2)\neq 0$ *only if* $\mathbb{F}$ *has characteristic two. In particular,* $P\in\mathcal{T}_3(\mathbb{F})$ *when* $m=3$.

(f) $m\geqslant 4$, $\mathbb{F}$ *has characteristic two, and* $\operatorname{Im}T=\langle w_1\oslash w_2, w_1\oslash w_3, w_2\oslash w_3\rangle$ *for some linearly independent vectors* $w_1\in\mathcal{U}_p$, $w_2\in\mathcal{U}_q$ *and* $w_3\in\mathcal{U}_r$ *with* $p,q\leqslant m+1-r$ *and either* $p\leqslant m+1-q$, *or* $p=q>\frac{m+1}{2}$ *and* $w_2=\alpha w_1+z$ *for some nonzero scalar* $\alpha\in\mathbb{F}$ *and some vector* $z\in\mathcal{U}_k$ *with* $1\leqslant k\leqslant m+1-p<\frac{m+1}{2}$ *such that* w_2,z *are linearly independent.*

(g) $m\geqslant 4$, $\mathbb{F}$ *has characteristic two, and there exist an invertible matrix* $P\in\mathcal{M}_m(\mathbb{F})$ *and some scalars* $\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{F}$ *with* $\lambda_4\neq 0$ *and* $(\lambda_0,\lambda_3)\neq 0$, *such that*

$$T(A)=P\left(\begin{array}{c|cccc} & \lambda_0 a_{ss} & a_{12}+\lambda_1 a_{tt} & a_{13}+\lambda_2 a_{tt} & \lambda_3 a_{ss}\\ & 0 & \lambda_4 a_{tt} & 0 & a_{13}+\lambda_2 a_{tt}\\ 0_{4,m-4} & 0 & 0 & \lambda_4 a_{tt} & a_{12}+\lambda_1 a_{tt}\\ & 0 & 0 & 0 & \lambda_0 a_{ss}\\ \hline 0 & & & 0_{m-4,4} & \end{array}\right)P^+$$

for all $A=(a_{ij})\in\mathcal{ST}_3(\mathbb{F})$, *where* $\{s,t\}=\{1,2\}$, $Pe_1\in\mathcal{U}_p$ *and* $Pe_j\in\mathcal{U}_{q_j}$ *with* $1\leqslant p\leqslant\frac{m+1}{2}$ *and* $1\leqslant q_j\leqslant m+1-p$ *for* $j=2,3,4$, *and either* $q_3\leqslant m+1-q_2$ *or* $q_2=q_3>\frac{m+1}{2}$ *and* $Pe_3=\alpha Pe_2+z$ *for some nonzero scalar* $\alpha\in\mathbb{F}$ *and some vector* $z\in\mathcal{U}_k$ *with* $1\leqslant k\leqslant m+1-q_2<\frac{m+1}{2}$ *such that* Pe_2,z *are linearly independent.*

Proof. We distinguish our proof into two parts:

Case I: $\mathbb{F}$ has characteristic two. Since T is a bounded rank-two linear preserver, then we have $T(e_1^2), T(e_1\oslash e_2), T(e_1\oslash e_3), T(e_1\oslash e_4)$ are linearly independent. Further, by Theorem 2.6 there are three possible forms:

$$T(e_1^2) = u_1 \oslash u_4 + \alpha_4 u_1^2, \quad T(e_1 \oslash e_i) = u_1 \oslash u_i + \alpha_i u_1^2 \tag{3.34}$$

for some $u_1 \in \mathcal{U}_p$, $u_4, u_i \in \mathcal{U}_q$ satisfying $1 \leqslant p \leqslant m+1-q \leqslant m$ and some $\alpha_i \in \mathbb{F}$ such that $1 \leqslant p \leqslant \frac{m+1}{2}$ whenever $(\alpha_2, \alpha_3, \alpha_4) \neq 0$, for $i = 2, 3$.

$$\left\langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \right\rangle = \langle u_5 \oslash u_6, u_5 \oslash u_7, u_6 \oslash u_7 \rangle \tag{3.35}$$

for some linearly independent vectors $u_5 \in \mathcal{U}_p$, $u_6 \in \mathcal{U}_q$ and $u_7 \in \mathcal{U}_r$ such that $p, q \leqslant m+1-r$ and either $p \leqslant m+1-q$, or $p = q > \frac{m+1}{2}$ and $u_6 = \alpha u_5 + z$ for some nonzero scalar $\alpha \in \mathbb{F}$ and some vector $z \in \mathcal{U}_k$ with $1 \leqslant k \leqslant m+1-p < \frac{m+1}{2}$ such that u_6, z are linearly independent.

$$\left\langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \right\rangle = \left\langle u_8^2, u_9^2, u_8 \oslash u_9 \right\rangle \tag{3.36}$$

for some linearly independent vectors $u_8, u_9 \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$. Further we note that if $\langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$ is a rank-two subspace then $m \geqslant 4$. Next there are two possible cases for $T(e_2^2)$:

Case I-A: Suppose that $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$. Then it is clear that $T(e_2^2) \notin \langle T(e_1^2), T(e_1 \oslash e_2) \rangle$ and so we have $\operatorname{Im} T$ is one of the following forms:

(a) $\operatorname{Im} T = u_1 \oslash U$ for some nonzero vector $u_1 \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant m+1-q \leqslant m$.

(b) $\operatorname{Im} T = u_1 \oslash U + \langle u_1^2 \rangle$ for some nonzero vector $u_1 \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant q \leqslant m+1-p$.

(c) $\operatorname{Im} T = \langle u_1 \oslash w_1 + \lambda_1 u_1^2, \ldots, u_1 \oslash w_k + \lambda_k u_1^2 \rangle$ for some linearly independent vectors $u_1 \in \mathcal{U}_p$ and $w_1, \ldots, w_k \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m+1-p$, and some scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ with $(\lambda_1, \ldots, \lambda_k) \neq 0$.

(d) $\operatorname{Im} T = \langle w_1 \oslash w_2, w_1 \oslash w_3, w_2 \oslash w_3 \rangle$ for some linearly independent vectors $w_1 \in \mathcal{U}_p$, $w_2 \in \mathcal{U}_q$ and $w_3 \in \mathcal{U}_r$ such that $p, q \leqslant m+1-r$ and either $p \leqslant m+1-q$, or $p = q > \frac{m+1}{2}$ and $w_2 = \alpha w_1 + z$ for some nonzero scalar $\alpha \in \mathbb{F}$ and some vector $z \in \mathcal{U}_k$ with $1 \leqslant k \leqslant m+1-p < \frac{m+1}{2}$ such that w_2, z are linearly independent.

(e) $\operatorname{Im} T = \langle x^2, y^2, x \oslash y \rangle$ for some linearly independent vectors $x, y \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$.

Case I-B: Suppose that $T(e_2^2) \notin \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$. Then T is a bounded rank-two linear preserver implies that $\{T(e_1^2), T(e_1 \oslash e_2), T(e_2^2)\}$ is linearly independent. Further, we note that $\langle T(e_1^2), T(e_1 \oslash e_2), T(e_2^2) \rangle$ is a 3-dimensional subspace of bounded rank-two matrices, and there are three possible forms for $T(e_1^2)$:

$$T(e_1^2) = v_1 \oslash v_4 + \beta_4 v_1^2,\ T(e_1 \oslash e_2) = v_1 \oslash v_2 + \beta_2 v_1^2,\ T(e_2^2) = v_1 \oslash v_3 + \beta_3 v_1^2 \tag{3.37}$$

for some $v_1 \in \mathcal{U}_p$, $v_j \in \mathcal{U}_{q_j}$ satisfying $1 \leqslant p \leqslant m+1-q_j \leqslant m$ and some scalars $\beta_j \in \mathbb{F}$, $j = 2, 3, 4$, such that $1 \leqslant p \leqslant \frac{m+1}{2}$ whenever $(\beta_2, \beta_3, \beta_4) \neq 0$.

$$\left\langle T(e_1^2), T(e_1 \oslash e_2), T(e_2^2) \right\rangle = \langle v_5 \oslash v_6, v_5 \oslash v_7, v_6 \oslash v_7 \rangle \tag{3.38}$$

for some linearly independent vectors $v_5 \in \mathcal{U}_p$, $v_6 \in \mathcal{U}_q$ and $v_7 \in \mathcal{U}_r$ such that $p, q \leqslant m+1-r$ and either $p \leqslant m+1-q$, or $p = q > \frac{m+1}{2}$ and $v_6 = \beta v_5 + z'$ for some nonzero scalar $\beta \in \mathbb{F}$ and some vector $z' \in \mathcal{U}_k$ with $1 \leqslant k \leqslant m+1-p < \frac{m+1}{2}$

such that v_6, z' are linearly independent.

$$\left\langle T(e_1^2), T(e_1 \oslash e_2), T(e_2^2)\right\rangle = \left\langle v_8^2, v_9^2, v_8 \oslash v_9\right\rangle \tag{3.39}$$

for some linearly independent vectors $v_8, v_9 \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$. Further we note that if $\langle T(e_1^2), T(e_1 \oslash e_2), T(e_2^2)\rangle$ is a rank-two subspace then $m \geqslant 4$. Then we have four possible cases to consider:

Case I-B-(i): $T(e_1^2), T(e_1 \oslash e_2)$ are of rank two. Then either (3.34) or (3.35) holds. Suppose that (3.34) holds. Then this implies that either (3.37) or (3.38) holds. We first consider the case where (3.37) holds. If either $\langle u_1^2\rangle \subset \langle T(e_1^2), T(e_1 \oslash e_2)\rangle$, or u_1, u_2, u_4 are linearly independent, then by Lemma 3.1, one can easily conclude that $v_1 \in \langle u_1\rangle$. Hence $\mathrm{Im}\, T$ is one of the forms $(a), (b), (c)$ listed in Case I-A. Now assume that (3.38) holds. Then $\alpha_2 = \alpha_4 = 0$ and $\langle u_1, u_2, u_4\rangle = \langle v_5, v_6, v_7\rangle$. Hence we write $T(e_2^2) = a_1 u_2 \oslash u_4 + b_1 u_1 \oslash u_2 + c_1 u_1 \oslash u_4$ for some scalars $a_1, b_1, c_1 \in \mathbb{F}$ with $a_1 \neq 0$. If $u_1, \ldots, u_4$ are linearly independent, then we define $Pe_1 = u_1, Pe_2 = u_4, Pe_3 = u_2, Pe_4 = u_3$. Otherwise, choose some $u_5 \in \mathcal{M}_{m,1}(\mathbb{F})$ such that u_1, u_2, u_4, u_5 are linearly independent, then we take $Pe_4 = u_5$ instead of $Pe_4 = u_3$. Hence we obtain

$$T(A) = P \left(\begin{array}{c|cccc} & \lambda_0 a_{11} & a_{12} + \lambda_1 a_{22} & a_{13} + \lambda_2 a_{22} & \lambda_3 a_{11} \\ & 0 & \lambda_4 a_{22} & 0 & a_{13} + \lambda_2 a_{22} \\ 0_{4,m-4} & 0 & 0 & \lambda_4 a_{22} & a_{12} + \lambda_1 a_{22} \\ & 0 & 0 & 0 & \lambda_0 a_{11} \\ \hline 0 & & & 0_{m-4,4} & \end{array}\right) P^{+}$$

for some scalars $\lambda_0, \ldots, \lambda_4 \in \mathbb{F}$ with $\lambda_4 \neq 0$ and $(\lambda_0, \lambda_3) \neq 0$, for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$.

Suppose that (3.35) holds. Then this implies that either (3.37) or (3.38) holds. We first consider the case where (3.37) holds. Then $\beta_2 = \beta_4 = 0$ and $\langle v_1, v_2, v_4\rangle = \langle u_5, u_6, u_7\rangle$. Hence we write $T(e_1 \oslash e_3) = a_2 v_2 \oslash v_4 + b_2 v_1 \oslash v_2 + c_2 v_1 \oslash v_4$ for some scalars $a_2, b_2, c_2 \in \mathbb{F}$ with $a_2 \neq 0$. If $v_1, \ldots, v_4$ are linearly independent, then we define $Pe_1 = v_1, Pe_2 = v_4, Pe_3 = v_2, Pe_4 = v_3$. Otherwise, choose some $v_5 \in \mathcal{M}_{m,1}(\mathbb{F})$ such that v_1, v_2, v_4, v_5 are linearly independent, then we take $Pe_4 = v_5$ instead of $Pe_4 = v_3$. Hence we obtain

$$T(A) = P \left(\begin{array}{c|cccc} & \lambda_0 a_{22} & a_{12} + \lambda_1 a_{11} & a_{13} + \lambda_2 a_{11} & \lambda_3 a_{22} \\ & 0 & \lambda_4 a_{11} & 0 & a_{13} + \lambda_2 a_{11} \\ 0_{4,m-4} & 0 & 0 & \lambda_4 a_{11} & a_{12} + \lambda_1 a_{11} \\ & 0 & 0 & 0 & \lambda_0 a_{22} \\ \hline 0 & & & 0_{m-4,4} & \end{array} \right) P^+$$

for some scalars $\lambda_0, \ldots, \lambda_4 \in \mathbb{F}$ with $\lambda_4 \neq 0$ and $(\lambda_0, \lambda_3) \neq 0$, for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. On the other hand, if (3.35) and (3.38) hold, then we have $\langle u_5, u_6, u_7 \rangle = \langle v_5, v_6, v_7 \rangle$. This therefore implies that $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$ and contradicts our earlier hypothesis of Case I-B.

We are only left with the description of matrix P. Note that P is invertible. Further, by similar argument as in the proof of Lemma 3.4, we can show that $Pe_1 \in \mathcal{U}_p$, $Pe_j \in \mathcal{U}_{q_j}$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q_j \leqslant m + 1 - p$, for all $j = 2, 3$ (including $j = 4$ if $\lambda_0 \neq 0$). Since $Pe_2 \oslash Pe_3 \in \mathcal{ST}_n(\mathbb{F})$, then by Lemma 1.3 we have either $q_3 \leqslant m+1-q_2$ or $q_2 = q_3 > \frac{m+1}{2}$ and $Pe_3 = \alpha Pe_2 + z$ for some nonzero scalar $\alpha \in \mathbb{F}$ and some vector $z \in \mathcal{U}_k$ with $1 \leqslant k \leqslant m + 1 - q_2 < \frac{m+1}{2}$ such that Pe_2, z are linearly independent.

Case I-B-(ii): $T(e_1^2), T(e_1 \oslash e_2)$ are of rank one. Then we have (3.36) and (3.39) hold. Hence $\langle u_8, u_9 \rangle^2 = \langle v_8, v_9 \rangle^2$ yielding

$$T(e_2^2) \in \big\langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \big\rangle$$

and so again contradicts the earlier hypothesis of Case I-B.

Case I-B-(iii): $T(e_1 \oslash e_2)$ is of rank one and $T(e_1^2)$ is of rank two and so u_1, u_3, u_4

are linearly independent. Then either (3.34) or (3.36) holds. Suppose that (3.34) holds. Then we have either (3.37) or (3.39) holds. We first consider the case where (3.37) holds. Thus $v_1 \in \langle u_1 \rangle$ and so we obtain $\operatorname{Im} T$ is one of the forms $(a), (b), (c)$ listed in Case I-A. Now assume that (3.39) holds. Then $\langle u_1, u_4 \rangle = \langle v_8, v_9 \rangle$. Hence $\langle u_1, u_4 \rangle^2 = \langle v_8, v_9 \rangle^2$. Thus we have $T(e_2^2) = a_4 u_4^2 + b_4 u_1 \oslash u_4 + c_4 u_1^2$ for some scalars $a_4, b_4, c_4 \in \mathbb{F}$ with $a_4 \neq 0$. Define $Pe_1 = u_1, Pe_2 = u_4, Pe_3 = u_3$, then we have

$$T(A) = P \left(\begin{array}{c|ccc} & a_{11} & a_{13} + \lambda_0 a_{22} & \lambda_1 a_{11} + \lambda_2 a_{13} + \lambda_3 a_{22} + \lambda_4 a_{12} \\ 0_{3,m-3} & 0 & \lambda_5 a_{22} & a_{13} + \lambda_0 a_{22} \\ & 0 & 0 & a_{11} \\ \hline 0 & & 0_{m-3,3} & \end{array} \right) P^+$$

for some scalars $\lambda_0, \ldots, \lambda_5 \in \mathbb{F}$ with $\lambda_4, \lambda_5 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$.

Now suppose that (3.36) holds. Then we have either (3.37) or (3.39) holds. We first consider the case where (3.37) holds. Then $\langle v_1, v_4 \rangle = \langle u_8, u_9 \rangle$. Thus $\langle v_1, v_4 \rangle^2 = \langle u_8, u_9 \rangle^2$. Hence we have $T(e_2^2) = a_6 v_4^2 + b_6 v_1 \oslash v_4 + c_6 v_1^2$ for some scalars $a_6, b_6, c_6 \in \mathbb{F}$ with $a_6 \neq 0$. Define $Pe_1 = v_1, Pe_2 = v_4, Pe_3 = v_3$, then we have

$$T(A) = P \left(\begin{array}{c|ccc} & a_{22} & a_{13} + \lambda_0 a_{11} & \lambda_1 a_{22} + \lambda_2 a_{13} + \lambda_3 a_{11} + \lambda_4 a_{12} \\ 0_{3,m-3} & 0 & \lambda_5 a_{11} & a_{13} + \lambda_0 a_{11} \\ & 0 & 0 & a_{22} \\ \hline 0 & & 0_{m-3,3} & \end{array} \right) P^+$$

for some scalars $\lambda_0, \ldots, \lambda_5 \in \mathbb{F}$ with $\lambda_4, \lambda_5 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. Now assume that (3.39) holds. Then $\langle u_8, u_9 \rangle = \langle v_8, v_9 \rangle$. But this implies that $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$, a contradiction.

Case I-B-(iv): $T(e_1^2)$ is of rank one and $T(e_1 \oslash e_2)$ is of rank two and so u_1, u_2, u_3 are linearly independent. Then either (3.34) or (3.36) holds. Suppose that (3.34) holds. Then we have either (3.37) or (3.39) holds. We first consider the case where (3.37) holds. Thus $v_1 \in \langle u_1 \rangle$ and so we obtain $\operatorname{Im} T$ is one of the forms $(a), (b), (c)$

listed in Case I-A. Now assume that (3.39) holds. Then $\langle u_1, u_2\rangle = \langle v_8, v_9\rangle$. Thus $\langle u_1, u_2\rangle^2 = \langle v_8, v_9\rangle^2$. Then we have $T(e_2^2) = a_8u_2^2 + b_8u_1 \oslash u_2 + c_8u_1^2$ for some scalars $a_8, b_8, c_8 \in \mathbb{F}$ with $a_8 \neq 0$. Define $Pe_1 = u_1, Pe_2 = u_2, Pe_3 = u_3$, then we have

$$T(A) = P\left(\begin{array}{c|ccc} & a_{11} & a_{12}+\lambda_0 a_{22} & \lambda_1 a_{11}+\lambda_2 a_{12}+\lambda_3 a_{22}+\lambda_4 a_{13} \\ 0_{3,m-3} & 0 & \lambda_5 a_{22} & a_{12}+\lambda_0 a_{22} \\ & 0 & 0 & a_{11} \\ \hline 0 & & 0_{m-3,3} & \end{array}\right)P^+$$

for some scalars $\lambda_0, \ldots, \lambda_5 \in \mathbb{F}$ with $\lambda_4, \lambda_5 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$.

Now suppose that (3.36) holds. Then we have either (3.37) or (3.39) holds. We first consider the case where (3.37) holds. Then $\langle v_1, v_2\rangle = \langle u_8, u_9\rangle$. Thus, $\langle v_1, v_2\rangle^2 = \langle u_8, u_9\rangle^2$. Hence we have $T(e_2^2) = a_{10}v_2^2 + b_{10}v_1 \oslash v_2 + c_{10}v_1^2$ for some scalars $a_{10}, b_{10}, c_{10} \in \mathbb{F}$ with $a_{10} \neq 0$. Define $Pe_1 = v_1, Pe_2 = v_2, Pe_3 = v_3$, then we have

$$T(A) = P\left(\begin{array}{c|ccc} & a_{22} & a_{12}+\lambda_0 a_{11} & \lambda_1 a_{22}+\lambda_2 a_{12}+\lambda_3 a_{11}+\lambda_4 a_{13} \\ 0_{3,m-3} & 0 & \lambda_5 a_{11} & a_{12}+\lambda_0 a_{11} \\ & 0 & 0 & a_{22} \\ \hline 0 & & 0_{m-3,3} & \end{array}\right)P^+$$

for some scalars $\lambda_0, \ldots, \lambda_5 \in \mathbb{F}$ with $\lambda_4, \lambda_5 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. Now assume that (3.39) holds. Then $\langle u_8, u_9\rangle = \langle v_8, v_9\rangle$. But this implies that $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3)\rangle$, a contradiction.

Case II: $\mathbb{F}$ has characteristic not two. Since T is a bounded rank-two linear preserver, then $\langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3)\rangle$ is a 3-dimensional subspace of bounded rank-two matrices. Hence by Theorem 2.6 we have either

$$T(e_1^2) = u_1 \oslash u_4,\ T(e_1 \oslash e_2) = u_1 \oslash u_2,\ T(e_1 \oslash e_3) = u_1 \oslash u_3 \tag{3.40}$$

for some $u_1 \in \mathcal{U}_p$, $u_i \in \mathcal{U}_{q_i}$ satisfying $1 \leqslant p \leqslant m+1-q_i \leqslant m$, $i = 2, 3, 4$, such that

$\{T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3)\}$ is linearly independent; or

$$\langle T(e_1^2), T(e_2 \oslash e_1), T(e_2 \oslash e_3)\rangle = \langle u_5^2, u_6^2, u_5 \oslash u_6\rangle \tag{3.41}$$

for some linearly independent vectors $u_5, u_6 \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$. Further we note that if $\langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3)\rangle$ is a rank-two subspace then $m \geqslant 4$. Next there are two possible cases for $T(e_2^2)$:

Case II-A: Suppose that $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3)\rangle$. Then it is clear that $T(e_2^2) \notin \langle T(e_1^2), T(e_1 \oslash e_2)\rangle$ and so we have $\operatorname{Im} T$ is one of the following forms:

(a) $\operatorname{Im} T = u_1 \oslash U$ for some nonzero vector $u_1 \in \mathcal{U}_p$ and some subspace U of $\mathcal{U}_q$ with $1 \leqslant p \leqslant m+1-q \leqslant m$.

(b) $\operatorname{Im} T = \langle x^2, y^2, x \oslash y\rangle$ for some linearly independent vectors $x, y \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$.

Case II-B: Suppose that $T(e_2^2) \notin \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3)\rangle$. Then we note that $\langle T(e_1^2), T(e_1 \oslash e_2), T(e_2^2)\rangle$ is a 3-dimensional subspace of bounded rank-two matrices. Hence by Theorem 2.6, we have either

$$T(e_1^2) = v_1 \oslash v_4, \quad T(e_1 \oslash e_2) = v_1 \oslash v_2, \quad T(e_2^2) = v_1 \oslash v_3 \tag{3.42}$$

for some $v_1 \in \mathcal{U}_p$, $v_j \in \mathcal{U}_{q_j}$ satisfying $1 \leqslant p \leqslant m+1-q_j \leqslant m$, $j = 2, 3, 4$, such that $\{T(e_1^2), T(e_1 \oslash e_2), T(e_2^2)\}$ is linearly independent; or

$$\langle T(e_1^2), T(e_1 \oslash e_2), T(e_2^2)\rangle = \langle v_5^2, v_6^2, v_5 \oslash v_6\rangle \tag{3.43}$$

for some linearly independent vectors $v_5, v_6 \in \mathcal{U}_s$ such that $1 \leqslant s \leqslant \frac{m+1}{2}$. Further we note that if $\langle T(e_1^2), T(e_1 \oslash e_2), T(e_2^2)\rangle$ is a rank-two subspace then $m \geqslant 4$. Then we have four possible cases to consider:

Case II-B-(i): $T(e_1^2), T(e_1 \oslash e_2)$ are of rank two. Then (3.40) and (3.42) both hold. We note that either $\langle u_1^2 \rangle \subset \langle T(e_1^2), T(e_1 \oslash e_2) \rangle$, or u_1, u_2, u_4 are linearly independent, one can easily conclude that $v_1 \in \langle u_1 \rangle$ and so we obtain $\operatorname{Im} T$ is of the form (a) listed in Case II-A.

Case II-B-(ii): $T(e_1^2), T(e_1 \oslash e_2)$ are of rank one. Then we have (3.41) and (3.43) both hold. Hence $\langle u_5, u_6 \rangle^2 = \langle v_5, v_6 \rangle^2$ yielding $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$, and so contradicts the hypothesis of Case II-B.

Case II-B-(iii): $T(e_1 \oslash e_2)$ is of rank one and $T(e_1^2)$ is of rank two and so u_1, u_3, u_4 are linearly independent. Suppose that (3.40) holds. Then we first consider the case where (3.42) holds. Thus $v_1 \in \langle u_1 \rangle$ and so we obtain $\operatorname{Im} T$ is of the form (a) listed in Case II-A. Now assume that (3.43) holds. Then $\langle u_1, u_4 \rangle = \langle v_5, v_6 \rangle$. Thus $\langle u_1, u_4 \rangle^2 = \langle v_5, v_6 \rangle^2$. Hence we have $T(e_2^2) = a_4 u_4^2 + b_4 u_1 \oslash u_4 + c_4 u_1^2$ for some scalars $a_4, b_4, c_4 \in \mathbb{F}$ with $a_4 \neq 0$. Define $Pe_1 = u_1, Pe_2 = u_4, Pe_3 = u_3$, then we have

$$T(A) = P \left(\begin{array}{c|ccc} & a_{11} & a_{13} + \lambda_0 a_{22} & \lambda_1 a_{22} + \lambda_2 a_{12} \\ 0_{3,m-3} & 0 & \lambda_3 a_{22} & a_{13} + \lambda_0 a_{22} \\ & 0 & 0 & a_{11} \\ \hline 0 & & 0_{m-3,3} & \end{array} \right) P^+$$

for some scalars $\lambda_0, \ldots, \lambda_3 \in \mathbb{F}$ with $\lambda_2, \lambda_3 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$.

Now suppose that (3.41) holds. We first consider the case where (3.42) holds. It follows that $\langle v_1, v_4 \rangle = \langle u_5, u_6 \rangle$. Thus $\langle v_1, v_4 \rangle^2 = \langle u_5, u_6 \rangle^2$. Then we have $T(e_2^2) = a_6 v_4^2 + b_6 v_1 \oslash v_4 + c_6 v_1^2$ for some scalars $a_6, b_6, c_6 \in \mathbb{F}$ with $a_6 \neq 0$. Define $Pe_1 =$

$v_1, Pe_2 = v_4, Pe_3 = v_3$, then we have

$$T(A) = P \left(\begin{array}{c|ccc} & a_{22} & a_{13} + \lambda_0 a_{11} & \lambda_1 a_{11} + \lambda_2 a_{12} \\ 0_{3,m-3} & 0 & \lambda_3 a_{11} & a_{13} + \lambda_0 a_{11} \\ & 0 & 0 & a_{22} \\ \hline 0 & & 0_{m-3,3} & \end{array} \right) P^+$$

for some scalars $\lambda_0, \ldots, \lambda_3 \in \mathbb{F}$ with $\lambda_2, \lambda_3 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. Now assume that (3.43) holds. Then $\langle u_5, u_6 \rangle = \langle v_5, v_6 \rangle$. But this implies that $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$, a contradiction.

Case II-B-(iv): $T(e_1^2)$ is of rank one and $T(e_1 \oslash e_2)$ is of rank two and so u_1, u_2, u_3 are linearly independent. Suppose that (3.40) holds. Then we consider the case where (3.42) holds. Thus $v_1 \in \langle u_1 \rangle$ and so we obtain $\operatorname{Im} T$ is of the form (a) listed in Case II-A. Now assume that (3.43) holds. Then $\langle u_1, u_2 \rangle = \langle v_5, v_6 \rangle$. Thus $\langle u_1, u_2 \rangle^2 = \langle v_5, v_6 \rangle^2$. Hence we have $T(e_2^2) = a_8 u_2^2 + b_8 u_1 \oslash u_2 + c_8 u_1^2$ for some scalars $a_8, b_8, c_8 \in \mathbb{F}$ with $a_8 \neq 0$. Define $Pe_1 = u_1, Pe_2 = u_2, Pe_3 = u_3$, then we have

$$T(A) = P \left(\begin{array}{c|ccc} & a_{11} & a_{12} + \lambda_0 a_{22} & \lambda_1 a_{22} + \lambda_2 a_{13} \\ 0_{3,m-3} & 0 & \lambda_3 a_{22} & a_{12} + \lambda_0 a_{22} \\ & 0 & 0 & a_{11} \\ \hline 0 & & 0_{m-3,3} & \end{array} \right) P^+$$

for some scalars $\lambda_0, \ldots, \lambda_3 \in \mathbb{F}$ with $\lambda_2, \lambda_3 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$.

Now suppose that (3.41) holds. We first consider the case where (3.42) holds. It follows that $\langle v_1, v_2 \rangle = \langle u_5, u_6 \rangle$. Thus $\langle v_1, v_2 \rangle^2 = \langle u_5, u_6 \rangle^2$. Then we have $T(e_2^2) = a_{10} v_2^2 + b_{10} v_1 \oslash v_2 + c_{10} v_1^2$ for some scalars $a_{10}, b_{10}, c_{10} \in \mathbb{F}$ with $a_{10} \neq 0$. Define $Pe_1 = v_1, Pe_2 = v_2, Pe_3 = v_3$, then we have

$$T(A) = P \left(\begin{array}{c|ccc} & a_{22} & a_{12} + \lambda_0 a_{11} & \lambda_1 a_{11} + \lambda_2 a_{13} \\ 0_{3,m-3} & 0 & \lambda_3 a_{11} & a_{12} + \lambda_0 a_{11} \\ & 0 & 0 & a_{22} \\ \hline 0 & & 0_{m-3,3} & \end{array} \right) P^+$$

for some scalars $\lambda_0, \ldots, \lambda_3 \in \mathbb{F}$ with $\lambda_2, \lambda_3 \neq 0$ for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. Now assume that (3.43) holds. Then $\langle u_5, u_6 \rangle = \langle v_5, v_6 \rangle$. But this implies that $T(e_2^2) \in \langle T(e_1^2), T(e_1 \oslash e_2), T(e_1 \oslash e_3) \rangle$, a contradiction.

Finally, we apply Lemma 3.4 to those forms obtained in Cases I-B-(iii), I-B-(iv), II-B-(iii) and II-B-(iv), to obtain the required result. The proof is complete. $\square$

The reader should note that for form **(g)**, even if we restrict m to 4, the matrix P will not necessarily be of upper triangular form. The following example illustrates this situation.

Example 3.10. Let $\mathbb{F}$ be a field with at least three elements and of characteristic two. Let $T : \mathcal{ST}_3(\mathbb{F}) \longrightarrow \mathcal{ST}_4(\mathbb{F})$ be the bounded rank-two linear preserver defined by

$$T(A) = P \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{13} \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{pmatrix} P^{+}$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_4(\mathbb{F}).$$

Clearly T is a bounded rank-two linear preserver. To see that $T(A)$ is still of upper triangular form, we may take $a_{11} = a_{12} = a_{13} = a_{22} = 1$ and countinue with the matrix multiplication.

The next example shows that if $\mathbb{F}$ has characteristic two, then it is possible to have $Pe_2, Pe_3 \in \mathcal{U}_k$ such that $k > \frac{n+1}{2}$.

Example 3.11. Let $\mathbb{F}$ be a field with at least three elements and of characteristic two. Let $T : \mathcal{ST}_3(\mathbb{F}) \longrightarrow \mathcal{ST}_4(\mathbb{F})$ be the bounded rank-two linear preserver defined by

$$T(A) = P \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{13} \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{pmatrix} P^{+}$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_4(\mathbb{F}).$$

Clearly T is a bounded rank-two linear preserver and it can be verified easily that $T(A)$ is of upper triangular form. Here, we have $Pe_2, Pe_3 \in \mathcal{U}_3$ where $3 > \frac{4+1}{2}$.

We give a few examples of bounded rank-two linear preservers $\mathcal{ST}_3(\mathbb{F}) \to \mathcal{ST}_4(\mathbb{F})$ to illustrate the forms **(b)** and **(f)** listed in Theorem 3.9.

Example 3.12. Let $\mathbb{F}$ be a field with at least three elements and of characteristic two. Let $\{e_1, \ldots, e_4\}$ be the standard basis of $\mathcal{M}_{4,1}(\mathbb{F})$.

(a) Let $T_1 : \mathcal{ST}_3(\mathbb{F}) \to \mathcal{ST}_4(\mathbb{F})$ be the linear map defined by

$$T_1(A) = a_{13}e_1^2 + a_{12}e_1 \oslash e_2 + (a_{11} + a_{22})\, e_2^2$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. Then T_1 is a bounded rank-two linear preserver with

$$\operatorname{Im} T_1 = \left\langle e_1^2, e_2^2, e_1 \oslash e_2 \right\rangle.$$

(b) Let $T_2 : \mathcal{ST}_3(\mathbb{F}) \to \mathcal{ST}_4(\mathbb{F})$ be the linear map defined by

$$T_2(A) = (a_{13} + a_{22})\, e_1 \oslash e_2 + (a_{12} + a_{22})\, e_1 \oslash e_3 + (a_{11} + a_{22})\, e_2 \oslash e_3$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$. Then T_2 is a bounded rank-two linear preserver with

$$\operatorname{Im} T_2 = \langle e_1 \oslash e_2, e_1 \oslash e_3, e_2 \oslash e_3 \rangle .$$

We note that $\langle T(e_1^2), T(e_1 \oslash e_2)\rangle$ is a 2-dimensional subspace of bounded rank-two matrices. Hence in view of Theorem 2.6, we have the following

Corollary 3.13. *Let $\mathbb{F}$ be an arbitrary field with $|\mathbb{F}| \geqslant 3$, and let m be an integer such that $m \geqslant 2$. If $T : \mathcal{ST}_2(\mathbb{F}) \longrightarrow \mathcal{ST}_m(\mathbb{F})$ is a bounded rank-two linear preserver then T is one of the following forms:*

(a) *there exist an invertible matrix $P \in \mathcal{M}_m(\mathbb{F})$ and scalars $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ with $\lambda_0, \lambda_3 \neq 0$, such that*

$$T(A) = P \left(\begin{array}{c|cc} 0_{2,m-2} & \begin{matrix} \lambda_0 a_{1s} + \lambda_1 a_{1t} \\ 0 \end{matrix} & \begin{matrix} \lambda_2 a_{1s} + \lambda_3 a_{1t} \\ \lambda_0 a_{1s} + \lambda_1 a_{1t} \end{matrix} \\ \hline 0 & \multicolumn{2}{c}{0_{m-2,2}} \end{array} \right) P^{+}$$

for all $A = (a_{ij}) \in \mathcal{ST}_2(\mathbb{F})$, where $Pe_1 \in \mathcal{U}_p$ and $Pe_2 \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m + 1 - p$, $\{s, t\} = \{1, 2\}$, and $(\lambda_1, \lambda_2) \neq 0$ only if $\mathbb{F}$ has characteristic two. In particular, $P \in \mathcal{T}_2(\mathbb{F})$ when $m = 2$.

(b) *$m \geqslant 3$ and $\operatorname{Im} T = \langle \alpha_1 u \oslash v + \alpha_2 u^2 + \alpha_3 v^2,\ \beta_1 u \oslash v + \beta_2 u^2 + \beta_3 v^2 \rangle$ is two-dimensional, for some linearly independent vectors $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{m+1}{2}$, and some fixed scalars $\alpha_1, \alpha_2, \alpha_3,\ \beta_1, \beta_2, \beta_3 \in \mathbb{F}$ with $(\alpha_2, \beta_2) \neq 0$ and $(\alpha_3, \beta_3) \neq 0$.*

(c) *$m \geqslant 3$ and $\operatorname{Im} T = u \oslash \langle v_1, v_2 \rangle$ for some linearly independent vectors $u \in \mathcal{U}_p$ and $v_1, v_2 \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m+1-p$.*

(d) *$m \geqslant 3$, $\mathbb{F}$ has characteristic two, and $\operatorname{Im} T = \langle u \oslash v_1 + \lambda_1 u^2,\, u \oslash v_2 + \lambda_2 u^2 \rangle$ for some linearly independent vectors $u \in \mathcal{U}_p$ and $v_1, v_2 \in \mathcal{U}_q$ with $1 \leqslant p \leqslant \frac{m+1}{2}$ and $1 \leqslant q \leqslant m+1-p$, and some scalars $\lambda_1, \lambda_2 \in \mathbb{F}$ with $(\lambda_1, \lambda_2) \neq 0$.*

(e) *$m \geqslant 4$, $\mathbb{F}$ has characteristic two, and $\operatorname{Im} T = \langle w_1 \oslash w_2, (w_1 + \lambda w_2) \oslash w_3 \rangle$ for some nonzero scalar $\lambda \in \mathbb{F}$ and some linearly independent vectors $w_1 \in \mathcal{U}_p$, $w_2 \in \mathcal{U}_q$ and $w_3 \in \mathcal{U}_r$ such that $p, q \leqslant m+1-r$ and either $p \leqslant m+1-q$, or $p = q > \frac{m+1}{2}$ and $w_2 = \alpha w_1 + z$ for some nonzero scalar $\alpha \in \mathbb{F}$ and some vector $z \in \mathcal{U}_k$ with $1 \leqslant k \leqslant m+1-p < \frac{m+1}{2}$ such that w_2, z are linearly independent.*

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