QUANTUM ENTANGLEMENT CRITERIA

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FACULTY OF SCIENCE
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ORIGINAL LITERARY WORK DECLARATION

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ABSTRACT

Entanglement of quantum is a quantum phenomenon which cannot be described using classical physics. This study reviews the quantum entanglement which discusses the various methods which are used to detect the entanglement. The properties of entanglement in the quantum system have been determined through the theoretical analysis of the quantities of nonclassical measure. That leads to the estimation of quantum entanglement with different parameters conditions. The study of the features and the nature of quantum entanglement enhance our understanding of the nature of entanglement processes. Furthermore, the entanglement is useful in the field of information processing where the entanglement is a basic ingredient of quantum information processing.

ABSTRAK

Keterbelitan kuantum adalah satu fenomena kuantum yang tidak boleh digambarkan dalam aspek fizik klasik. Kajian ini mengkaji keterbelitan kuantum yang membincangkan pelbagai kaedah yang digunakan untuk mengesan keterbelitan. Teori secara analisis dalam aspek bukan klasik menentukan sifat keterbelitan dalam sistem kuantum. Penggunaan syarat parameter yang berbeza dibincangkan dalam menganggar keadan keterbelitan di dalam sistem. Kajian tentang ciri-ciri dan sifat keterbelitan kuantum meningkatkan pemahaman kita tentang sifat dalam proses keterbelitan. Tambahan pula, keterbelitan sangat berguna dalam bidang pemprosesan maklumat di mana keterbelitan adalah ramuan asas dalam pemprosesan maklumat bagi kuantum.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Quantum optics is one branch of research areas in physics. It has been a dominant research area for at least two decades. The research on quantum theory which involves the interaction between two particles obtains distinguished result of the research from classical field. The quantum optics is developed from a study of the interactions of photon and matter which leads to interrelated results.

Experiments of quantum optics involve a light as the main resource because the photons are travelling in the form of a wave. The radiation of light in the experiment affects the interaction between photons. This is reflected in the photons exchanging the energy which is carried by each photon. A photon which travels in a light beam represents a quantum of discrete energy in that free moving photon.

The description of radiation of light leads to formulation of the energy quanta. The theoretical description enables us to study the nonclassical properties of light that cannot be expressed by the classical electromagnetic field. However, it can be explained with the quantized form of electromagnetic field. Therefore, the energy of quantum is formulated in relation to quantization of the electromagnetic field.

In the quantum electrodynamics, the quantization of the electromagnetic field employs annihilation and creation operators which is lowering and increasing number of quanta respectively. Thus, the quantum state can be measured based on photon interaction which involves the energy exchange. Therefore, the quantization of the electromagnetic field is very important to describe the quantum optical phenomena.

The quantum state of light provides the main tool in order to describe optical phenomena such as quantum entanglement. The concept of this optical phenomenon involves interaction of particles with quantized light. The quantum entanglement is identified from its resource which is a quantum state. There are many quantities which are introduced to describe entanglement of quantum systems based on the theoretical research of previous researchers.

The theoretical research on quantum entanglement criteria has undergone rapid development. This development is due to the importance and advantages of entanglement in quantum information and data security. This study contains the review of quantum entanglement criteria, and also its features, properties and requirements. The entanglement exists after interaction of particles which can be regarded as correlations between two particles. Therefore, the entangled and correlated states exist simultaneously.

This thesis has collected a number of well known quantities present in the criteria of quantum entanglement. All the criteria provide the conditions of parameters to be observed in order to get the entangled state. There are twelve criteria to detect the entanglement in a system which are; entropy, Peres-Horodecki, squeezing, photon antibunching, sub Poissonian, Duan criterion, Hillery-Zubairy criterion, Cauchy-Schwarz inequality, Bell's theorem, Greenberger-Horne-Zeilinger theorem, negative Wigner function and logarithmic negativity. Subsequently, some quantities are related because of the several common parameter used.

The study of the properties, requirements, and features of quantum entanglement gives the essential facts about the quantum systems. Additionally, the correlation and relationship between different measures of quantum entanglement provide insight on the uniqueness properties nature of entanglement. The quantum entanglement is a very important resource for quantum information processing. Nowadays, the application of quantum entanglement is not limited to quantum information processing only, but it can be developed in other fields like medicine, technology and engineering.

1.2 Objectives

The main objective of this research is to review the criteria of detecting entanglement of a system theoretically. Also, the theories that describe the strength of an entangled quantum system are discussed in this paper. From previous studies, different quantities were discovered with each of them having different entanglement criteria in terms of various parameters. The collections of the quantities in this research bring out valuable knowledge to be shared in the area of quantum optics. The explanation of the necessary and sufficient conditions of each entanglement criteria is very important in evaluating the formation of entanglement. Additionally, the knowledge of each entanglement criteria can describe the generation of particles in a quantum system.

The second objective of this research is to analyze the entanglement quantities according to the conditions of each quantity. The condition leads to either necessary or sufficient condition in determining the entanglement. Moreover, this contributes to the analysis of the requirement of entanglement which has relations in terms of the parameters that are involved in a mathematical framework. Some entanglement quantities use identical parameters in order to detect the entanglement of the quantum system. Furthermore, the connection between different entanglement quantities opens up prospect for a more holistic and integrated picture of entanglement.

The next objective is to develop the connection between the entanglement quantities. After analyzing the entanglement quantities, there will be a relationship which arises from the conditions of entanglement for certain quantities. It is possible to link up several criteria which are sharing the same parameters.

1.3 Outline

This thesis is divided into four chapters. In the Chapter 2, it contain the fundamentals of quantum systems. This chapter explained about the fundamentals of quantum systems before it can be applied to the quantities of entanglement. The quantization of energy described the basic measure of quantum systems through the interaction of particles from the quantization of light. Then, the state of quantum systems is explained in detail in order to determine the entanglement.

After that, in the Chapter 3, it contains the main part of thesis which contain the quantities of nonclassical measures. In this chapter has been divied into twelve sections as it contains the review of twelve entanglement quantities within the mathematical framework and explanation. Each quantities of entanglement provides the conditions that must be satisfied in order to detect the entanglement in a quantum system. All sections also give the discussion of entanglement quantities in order to provide the excellent review of entanglement criteria.

Meanwhile, Chapter 4 contains the relationships between entanglement criteria. It has five sections in explaining the relationships between entanglement criteria. The contents of this chapter discusses about the features, properties, requirement of quantum entanglement, correlations and connections between the entanglement criteria. The final chapter is Chapter 5 gives wrapping up about the criteria of quantum entanglement and the brief applications of quantum entanglement in the process of transmission information.

1.4 Literature Review

In the sense of language, the word quantum is from Latin language which means how much. This word used to describe the energy of atomic particles. Therefore, a quantum is defined as a discrete quantity of energy is predicted and observed which is related to the frequency of the radiation it represents. From theoretical basis of physics, a quantum has a relation with energy and matters.

A quantum is defined as an individual bundle or a packet of energy in some situations behaves much like particles of matter. Thus, the free motion particles are found in the certain wavelike properties and then it spread out to some degree. Therefore, quantum theory is important in quantify the energy of elementary particles. In fact, the energy is held to be emitted and absorption in tiny and discrete amount. The quantum theory incorporates with frequency of light which correspond to the amount of energy.

Historically, the research of quantum optic is well established by invention of new theories from previous physicians. The quantum optics field has been analyzed since Einstein era as a most important point in this field. Before that, the chronology of quantum theory begin by Max Planck at the year of 1901 who introduced Planck's constant in order to describe the discrete energy which exist in the black body radiation (Fox, 2006). Then, Albert Einstein applied the Planck's constant into the photoelectric derivation four years after that, at the year of 1905. He also illustrated the quantization of light which has photon travels in the light beam and carries the energy.

After that, Sir Geoffrey Ingram Taylor who known as Taylor found the single interference pattern from his observation in the photography plate four years laters (Taylor, 1909). It was known as quantum superposition where the moving photon travels according to quantization of light. At the same year, Albert Einstein discovers the fluctuation radiation which used in determining the energy fluctuation of quantum (Fox, 2006). Hence, Paul Adrien Maurice Dirac who known as Dirac has strengthened the quantum theory after his seminal paper discuss about the relativistic of emission and absorption radiation in the interaction of atom and field (Dirac, 1927).

The next studies are related to quantum optics established through experimental and theoretical methods. For example, the experiment of photodetection introduced by Roy Jay Glauber at 1963 as an attempt to measure the optical phenomena according to the studies of photon statistics (Glauber, 1963a). After that, the experiment by Kimble, Dagenais and Mandel at 1977 demonstrated that photoelectric count in the resonance fluorescence (Kimble & L., 1977). It resulting the nonclassical properties of quantum. The experiment from quantum theories was proven the significant of quantum optics in physics field.

The development of quantum theories has been applied into the telecommunication area. It was proven by Charles Henry Bennett who made an experiment about quantum cryptography using any nonorthogonal states (Bennett, 1992). That experiment have shown the communication between two parties whose sharing the secret key using the concept of encrypt the messages to avoid evesdropper and then, decrypt it by the receiver the information. In short, the researches on this area are very important as growing the development of information technology area. Additionally, the inventions of quantum optics contribute to development of information processing.

The quantum optics is one field of research from the widest fields of physics. This area of research is combination of optical physics and quantum mechanical (Scully & Zubairy, 1997). In fact, the optical physic was fit with classical physics for electromagnetic waves. However, the quantum mechanics always deal with nonclassical properties of quantum theory. The researches of quantum optics deal with application of quantum theory. Basically, quantum optics is study the natures and effects of light which have been explained by (Fox, 2006).

The research of quantum optics is concerning the light as a beam of photon because it travels within the speed of light. Additionally, this research also inclusive the interactions of light with matter due to quantization of light (Sakurai, 2011). In relation to quantum mechanics particles, the quantization of light is spread over infinite region. Therefore, light as quantized photon become important properties in the research.

The main topic of this research is focus on the quantum entanglement phenomena. The research on quantum entanglement is growing rapidly due to its advantageous which can be applying into transmission information processing. Before that, the word of entanglement defined as action or fact of entangling or being entangled which is a complicated or compromising relationship or situation. Therefore, the quantum entanglement defined that correlation between two particles after interact and then separate by each other.

The quantum entanglement is one of study area in the field of quantum optics. Generally, quantum entanglement appears universally in the atomic world. It is because the idea of quantum entanglement is relevant for elementary particles like atom and molecule. Theoretically, quantum entanglement happens when a system which has two composite particles in the quantum state are linked together. Initially, the individual particle is in the separate state, but, after a direct interaction between the particles, it become entangled (Ball, 2011). The direct interaction is occurring without any physical contact of separated particles.

The interactions between two particles lead to entanglement because of each particle carries the energy. The number of particles is unpredictable to be measure because of the series of particles will tend to more closely to half up and half down. This condition is due to the distance between two particles is irrelevant and the behavior of that particles. The quantum entanglement can be happen in any interaction between any particles such as atom, photon, and electrons, as large as bulky ball (Nairz & Zeilinger, 2002) or as small as a diamond (Ball, 2011). In the same manner, the interaction between more than two particles also can shows entanglement in the quantum system. Instead of them, the entanglement also exists in the interaction between atom and field (Entezar, 2009).

The significant properties of free motion particles produce the interactions which have effect to both particles. It is because, each moving particle has important factor which are momentum, position, spin and polarization. Therefore, the interactions of particles might be separable or inseparable due to the quantization of light (Gerry & Knight, 2005). For that reason, the particles in a quantum system will tend to more closely to half up and half down in order to be a pair of entangled state. However, the spinning of that particles are undetermined until such time some physical interference appears.

The inseparable particles after that interaction are identified as entanglement. In the process of interaction between two particles, a pair particle is said to be entangled. Then, it will generate the energy according to the quantum superposition (Scully & Zubairy, 1997). Due to the quantization of light, each moving particle in the light beam carries energy which is known as a quantum. The quantum energy was described in relation to electromagnetic field where the moving particles are corresponding to the superposition of wave function (Gerry & Knight, 2005). Thus, the energy of quantum defined in the discrete value due to the definition of quantum itself. Besides, the calculation of quantum energy provide the highway in the describing the quantum state.

The quantum system is said to be entangled when they have correlation between that particles. But, in the case of quantum system, the correlation between two particles leads to nonclassical correlation. The nonclassical correlation occurs when the particles with the different behavior in terms of quantum state are coupled together. This condition demonstrate that the interconnectedness of particles in the concept of quantum entanglement. In addition, the correlated particles are physically in the powerful state because this system consists of two equal and distinguishable binary subsystems (Zander & Plastino, 2006). The entanglement demonstrates a strong quantum correlation where the linked particles are sharing the information which contained in the each particle

On top of that, the nonclassical correlation which also known as non-local correlation which is included in the principle of quantum entanglement. This is because of it happen between particles which come from the separated quantum system. Some physicists said that the entanglement is a strange phenomenon because of the separated particles is linked intrinsically. This also related to the famous theory of entanglement according to Albert Einstein. He defined the entanglement as "spooky action at a distance". This theory can be interpreted as particles separated at a distance is linked together instantaneously and influenced by one another. Therefore, when an action is performed on one particle, the other particle is responds immediately.

Therefore, the study of quantum entanglement required the knowledge of non-classical properties of physics theory. Additionally, the quantum entanglement is integrated based on quantum theory. A perfect pair of particles can be described as quantum entanglement by the equal properties such as the quantum state with respect to position, momentum or polarization. Hence, the quantum entanglement can be described with respect to the violation of separability and locality of quantum state. The measure of quantum state is very important as a vector space to depict the quantum entanglement.

The meaning of entanglement is usually connected to non-locality and hidden variables which causes a lot of confusion. These two proposed of mechanisms was influenced to the result of experiments of entanglement. And hence, it was accepted as the properties of quantum mechanics. For example, the Schrodinger's cat thought experiment shows that the possible outcome can help to predict the superposition of two coherent states (Gerry & Knight, 2005). The result of possible outcomes show that, whether the cat's state is either dead or alive corresponds to the decoherence of the entangled coherent state. It shows that, an entangled state must hold a superposition of distinguished states.

The detection of entanglement is very important to be identified because of the effects of the interaction arise another situation which give advantages to the system. On the other words, the particles in the entangled state were operating in the harmonious unity. Instead of that, there are present the optical phenomena that lead to entanglement. Many experiments were proven the entanglement phenomena in the system involving interactions of any particles (Amselem & Bourennane, 2009) (Menzel, Candia, & etc.al., 2012) (Riziko, Kato, & etc.al., 2013)

Theoretically, the entanglement can be detected based on several quantities with respect to parameter involved in that case. There have twelve quantities describing the quantum entanglement within the mathematical framework. They are entropy, Peres-Horodocki, photon antibunching, sub Poissonian, squeezing, Chauchy-Schwarz inequality, Duan criterion, Hillery-Zubairy criterion, Bell theorem, GHZ equality, logarithmic negativity and negative Wigner function. In brief, these quantities give precise condition of quantum entanglement.

The entropy and Peres-Horodecki are two quantities that described the entanglement with respect to the density operator. The entropy has two models which the first model introduced by Shannon and it known as Shannon entropy (Shannon, 1948). The second model is established model from the first model at its mathematical framework (Gerry & Knight, 2005). It known as von Neumann entropy also it forms by the name of discoverer of this model. The entropy is very familiar in this field because this method used in determining the entanglement. Meanwhile, the Peres-Horodecki described the inseparability of particles according to partial transposition of density operators (Simon, 2000).

Next, the photon antibunching is a method which described the entanglement from the implementation of second order correlation (Scully & Zubairy, 1997). This quantity has proven by experiment of photodetector experiment in order to interpret the possible photons are detected within two times correlations. The idea of second order correlation also has been employed into sub Poissonian and Cauchy-Shwarz inequality based on photon distribution (Gerry & Knight, 2005). These quantities provided the interpretation of the entanglement and also the theoretical condition when entanglement is occurs.

Another key point of entanglement quantity is squeezing as a famous method in describing the entanglement. This quantity has a relation with quantum state in the light radiation. It is because, there exist simple harmonic oscillator of two operators of coherent states which easier to explain the entanglement condition. It involved uncertainty principle in its mathematical framework (Gerry & Knight, 2005). Duan criterion and Hillery-Zuabiry criterion are two techniques which named by its investigators. These two quantities employed two modes squeezing in order to describe the entanglement in the quantum system (Sete & Ooi, 2012).

After that, the two quantities which have a relation with entangled state are Bell theorem and GHZ equality. These quantities used to prove the entangled state based on the interpretation of quantum state (Scully & Zubairy, 1997). Generally, Bell theorem used to interpret the bipartite state, while GHZ equality used to interpret tripartite state or multipartite state.

The last two quantities of describing the entanglement are negative Wigner function and logarithmic negativity. The similarity of these quantities is the entanglement is detected at the negative region as nonclassical properties of quantum theory (Wolfgang, 2001) and (Vedral, 2006). In addition, these two quantities provided the condition according to parameter used in determining entanglement. In short, all these quantities are discussed in detail within the mathematical framework.

The study all these twelve entanglement criteria has been referred to many related journal papers which provide useful input related to entanglement. On top of that, the continous variable of entanglement has been used to apply in the single atom Raman laser as disscuss by Eyob and Ooi at (Sete & Ooi, 2012). This paper also emphasize the squeezing properties together with steady state entanglement behavior of cavity radiation. The significant result from this paper show that the cavity field exhibit transient and also steady state entanglement which proven by three criteria which are Hillery-Zubairy criterion, logarithmic negativity and Duan-Giedke-Cirac-Zoller (DGCZ) criterion. Another paper by Ooi (2007) suggested the two-photon laser as the source of entanglement based on phase controlled (C. H. R. Ooi, 2007a). This paper helps in digging the deep understanding of quantum entanglement quantities.

Other than that, the classical correlation properties are widely used in the laser field which certainly provides the input for nonclassical correlation that is closely related to measure the entanglement. According to Ooi and Gong (2012) suggested that quantum correlation of photon pair which driven by laser field (C. H. R. Ooi & Gong, 2012). The result of nonclassical correlation effects to the features of nanoparticles of microcavity. Besides, Ooi (2011) conducted the photon correlation in the arbitrary laser in order to analyze the laser pulses sequence, pulses duration, chirping and initial quantum state (C. H. R. Ooi, 2011). In general, these papers enlighten the recent result of research from the advanced researchers. It gives significant input because of it does contain the new innovation in the research area.

The coherent state is very important in measuring the entanglement as a point of reference. Ooi et. al. (2007) discusses the effect of coherence that give rise interesting features in the properties of emitted photon (C. H. R. Ooi, Kim, & Lee, 2007). This study also compares the direction of photon correlation through symmetric and antisymmetric phase entangled state. The nonclasical properties of physics is widely discuss in the papers of (C. H. R. Ooi, 2007b) and (H.-S. N. Ooi C. H. Raymond & Singh, 2012). These papers discuss in detail the nonclassical properties such as photon antibunching, sub Poissonian and negative Wigner function which closely related to the entanglement criterion. There are many others paper which are equally important input in this study of quantum entanglement criteria. Hence, there also discuss the quantum behaviors and also given situation of interaction that might influence to entanglement process.

The growing interest on entanglement quantities as a resource for quantum information processing had influenced development work to focus on known quantities like entropy, positive partial transposition, two mode squeezing and others (Guhne & Toth, 2009). Although quantum entanglement is often characterized as weird, it could be useful in transmitting information (Ball, 2011). The quantum entanglement corresponds as an essential property which provides advantages over quantum information and its complement.

CHAPTER 2

FUNDAMENTALS OF QUANTUM SYSTEMS

2.1 Quantization of energy

A quantum is a discrete amount of particles which is involved in the quantization of light. Photons which travel in the light beam hold a quantized energy known as quanta. Then, there are interactions between the elementary particle and also involving electromagnetic wave. The quantum is formed by the quantized energy of light. The quantization of energy is a phenomenon which resulted from the effects the photon emission in the light beam radiation. Each photon carries an energy of quantum which is identified based on the frequency of light. The energy of quantum, E is defined as

$$E = \hbar f \tag{2.1}$$

where $\hbar$ is the Planck constant and f is the frequency of light. In other words, the energy depends on the wavelength as the following equation

$$E = \hbar \frac{c}{\lambda}. \tag{2.2}$$

In the other words, the frequency of light is interpreted as $f = \frac{c}{\lambda}$ where c is speed of light and λ is wavelength of the light. In fact, the quantization of light is described based on the quantization of electromagnetic field (Gerry & Knight, 2005) due to photon absorption and photon emission of light radiation process. Additionally, the electromagnetic waves carry the electric and magnetic properties.

Then, the electromagnetic field was established according to Maxwell equation which described the light radiation (Scully & Zubairy, 1997), (Meystre & Sargent, 2007). The Maxwell equations define the changing of electric field and magnetic fields. At the vacuum state, the Maxwell equation reduced to

$$\bigtriangledown \cdot \overrightarrow{E} \;=\; 0 \tag{2.3}$$

$$\nabla \cdot \overrightarrow{B} = 0 \tag{2.4}$$

$$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{2.5}$$

$$\nabla \times \overrightarrow{B} = \mu_0 \varepsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} \tag{2.6}$$

where $\overrightarrow{E}$ is electric field and $\overrightarrow{B}$ is magnetic field. Meanwhile, μ_0 is defined as permeability of free space and ε_0 is defined as permittibility of free space. These equations show that linear differential equation provided the wave equation by taking the curl of electric field

$$\nabla \times \left(\nabla \times \overrightarrow{E}\right) = \nabla \times \left(-\frac{\partial \overrightarrow{B}}{\partial t}\right). \tag{2.7}$$

The left hand side of equation (2.7) is derived as

$$\begin{aligned} \nabla \times \left(\nabla \times \overrightarrow{E}\right) &= \nabla \left(\nabla \cdot \overrightarrow{E}\right) - \nabla^2 \overrightarrow{E} \qquad (2.8) \\ &= -\nabla^2 \overrightarrow{E}. \qquad (2.9) \end{aligned}$$

In the meantime, the right hand side of equation (2.7) is

$$\begin{aligned} \nabla \times \left(-\frac{\partial \overrightarrow{B}}{\partial t}\right) &= -\frac{\partial}{\partial t}\left(\nabla \times \overrightarrow{B}\right) \qquad (2.10) \\ &= -\mu_0 \varepsilon_0 \frac{\partial^2 \overrightarrow{E}}{\partial t^2}. \qquad (2.11) \end{aligned}$$

After combining both sides from equation (2.9) and (2.11), we obtained the following wave equation

$$\nabla^2 \overrightarrow{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \overrightarrow{E}}{\partial t^2}. \tag{2.12}$$

The relation describes the propagation of electric field with the speed of light, c,

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \tag{2.13}$$

where μ_0 is the ermeability of free space and ε_0 the permittibility of free space.

The wave equation (2.12) for electric field also has its counterpart for the magnetic field

$$\nabla^2 \overrightarrow{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \overrightarrow{E}}{\partial t^2} = 0 \tag{2.14}$$

$$\nabla^2 \overrightarrow{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \overrightarrow{B}}{\partial t^2} = 0. \tag{2.15}$$

It was demonstrated that the quantization of light has a firm relation between speed of light and electricity and also magnetism. It is because the electric and magnetic energy can be transformed into an equation similar to quantum harmonic oscillators. The equation (2.14) describes the electromagnetic wave propagating in the vacuum state. One possible solution of the wave equation is a plane wave propagating in space, r and time, t (Meystre & Sargent, 2007)

$$\overrightarrow{E}(r,t) = \overrightarrow{E}_0 (\mathbf{K} \cdot \mathbf{r} - \nu t) \tag{2.16}$$

where $\overrightarrow{E}_0$ is a constant of electric field and K is defined as constant vector in the direction of propagation. If there are two wavefunctions such as $\overrightarrow{E}_1(r,t)$ and $\overrightarrow{E}_2(r,t)$, its lead to the principle of superposition,

$$\overrightarrow{E}(r,t) = a_1 \overrightarrow{E}(r,t) + a_2 \overrightarrow{E}(r,t) \tag{2.17}$$

where a_1 and a_2 defined as a constant. After taking into account the superposition principle, the solution of electromagnetic wave defined as

$$\overrightarrow{E}(r,t) = \sum_j \overrightarrow{E}_j (\mathbf{K} \cdot \mathbf{r} - \nu t). \tag{2.18}$$

The quantization of light depends on the radiation field in a one dimension cavity of free space. That operator is written separately due to the superposition waves as the following

$$\overrightarrow{E}^{+}(r,t) = \sum_j \varepsilon_j \xi_j \hat{a}_j e^{-i\nu_j t + i\mathbf{K}\cdot\mathbf{r}} \tag{2.19}$$

$$\overrightarrow{E}^{-}(r,t) = \sum_k \varepsilon_j \xi_j \hat{a}_j^{\dagger} e^{i\nu_j t - i\mathbf{K}\cdot\mathbf{r}}. \tag{2.20}$$

The operator $\overrightarrow{E}^{+}(r,t)$ consists of annihilation operator, $\hat{a}_j$ and its complement operator $\overrightarrow{E}^{-}(r,t)$ consist of creation operator, $\hat{a}_j^{\dagger}$. This show that, it has been distinguished the decomposition of wavefunction in that equations. The exponent term in the equation (2.19) and (2.20) demonstrated the superposition of electric field.

Combining equation (2.19) and equation (2.20), the superposition of electromagnetic wave can be simplified as

$$\overrightarrow{E}(r,t) = \overrightarrow{E}^{+}(r,t) + \overrightarrow{E}^{-}(r,t) \tag{2.21}$$

$$= \sum_j \varepsilon_j \xi_j \left(\hat{a} e^{-iv_j t + i\mathbf{K}\cdot\mathbf{r}} + \hat{a}_j^\dagger e^{iv_j t - i\mathbf{K}\cdot\mathbf{r}} \right) \tag{2.22}$$

$$= \sum_j \varepsilon_j \xi_j \hat{a}_j e^{-iv_j t + i\mathbf{K}\cdot\mathbf{r}} + H.c. \tag{2.23}$$

where $H.c.$ defined as Hermitian conjugate. Besides, the parameter ξ_j defines electric field per photon in terms of complex function of frequency, v_j

$$\xi_j = \sqrt{\frac{\hbar v_j}{2V\varepsilon_0}} \tag{2.24}$$

In particular, the single mode electric field is polarized in the x direction of the cavity field using the sinusoidal function and also the operators of annihilation, $\hat{a}$ and creation, $\hat{a}^\dagger$. It has the form

$$\overrightarrow{E}_x(z,t) = \xi_j \left(\hat{a}_j e^{-iv_j t} + \hat{a}_j^\dagger e^{iv_j t} \right) \sin kz \tag{2.25}$$

$$= \sqrt{\frac{\hbar v_j}{2V\varepsilon_0}} \left(\hat{a}_j e^{-iv_j t} + \hat{a}_j^\dagger e^{iv_j t} \right) \sin kz \tag{2.26}$$

$$= \sqrt{\frac{2v_j^2 m_j}{V\varepsilon_0}} q(t) \sin(kz) \tag{2.27}$$

$$= A_j q_j(t) \sin(kz) \tag{2.28}$$

where $q_j(t)$ is denoted as normal mode amplitude with the dimension of a length, $k_j = \frac{j\pi}{L}$ which L is defined the length of cavity resonator with mode $j = 1, 2, ...$ (Scully & Zubairy, 1997). The transverse area, A defined from the travelling plane waves which is formulated within mass of photon, m_j and frequency, $v_j = \frac{j\pi c}{L}$

$$A_j = \sqrt{\frac{2v_j^2 m_j}{V\varepsilon_0}} \tag{2.29}$$

In the same manner, the magnetic field polarized in the y direction in order to accomplish the harmonic oscillation from equation (2.15). It defined as

$$\overrightarrow{B}_y(z,t) = \frac{\mu_0\varepsilon_0}{k_j} \sqrt{\frac{2v_k^2 m}{V\varepsilon_0}} \dot{q}_j(t) \cos(k_j z) \tag{2.30}$$

$$. = \frac{\mu_0\varepsilon_0}{k_j} A_j \dot{q}_j(t) \cos(k_j z). \tag{2.31}$$

The $\dot{q}_j(t)$ denoted as canonical of normal mode amplitude $q_j(t)$.

From the quantization of single mode field, the Hamiltonian electromegnetic field equation is determined from the total of electromagnetic wave function which is denoted as total energy. Thus, the classical Hamiltonian formula is defined as

$$H = \frac{1}{2} \int dV \left(\varepsilon_0 \overrightarrow{E}_x^2(z,t) + \frac{1}{\mu_0} \overrightarrow{B}_y^2(z,t) \right). \tag{2.32}$$

where the integration over the volume element, dV. Therefore, the total electromagnetic wavefunction from equations (2.25) and (2.31) is formulated as

$$\varepsilon_0 \overrightarrow{E}_x^2(z,t) + \frac{1}{\mu_0}\overrightarrow{B}_y^2(z,t) = \varepsilon_0 A_j^2 q_j^2(t)\sin^2(k_j z) \tag{2.33}$$

$$+\frac{1}{\mu_0}\left(\frac{\mu_0\varepsilon_0}{k_j}\right)^2 A_j^2 \dot{q}_j^2(t)\cos^2(k_j z) \tag{2.34}$$

$$= \varepsilon_0 q_j^2(t) + \frac{\mu_0\varepsilon_0^2}{k_j^2}\dot{q}_j^2(t) \tag{2.35}$$

$$= \varepsilon_0\left(q_j^2(t) + \frac{\mu_0\varepsilon_0}{k_j^2}\frac{p_j^2(t)}{m_j^2}\right) \tag{2.36}$$

$$= \varepsilon_0\left(q_j^2(t) + \frac{1}{k_j^2 c^2}\frac{p_j^2(t)}{m_j^2}\right) \tag{2.37}$$

$$= \varepsilon_0\left(q_j^2(t) + \frac{1}{\omega^2}\frac{p_j^2(t)}{m_j^2}\right). \tag{2.38}$$

Consider that the frequancy, $\omega = kc = \frac{k}{\sqrt{\mu_0\varepsilon_0}}$, and $\dot{q}_j(t) = \frac{p_j(t)}{m_j}$ is the canonical momentum. Then equation (2.32) can be simpified as the following

$$H = \frac{\varepsilon_0}{2}\int dV\left[\omega^2 q_j^2(t) + \frac{p_j^2(t)}{m_j^2}\right] \tag{2.39}$$

$$= \frac{1}{2}\sum_j .\omega^2 q_j^2(t) + p_j^2(t) \tag{2.40}$$

As a result, the electromagnetic wavefunction contribute to the Hamiltonian of quantization field. In this case, the quantization of classical system was satisfied the commutation relation, $[p_j, q_j] = i\hbar$ (Scully & Zubairy, 1997). Thus, the classical Hamiltonian for radiation field in the equation (2.40) expressed the sum of independent oscillator energies of electric and magnetic field.

Basically, the Hamiltonian equation is described by total of potential energy, V and kinetic energy, T

$$H = V + T \tag{2.41}$$

$$= \frac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m} \tag{2.42}$$

where m represents the mass of quantum, $\hat{x}$ is position operator and $\hat{p}$ is momentum operator. The momentum operator, $\hat{p}$ is expressed by

$$\hat{p} = -i\hbar\frac{\partial}{\partial\hat{x}}. \tag{2.43}$$

Under those circumstances, the position, $\hat{x}$ and momentum, $\hat{p}$ denoted as commutator for the commutation relation, $[\hat{x},\hat{p}] = i\hbar$. This commutation relation can be proved by considering the product of eigenstate, $|\psi\rangle$ within the constant momentum, $\hat{p}_0$ as the following

$$\begin{aligned}
[\hat{x},\hat{p}]\,|\psi\rangle &= (\hat{x}\hat{p}-\hat{p}\hat{x})\,|\psi\rangle && (2.44)\\
&= (\hat{p}-\hat{p}_0 I)\cdot\hat{x}\,|\psi\rangle && (2.45)\\
&= i\hbar\,|\psi\rangle\,. && (2.46)
\end{aligned}$$

From equation (2.41), the Hamiltonian equation which is also known as the energy for quantum harmonic oscillator is dependent on position, $\hat{x}$ which has the form as

$$H = \frac{1}{2}m\omega^2\hat{x}^2 - \frac{\hbar^2}{2m}\frac{d^2}{d\hat{x}^2} \tag{2.47}$$

where ω denoted as frequency of light.

$$H\psi = -\frac{\hbar^2}{2m}\bigtriangledown^2\psi(r,t) + V(r)\psi(r,t). \tag{2.48}$$

The equation (2.47) have similar operator when it compared to the Schrodinger equation (2.48) of single particles and time dependent with potential energy, V. The energy of quantum was measured in terms of $\hbar\omega$ which it can determine the distance apart between the dimensions. The distance found by nondimensionalization which is defined as constant position, $\hat{x}_c$

$$\hat{x}_c = \sqrt{\frac{\hbar}{m\omega}}. \tag{2.49}$$

The position, $\hat{x}$ and momentum, $\hat{p}$ equation are constructed after pluging in the constant position, $\hat{x}_c$ and it formulated with respect to the annihilation, $\hat{a}$ and creation, $\hat{a}^\dagger$ operators (Scully & Zubairy, 1997)

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right) && (2.50)\\
\hat{p} &= i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}-\hat{a}^\dagger\right). && (2.51)
\end{aligned}$$

After that, the annihilation, $\hat{a}$ and creation operator, $\hat{a}^\dagger$ can be formed from the equation (A.1) and (A.2) which can be referred to the Appendix A.1. Thus, the annihilation, $\hat{a}$ and creation, $\hat{a}^\dagger$ depicted as

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right) \tag{2.52}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{2.53}$$

These operators satisfy the principles of commutation relation, $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$. The step by step derivation has been shown in the Appendix A.3.

Thus, the quantized energy which is known as Hamiltonian, H is easily measured based on the operator of annihilation, $\hat{a}$ and creation, $\hat{a}^{\dagger}$. It clarified as

$$H = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right). \tag{2.54}$$

The equation (2.54) is formulated as the energy with respect to the number of state which it intrepreted as $\hat{n} = \hat{a}^{\dagger}\hat{a}$. Thus, it can be replaced the product of operators annihilation, $\hat{a}$ and creation, $\hat{a}^{\dagger}$ into the quantized energy formula.

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{2.55}$$

The quantized energy is very important in measuring the properties of quantum system such as quantum state. According to mathematical framework of quantum mechanics, a quantum state corresponds to the state vector. The measurement of quantum state depends on the energy or momentum of quantum. Besides, the probabilities can be predicted by the quantum state.

The energy of quantum state $|n\rangle$ is obtained when the Hamiltonian acts on the eigenstate,

$$\begin{aligned} H|n\rangle &= \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)|n\rangle & (2.56) \\ &= E_n|n\rangle & (2.57) \end{aligned}$$

where the eigenvalue of energy, E_n is expressed in terms of the state number, n as in the equation (2.55). Then, the energy is generated from the ground level at $n = 0$ into the excited level, $n = N$ (Gerry & Knight, 2005). The generation of energy is formulated when multiplying the equation (2.56) with creation operator, $a^{\dagger}$

$$\hbar\omega\hat{a}^{\dagger}\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)|n\rangle = E_n\hat{a}^{\dagger}|n\rangle. \tag{2.58}$$

The commutation relations for annihilation, $\hat{a}$ and creation operators, $\hat{a}^{\dagger}$, with Hamiltonian, H are defined as

$$\begin{aligned} \left[H, \hat{a}^{\dagger}\right] &= \hbar\omega\hat{a}^{\dagger} & (2.59) \\ [H, \hat{a}] &= -\hbar\omega\hat{a}. & (2.60) \end{aligned}$$

After applying the equations (2.59) and (2.60) into equation (2.58), we have the eigenvalue of energy

$$\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)\hat{a}^{\dagger}|n\rangle = (E_n+\hbar\omega)\,\hat{a}^{\dagger}|n\rangle \tag{2.61}$$

$$H\left(\hat{a}^{\dagger}|n\rangle\right) = (E_n+\hbar\omega)\left(\hat{a}^{\dagger}|n\rangle\right). \tag{2.62}$$

The equation (2.61) demonstrates that the energy eigenvalue gains a quantum of energy $\hbar\omega$. However, the equation (2.58) generated by multiplying with annihilation operator, $\hat{a}$, gives the eigenvalue that is losing one quantum of energy

$$\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)\hat{a}|n\rangle = (E_n-\hbar\omega)\,\hat{a}|n\rangle \tag{2.63}$$

$$H\left(\hat{a}|n\rangle\right) = (E_n-\hbar\omega)\left(\hat{a}|n\rangle\right). \tag{2.64}$$

The ground state, which is the lowest level, $n=0$, has the eigenstate denoted as $|0\rangle$. The quantum state cannot be lower than that,

$$H\left(\hat{a}|0\rangle\right) = (E_0-\hbar\omega)\left(\hat{a}|0\rangle\right) \tag{2.65}$$

$$= 0. \tag{2.66}$$

$$\tag{2.67}$$

So, the result of eigenvalue is equal to zero because of the product of annihilation operator, $\hat{a}$ on the ground state gives, $\hat{a}|0\rangle = 0$. Therefore, if the energy of quantum defined based on equation (2.56), the eigenvalue has the form

$$H|0\rangle = \hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)|0\rangle \tag{2.68}$$

$$= \frac{1}{2}\hbar\omega|0\rangle. \tag{2.69}$$

The equation (2.68) shows that the lowest energy eigenvalue at E_0 (Gerry & Knight, 2005) is defined as

$$E_0 = \frac{1}{2}\hbar\omega. \tag{2.70}$$

The quantized energy eigenvalues depends on the number state can be quantified as

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right), n = 0,1,2,... \tag{2.71}$$

The number state is considered as eigenstate of energy which determines the energy of quantum for different level of state. The energy of equation (2.71) is illustrated as in Figure (2.1). It shows the lowest energy, E_0 and elevated like a stairs to achieve the highest lavel E_n.

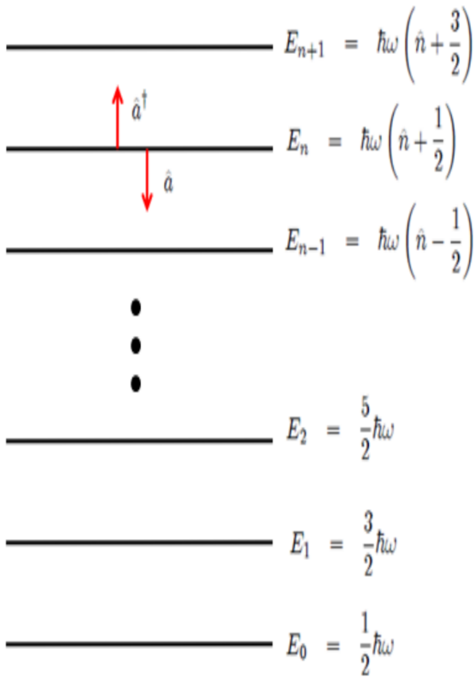

Figure 2.1: The quantized energy level for the number states

It illustrates that the number operator is determined by the annihilation, $\hat{a}$ and creation, $\hat{a}^\dagger$ operators

$$\hat{n} = \hat{a}^\dagger \hat{a}. \tag{2.72}$$

The number state also can be defined as eigenstate of number operator

$$\hat{a}^\dagger \hat{a} \left| n \right\rangle = n \left| n \right\rangle. \tag{2.73}$$

From the equation (2.73), the annihilation, $\hat{a}$ and creation, $\hat{a}^\dagger$ operators is expressed with respect to the number of photon. For detail calculations please refer to the Appendix A.2. The normalization of eigenstate is expressed as

$$\left\langle n \middle| n \right\rangle = 1. \tag{2.74}$$

The annihilation operator, $\hat{a}$, is identified as losing the number of photons which defined as the following

$$\hat{a} \left| n \right\rangle = \sqrt{n} \left| n-1 \right\rangle. \tag{2.75}$$

The creation operator $\hat{a}^\dagger$ gives the gain in the number of photon as

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{2.76}$$

The significance of quantization energy of the electromagnetic wave correspond to the interaction of particles. Hence, the quantization of energy is a process based on the radiation of electromagnetic field. The interactions lead to the exchange the energy of interacting particles.

The direct interaction between light and elementary particles like atom usually will create the entangled state of atoms. The interactions between the atoms in the radiation of light affect to the particles. Consider the two level atoms which have excited state and ground state. The effect of interaction within light cause emission and absorption of energy (Fox, 2006). The process of absorption and emission by two level atoms are illustrated in the figures below.

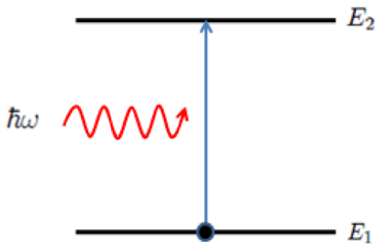

Figure 2.2: Absorption of photon by two level atom

According to the Fig. (2.2), initially, the atom at the ground state and when the light beam is turned on, it absorbs photon from the light beam and goes to an excited state.

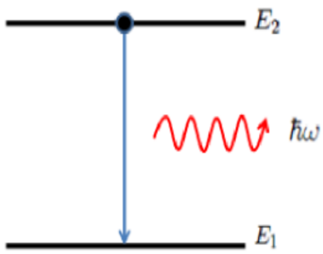

Figure 2.3: Emission of photon by two level atom

The emission process by two level atom happens when the atom is in the excited state at first and after the light is radiated, the photon is emitted as shown in the Figure (2.3).

Based on quantization of light, the frequency of radiation affects the transition of quantum energy of photon. The difference in energy can be measured after deducting the energy at the second level into the energy at the first level. It gives the result of photon energy which is the product of Planck constant, $\hbar$ and frequency of radiation, ω.

$$\triangle E = E_2 - E_1 \tag{2.77}$$

$$E_{photon} = \hbar\omega. \tag{2.78}$$

The energy of photon, E_{photon} is equal to the difference of energy, $\triangle E$. Accordingly, the interaction of particles cause an entanglement of the quantum system by sharing of information and exchange of energy.

2.2 Quantum state

2.2.1 Pure state and mixed state

Quantum state defines the state of a quantum system through a vector in a Hilbert space. Theoretically, the state vector defined contains statistical information of the quantum state and it is very important in the explanation or interpretation of quantum theory. The energy of a quantum state, generated from mathematical derivation using the quantum mechanical principles, is the set discrete numbers, corresponding to the set of eigenstate $\{n\}$. Each quantum state can be described by kets, $|\cdot\rangle$ and bras, $\langle\cdot|$ notations, refered to as the Dirac notation.

Generally, the quantum state is a linear combination of multiple different eigenstates. The eigenvalue corresponds to the possible values observed. The linear combination of eigenstates which depends on time has the form

$$|\psi_n(t)\rangle = \sum_n C_n(t)\,|\phi_n\rangle \tag{2.79}$$

where $C_n(t)$ denotes the time dependent coefficient of the $|\phi_n\rangle$ state. The coefficient of eigenstate can be described as the probability distribution from the observation of the ensemble of eigenstates.

In the same way, the quantum state can be defined in terms of annihilation and creation operators with respect to the eigenstate. From equation (2.75) and (2.76), we can formulate a quantum state by the superposition of two eigenstates (Scully & Zubairy, 1997).

Let us illustrate this concept and show how the quantum operation works by using the annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators of hormonic oscillators acting on the number state $\{|n\rangle\}$

$$\begin{aligned} \hat{a}\,|n\rangle &= |n-1\rangle \qquad (2.80) \\ \hat{a}^\dagger\,|n\rangle &= |n+1\rangle\,. \qquad (2.81) \end{aligned}$$

For extended multimode field, the eigenstate also extended with respect to the order of the number states (Scully & Zubairy, 1997),

$$\left|n_{k_1}, n_{k_2}, \ldots, n_{k_l}, \ldots\right\rangle = |\{n_k\}\rangle\,. \qquad (2.82)$$

Therefore, the annihilation and creation operators also carry subscripts with the index, k_l for $l = 1,2,3,...$

$$\begin{aligned} \hat{a}_{k_l}\left|n_{k_1}, n_{k_2}, \ldots, n_{k_l}, \ldots\right\rangle &= \sqrt{n_{k_l}}\left|n_{k_1}, n_{k_2}, \ldots, n_{k_l}-1, \ldots\right\rangle \qquad (2.83) \\ \hat{a}^\dagger_{k_l}\left|n_{k_1}, n_{k_2}, \ldots, n_{k_l}, \ldots\right\rangle &= \sqrt{n_{k_l}+1}\left|n_{k_1}, n_{k_2}, \ldots, n_{k_l}+1, \ldots\right\rangle. \qquad (2.84) \end{aligned}$$

As a result, the state vector is obtained as

$$\begin{aligned} |\psi\rangle &= \sum_{n_{k_1}}\sum_{n_{k_2}}\cdots\sum_{n_{k_l}}\cdots c_{n_{k_1},n_{k_2},\ldots,n_{k_l},\ldots}\left|n_{k_1}, n_{k_2}, \ldots, n_{k_l}, \ldots\right\rangle \qquad (2.85) \\ &= \sum_{\{n_k\}} c_{\{n_k\}}\,|\{n_k\}\rangle\,. \qquad (2.86) \end{aligned}$$

Basically, the density operators are widely used in the explaination of quantum entanglement. The density operator originates from a vector state. A state vector, $|\psi\rangle$ of a quantum system is called pure state. This vector state represent the single state which can be used to determine the density operator. The state of quantum can be used to measure the density operator based on the product of vector and its conjugate, ψ^*

$$\rho = (\psi^*)(\psi) \tag{2.87}$$

$$= \langle\psi|\,|\psi\rangle \tag{2.88}$$

where a quantum state, $|\psi\rangle$ defined in terms of quantum information theory as

$$\psi = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \tag{2.89}$$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,. \tag{2.90}$$

The equation (2.89) shows the quantum state in terms of Dirac notation where α and β are defined as complex numbers (Vedral, 2006). Thus, the complex conjugate of quantum state in equation (2.89) has the form of

$$\psi^\dagger = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} \tag{2.91}$$

$$\langle\psi| = \alpha^*\,\langle 0| + \beta^*\,\langle 1|\,. \tag{2.92}$$

The lable $|0\rangle$ and $|1\rangle$ are arbitrary orthogonal normalized state. It defined as the following

$$|0\rangle = \begin{pmatrix}1 \\ 0\end{pmatrix} \tag{2.93}$$

$$|1\rangle = \begin{pmatrix}0 \\ 1\end{pmatrix} \tag{2.94}$$

$$\langle 0| = \begin{pmatrix}1 & 0\end{pmatrix} \tag{2.95}$$

$$\langle 1| = \begin{pmatrix}0 & 1\end{pmatrix} \tag{2.96}$$

Consequently, the density operator is illustrated from the product of quantum state and its complex conjugate as clarify in the equation (2.88). The density operator is clarified as

$$\rho = |\psi\rangle\,\langle\psi| \tag{2.97}$$

$$= \begin{pmatrix}\alpha \\ \beta\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} \tag{2.98}$$

$$= \begin{pmatrix}|\alpha|^2 & \alpha^*\beta \\ \beta^*\alpha & |\beta|^2\end{pmatrix} \tag{2.99}$$

Alternatively

$$\begin{aligned}\rho &= (\alpha\,|0\rangle + \beta\,|1\rangle)\,(\alpha^*\,\langle 0| + \beta^*\,\langle 1|)\,. && (2.100)\\ &= \alpha\alpha^*\,|0\rangle\,\langle 0| + \alpha^*\beta\,|1\rangle\,\langle 0| + \beta^*\alpha\,|0\rangle\,\langle 1| + \beta\beta^*\,|1\rangle\,\langle 1| && (2.101)\end{aligned}$$

Based on the normalization of quantum state, the result of inner product of bras and kets notations are

$$\begin{aligned}\langle 0|\,|0\rangle &= \langle 1|\,|1\rangle = 1 && (2.102)\\ \langle 0|\,|1\rangle &= \langle 1|\,|0\rangle = 0. && (2.103)\end{aligned}$$

Therefore, the trace of the density operator is illustrated as

$$\begin{aligned}Tr\rho &= \alpha\alpha^*\,(1) + \alpha^*\beta\,(0) + \beta^*\alpha\,(0) + \beta\beta^*\,(1) && (2.104)\\ &= |\alpha|^2 + |\beta|^2. = 1 && (2.105)\end{aligned}$$

However, the state is said to be a mixed state when there involved the mixture of pure states in a quantum system. Therefore, the mixed state is defined as,

$$\rho = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i| \qquad (2.106)$$

where p_i is probability of ith state of the ensemble quantum state $|\psi_i\rangle$. In other words, it must satisfies the following relation

$$\begin{aligned}0 &\leq p_i \leq 1 && (2.107)\\ \sum_i p_i &= 1. && (2.108)\end{aligned}$$

The analogy from equation (2.106), the mixture or blend of ensemble pure state will produce a mixed state with the corresponding probability, p_i.

The Figure (2.4) demonstrates the theory of ensemble pure states are mixed before it produce a new state known as a mixed state.

In order to determine the state of quantum system, consider the expectation value of quantum variable A

$$\langle A\rangle = Tr\{A\rho\} = Tr\{\sum_i p_i A\,|\psi_i\rangle\,\langle\psi_i|\} \qquad (2.109)$$

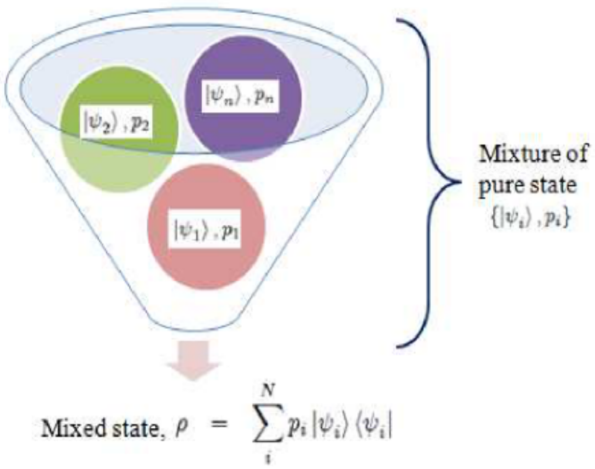

Figure 2.4: A mixture of an ensemble of pure states

Since the quantum system A expressed with the eigenket, $\left|\alpha_j\right\rangle$ and eigenvalue, a_j, equation (2.109) can be expressed in the form of

$$\begin{aligned} \langle A\rangle &= \sum_{ij} p_i a_j \left\langle\alpha_j\right| A \left|\psi_i\right\rangle \left\langle\psi_i\right| \left|\alpha_j\right\rangle & (2.110) \\ &= \sum_{ij} p_i a_j \left|\left\langle\alpha_j\right|\psi_i\rangle\right|^2 & (2.111) \end{aligned}$$

In this case, the quantum system A is self adjoint operator and it has the relation as the following equation

$$\left\langle\alpha_j\right| A \left|\psi_i\right\rangle = \left\langle\psi_i\right| A^\dagger \left|\alpha_j\right\rangle \quad (2.112)$$

Then, the expected value of the quantum system A with respect to the density operator is resulting to the trace product of density operator and quantum system A. This is shown in the following derivation

$$\begin{aligned} \langle A\rangle_\rho &= Tr\left(\sum_i p_i \left|\psi_i\right\rangle\left\langle\psi_i\right| A\right) & (2.113) \\ &= Tr(\rho A) & (2.114) \end{aligned}$$

From the derivation of expectation value of quantum system A, the equation (2.114) shows that the tracing of product of density operator and quantum system. The result from the tracing is independent of the representation. Based on this operation, it can measure the trace of the density operator. Therefore, the tracing of density operator equal to

$$\begin{aligned} Tr\rho &= \sum_{ij} p_i \langle\psi_i\left|\alpha_j\right\rangle\left\langle\alpha_j\right|\psi_i\rangle & (2.115) \\ &= 1. & (2.116) \end{aligned}$$

In order to show the comparison of pure state and mixed state, take the square of density operator for each quantum state and applying the tracing on them.

$$\rho^2 = \rho \cdot \rho. \tag{2.117}$$

For pure state, obtained the result of tracing equally to the equation (2.116)

$$\rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| \tag{2.118}$$

$$Tr\rho^2 = Tr\rho \tag{2.119}$$

$$= 1. \tag{2.120}$$

The result of mixed state tracing is different from the result of pure state. In this case, the result of tracing should be less or equal to 1. This is due to the probability outcome of the ensembles of quantum state.

$$\rho^2 = \sum_i \sum_j p_i p_j |\psi_i\rangle\langle\psi_i||\psi_j\rangle\langle\psi_j| \tag{2.121}$$

$$Tr\rho^2 \leq 1. \tag{2.122}$$

The calculation of the tracing of square density operator can be refered to Appendix B.1. Therefore, from the equations (2.119) and (2.122), distinguishes the properties of pure state and mixed state. The measurement of quantum state which depends on the density operator is important to study quantum entanglement.

2.2.2 Coherent state

In the concept of physics, there is a quantum state where the quantum harmonic oscillators are mostly closed to the classical harmonic oscillators. This state known as coherent state which used in determine the quantum state. In the coherent state of the electromagnetic field describes a maximal coherence and at the same time it also describes closely the classical behavior. In the radiation of light, the coherent state is identified when it has property equivalent to the classical amplitude and phase. This is recognized as dimensionless state of the complex number, α (Fox, 2006). It defined as

$$\alpha = X_1 + iX_2 \tag{2.123}$$

where X_1 and X_2 are dimensionless quadratures of the field which depend on the phase change. It defined as the following

$$X_1 = |\alpha| \cos\phi \tag{2.124}$$

$$X_2 = |\alpha| \sin\phi. \tag{2.125}$$

Furthermore, the dimensionless complex number, α can be expressed in terms of amplitude, $|\alpha|$ and phase, ϕ

$$\alpha = |\alpha| \exp i\phi \tag{2.126}$$

where the amplitude, $|\alpha|$ is defined as

$$|\alpha| = \sqrt{X_1^2 + X_2^2}. \tag{2.127}$$

In another view, the coherent can be described in terms of annihilation and creation operators when the quadrature operators have the form of

$$X_1 = \frac{1}{2}\left(\hat{a}^\dagger + \hat{a}\right) \tag{2.128}$$

$$X_2 = \frac{1}{2}\left(\hat{a}^\dagger - \hat{a}\right). \tag{2.129}$$

By the same token, the annihilation and creation operators have the form like the equation (2.75) and equation (2.76). The superposition of the number state leads to coherent state as follows

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{2.130}$$

where C_n is the coefficient of the number state. The coherent state is the eigenstate of the annihilation operator as shown in the following

$$\hat{a}|\alpha\rangle = \hat{a}\sum_{n=0}^{\infty} C_n |n\rangle \tag{2.131}$$

$$= \sum_{n=0}^{\infty} C_n \sqrt{n} |n-1\rangle \tag{2.132}$$

$$= \sum_{n=0}^{\infty} \alpha C_n |n\rangle \tag{2.133}$$

$$= \alpha \sum_{n=0}^{\infty} C_n |n\rangle \tag{2.134}$$

$$= \alpha |\alpha\rangle \tag{2.135}$$

Taking the adjoint resulted in the following equation

$$\langle\alpha| \hat{a}^\dagger = \alpha^* \langle\alpha|. \tag{2.136}$$

where the coherent state corresponds to the eigen-bra of annihilation operator with the eigenvalue resulting its conjugate, α^* (Fox, 2006). In order to connect the coherent state with number states, we use the recurrence relation

$$
\begin{aligned}
C_n &= \frac{\alpha}{\sqrt{n}} C_{n-1} && (2.137)\\
&= \frac{\alpha^2}{\sqrt{n(n-1)}} C_{n-2} && (2.138)\\
&= \frac{\alpha^n}{\sqrt{n!}} C_0. && (2.139)
\end{aligned}
$$

Therefore, the coherent state is defines as

$$
|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \tag{2.140}
$$

One of the properties of coherent state of the radiation field is that the mean number of photons measured is the eigenvalue squared (2.135) and (2.136)

$$
\begin{aligned}
\langle\alpha| \hat{a}^\dagger \hat{a} |\alpha\rangle &= |\alpha|^2 && (2.141)\\
\langle\hat{n}\rangle &= |\alpha|^2 . && (2.142)
\end{aligned}
$$

Additionally, the constant C_0 is identified in the equation (2.139) by using the normalization condition $\langle\alpha|\alpha\rangle = 1$. Thus the product of the eigen-bra and eigen-ket of the coherent state can be written as

$$
\langle\alpha| |\alpha\rangle = N(\alpha) \sum_{m=0}^{\infty} \langle m| \frac{\alpha^m}{\sqrt{m!}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \tag{2.143}
$$

As derived in the Appendix B.2, the normalization constant is

$$
N(\alpha) = \exp\left(\frac{-|\alpha|^2}{2}\right) \tag{2.144}
$$

Based on the certain entanglement theories, the coherent state can be a reference point for measuring the entangled state. Therefore, the coherent state is very important in quantum theory.

CHAPTER 3

QUANTITIES OF NONCLASSICAL MEASURES

Entanglement has been widely recognized as a fundamental aspect of quantum theory. It is also remarkable as a key factor of information processes which is impossible to implement on classical systems. The non classical properties of quantum theories are significant in the process of quantum information and communication. This research area regarding quantum theory leads to emphasize the entanglement process in a quantum system.

Entanglement occurs when two particles interact physically and connected together. The nature of entanglement requires correlations between particles under influence from the process of entanglement. The characteristic of entanglement requires theoretical analysis of entanglement for a deepest understanding.

The existence of entanglement in a system is described by several quantities which are denoted as quantum entanglement criteria. Every quantity provides a mathematical framework which is useful to quantify such a quantum entanglement process. There are various quantities related to entanglement which are expressed in terms of density operator, product of operators and quantum state. All the quantities describe entanglement based on the certain criteria. Additionally, each quantity has a condition for which the entanglement should be detected in the system. Thus, the entire review presents a wide scope on the conditions of entangled states.

In this section, we review the twelve criteria of entanglement. Each quantity has a condition that should be satisfied for detecting the entanglement. We also discuss the relationships among the criteria used in order to better understand the entanglement in a given system.

3.1 Entropy

The entropy is a quantity to measure the uncertainty which is associated with random variables based on the concept of information processing. The entropy is related to thermodynamics and statistical mechanics. In the sense of information, the entropy measures the uncertainty of associated values of the random variables which carried the messages. However, in the mathematical sense, the entropy is related to the asymptotic behavior of probabilities.

There are two well known entropy theories which are Shannon entropy and von Neumann entropy. The Shannon entropy is measured based on the concept of information theory, as introduced by Claude E. Shannon from his research paper, "A Mathematical Theory of Communication" (Shannon, 1948). Additionally, the Shannon entropy predicts the possible outcome of lossless compression in any communication. After that, the entropy theory was extended from the concept of classical entropy to the field of quantum mechanics. It was introduced into quantum physics by John von Neumann. The von Neumann entropy is known in statistical mechanics theory where it is calculated based on the density operator of quantum states.

The von Neumann entropy computes the formalism of density operators in the framework of the states and operations in the Hilbert space. Basically, the quantum states is identified from a set of wave functions, $|\psi\rangle$ which depend on the quantum numbers $n_1, n_{2,} ... n_N$. The measurement of quantum state associated with probabilities, p_k in order to predict the outcomes. The probabilities perform in the quantum system is described in terms of Hilbert space.

The system of von Neumann entropy involved the interaction of two photons A and B which also known as bipartite system. The procedure in measuring the classically correlated of the system is referred to the von Neumann entropy. Theoretically, the correlated system of two photons has a good state to be determined because of the system consisting of two equal and distinguishable binary subsystems (Zander & Plastino, 2006). Therefore, the correlated system of mixed state can detect the entanglement where it was discussed in many papers (Meik & Armin, 2009) and (Rajagopal & Rendell, 2005).

The density operator serves as a significant parameter for entropy quantity in describing the detection of entanglement. According to Shannon entropy, the probability distribution $\{p_1, p_2, ..., p_N\}$ of possible outcomes defined as

$$S = -\sum_{i}^{n} p_k \log p_k \tag{3.1}$$

where p_k is probability distribution for $k = 1, 2, \ldots N$. This quantity shows that discrete random variable within possible value $\{p_1, p_2, ..., p_N\}$. According to the quantum information theory, the probability, $\{p_k\}$, defined as eigenvalue of density operators. This theory has been implemented into von Neumann entropy with respect to the statistical thermodynamics. The von Neumann entropy is defined as

$$S = -k_B \sum p_k \ln p_k \tag{3.2}$$

where k_B defined as Boltzman constant such as in the thermodynamics (Gerry & Knight, 2005). That was identified as the general equation of von Neumann entropy. Then, it replaces the probability, p_k, base on equation (2.114) the trace of density operator, ρ

$$S(\rho) = -k_B Tr(\rho \ln \rho). \tag{3.3}$$

Therefore, the entropy is a quantity which is described the entanglement based on the von Neumann entropy because it is useful to be applied in the quantum information theory. The systems which involved in detecting the entanglement by entropy can be considered as interaction between a pair of any particles. In order to detect the entanglement, the two systems A and B

$$S(A) = -Tr_A(\rho_A \ln \rho_A) \tag{3.4}$$

$$S(B) = -Tr_A(\rho_B \ln \rho_B). \tag{3.5}$$

These two systems are determined as correlated when the observation of $A(B)$ system projects the other system $B(A)\dot{T}$herefore, the projection of the systems will produce a new state which is known as a mixed state. Basically, a mixed state contains information shared by two systems which precise than a pure state. The von Neumann entropy, $S(\rho)$ enable one to detect the entanglement based on the quantity of quantum information theory (Lang & Shaji, 2011).The classical correlation entropy, $S(A:B)$

has been introduced by (Lang & Shaji, 2011) and (Rajagopal & Rendell, 2005) denoted as

$$S(A:B) = S(A) + S(B) - S(A,B) \tag{3.6}$$

where the entropy of two systems A and B was defined in the equations (3.4) and (3.5) and then, the joint von Neumann entropy defined as

$$S(A,B) = -Tr\left(\rho_{AB} \ln \rho_{AB}\right). \tag{3.7}$$

Quantum mutual information shown a correlated system according to joint von Neumann entropy $S(A,B)$. The correlated system is based on the classical correlation entropy which is shown in the equation (3.6). If the system A and B are statistically uncorrelated, obtained the correlated system equal to zero, $S(A:B) = 0$. In contrast, if the value of correlated system is bigger than zero, $S(A:B) > 0$, then, it proves that the system is statistically correlated (Rajagopal & Rendell, 2005). From this condition, the correlated system which leads to entanglement is easy to identify. The system is said to be entangled when,

$$\begin{aligned} S(A:B) &> 0 && (3.8)\\ S(A)+S(B)-S(A,B) &> 0 && (3.9)\\ S(A)+S(B) &> S(A,B). && (3.10) \end{aligned}$$

Thus, the system should be entangled when it satisfied the necessary condition of quantum entanglement, $S(A)+S(B) > S(A,B)$. The entropy criteria shows that total entropy of two systems, A and B should be greater than joint von Neumann entropy in presence of entanglement. The condition of entropy A and B in order to describe the entanglement has been illustrated in the following figure.

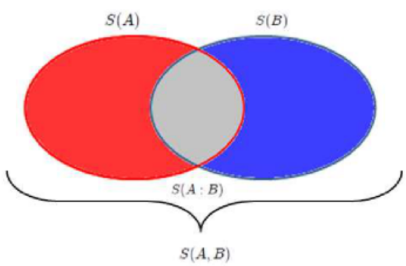

Figure 3.1: The correlated region of joint von Neumann entropy.

Figure (3.1) shows the joint von Neumann entropy $S(A:B) > 0$ in gray shaded area as correlated by two system of elliptical red shaded, $S(A)$ and elliptical blue shaded, $S(B)$.

The quantum entanglement based on entropy method occurs from the correlation of two systems. The correlation of entropy is significant to determine the maximal entanglement. The entropy is important quantity to detect the entanglement. It is because the entropy is the most accepted method when applied to other methods of quantifying the entanglement.

3.2 Peres-Horodecki criterion

The second quantity is the Peres-Horodecki criterion which is also known as positive partial transposition, enables to detect the quantum entanglement of a system (Simon, 2000). This criterion was established by Asher Peres, (1996). He has proven the necessary condition for separability by partial transposition of density matrix and obtaining non-negative eigenvalue (Peres, 1996). In the similar way, Horodecki and etc., (1996) provide the necessary and sufficient condition of separability by taking into account the positive partial transposition (H. P. Horodecki M. & Horodecki, 1996). Thus, a Peres-Horodecki criterion refers to partial transposition of density operator in order to detect the separability of a quantum system.

The Peres-Horodecki criterion is not complete without involving the projection of subsystem into another due to the fundamental problem of operation which is arise from the separable states. In this case, the separability of a system can be analyzed according to the trace of any density operator as equation (2.114) that should be positive,

$$Tr\rho \geq 0 \quad or \quad Tr\rho = 1 \tag{3.11}$$

for any projection (H. P. Horodecki M. & Horodecki, 1996). Consequently, the projection was referred to the positive mapping of the set of operators A_1 and A_2 which are acting on Hilbert space H_1 and H_2 respectively. The set of positive operators is determined as

$$\Lambda(A) \geq 0 \tag{3.12}$$

$$for \quad (A) \geq 0 \tag{3.13}$$

Due to the positive mapping operator; Λ implies to the linear positive mapping from A_1 to A_2 which is $L(A_1, A_2)$ defined as

$$\Lambda \in L(A_1, A_2). \tag{3.14}$$

It produced completely positive result. After that, the positive mapping implies into the tensor product of density operator,

$$(\Lambda\rho) \otimes \tilde{\rho} \geq 0 \tag{3.15}$$

which is induced to the property of separable state. The system is said to be separable when the tensor product of two density operators equals to the composite density operator as the following

$$\rho = \rho_A \otimes \rho_B \tag{3.16}$$

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B. \tag{3.17}$$

For two quantum subsystems A with N dimensions and B with M dimensions, the composite density operator is expanded into,

$$\rho = \sum_{ij}^{N} \sum_{kl}^{M} \rho_{ij,kl} |i\rangle \langle j| \otimes |k\rangle \langle l| \tag{3.18}$$

$$\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B. \tag{3.19}$$

From the equation (3.18), there involved two separable density operators into the summation of direct product. The result of density operator, ρ on the equation (3.19) applied in the transposition operation. The transpose of density operator is non-negative matrix with a unit trace where the eigenvalue should always be positive (Peres, 1996). For a given separable density matrix such as an equation (3.18), it is easy to calculate the partial transpose with respect to one subsystem,

$$\rho^{T_A} = \sum_k p_k \left(\rho_k^A\right)^T \otimes \rho_k^B \tag{3.20}$$

$$= \sum_k p_k \tilde{\rho}_k^A \otimes \rho_k^B \geq 0. \tag{3.21}$$

Therefore, the equation (3.21) is simplified into the following equation

$$\rho^{T_A} \geq 0 \quad or \quad \rho^{T_B} \geq 0. \tag{3.22}$$

The Peres-Horodecki criterion states that if the state of density operator, ρ is separable, then, its partial transpose, ρ^{T_A} is valid density operator in relation to the positive semi-definiteness, $\rho^{T_A} \geq 0$. This criterion also applies for ρ^{T_B}. The positivity of partial transpose is a basic condition of separability. In addition, the separability of density operator for a low dimensional system like bipartite system with 2×2 or 2×3 dimension can be applied to density operator, ρ which is also obtained in separable system. Thus, the inseparability of density operator, the violation of positive partial transpose is applied to quantify the entanglement (H. P. Horodecki M. & Horodecki, 1996). Thus, the inseparability or entanglement is detected when the conditions of density operator

$$\rho \neq \sum_k p_k \rho_k^A \otimes \rho_k^B \tag{3.23}$$

$$\rho^{T_A} < 0 \quad or \quad \rho^{T_B} < 0 \tag{3.24}$$

are satisfied.

After all, the Peres-Horodecki criterion is very useful to detect the entanglement of bipartite quantum state because it involves the separability of two systems.The discussion of this criterion has demonstrated that the partial transposes of two separable density matrices formulates to positive eigenvalue and develop into the inseparability of subsystem. If the eigenvalue is negative, the state can be entangled but according to the sufficient condition of separability.

3.3 Squeezing

The effects of quantization of light exhibits a relation between the light of harmonic oscillator in the elementary of quantum mechanics (Fox, 2006). The properties of light in the vacuum field correspond to the coherent state of light. This condition is due to quantum mechanical coherent state which is equivalent to the state of classical electromagnetic waves.

The study of the quantization of light for the different states required to satisfy the uncertainty principle with respect to the number of phase and number of photon. This was explaining that photon number distribution correspond to the amplitude of squeezed state. In the same manner, the uncertainties of phase correspond to the phase of squeezed state. In the view of the quantization of light which carries the energy, it was established into the quantization of harmonic oscillators. It interpreted the correlation at first state and the second state of amplitude and phase. Under those circumstances, the single mode field involved the determination of uncertainty relation between amplitude and phase.

It is important to realize that the quantum uncertainty for squeezed state involves shot noise and the uncertainty of phase (Scully & Zubairy, 1997). The shot noise also known as quantum noise defined as uncertainty by the changing of physical quantity of quantum. Therefore, the properties of nonclassical of light lead to squeezed state because it has two equal quadratures at minimum uncertainty state which is identified as coherent state.

The condition of squeezed state must always equal to the uncertainty relation of quadrature squeezing and should satisfy the commutation relation. The quadrature of squeezing can be measured according to the number of state, n. The squeezing of light is verified from the correlation between orthogonal quadrature of two separated annihilation operator, $\hat{a}$ and creation operators $\hat{a}^{\dagger}$. Therefore, the squeezing is easily recognized according to the product of operators in a state where basically defined in the equation (2.72) as number of photon, $\hat{n} = \hat{a}^{\dagger}\hat{a}$ (Gerry & Knight, 2005).

The commutation relation for the number of photon-phase,

$$\left[\hat{n}, \hat{\phi}\right] = i \tag{3.25}$$

and it should satisfy the product uncertainty relation to measure the squeezed state,

$$\begin{aligned} \triangle\hat{n}\triangle\hat{\phi} &\geq \frac{\left|\left[\hat{n}, \hat{\phi}\right]\right|^2}{2} && (3.26) \\ &\geq \frac{1}{2}. && (3.27) \end{aligned}$$

The equation (3.27) emphasize the quadrature product of number of photon and phase.The coherent light or the stability of light for squeezing defined as $|\alpha\rangle$ which can identified the mean number of photon such as $|\alpha|^2 = \bar{n}$. The mean number of photon in the coherent state, $\bar{n}$ used to measure the variance of squeezing in terms of the number of photon (Gerry & Knight, 2005) which is described as,

$$\left\langle (\triangle\hat{n})^2 \right\rangle = \langle\hat{n}\rangle + \left(\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2\right). \tag{3.28}$$

After that, it can generate in terms of annihilation and creation operator in order to compare the amplitude of squeezed state and coherent state.

$$\left\langle (\triangle\hat{n})^2 \right\rangle = \langle \hat{n} \rangle + \left(\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle - \left\langle \hat{a}^\dagger \hat{a} \right\rangle^2 \right). \tag{3.29}$$

From the equation (3.28), the variance of squeezing lies in the mean number of photon. It enables to identify the condition of squeezing. The state is identified as squeezed state when the variance is narrower than the mean number of photon in the coherent state. It written as the following

$$\left\langle (\triangle\hat{n})^2 \right\rangle < \langle \hat{n} \rangle . \tag{3.30}$$

In the other view, the quadrature of squeezing is considered as nonclassical effective because of the coherent state is minimizes at the identical two orthogonal quadrature operators. Thus, the quadrature operators of single mode field are denoted as

$$\hat{c}_+ = \frac{1}{2}\left(\hat{a}^\dagger + \hat{a}\right) \tag{3.31}$$
$$\hat{c}_- = \frac{i}{2}\left(\hat{a}^\dagger - \hat{a}\right). \tag{3.32}$$

Consequently, the quadrature operators must be satisfied the commutation relation,

$$\begin{aligned}
[\hat{c}_+, \hat{c}_-] &= \frac{i}{4}\left[\left(\left(\hat{a}^\dagger + \hat{a}\right)\left(\hat{a}^\dagger - \hat{a}\right) - \left(\hat{a}^\dagger - \hat{a}\right)\left(\hat{a}^\dagger + \hat{a}\right)\right)\right] && (3.33)\\
&= \frac{i}{4}\left[\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}) - \left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}\right)\right] && (3.34)\\
&= \frac{i}{4}\left(2\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\hat{a}\right) && (3.35)\\
&= \frac{i}{2}\left[\hat{a}, \hat{a}^\dagger\right] && (3.36)\\
&= \frac{i}{2}. && (3.37)
\end{aligned}$$

The product of quadrature variance must be satisfied the following inequality (Gerry & Knight, 2005),

$$\begin{aligned}
\left\langle (\triangle\hat{c}_+)^2 \right\rangle \left\langle (\triangle\hat{c}_-)^2 \right\rangle &\geq \frac{|[\hat{c}_+, \hat{c}_-]|^2}{4} && (3.38)\\
&\geq \frac{1}{16} && (3.39)
\end{aligned}$$

where the quadrature variance for each operator denoted as,

$$\left\langle (\triangle\hat{c}_+)^2 \right\rangle = \left\langle (\triangle\hat{c}_+)^2 \right\rangle - \langle \triangle\hat{c}_+ \rangle^2 \tag{3.40}$$

$$= \frac{1}{4}\left(2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle + 1\right) \quad (3.41)$$

$$\left\langle (\triangle\hat{c}_{-})^{2}\right\rangle = \left\langle (\triangle\hat{c}_{-})^{2}\right\rangle - \langle \triangle\hat{c}_{-}\rangle^{2} \quad (3.42)$$

$$= \frac{1}{4}\left(2\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle + 1\right). \quad (3.43)$$

Thus, we obtain the equal quadrature variance for n photon's state given by,

$$\left\langle (\triangle\hat{c}_{+})^{2}\right\rangle = \left\langle (\triangle\hat{c}_{-})^{2}\right\rangle = \frac{1}{4}(2n+1) \geq \frac{1}{4}. \quad (3.44)$$

It then leads to the product of quadrature variance that should be bigger than $\frac{1}{16}$ as shown in the equation (3.39).

On the other view, the squeezing of field can be referred to as a coherent state with respect to the position and momentum of the quantized harmonic oscillator. Therefore the uncertainties of the two quadratures for coherent state have an identical value such as

$$\langle \triangle\hat{c}_{+}\rangle = \langle \triangle\hat{c}_{-}\rangle = \frac{1}{2} \quad (3.45)$$

resulting in the minimum uncertainty state (Fox, 2006). The squeezed state is detected must have less uncertainty in one quadrature value than a coherent state.

Then, the squeezing of light corresponds to quantum entanglement in the system which is identified as a perfect squeezing the single mode. Therefore, the perfect squeezing can be identified when one of the following conditions (Sete & Ooi, 2012) are satisfied either

$$\triangle\,\hat{c}_{-}^{2} \ll 1, \triangle\hat{c}_{+}^{2} \approx 0 \quad or \quad \triangle\,\hat{c}_{+}^{2} \ll 1, \triangle\hat{c}_{-}^{2} \approx 0 \quad (3.46)$$

where $\triangle\hat{c}_{+} \neq \triangle\hat{c}_{-}$. These conditions demonstrated that a squeezed state of $\hat{c}_{+}$ or $\hat{c}_{-}$ happen by reducing the quantum noise of its complement, for example reducing the uncertainty of amplitude at the expense of the uncertainty of phase (Scully & Zubairy, 1997). In short, the quadrature of squeezed state is connected to the detection of entanglement of photons in the squeezing of light. This phenomenon is demonstrated in the figure below.

Figure (3.2) depicts that the coherent state shown as the blue shaded area in quadrature diagram has been squeezed into the same area represented by red ellipse. This figure also illustrates that the entanglement happens through the difference in the uncertainty of two quadratures.

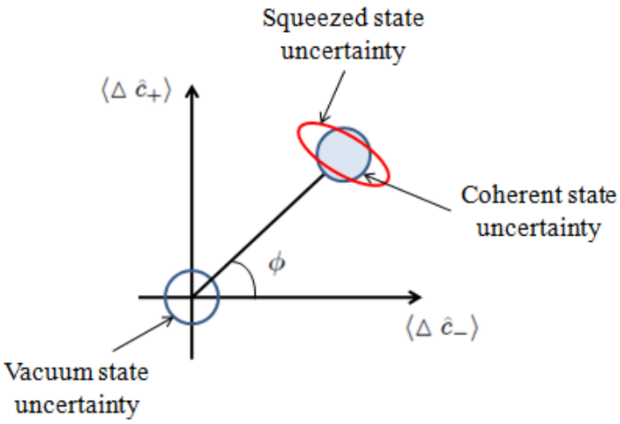

Figure 3.2: The squeezed state uncertainty.

3.4 Photon antibunching

Photon antibunching phenomena are exhibited when the light contains quantum properties of photons. This phenomena exist in the correlation function of scattered photon which is vanishes when time delay equal to zero (Villaeys, 1980). This phenomenon appears in the light produced in the single atom resonant fluorescence (Davidovich, 1996). The photon antibunching is identified as an important nonclassical properties in the radiation field. The characteristic of photon antibunching phenomena describes whether the photon in the light beam tends to group together or stay apart in time. For the group of photons, there exists correlations between the photons with different times, t and $t+\tau$. This phenomenon is shown directly in two time correlations measurement and have a relation with probability.

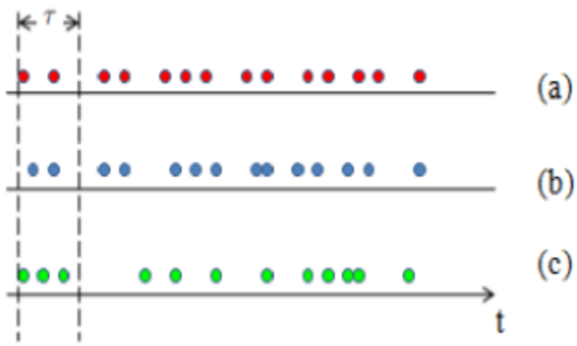

Figure 3.3: The photon detection of (a) photon antibunching, (b) random and (c) photon bunching

Figure (3.3) illustrates a comparison of photon detection with respect to time evolution between photon antibunching, random and photon bunching. From the observation, photon antibunching presents the correlation when the photon tends not to group together.

The second order correlations is discovered by Roy Jay Glauber (Glauber, 1963b). Hence, quantum correlation is very important in the studying of photon statistics due to joint probability of correlations. The quantum correlation was extended to the counting rate of photodetectors. It concerns the matter of probability detecting the second photon which decreases as the time delay. In order to derive the second order correlation, let's consider the single mode field in term of number of photons. It is denoted as,

$$g^{(2)}(\tau) = \frac{\left\langle \hat{a}^{\dagger}(t)\,\hat{a}^{\dagger}(t+\tau)\,\hat{a}(t+\tau)\,\hat{a}(t)\right\rangle}{\langle \hat{a}^{\dagger}\hat{a}\rangle^{2}} \tag{3.47}$$

$$= \frac{\left\langle \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\right\rangle}{\langle \hat{a}^{\dagger}\hat{a}\rangle^{2}}. \tag{3.48}$$

For the coherent state, at $t = 0$, the second order correlation of joint probability is denoted as

$$g^{(2)}(0) = 1. \tag{3.49}$$

For the second order correlation which depend on time delay, obtained the result is $g^{(2)}(\tau) = 1$. The probability of joint detection coincides with the probability of independent detection which denoted as coherent state (Davidovich, 1996). In the case when the time delay approaches infinity, $\tau \to \infty$, the correlation of first photodetection dies out, $g^{(2)}(\tau) = 0$. Therefore, if

$$g^{(2)}(\tau) < g^{(2)}(0) \tag{3.50}$$

the joint probability of detecting a second photon is decreasing with time delay which is identified as photon bunching. Instead of this, if

$$g^{(2)}(\tau) > g^{(2)}(0) \tag{3.51}$$

the joint probability of detecting the second photon is increasing within time delay, then this phenomenon indicates as photon antibunching.

Photon antibunching is included in the criteria of detecting entanglement where it is identified based on second order correlation of time dependent, $g^{(2)}(t)$. The photon antibunching is said to be entangled if the field satisfied the inequality, $g^{(2)}(\tau) > g^{(2)}(0)$ where the probability of detecting the second photon increases with time delay (Gerry & Knight, 2005). Photon antibunching is also classified to be in nonclassical property where it exhibits a correlation between photons with respect to time delay. But, for nonclassical field state, we should have $g^{(2)}(0) < 1$ due to the violation of classical result. If the time delay, τ approach to infinity, $\tau \to \infty$, it affects to the result of second order correlation which is $g^{(2)}(\tau) \to 1$.

Thus, the circumstance of $g^{(2)}(0) < 1$ must be followed in order to demonstrate photon antibunching at the finite time delay, τ. Therefore, photon antibunching phenomena and the quantum entanglement are observed when

$$g^{(2)}(0) < 1 \tag{3.52}$$

$$g^{(2)}(0) < g^{(2)}(\tau). \tag{3.53}$$

The photon antibunching is related to sub Poissonian statistics with respect to the correlation of joint probability and the variance of photon number in the single mode field. Therefore, this relation influences both criteria providing a firm evidence to detect the entanglement in a system.

3.5 Sub Poissonian criterion

The nonclassical of light identified through two photon correlation may also show photon antibunching and sub Poissonian statistics. Therefore, photon antibunching has a close relation with sub Poissonian statistic in terms of the nonclassical properties. The sub Poissonian statistics based on the distribution of photon number describes the correlation based on photon counting. In the same way as photon antibunching, sub Poissonian can be connected to the second order correlation of coherence state.

The detection of photon from the light beam is identified based on the interaction between subsystems which influence the detection of correlation between the atom and the field. The efficiency of detection leads to the growing of atomic distribution and also the field of statistics. From the photon antibunching method, the second order correlation can be defined in terms of the mean number of photons, $\langle \hat{n} \rangle$ and the variance, $\left\langle (\triangle \hat{n})^2 \right\rangle$ as expressed in the equations (3.28) and (3.48) (Gerry & Knight, 2005),

$$
\begin{aligned}
g^{(2)}(\tau) &= \frac{\left\langle \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a} \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2} && (3.54) \\
&= 1 + \frac{\left\langle (\triangle \hat{n})^2 \right\rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} && (3.55)
\end{aligned}
$$

The sub Poissonian criterion holds the condition that the photon number distribution must be narrower than Poissonian. From the equation (3.49), the coherent state gives $g^{(2)}(0) = 1$. In this case, the coherent state known as Poissonian state due to the variance has the same average photon number, $\left\langle (\triangle \hat{n})^2 \right\rangle = \langle \hat{n} \rangle$ (Gerry & Knight, 2005). Therefore, the sub Poissonian can be identified when $g^{(2)}(0) < 1$ unless the correlation $g^{(2)}(\tau)$ is not dependent on τ. Consequently, if $g^{(2)}(\tau) < 1$ for all τ, this condition exhibits the sub Poissonian statistics.

Hence, the squeezing criteria also has a close relation to sub Poissonian criterion in terms of the amplitude of phase squeezing which is denoted as sub Poissonian statistics (Fox, 2006). It expressed according to the correlated number fluctuation of photons which holds a condition of nonclassical amplitude of squeezing. Since the variance of squeezing in the equation (3.28), we have the sub Poissonian condition in terms of number of photons

$$\left\langle (\triangle \hat{n})^2 \right\rangle > \langle \hat{n} \rangle . \tag{3.56}$$

This is recognized as sub Poissonian statistics because it has narrower number of photons distribution than Poissonian (Christopher, 1995). Mandel (1979), has shown the ratio of photon distribution which estimated the negative value (Mandel, 1979),

$$\frac{\left\langle (\triangle \hat{n})^2 \right\rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle} < 0. \tag{3.57}$$

That expression is rewriten as Mandel Q parameter for the field state (Gerry & Knight, 2005),

$$Q_f = \frac{\left\langle (\triangle \hat{n})^2 \right\rangle}{\langle \hat{n} \rangle} - 1. \tag{3.58}$$

Based on sub Poissonian condition, the entanglement is detected in the range of

$$-1 \leq Q_f < 0 \tag{3.59}$$

because of the fluctuation photon number, $\left\langle (\triangle\hat{n})^2 \right\rangle$ must be less than the average photon number. The anticorrelated state exists when the distribution of photon number of variance is greater than mean of photon number where the Mandel Q parameter represents the sub Poissonian statistic. The maximal sub Poissonian statistic occurs when $Q_f = -1$ due to the measure of nonclassical of sub Poissonian properties (Kim, 1999) and (Faghihi & Tavassoly, 2012). Therefore, sub-Poissonian statistics of field state demonstrated that the nonclassical properties are able to detect entanglement between atom and field in the quantum system.

3.6 Cauchy-Schwarz

The Cauchy-Schwarz inequality also applied to the second order correlation for detecting the entanglement. This method requires the expectation value of cross correlation between two modes which are bounded by autocorrelation (Kheruntsyan & al, 2012). In general, Cauchy-Schwarz inequality denoted as,

$$\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle \left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle \geq \left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b} \right\rangle^2. \tag{3.60}$$

The correlation between mode $\hat{a}$ and mode $\hat{b}$ indicated to the violation of inequality (3.60) which is represent the firm nonclassical correlation. The Cauchy-Schwarz inequality can detect the entanglement when it satisfied the second order correlation function at time, $t = 0$ (Christopher, 1995). The second order correlation of each mode defined as,

$$g_a^{(2)}(0) = \frac{\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2} \tag{3.61}$$

$$g_b^{(2)}(0) = \frac{\left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle}{\left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2}. \tag{3.62}$$

The derivation of the product of second order correlation at zero time correlation function equation (3.61) and equation (3.62) can be refer to the Appendix C.1. Then, the result of its product defined as

$$g_a^{(2)}(0)\,g_b^{(2)}(0)=\frac{\left\langle \hat{a}^{\dagger 2}\hat{b}^{\dagger 2}+\hat{a}^{2}\hat{b}^{\dagger 2}+\hat{a}^{\dagger 2}\hat{b}^{2}+\hat{a}^{2}\hat{b}^{2}\right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle^{2}\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle^{2}}. \tag{3.63}$$

The second order correlation of cross correlation function identified as,

$$g_{ab}^{(2)}(0) = \frac{\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b}\right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle} \tag{3.64}$$

$$\left[g_{ab}^{(2)}(0)\right]^{2} = \frac{\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b}\right\rangle^{2}}{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle^{2}\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle^{2}} \tag{3.65}$$

From the equation (3.60), we obtained the inequality to identified the entanglement

$$\frac{\left\langle \hat{a}^{\dagger 2}\hat{a}^{2}\right\rangle\left\langle \hat{b}^{\dagger 2}\hat{b}^{2}\right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle^{2}\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle^{2}} \geq \frac{\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b}\right\rangle^{2}}{\left\langle \hat{a}^{\dagger}\hat{a}\right\rangle^{2}\left\langle \hat{b}^{\dagger}\hat{b}\right\rangle^{2}} \tag{3.66}$$

$$g_a^{(2)}(0)\,g_b^{(2)}(0) \geq \left[g_{ab}^{(2)}(0)\right]^{2}. \tag{3.67}$$

Thus, obtained Cauchy-Schwarz criteria in terms of second order correlation as

$$g_a^{(2)}(0)\,g_b^{(2)}(0) \geq \left[g_{ab}^{(2)}(0)\right]^{2} \tag{3.68}$$

$$\sqrt{g_a^{(2)}(0)\,g_b^{(2)}(0)} \geq g_{ab}^{(2)}(0). \tag{3.69}$$

Under this circumstance, the entanglement is detected when inequalities (3.68) is satisfied.

The optical phenomena like sub Poissonian, photon antibunching and Cauchy-Schwarz inequality shared the nonclassical properties which are in terms of second order correlation in detecting the entanglement. The entanglement present in these optical phenomena when the nonclassical correlation exist in the two mode fields. This situation shows that correlation and entanglement have close relation regarding to the autocorrelation between two systems (Kheruntsyan & al, 2012).

3.7 Duan criterion

The Duan criterion can be defined based on the continuous variables of entanglement system. The continuously variables of two modes states determined the nonclassical of quantum mechanics which represents the generation of correlation. Basically, the two modes radiation deals with continuous variables which are similar to the radiation of two modes squeezing. In the setting of continuous variables, we need the Gaussian state of continous variables in obtaining the inseparable criterion.

According to the Duan's criterion (Sete & Ooi, 2012), the two modes squeezing have relation to the entanglement of cavity field. Duan criterion was introduced as the maximally entangled continuous variables which can be expressed as a co-eigenstate of a pair of EPR type operators (Duan & Zoller, 2000). It is expressed as the following

$$u = |a|x_1 + \frac{1}{a}x_2 \tag{3.70}$$

$$v = |a|p_1 - \frac{1}{a}p_2 \tag{3.71}$$

where a is a nonzero real number. The Duan criterion measures the total variance as shown in the following equation

$$D = \triangle u^2 + \triangle v^2. \tag{3.72}$$

For inseparable states, the total variance should satisfy a lower bound. Under this circumstance, Duan criterion is bounded by $a^2 + \frac{1}{a^2}$. Hence, the total variance for inseparable state has the form

$$\left\langle \triangle u^2 \right\rangle + \left\langle \triangle v^2 \right\rangle < a^2 + \frac{1}{a^2} \tag{3.73}$$

where the maximal entangled of continuous variables if $a = 1$ due to reduction to zero. Thus, the equation (3.73) simplified as

$$\left\langle \triangle u^2 \right\rangle + \left\langle \triangle v^2 \right\rangle < 2. \tag{3.74}$$

Therefore, the state is entangled when equation (3.74) satisfies the Duan criterion condition that is $D < 2$.

In the same manner, Duan criterion demonstrates that entanglement of continuous variables are detected through generating of two modes squeezing (Duan & Zoller, 2000). Thus, it can refer to the two modes squeezing based on the equation (3.28). The coherent state happen if there has an identical quadrature variance like equation (3.45). Therefore, at the steady state of two modes squeezing, the quadrature variance is

$$\triangle c_{\pm}^2 = 1 + \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle \pm 2Re\left[\langle \hat{a}_1, \hat{a}_2 \rangle\right]. \tag{3.75}$$

To preserve the condition introduced by Duan criterion, the entanglement can be detected through the generating the two modes squeezing,

$$\triangle u^2 + \triangle v^2 = 2\triangle c_-^2 \tag{3.76}$$

Let the total of two quadrature variables, equation (3.76) and equation (3.75) substitute into the equation (3.72). Then, the total quadrature variance obtained as the following

$$\begin{aligned} 2\triangle c_-^2 &\geq 0 & (3.77)\\ 2\left(1+\langle \hat{n}_1\rangle + \langle \hat{n}_2\rangle - 2Re\left[\langle \hat{a}_1, \hat{a}_2\rangle\right]\right) &\geq 0 & (3.78)\\ 2\left(\langle \hat{n}_1\rangle + \langle \hat{n}_2\rangle - 2Re\left[\langle \hat{a}_1, \hat{a}_2\rangle\right]\right) &< 2. & (3.79) \end{aligned}$$

From the equation (3.79), it shows that the result of total quadrature variance is less than 2 which is similar to the equation (3.74). Therefore, the two mode squeezing of cavity radiation is known to be entangled if the quantum fluctuations satisfied the condition of

$$\triangle u^2 + \triangle v^2 < 2. \tag{3.80}$$

This condition influence to the detecting the entanglement between the ensembles of atoms. The entanglement of Duan criterion produces a relationship between two modes of squeezing (Sete & Ooi, 2012).

3.8 Hillery-Zubairy criterion

Next we reviewed the criterion that was discovered by Mark Hillery and M. Suhail Zubairy. They introduced two modes states based on the electromagnetic field where they define the operators as the following (Hillery & Zubairy, 2006),

$$\begin{aligned} L_1 &= \hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b} & (3.81)\\ L_2 &= i\left(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}\right) & (3.82)\\ L_3 &= \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}. & (3.83) \end{aligned}$$

In this case, Hillery and Zubairy have proposed a relation from two modes squeezing and Cauchy-Schwarz inequality to determine the entanglement. They were applied to

Special Unitary group (2) in Lie algebra which is expressed as the following

$$J_i = \frac{L_i}{2}, \quad for \quad i = 1,2,3 \tag{3.84}$$

which leads to the commutation relation that should be satisfied

$$[J_1, J_2] = iJ_3. \tag{3.85}$$

The uncertainty principle of variables according to the general version should also satisfy the product of quadrature variance,

$$\left(\triangle L_1^2\right)\left(\triangle L_2^2\right) \geq \frac{|[J_1, J_2]|^2}{4}. \tag{3.86}$$

The inequality (3.86) shows the condition that the product of quadrature variance must be greater or equal to the commutation relation. The total of quadrature variance obtained from the uncertainties of L_1 and L_2 as shown

$$(\triangle L_1)^2 + (\triangle L_2)^2 = 2\left[\langle (N_a+1) N_b\rangle + \langle N_a (N_b+1)\rangle - 2\left|\left\langle \hat{a}\hat{b}^\dagger\right\rangle\right|^2\right] \tag{3.87}$$

$$= 2\left[\langle N_a+1\rangle \langle N_b\rangle + \langle N_a\rangle \langle N_b+1\rangle - 2\left|\langle \hat{a}\rangle \left\langle \hat{b}^\dagger\right\rangle\right|^2\right] \tag{3.88}$$

where $N_a = \hat{a}^\dagger\hat{a}$ and $N_b = \hat{b}^\dagger\hat{b}$. The Schwarz inequality has been applied in the equation (3.88) as defined by (Hillery & Zubairy, 2006) in the following

$$|\langle \hat{a}\rangle|^2 \leq \langle N_a\rangle \quad and \quad \left|\langle \hat{b}\rangle\right|^2 \leq \langle N_b\rangle. \tag{3.89}$$

Therefore, the equation (3.88) becomes

$$(\triangle L_1)^2 + (\triangle L_2)^2 \geq 2\left(\langle N_a\rangle + \langle N_b\rangle\right). \tag{3.90}$$

In order to examine the condition of inequality (3.90) it can be shown that the state is entangled if the following condition is satisfied

$$\langle N_a N_b\rangle < \left|\left\langle \hat{a}\hat{b}^\dagger\right\rangle\right|^2. \tag{3.91}$$

So, the inequality (3.91) can be employed into Hillery-Zubairy case by using the photon number of two modes entanglement (Sete & Ooi, 2012) when the following inequality is satisfied

$$\langle n_1\rangle \langle n_2\rangle < |\langle \hat{a}_1\hat{a}_2\rangle|^2. \tag{3.92}$$

The Hillery-Zubairy criterion is relevant for multipartite system with respect the inequality (3.92).

3.9 Bell's theorem

The probability outcomes of wave functions based on the coherent superposition which allows the collapsed wave function to take a single definite state. The interpretation of the mechanism of possible outcomes for definite state has been facing difficulty because there exists the negative probabilities in the local theory (Scully & Zubairy, 1997).

Thus, the argument of Einstein, Podolsky and Rosen (EPR) concerning nonlocal aspectss of quantum mechanics has identified quantum mechanics as an incomplete theory. They concluded that quantum mechanical description should have supplement of postulating the existence of hidden variables using statistical prediction. Through mathematical concept, the prediction of quantum mechanics is valid but contains physically unrealistic postulates. The EPR paradox presents the situation of general probabilistic scheme showing that quantum theory seems incomplete.

However, John Stewart Bell succeeded in replacing the postulate by a reasonable condition of locality (Cluster, Horne, & etc.al., 1969). Nonlocality in the sense of Bell's equalities is well known and it is utilized as a resource in many aspects. Bell showed that local hidden variable theory imposes experimental constraints on the statistical measurements of separated systems. However, this constraint, known as Bell inequalities, can be violated by the use of entangled state.

Bell theorem is used to detect the entanglement according to the correlated state which results in the joint probability (Scully & Zubairy, 1997). This theorem has been proven by an experiment using Stern-Gerlach apparatus (SGA). The experiment involves anticorrelation of spin projection and resulted in the different joint probability. In fact, the anticorrelation of spin projection is observed in that experiment. The joint probabilities of ab, bc and ac are denoted as

$$P_{ab} = P(\alpha_{12}|\beta_{12}) + P(\alpha_{21}|\beta_{21}) \tag{3.93}$$

$$P_{bc} = P(\beta_{12}|\gamma_{12}) + P(\beta_{21}|\gamma_{21}) \tag{3.94}$$

$$P_{ac} = P(\alpha_{12}|\gamma_{12}) + P(\alpha_{21}|\gamma_{21}) \tag{3.95}$$

where α_{ij}, β_{ij}, and γ_{ij} represent the notations of spin projection on the site 1 and site 2 for the angles orientation of θ_a, θ_b and θ_c respectively. Then, the result of two different joint probabilities for detection of both photons can be calculated by the correlation function. It shows in the following calculation

$$
\begin{aligned}
P_{ab}+P_{bc} &= P(\alpha_{12}|\beta_{12})+P(\alpha_{21}|\beta_{21})+P(\beta_{12}|\gamma_{12})+P(\beta_{21}|\gamma_{21}) &(3.96)\\
&= P(\alpha_{12}|\gamma_{12})+P(\alpha_{21}|\gamma_{21})+P(\beta_{12}|\beta_{12})+P(\beta_{21}|\beta_{21}) &(3.97)\\
&= P_{ac}+P(\beta_{12}|\beta_{12})+P(\beta_{21}|\beta_{21}) &(3.98)
\end{aligned}
$$

The equation (3.98) shows that the probabilitic outcome of anticorrelation of spin projection must be positive. This condition demonstrated that the pairs of particles passing through the apparatus would be in the entangled state based on the assumptions of locality and reality of Bell's theorem. The outcome of the joint probability can be any possible measurement. Therefore, the correlated state of Bell's theorem is said to be entangled when the total joint of probability is greater than the one joint probability (Scully & Zubairy, 1997). It can be simplified as the following

$$P_{ab}+P_{bc} > P_{ac}. \tag{3.99}$$

The condition of Bell's theorem, equation (3.99) must be satisfied in order to detect the entanglement in terms of joint probability. Moreover, Bell's inequality shows a perfect measurement which can be applied into quantum key distribution and quantum communication.

3.10 Greenberger-Horne-Zeilinger (GHZ) theorem

Greenberger-Horne-Zeilinger is well known as GHZ theorem which has interpretation for maximal entangled state for multipartite state. This theorem involves at least three subsystems which are extremely well-known. The standard measure of maximal entangled state of three photons is defined as

$$|\psi\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \tag{3.100}$$

To prove GHZ theorem, the test is certainly similar to the test of Bell inequality except the amount of particles because it used three particles correlation experiment. This theorem used the concept of probability to measure the possible outcome of maximally entangled state. From the experiment, the three photons in an entangled state is in superposition with respect to the coordinate system. The prior measurement is the polarization of photons with 50% for each orientation.

This quantity identifies the entanglement of tripartite state, (Scully & Zubairy, 1997)

$$|\psi\rangle_3 = \frac{1}{\sqrt{2}}\left(|\uparrow_1\uparrow_2\uparrow_3\rangle - |\downarrow_1\downarrow_2\downarrow_3\rangle\right) \tag{3.101}$$

According to the GHZ experiment, detecting the entangled state involves a fair sampling of probability of three states. The tripartite state $|\psi\rangle_3$ can be verified as entangled after operated by the eigenstate operators $\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}$, $\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}$ and $\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)}$. The product of eigenstate and spin states is identified using

$$\begin{aligned}
\sigma_x|\uparrow\rangle &= |\downarrow\rangle \\
\sigma_x|\downarrow\rangle &= |\uparrow\rangle \\
\sigma_y|\uparrow\rangle &= i|\downarrow\rangle \\
\sigma_y|\downarrow\rangle &= -i|\uparrow\rangle .
\end{aligned}$$

For example, the product of eigenstate and its operator are obtained as

$$\begin{aligned}
\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}|\psi\rangle_3 &= \frac{1}{\sqrt{2}}\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}\left(|\uparrow_1\uparrow_2\uparrow_3\rangle - |\downarrow_1\downarrow_2\downarrow_3\rangle\right) && (3.102)\\
&= \frac{1}{\sqrt{2}}\left[i^2|\downarrow_1\downarrow_2\downarrow_3\rangle - \left(i^2\right)|\uparrow_1\uparrow_2\uparrow_3\rangle\right] && (3.103)\\
&= |\psi\rangle_3 && (3.104)
\end{aligned}$$

The product of eigenstate, $|\psi\rangle_3$ and eigenstate operators $\left(\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}\right)$ give the result of eigenvalue equal to $+1$. Similarly, the product of eigenstate, $|\psi\rangle_3$ with the other two eigenstate operators $\left(\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}\right)$ and $\left(\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)}\right)$ also produced the same eigenvalue of $+1$. Therefore, the value of eigenstate operator of $\sigma_x^{(3)}$ is equal to $+1$ if $\sigma_y^{(1)}$ and $\sigma_y^{(1)}$ equal to $+1$. However, if $\sigma_y^{(1)}$ and $\sigma_y^{(2)}$ equal to $+1$ and -1 respectively, it will cause the value of $\sigma_x^{(3)}$ to be -1 (Scully & Zubairy, 1997). These probability outcomes of eigenstate operators have been simplified in the tabular form. The Table (3.1) shows the probability outcomes of three eigenstate products.

In the entangled state case, the contradicted value of eigenstate operators should be assigned the value -1 because the outcome of the hidden variable theory is pre-

Eigenstates	Possible outcome
$\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}\|\psi\rangle$	+1
$\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}\|\psi\rangle$	+1
$\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)}\|\psi\rangle$	+1
$\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}\|\psi\rangle$	−1

Table 3.1: The table probability outcome of GHZ equality.

dicted to be always $+1$.

$$\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}|\psi\rangle_3 = -|\psi\rangle . \tag{3.105}$$

Therefore, the state is said to be entangled if and only if the product of the eigenstate, $|\psi\rangle_3$ and eigenstate operators, $\left(\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}\right)$ equal to -1(Scully & Zubairy, 1997). It should satisfy this equality

$$\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}|\psi\rangle = -|\psi\rangle . \tag{3.106}$$

The equality (3.106) defined as the condition of tripartite state in the detection of entanglement.

3.11 Negative Wigner function

Phase space is a fundamental concept in classical mechanics which is described by the probability distribution function. In the quantum mechanical systems, there are three famous quasiprobability distributions called the Glauber-Sudarshan P, Husimi Q and the Wigner functions. However, these quantities take negative values as the result of quantum properties. The Wigner function is a method to interpret quantum properties using a classical probability distribution in phase space. The general features of Wigner function in phase space formulation is its relationship to the density operator in statistical quantum theory (Abir & Singh, 2011) and (Wolfgang, 2001). It shows as the following equation

$$W(x,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\xi \exp\left(-\frac{i}{\hbar}p\xi\right)\left\langle x+\frac{1}{2}\xi\,|\rho|\,x-\frac{1}{2}\xi\right\rangle . \tag{3.107}$$

The density operator gives density matrix element of the correlated system (H.-S. N. Ooi C. H. Raymond & Singh, 2012) and hence the Wigner function. The classical probability distribution requires the Wigner function $W(x,p)$ should be positive. Instead of that, the Wigner function can be negative which is classified as the nonclassicality of the probability distribution. Thus, the negative domain of Wigner function arises from the interference of waves such as an intimate connection between the two wave functions.

The entanglement of a system can be determined according to the negativity of Wigner function. The Wigner function becomes delocalized in position due to the entanglement. The negativity of Wigner function has been linked to the nonlocality according to Bell inequality. The normalization condition for the Wigner function is written as

$$\int\int W(x,p)dxdp = 1. \tag{3.108}$$

Based on the normalization in equation (3.108), the magnitude of the Wigner function is bounded by

$$|W(x,p)| \leq \frac{1}{\pi\hbar}. \tag{3.109}$$

Then, the negative Wigner function $W(x,p)$ is based on the position and momentum operator defined as,

$$\begin{aligned} \delta(\psi) &= \int\int [W(x,p) - |W(x,p)|]\,dxdp & (3.110) \\ &= 1 - \int\int |W(x,p)|\,dxdp & (3.111) \end{aligned}$$

where only the negative regions of $W(x,p)$ contribute to the integration. For this reason, when Wigner function is positive everywhere $W(x,p) = |W(x,p)|$, therefore $\delta(\psi) = 0$ is for the case of coherent state. However, when the Wigner function has negative values at some regions, $W(x,p) - |W(x,p)|$ also has negative values in the same regions, the integration over the regions of negative values gives $\delta(\psi) < 0$. Therefore, this is a required condition to detect the entanglement of the quantum systems.

3.12 Logarithmic negativity

The entanglement of continuous variables is measured for the Gaussian state which can be the best characterized by the logarithmic negativity. In the same manner, continuous variables is evaluated in terms of the simplistic eigenvalues of the covariance matrix and computed from the Wigner function (Abir & Singh, 2011). The logarithmic negativity can detect the entanglement for two modes states based on the negativity of partial transposition (Vidal & Werner, 2002). The negative partial transpose is non-increasing with respect to the entanglement monotone in order to measure the degree of entanglement. The logarithmic negativity criterion has been proposed to have a simplectic eigenvalue of partial transpose (Isar, 2008). In this case, V represents the smallest eigenvalue of simplectic matrix.

$$V = \sqrt{\frac{\sigma + \sqrt{\sigma^2 - 4\det\Upsilon}}{2}}. \tag{3.112}$$

The invariance is constructed from the element of covariance matrix which is denoted as

$$\sigma = \det A_1 + \det A_2 - 2\det A_{12} \tag{3.113}$$

$$\Upsilon = \begin{pmatrix} A_1 & A_{12} \\ A_{12}^T & A_2 \end{pmatrix}. \tag{3.114}$$

For the cavity mode, it initially in a vacuum state, the covariant matrix has the form

$$\Upsilon = \begin{pmatrix} m & 0 & c & 0 \\ 0 & m & 0 & -c \\ c & 0 & n & 0 \\ 0 & -c & 0 & n \end{pmatrix}. \tag{3.115}$$

Then the element matrix of diagonal form measured based on the annihilation and creation operators of two modes states

$$m = \left\langle \hat{a}_1^\dagger, \hat{a}_1 \right\rangle + \left\langle \hat{a}_1, \hat{a}_1^\dagger \right\rangle \tag{3.116}$$

$$n = \left\langle \hat{a}_2^\dagger, \hat{a}_2 \right\rangle + \left\langle \hat{a}_2, \hat{a}_2^\dagger \right\rangle \tag{3.117}$$

$$c = \langle \hat{a}_1, \hat{a}_2 \rangle + \left\langle \hat{a}_1^\dagger, \hat{a}_2^\dagger \right\rangle. \tag{3.118}$$

From the covariance matrix, it can be seen that the element matrices of m and n are symmetric (Vidal & Werner, 2002). Therefore, the pure state is also defined as symmetric and fulfills the self adjoint of $c = -c = \sqrt{m^2 - 1}$. Thus, the correlations of invariant are determined by four local symplectic invariants which are

$$
\begin{aligned}
\det\Upsilon &= \left(mn - c^2\right)\left(mn - (-c)^2\right) && (3.119)\\
\det A_1 &= m^2 && (3.120)\\
\det A_2 &= n^2 && (3.121)\\
\det A_{12} &= c\,(-c)\,. && (3.122)
\end{aligned}
$$

Finally, the logarithmic negativity for two modes states is measured by taking the maximum commutation because both corresponds at the smallest eigenvalue. The negativity is completely defined by symplectic spectrum of transpose of covariance matrix (Isar, 2008) as the following

$$
\begin{aligned}
E_N &= -\frac{1}{2}\log_2[4f(\Upsilon)] && (3.123)\\
f(\Upsilon) &= \frac{1}{2}(\det A_1 + \det A_2) - \det A_{12} && (3.124)\\
&\quad -\sqrt{\left[\frac{1}{2}(\det A_1 + \det A_2) - \det A_{12}\right]^2 - \det\Upsilon} && (3.125)
\end{aligned}
$$

This is simplified in the following equation

$$E_N = \max\left[0, -\log_2 V\right]. \tag{3.126}$$

The entanglement is detected in this criterion when logarithmic negativity, E_N is positive, $E_N > 0$ and covariance matrix is negative, $V < 1$. If $E_N \leq 0$, then the state is separable (Isar, 2008). Therefore, the lowest eigenvalue of the covariance matrix with respect to logarithmic negativity can identify the entanglement.

All major criteria of entanglement have been explained theoretically. Each criterion has its significant in order to clarify the conditions of parameter entanglement. Moreover, the conditions provided the relation of illustration when the entanglement is detected through experiment. The entanglement of quantum system can be measured in two ways; which are concurrence and formation of entanglement. All the entanglement criteria demonstrated in terms of their conditions are illustrated in the following figure.

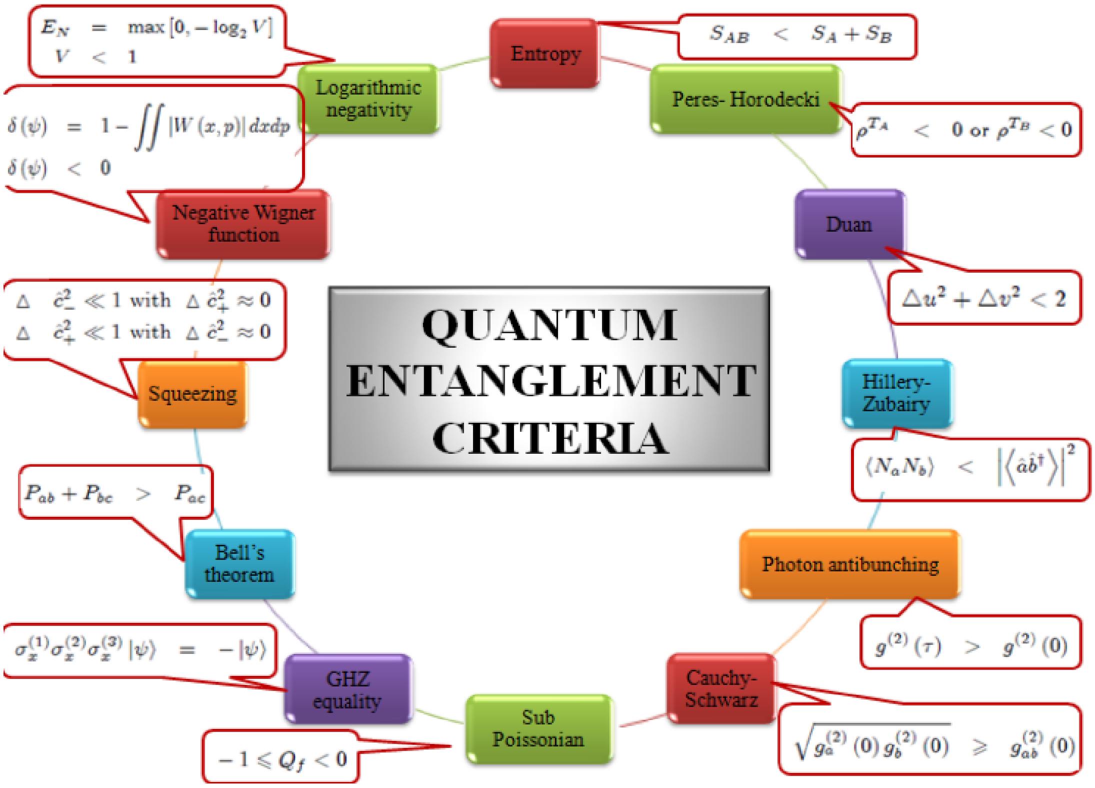

Figure 3.4: Quantum entanglement criteria.

The Figure (3.4) provides the condition for each quantity in order to detect the entanglement. All twelve quantities with their conditions are illustrated in Figure (3.4) to provide a view of parameters used. The combination of all quantities of entanglement in Figure (3.4) gives comparison of the final condition that each quantities must be satisfied.

CHAPTER 4

RELATIONSHIPS BETWEEN ENTANGLEMENT CRITERIA

The relationship between entanglement criteria has covered five important factors of quantum entanglement. There are the features of quantum entanglement, properties of quantum entanglement, relationships of entanglement criteria, correlations and also requirements of quantum entanglement. The purpose of this theoretical analysis is to connect certain criteria which shared the same parameters and properties. Moreover, the analysis provides the significant information of quantum system to be applied in the process of transmitting the information. It can also broaden the usage of theories in detecting the entanglement in the though situations.

4.1 Features of quantum entanglement

The entanglement can be quantified through the concurrence and entanglement of formation. After entanglement has been detected, it can be quantified the degree of entanglement based on two methods which are concurrence and entanglement of formation. Moreover, these two methods are related because of concurrence provides the estimation for the entanglement of formation (Ming & Li-Jost, 2010). In quantum information science, concurrence is an entanglement monotone which is defined for a mixed state of two qubits. For example, when the pure state, $|\psi\rangle$, equation (2.90) apply in the tensor product of Hilbert space, $H_A \otimes H_B$, the concurrence is defined as

$$C\left(|\psi\rangle\right) = \sqrt{2\left(1 - Tr\left[\rho_A^2\right]\right)} \tag{4.1}$$

where $\rho_A = Tr_B\left[|\psi\rangle\langle\psi|\right]$. The concurrence is extended to the mixed state, ρ in the equation (2.106) by the convex roof

$$C\left(\rho\right) = \min_{\{p_i|\psi_i\rangle\}} \sum_i p_i C\left(|\psi_i\rangle\right) \tag{4.2}$$

where $p_i \geq 0$ and $\sum_i p_i = 1$. Concurrence is a measurement of entanglement between any two systems which is required minimal physical resource to prepare a quantum state. The quantity of concurrence describes the quantum phase transition in the interaction of quantum many body system (Li & Fei, 2011).

The entanglement of formation for pure state

$$|\psi\rangle = \sum_{ij} a_{ij} |ij\rangle \in H \otimes H \tag{4.3}$$

$$E(|\psi\rangle) = -Tr(\rho_1 \log_2 \rho_1) = -Tr(\rho_2 \log_2 \rho_2) \tag{4.4}$$

where

$$\rho_1 = A(A^\dagger) = Tr_2 |\psi\rangle\langle\psi| \tag{4.5}$$

$$\rho_2 = \left(A^\dagger A\right)^* = Tr_1 |\psi\rangle\langle\psi| \tag{4.6}$$

In this case, $(A)_{ij} = a_{ij}$ and $AA^\dagger$ is based on the Werner state.

For a mixed state, the entanglement of formation is defined as

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \tag{4.7}$$

$$E(\rho) = \min \sum p_i E(|\psi_i\rangle) \tag{4.8}$$

According to (Vedral, 2006), the formation of entanglement is denoted as,

$$E(\rho) = \varepsilon(C(\rho)) \tag{4.9}$$

where $\varepsilon(C(\rho))$ is a function of entanglement which defined as,

$$\varepsilon(C(\rho)) = h\left(\frac{1+\sqrt{1-C(\rho)^2}}{2}\right) \tag{4.10}$$

where $h = -x\log_2 x - (1-x)\log_2(1-x)$ is binary entropy function (Jaeger, Alexander, & etc.al., 2003). Generally, the function $E(\rho)$ is monotonically increasing which affecting from $0 \leq C(\rho) \leq 1$ (Wootters, 2001).

The important things of entanglement criteria which based on concurrence are because of existing correlation between two subsystems which perform separately at first. The strongest correlation is a reason of formation of entanglement can approach to the maximum possible of concurrence. Therefore, the entanglement can be defined as the corresponding to the concurrence value where for any separable state, the measure of entanglement should be zero. This is the first feature of entanglement measures for mixed state (Vedral, 2006). Generally, the formation of entanglement also can be constructed from the minimum of all average entanglement of subensembles of pure state and also taken over all mixed state,

$$E(\rho) = \min \sum_i p_i S\left(\rho_i^A\right) \tag{4.11}$$

Thus, the features of entanglement as (Vedral, 2006) have mentioned that the entanglement of subensemble state ρ_i cannot exceed the expected entanglement in ρ state.

$$E(\rho) \geq \sum_i p_i E(\rho_i) \tag{4.12}$$

The relation of the concurrence and formation of entanglement of pure state and mixed state has been simplified in the following table.

	Density matrix	Concurrence	Entanglement of formation
Pure state	$\rho_A = Tr_B[\|\psi\rangle\langle\psi\|]$ $\rho_B = Tr_A[\|\psi\rangle\langle\psi\|]$	$C(\|\psi\rangle) = \sqrt{2(1 - Tr[\rho_A^2])}$	$E(\|\psi\rangle) = -Tr(\rho_A \log_2 \rho_A)$ $= -Tr(\rho_B \log_2 \rho_B)$
Mixed state	$\rho = \sum_i^N p_i \|\psi_i\rangle\langle\psi_i\|$	$C(\rho) = \min_{\{p_i\|\psi_i\rangle\}} \sum_i p_i C(\|\psi_i\rangle)$	$E(\rho) = \min \sum p_i E(\|\psi_i\rangle)$

Table 4.1: Table of relation of concurrence and formation of entanglement.

Table (4.1) demonstrated the concurrence and formation of entanglement are related because of the concurrence corresponds as a resource of formation of entanglement for pure state as well as for mixed state.

According to (William, 1998), the entanglement has a limitation when fixing the eigenvalues of ρ in one sort of constraint on the state. In addition, the limitation arises when entanglement is shared among other state such as bipartite state. If they're in the n qubits system, the entanglement of a pair qubit will be increased to compete the other pairs. The exact value of concurrence has been estimated based on lower bound and upper bound of entanglement of formation (Zhu & Fei, 2012). The lower bound and upper bound correspond to estimate the minimum and maximum value of concurrence respectively.

The entanglement of formation can be quantified for any $m \otimes n$ $(m \leq n)$ when it satisfied the condition

$$\varepsilon (C(\rho)) \leq E(\rho) \leq \eta (C(\rho)) \tag{4.13}$$

After considering that $\varepsilon (c)$ and $\eta (c)$ correspond to maximum convex function and minimum concave function respectively (Zhu & Fei, 2012). This condition proven that the concurrence is monotonically increasing within the limitation. Therefore, the entanglement of a system is able to quantify when preserved that condition.

4.2 Properties of quantum entanglement

The criteria of entanglement for each method have their strengths and weaknesses to determine the entanglement condition. In the concept of theory, the condition of nonclassical optics introduced the various parameters which provides the entanglement process. Therefore, the necessary and sufficient conditions of entanglement for each criterion bring to the better view of entanglement.

The first criterion which is entropy shows the sufficient condition of entanglement. The entropy is known as sufficient condition of entanglement because of the entropy can measure the maximal entanglement by approaching the quantum correlation. This method also adapts easily to the situation with independent requirement (Chaves, 2013). In addition, the entropy is able to indicate the maximum quantum correlation between the two systems like the equation (3.7) (Gerry & Knight, 2005). The entropy can also identify the entanglement of pure state when it violates the separable of product state as defined in the equation (3.23). Besides, the entropy is mostly used to measure the entanglement according to the formation of entanglement. The entropy is the most accepted method when it applies into other methods of quantifying the entanglement.

For the second entanglement criterion, Peres-Horodecki criterion demonstrated the necessary condition of entanglement. This criterion discovers the separability of bipartite state from the operation of partial transpose. The positive eigenvalue is resulting from the linear partial transpose operation which shows the necessary condition of separable quantum state (P. Horodecki, 1997). However, if the eigenvalue is negative, the state can be entangled but according to sufficient condition of separability. By the same token, the inseparable density operator according to partial transpose operation is necessary to have a negative eigenvalue as it's depicted in the equation (3.24).

In like manner, the squeezing demonstrated the necessary condition of entanglement. The squeezing of light is depicted based on two ways either number of photon or coherent state. Expressively, these two ways required the quadrature variance satisfied the minimum uncertainty relation (Scully & Zubairy, 1997). Generally, the two quadrature variance provide the elements of squeezed state that explain the amplitude is squeezing. The squeezed state is said to be entangled when the two different quadrature states at the minimum is uncertainty. For this reason, the squeezing criterion is required the uncertainty of quadrature unit and it must be satisfied the equation of (3.44). This condition shows the minimum area for the phasor uncertainty to emphasize the entanglement of the system. The squeezed state plays an important role when it applies into the others theories to detect the entanglement.

After that, the photon antibunching can defined the entanglement based on photon distribution of second order correlation. Physically, the entanglement exhibits in the photodetector when the photons appear and tend to be a group and they will be resulting a correlation. This criterion demonstrates the necessary condition based on the photon antibunching principles. It's required the condition of second order correlation based on probability distribution with respect to time dependent. The second order correlation at time delay is greater than second order at the dead time. It is shown at the equations (3.52) and (3.53).This condition demonstrates the entangled state because of there exists correlation between successive photon emission events.

Next criterion is sub Poissonian criterion is equal to the photon antibunching where it holds the necessary condition of entanglement. The key point of this view is the expand from the second order correlation with respect to the number of photon distribution by the equation (3.56). Hence, the sub Poissonian distribution is identified when the distribution is narrower than Poissonian. In this case, the coherent state is measured as Poissonian distribution. On the other words, sub Poissonian criteria are represented by Mandel Q parameter where they must satisfy the range of equation

(3.59) to detect the entanglement. The similar properties of necessary condition also go to Cauchy-Schwarz inequality where this method required to satisfy the crosscorrelation between the two systems as the equation (3.68) .

Additionally, the Duan criterion and Hillery-Zubairy criterion are classified as necessary condition of entanglement due to the requirement of their criteria. The Duan criterion requires to satisfy a lower bound of total variance such the equation (3.72) in order to define the inseparable state. Furthermore, the final necessity of Duan criteria are corresponding to the entangled state which is defined as an equation (3.80). After that, the Hillery-Zubairy criterion is categorized in the necessary condition of entanglement because of it applied two modes states based on electromagnetic field. Then, this method is required to satisfy the uncertainty principle and also the commutation relation like a squeezing. Moreover, this method applies Cauchy-Schwarz inequality and obtains the final equation (3.92) as the necessary condition of entanglement.

However, the Bell's theorem and GHZ equality classified as sufficient condition of entanglement. These are due to their quantum properties which can measure the entangled state theoretically. For the Bell's theorem, when it is satisfied the joint probability between two subsystems such as bipartite system, it can measure easily the entanglement of quantum state due to the exist of the correlation. However, to point out the efficient condition, Bell's theorem used three subsystems and produces three joint probabilities in order to develop a condition of entanglement which is defined as an equation (3.99). It is similar to GHZ equality where it is enough to identify the entanglement because it involves the tripartite system which exists nonclassical correlation in the system. Nevertheless, this method must satisfy the circumstance of equation (3.106) as a proof to detect the entanglement of the tripartite system.

The two last criteria which are negative Wigner function and logarithmic negativity set as the necessary conditions of entanglement due to both criteria must satisfy the negative result of the function in order to identify the nonclassical physics and then they are able to detect the entanglement. Therefore, the criteria which hold the either necessary or sufficient condition of entanglement are simplified in the following table.

Table (4.2) shows the checklist the properties of all entanglement criteria with

Properties / Criteria	Necessary condition	Sufficient condition	Density operator	No. of Photon	Correlation
Entropy		✓	✓		
Peres-Horodecki	✓		✓		
Squeezing	✓			✓	✓
Photon antibunching	✓			✓	✓
Sub Poissonian	✓			✓	✓
Cauchy-Schwarz inequality	✓			✓	✓
Duan criterion	✓			✓	✓
Hillery-Zubairy criterion	✓			✓	✓
Bell's theorem		✓		✓	
GHZ equality		✓		✓	
Negative Wigner function	✓		✓		
Logarithmic negativity	✓			✓	

Table 4.2: The checklist of quantum entanglement properties.

respect to the parameters and correlation. The important parameters in measuring the entanglement uses to have two possible either the density operator or number of photons. These two parameters are equally important as the main factor in the detection of quantum entanglement. The density operator has been used in the entropy, Peres-Horodecki criterion and negative Wigner function. Other than those theories, they measure the entanglement with respect to the number of photons. Furthermore, there exhibits correlation in the most of entanglement criteria which parallel to the nonclassical properties of quantum theory.

4.3 Relationships between entanglement criteria

The study of entanglement criteria leads to the analysis of similarity between those criteria. The similarity of fundamental parameter is used in the derivation which causes the establishment of detecting entanglement. Furthermore, those similarities are possible to discover the strong entangled state. Hence, the connection of the certain criteria is related to the nonclassical properties of physics. The main concern of this analysis is purposely to gain the effective criteria of quantum entanglement.

The fundamental parameter used in detecting the entanglement is density operator. From the study, there have three methods which are using the density operator as their parameter. Those methods are entropy, Peres-Horodecki criterion and negative Wigner function. The significant of the density operator in these methods measured the outcome of quantum state either it is defined as separable or inseparable state. Moreover, the density operator is defined as a sufficiency of separability of quantum state.

Theoretical analysis from the view of the similarity of the entropy and negative Wigner function methods bring to a link to strengthen the detection of entanglement. According to the properties of entropy, the entanglement is generated from the additive operation of a reduced density operator. This key point has been applied into the Wigner function method. Under this circumstance, the linear entropy takes place when the reduced density operator applied in the Wigner function (Bastidas, Reina, & etc.al., 2010). It is because of the entanglement is identified after decomposition of Wigner function. So, this connection enlightens the criteria of entanglement with respect to the density operator.

In the same way, the density operator is relevant when it applies into logarithmic negativity. This is due to the fact that the entanglement can be clarified using the partial transpose of the density operator. The negativity of the density operator measures the inseparability as it has been explained in the Peres-Horodecki criterion (Vidal & Werner, 2002). After that, the negative density operator has further applied in logarithm operation and it is considered as logarithmic negativity. This relation makes the entangled state to be confirmed.

The most of entanglement criteria sharing the basic parameter are defined with respect to the number of photons. There are squeezing, photon antibunching, sub Poissonian criterion, Cauchy-Schwarz, Duan criteria and Hillery-Zubairy criterion. Besides, these methods have relation due to the correlation between two subsystems exist in their derivation. For example, Duan criterion is related to squeezing criteria because of it sets up the continuous variables of two modes of squeezing state which is considered as EPR-like operators (Duan & Zoller, 2000). The operators of Duan criterion also have satisfied the commutation relation like squeezing. This case has proven that there exists the connection according to the squeezing.

The similar parameter which is second order correlation but different method provides a link between photon antibunching, sub Poissonian and Cauchy-Schwarz theories. However, these methods are distinguished by their theory. For example, photon antibunching defined the entanglement depend on the possible outcome of a second order correlation in terms of time evolution. The sub Poissonian clarifies the entanglement based on second order correlation of number of photon distribution. Meanwhile, Cauchy-Schwarz involves crosscorrelation of two subsystems in its derivation of entanglement. Because of that, the second order correlation of photon antibunching is relevant to be applied into sub Poissonian and Cauchy-Schwarz criteria (Christopher, 1995). In short, the second order correlation connects these three methods of the purpose to detect the entanglement.

4.4 Correlations

The correlation between particles shows the interconnectedness such as interdependence between each other. This concept is related to the nature of entanglement. Therefore, it is very important in identifying the entangled state in a quantum system. The correlation can also measure the separability of particles. It can be determined with respect to the density operator.

The correlated particles are identified when two particles are dependent on each other. This can be measured according to the tensor product of two density operators. After considering the density operators ρ^A and ρ^B are subset of the Hilbert space, H_A and H_B respectively

$$\rho^A \in H_A \tag{4.14}$$

$$\rho^B \in H_B \tag{4.15}$$

Therefore, its use to measure the complex Hilbert space by the tensor product operator (Guhne & Toth, 2009). The result of tensor product of two density operators obtained the density operator of composite system,

$$
\begin{aligned}
H &= H_A \otimes H_B && (4.16)\\
\rho^A, \rho^B &\in H && (4.17)\\
\rho^{AB} &= \rho^A \otimes \rho^B && (4.18)
\end{aligned}
$$

The equation (4.18) shows that both density operators are independent of each other. This procedure shows that both density operators can be measured by linear combination and resulting that the density operators are not depending value. Thus, it can be said that the density operators ρ^A and ρ^B are uncorrelated (Audretsch, 2007). This is one condition of separable state by definition. It has formed

$$\rho^{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{4.19}$$

In addition, the uncorrelated state is identified based on product state which is acting completely independently. Under those circumstances, the uncorrelated state clearly shows separable state.

In contrast, if both density operators are dependent on each other, the product of density operator is not a composite density matrix as shown

$$\rho^{AB} \neq \rho^A \otimes \rho^B \tag{4.20}$$

The equation (4.20) shows that the violation of separable state in both density operators. It is identified as correlated state because it cannot be written as a convex combination of factored state. In this case, the correlated state illustrated the interdependence of two density operators ρ^A and ρ^B and it is resulting the entangled state.

In particular, the correlation can measure the bipartite state, ρ^{AB} either entangled or separable that can be from the classical and quantum source (Sarandy, 2009). Theoretically, the classical correlated occurs when the quantum state lead to correlation which is observed in the classical system (Audretsch, 2007). After considering the classical communication between two systems, it is able to identify the classical correlated of composite density matrix, ρ^{AB} is based on local operator and classical communication (LOCC). The LOCC involves two distinguish Bell pairs such as

$$|\psi\rangle_1 = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \tag{4.21}$$

$$|\psi\rangle_2 = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right) \quad (4.22)$$

In this case, the two parties of A and B are communicating through a classical system in order to coordinate the local operation. Basically, the A party prepares the system S^A in the state ρ_i^A and communicate with B party who prepares the different system S^B in the state ρ_i^B. This operation is repeated many times randomly. Thus, it can be formulated as composite state by the construction a statistical mixture of product states

$$\rho^{AB} = \sum_{i=1}^{m} p_i \rho_i^A \otimes \rho_i^B \quad (4.23)$$

where the probability is always in the range of $0 < p_i < 1$ and the total of probability, $\sum_{i=1}^{m} p_i = 1$. Then, it is formulated with respect to LOCC as the following

$$\begin{aligned}
\rho^A \otimes \rho^B &= \sum_{i=1}^{m} p_i \, |\psi\rangle_A \langle\psi| \otimes |\psi\rangle_B \langle\psi| && (4.24)\\
&= \sum_{i=1}^{m} p_i \left(\left|0^A\right\rangle \left\langle 0^A\right| + \left|1^A\right\rangle \left\langle 1^A\right|\right) \otimes \left(\left|0^B\right\rangle \left\langle 0^B\right| + \left|1^B\right\rangle \left\langle 1^B\right|\right) && (4.25)\\
&= p_1 \left(\left|0^A\right\rangle \left\langle 0^A\right| \otimes \left|0^B\right\rangle \left\langle 0^B\right|\right) + p_2 \left(\left|1^A\right\rangle \left\langle 1^A\right| \otimes \left|1^B\right\rangle \left\langle 1^B\right|\right) && (4.26)\\
&= p_1 \left|0^A, 0^B\right\rangle \left\langle 0^A, 0^B\right| + p_2 \left|1^A 1^B\right\rangle \left\langle 1^A 1^B\right| && (4.27)
\end{aligned}$$

The equation (4.24) demonstrates that classical correlated which is produced by a proper mixture of the product state. Additionally, the product state is constantly influenced by independent state of density operators. As a result, the classical correlated which is manipulated locally by LOCC is illustrated as unentangled state (Vedral, 2006). It is due to the fact that LOCC is able to convert the entangled state into disentangled state.

On the contrary, the non-classical correlated which is violated the classical correlated as in the following equation

$$\rho^{AB} \neq \sum_{i=1}^{m} p_i \rho_i^A \otimes \rho_i^B \quad (4.28)$$

Thus, non classical correlated leads to entangled state because the composite density operator contains quantum correlation (Audretsch, 2007). For this reason, the composite density operator which is originally from the reduced density operator of entangled pure state after considering the entangled pure state. It is proven based on Schmidt decomposition of bipartite state which is formed by $A \cup B$ and decribed by the tensor product of Hilbert space, H_A and H_B respectively

$$H = H_A \otimes H_B \tag{4.29}$$

Thus, the pure state of the bipartite system has the form of

$$\left|\psi^{AB}\right\rangle = \sum_{ij=1}^{k} \chi_{ij} \left|i^{A}\right\rangle \otimes \left|j^{B}\right\rangle \tag{4.30}$$

where $\left|i^{A}\right\rangle$ and $\left|j^{B}\right\rangle$ belong to H_A and H_B respectively. And also $i \leq j$ for $k = \min\left(\dim H_A, \dim H_B\right)$. Therefore, the composite density operator is identified as

$$\rho^{AB} = \left|\psi^{AB}\right\rangle \left\langle \psi^{AB}\right| \tag{4.31}$$

Based on the equation (4.31), one obtains the reduced density operator for each subsystem as

$$\rho^{A} = Tr_B\left(\rho^{AB}\right) \tag{4.32}$$

$$\rho^{B} = Tr_A\left(\rho^{AB}\right) \tag{4.33}$$

It clearly shows that the nonclassical correlated measured the entangled state which is not a mixture of product state as it is presented in the equation (4.28).

Figure (4.1) shows how the correlation process with respect to the density operator determines the entangled state. Although the correlation is important in explaining the entanglement, the correlation is not necessary classical in order to identify the entanglement. In short, the classical correlated of quantum system influence to be separable state as an independent state. In contrast, the nonclassical correlated state is measuring the entanglement by its quantum correlation in the system.

4.5 Requirement of entanglement criteria

The quantum state is said to be entangled state when it fulfills the requirements of entanglement criteria. There have three requirements of entanglement which generally have a connection with entanglement criteria.

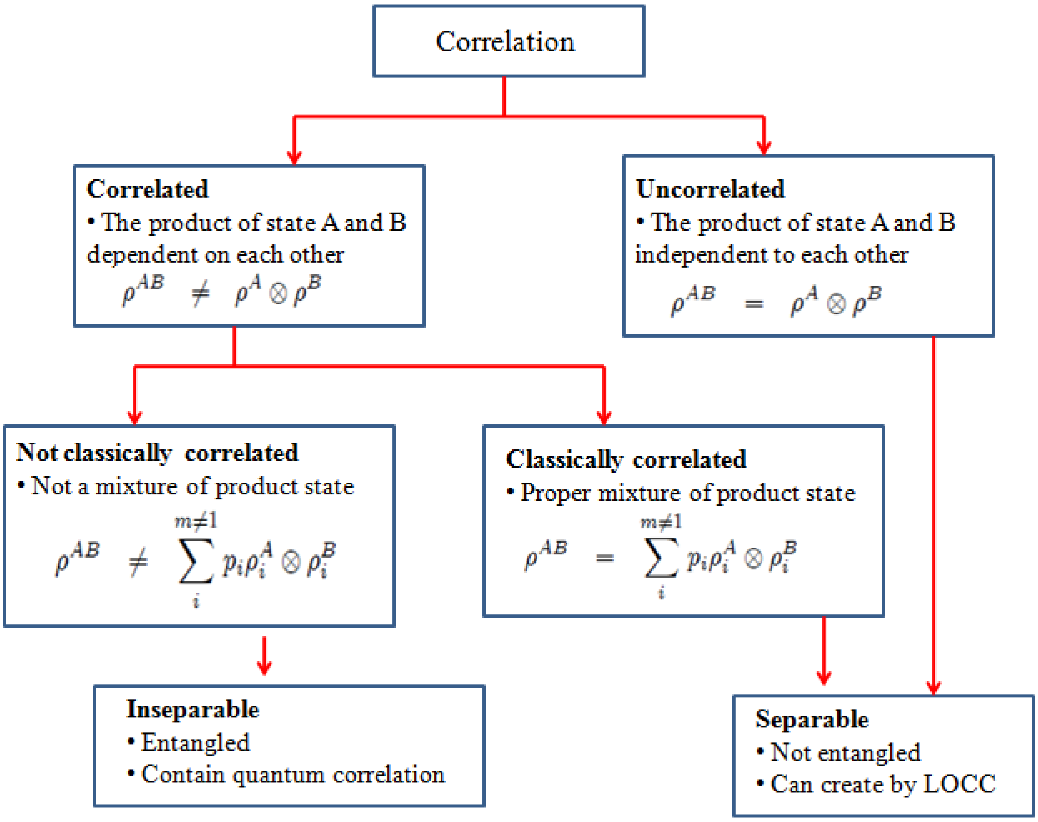

Figure 4.1: The chart of the correlation process determining the entangled state.

The first requirement of entanglement related to separability. The entanglement, $E(\rho)$ can be measured as vanished when the density operator, ρ is separable. This requirement has been proven by the Peres-Horodecki criteria. The separable density operator is measured by the tensor product of two density operators, where it is written as $\rho^{AB} = \rho^{A} \otimes \rho^{B}$. This is demonstrating that the density operator ρ^{A} and ρ^{B} are independent or separable and then they are resulting the composite density operator, ρ^{AB}. In short, for any separable state of density operator, ρ, the entanglement must be zero, $E(\rho) = 0$ (Vedral, 2006). Otherwise, the entanglement can be measured as $E(\rho) > 0$.

In the prior of entanglement of quantum state, the entanglement cannot be created by local operators and aided as classical communication (LOCC). According to the concept of LOCC, it operates the correlated state to manipulate the entangled state in order to separate it. Therefore, it demonstrates that LOCC is able to separate the disentangled state from entangled state (Vedral, 2006). This operator is treated as a measurable degree of entanglement in which the entangled point is corresponded as reference point. In all cases, it is resulting the less amount of entanglement.

The entanglement of a system can be measured by additional operator of subsystem. The entanglement must be satisfied additivity according to interdependent of subsystems (Audretsch, 2007). This requirement can be shown based on entropy criterion

$$\begin{aligned} S(A,B) &= S\left(\rho^{AB}\right) && (4.34) \\ &= S\left(\rho^{A}\otimes\rho^{B}\right) && (4.35) \\ &= S\left(\rho^{A}\right)+S\left(\rho^{B}\right) && (4.36) \end{aligned}$$

Regardless of the fact reduced density operator for both systems are defined according to von Neumann entropy as in

$$\begin{aligned} S\left(\rho^{A}\right) &= Tr_B\left(\ln\rho^{A}\right) && (4.37) \\ S\left(\rho^{B}\right) &= Tr_A\left(\ln\rho^{B}\right) && (4.38) \\ S\left(\rho^{AB}\right) &= -Tr\left(\rho_{AB}\ln\rho_{AB}\right) && (4.39) \end{aligned}$$

Thus, it shows that

$$S\left(\rho^{AB}\right) \leq S\left(\rho^{A}\right)+S\left(\rho^{B}\right) \tag{4.40}$$

According to equation (4.40), there have two possible conditions which are equal sign hold if the subsystem $S\left(\rho^{A}\right)$ and $S\left(\rho^{B}\right)$ are independent, $\rho^{AB}=\rho^{A}\otimes\rho^{B}$. Otherwise, the subsystem is dependent on each other and it is demonstrated as inseparable or entangled state. This is markedly that the entanglement must satisfy the additivity in all cases because of the subsystem are sharing the information. In that case, these requirements provide the significant properties of entanglement criteria.

CHAPTER 5

CONCLUSION

The quantum entanglement exhibit correlation between two particles or subsystems which is classified as nonclassical properties due to the fact that it cannot be explained in the classical physics. Additionally, the entanglement is fascinating and also plays an important role of physical phenomena. This provides the resources of quantum information and communication theory. Above all the entanglement criteria, they provided the condition which quantum theory must be satisfied. In order to emphasize the importance of quantum entanglement, the provided conditions are available when they are applied into other theories.

The quantum entanglement is the key ingredient of quantum information and quantum communication and also an essential basis of quantum computing and cryptography. Investigation of quantum entanglement is currently a very active area. Research is being done on measures for quantifying entanglement precisely, on the entanglement of many-particle systems, and on the manipulations of entanglement and its relation to thermodynamics. In the sense of transmitting information, the entangled system is very useful to transmit information because the process involves sharing the information without any intercept in the communication. In this case, the two parties shared the secret key which they use it to communicate secretly and exchange the messages. Therefore, two parties must have a strong correlation in transmitting the messages while there is an eavesdropper who attempts to hear the communication.

Entanglement can be applied in many fields such as quantum cryptography, quantum computer and quantum information. Therefore this paper has described the correlation, properties of entanglement and requirement of entanglement. For example, quantum cryptography lies in the intersection of quantum mechanics and information theory and due to the concept of shared information which is being converted to share the secret key. In addition, using an entangled photon pair prevents unintended information leakage in unused degrees of freedom. It is because all the information in the message is contained in the quantum correlations between the entangled pairs. In reality, there are technological challenges in protecting the valuable information. The need of transmitting information through secret communication is an important issue for military strategies during war time. Thus, it leads to the development of code breaking technique. In short, quantum entanglement has many advantages and challenges in application of global quantum communication in future.

Appendices

APPENDIX A

ANNIHILATION AND CREATION OPERATOR

A.1 Annihilation and creation operator with respect to the position and momentum

The position, $\hat{x}$ and momentum, $\hat{p}$ is constructed depend on annihilation, $\hat{a}$ and creation, $\hat{a}^\dagger$ operator

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right) \tag{A.1}$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}-\hat{a}^\dagger\right) \tag{A.2}$$

The creation operator, $\hat{a}^\dagger$ is measured from the derivation of equation (A.1) and (A.2),

$$\hat{x}\sqrt{\frac{2m\omega}{\hbar}} = \hat{a}+\hat{a}^\dagger \tag{A.3}$$

$$\frac{\hat{p}}{i}\sqrt{\frac{2}{m\omega\hbar}} = \hat{a}-\hat{a}^\dagger \tag{A.4}$$

$$\left(\hat{a}+\hat{a}^\dagger\right)-\left(\hat{a}-\hat{a}^\dagger\right) = \left(\hat{x}\sqrt{\frac{2m\omega}{\hbar}}\right)-\frac{\hat{p}}{i}\sqrt{\frac{2m\omega}{\hbar}} \tag{A.5}$$

$$2\hat{a}^\dagger = \sqrt{\frac{2m\omega}{\hbar}}\left(\hat{x}-\frac{\hat{p}}{im\omega}\right) \tag{A.6}$$

$$\hat{a}^\dagger = \frac{1}{2}\sqrt{\frac{2m\omega}{\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right) \tag{A.7}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right) \tag{A.8}$$

In similar way for the annihilation operator, $\hat{a}$, it measures as

$$\left(\hat{a}+\hat{a}^\dagger\right)+\left(\hat{a}-\hat{a}^\dagger\right) = \left(\hat{x}\sqrt{\frac{2m\omega}{\hbar}}\right)+\frac{\hat{p}}{i}\sqrt{\frac{2m\omega}{\hbar}} \tag{A.9}$$

$$2\hat{a} = \sqrt{\frac{2m\omega}{\hbar}}\left(\hat{x}+\frac{\hat{p}}{im\omega}\right) \tag{A.10}$$

$$\hat{a} = \frac{1}{2}\sqrt{\frac{2m\omega}{\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right) \tag{A.11}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right) \tag{A.12}$$

Thus, the creation, $\hat{a}^\dagger$ and annihilation, $\hat{a}$ defined as

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{A.13}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{A.14}$$

A.2 Annihilation and creation operator with respect to the number of photon

The number of operator is identified with respect to the creation $\hat{a}^\dagger$ and annihilation, $\hat{a}$ operators

$$\hat{n} = \hat{a}^\dagger \hat{a} \tag{A.15}$$

The number of state also defined as eigenstate energy of the number operator

$$\hat{a}^\dagger \hat{a} |n\rangle = \hat{n} |n\rangle \tag{A.16}$$

The product of creation operator and eigenstate of number operator obtained

$$\hat{a} |n\rangle = c_n |n-1\rangle \tag{A.17}$$

where c_n is a constant of normalization. The normalization formulated from inner product of eigenstate

$$\left(\langle n| \hat{a}^\dagger\right)(\hat{a}|n\rangle) = \langle n| \hat{a}^\dagger \hat{a} |n\rangle \tag{A.18}$$

$$= \hat{n} \tag{A.19}$$

where $\langle n|$ defined as complex conjugate for the state of $|n\rangle$. Meanwhile, after substitute equation (2.73), the normalization constant is identified

$$\left(\langle n| \hat{a}^\dagger\right)(\hat{a}|n\rangle) = \langle n-1| c_n^* c_n |n-1\rangle \tag{A.20}$$

$$= |c_n|^2 \tag{A.21}$$

Thus, the normalization constant equally to the number of state

$$|c_n|^2 = \hat{n} \tag{A.22}$$

$$c_n = \sqrt{\hat{n}} \tag{A.23}$$

For creation operator, it identified as loosing the number of state which defined as

$$\hat{a} |n\rangle = \sqrt{\hat{n}} |n-1\rangle \tag{A.24}$$

Then, the annihilation operator described as gaining the number of state which it defined as

$$\hat{a}^{\dagger}|n\rangle = c_n^*|n+1\rangle \tag{A.25}$$

$$= \sqrt{\hat{n}+1}\,|n+1\rangle \tag{A.26}$$

Thus both operation simplified as

$$\hat{a}|n\rangle = \sqrt{\hat{n}}\,|n-1\rangle \tag{A.27}$$

$$\hat{a}^{\dagger}|n\rangle = c_n^*|n+1\rangle \tag{A.28}$$

A.3 Commutation relation

The creation, $\hat{a}^{\dagger}$ and annihilation, $\hat{a}$ operators must be satisfied the principles of commutation relation such as $\left[\hat{a},\hat{a}^{\dagger}\right] = 1$. It derived as shown below

$$\left[\hat{a},\hat{a}^{\dagger}\right] = \left(\sqrt{\frac{m\omega}{2\hbar}}\right)^2 \left[\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right),\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)\right] \tag{A.29}$$

$$= \frac{m\omega}{2\hbar}\left(\left[\hat{x},-\frac{i}{m\omega}\hat{p}\right]+\left[\frac{i}{m\omega}\hat{p},\hat{x}\right]\right) \tag{A.30}$$

$$= \frac{-im\omega}{2\hbar}\left(\left[\hat{x},\frac{1}{m\omega}\hat{p}\right]+\left[\hat{x},\frac{1}{m\omega}\hat{p}\right]\right) \tag{A.31}$$

$$= \frac{-i}{2\hbar}\left([\hat{x},\hat{p}]+[\hat{x},\hat{p}]\right) \tag{A.32}$$

$$= -\frac{i}{2\hbar}\left([\hat{x},\hat{p}]+[\hat{x},\hat{p}]\right) \tag{A.33}$$

$$= -\frac{i}{2\hbar}(2i\hbar) \tag{A.34}$$

$$= -i^2 \tag{A.35}$$

$$= 1 \tag{A.36}$$

APPENDIX B

DENSITY OPERATOR

B.1 The trace of density operator

$$
\begin{aligned}
Tr\rho &= \sum_{ij} p_i \langle \psi_i \left| \alpha_j \right\rangle \left\langle \alpha_j \right| \psi_i \rangle && \text{(B.1)} \\
&= \sum_{ij} p_i \langle \psi_i \left| \alpha_j \right\rangle \left\langle \alpha_j \right| \psi_i \rangle && \text{(B.2)} \\
&= \sum_{ij} p_{ij} && \text{(B.3)} \\
&= 1 && \text{(B.4)}
\end{aligned}
$$

In order to show the comparison of pure state and mixed state, taking square of density operator

$$\rho^2 = \rho \cdot \rho \qquad \text{(B.5)}$$

For the pure state

$$
\begin{aligned}
\rho^2 &= |\psi\rangle\langle\psi|\psi\rangle\langle\psi| && \text{(B.6)} \\
&= |\psi\rangle\langle\psi| && \text{(B.7)} \\
&= \rho && \text{(B.8)} \\
Tr\rho^2 &= Tr\rho = 1 && \text{(B.9)}
\end{aligned}
$$

For the mixed state

$$
\begin{aligned}
\rho^2 &= \sum_i \sum_j p_i p_j |\psi_i\rangle\langle\psi_i||\psi_j\rangle\langle\psi_j| && \text{(B.10)} \\
Tr\rho^2 &= \sum_n \left\langle \varphi_n \left| \rho^2 \right| \varphi_n \right\rangle && \text{(B.11)} \\
&= \sum_n \sum_i \sum_j p_i p_j \langle\varphi_n|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\varphi_n\rangle && \text{(B.12)} \\
&= \sum_i \sum_j p_i p_j \left|\left\langle\psi_i|\psi_j\right\rangle\right|^2 \leq 1 && \text{(B.13)}
\end{aligned}
$$

Therefore, from the equations (2.119) and (2.122), it is proven that the distinguish properties of pure state and mixed state.

B.2 Normalization of quantum state

The superposition of a set of number states leads to the coherent state.

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{B.14}$$

Thus the coherent state can be written as

$$\begin{aligned} C_n &= \frac{\alpha^n}{\sqrt{n!}} N(\alpha) && \text{(B.15)} \\ |\alpha\rangle &= N(\alpha) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle && \text{(B.16)} \end{aligned}$$

It is important to realize that coherent state should be normalized,

$$\begin{aligned} \langle\alpha|\alpha\rangle &= 1 && \text{(B.17)} \\ &= |N(\alpha)|^2 \sum_{m,n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} \langle m|n\rangle && \text{(B.18)} \\ &= |N(\alpha)|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} && \text{(B.19)} \\ &= |N(\alpha)|^2 \exp\left(|\alpha|^2\right) && \text{(B.20)} \end{aligned}$$

Thus, the normalization clarify as

$$\begin{aligned} |N(\alpha)|^2 \exp\left(|\alpha|^2\right) &= 1 && \text{(B.21)} \\ |N(\alpha)|^2 &= \frac{1}{\exp\left(|\alpha|^2\right)} && \text{(B.22)} \\ N(\alpha) &= \sqrt{\frac{1}{\exp\left(|\alpha|^2\right)}} && \text{(B.23)} \\ N(\alpha) &= \exp\left(\frac{-|\alpha|^2}{2}\right) && \text{(B.24)} \end{aligned}$$

APPENDIX C

CAUCHY-SCHWARZ INEQUALITY

C.1 Crossorrelation derivation for Cauchy-Schwarz

The correlation defined as,

$$g_a^{(2)}(0) = \frac{\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2} \tag{C.1}$$

$$g_b^{(2)}(0) = \frac{\left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle}{\left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2} \tag{C.2}$$

The product of second order correlation at zero time correlation function

$$g_a^{(2)}(0)\, g_b^{(2)}(0) = \left(\frac{\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2} \right) \left(\frac{\left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle}{\left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2} \right) \tag{C.3}$$

$$= \frac{\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle \left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2 \left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2} \tag{C.4}$$

$$= \frac{\left\langle \hat{a}^{\dagger 2}\hat{b}^{\dagger 2} + \hat{a}^2\hat{b}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{b}^2 + \hat{a}^2\hat{b}^2 \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2 \left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2} \tag{C.5}$$

For the second order correlation of cross correlation function identified as,

$$g_{ab}^{(2)}(0) = \frac{\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b} \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle \left\langle \hat{b}^{\dagger}\hat{b} \right\rangle} \tag{C.6}$$

$$\left[g_{ab}^{(2)}(0) \right]^2 = \left(\frac{\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b} \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle \left\langle \hat{b}^{\dagger}\hat{b} \right\rangle} \right)^2 \tag{C.7}$$

$$= \frac{\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b} \right\rangle^2}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2 \left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2} \tag{C.8}$$

From the equation (3.60), obtained the inequality

$$\frac{\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle \left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2 \left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2} \geq \frac{\left\langle \hat{a}^{\dagger}\hat{a}\hat{b}^{\dagger}\hat{b} \right\rangle^2}{\left\langle \hat{a}^{\dagger}\hat{a} \right\rangle^2 \left\langle \hat{b}^{\dagger}\hat{b} \right\rangle^2} \tag{C.9}$$

$$g_a^{(2)}(0)\, g_b^{(2)}(0) \geq \left[g_{ab}^{(2)}(0) \right]^2 \tag{C.10}$$

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