ABSTRAK

Katakan G ialah suatu graf. *Nombor persilangan* bagi graf G, ditandakan $cr(G)$, adalah bilangan persilangan tepi-tepinya yang minimum daripada lukisan-lukisan G pada satah. *Kepencongan* bagi graf G, ditandakan $sk(G)$, adalah bilangan minimum tepi dalam G yang perlu dihapuskan untuk menghasilkan satu graf satahan.

Dalam Bab 1 tesis ini, sebahagian sifat asas dan takrif mengenai graf akan diberikan.

Dalam Bab 2, kami membentangkan sebahagian keputusan-keputusan yang telah diketahui mengenai nombor persilangan bagi graf dan juga satu kriteria penyatahan bagi sesuatu graf.

Dalam Bab 3, kami memberikan satu tinjauan mengenai kepencongan bagi graf dan memperkenalkan sebahagian graf yang bersifat $\pi - skew$.

Keputusan utaman dalam tesis ini dibentangkan dalam bab terakhir. Kami memperolehi beberapa hasil mengenai kepencongan bagi cantuman dua graf. Kemudian, kami menggunakan keputusan tersebut untuk menentukan secara lengkapnya bagi kepencongan graf k-bahagian lengkap untuk $k = 2, 3, 4$.

ABSTRACT

Let G be a graph. The crossing number of G, denoted as $cr(G)$, is the minimum number of crossings of its edges among all drawings of G in the plane. The skewness of a graph G, denoted as $sk(G)$, is the minimum number of edges in G whose deletion results in a planar graph.

In Chapter 1 of this thesis, some preliminaries and definitions concerning graphs are given.

In Chapter 2, we present some known results on crossing numbers of graphs and also a planarity criterion for a graph.

In Chapter 3, we provide a survey on skewness of graphs and introduce some graphs which are $\pi - skew$.

The main results of this thesis are presented in the last chapter. We prove some results concerning the skewness for the join of two graphs. We then use these results to determine completely the skewness of complete k-partite graphs for $k = 2, 3, 4$.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor, Professor Chia Gek Ling for supervising my research. His guidance, insightful comments and corrections are invaluable to me throughout my study.

A special thank to Professor Angelina Chin Yan Mui and Madam Budiyah binti Yeop for their support and advice. Last but not the least, I would like to thank my family for their unshakable support and faith.

Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction

In recent years, graph theory has been one of the most rapidly growing areas of mathematics. Graph theory and its applications can be found not only in other branches of mathematics such as algebra, geometry, combinatorics and topology, but also in other scientific disciplines, ranging from computer sciences, engineering, management sciences and life sciences.

Much effort has gone into finding the crossing number and skewness of graphs. Building from these researches, this thesis will continue from these work, particularly of that published in [4]. In this thesis, we determine completely the skewness of complete k-partite graphs for $k = 2, 3, 4$.

1.2 Preliminaries and Definitions

In this section, the basic definitions which will be frequently referred throughout this thesis are presented. For those terms and definitions not included here, the reader is referred to [2], [40] and [41].

A *graph* $G = (V, E)$ consists of a finite non-empty *vertex set* $V(G)$ and a finite (possibly empty) *edge set* $E(G)$. The elements in $V(G)$ (respectively $E(G)$) are called *vertices* (respectively *edges*). The number of vertices of G is the *order* of G and is denoted as $|V(G)|$. The number of edges of G is the *size* of G and is denoted as $|E(G)|$.

We say that a graph G is *finite* if both its vertex set and edge set are finite. A graph with no vertices (and hence no edges) is the *empty graph*. In this thesis, we confine our attention to finite graphs. Any graph with only one vertex is *trivial*.

Two vertices u and v of a graph G are *adjacent* if there is an edge uv joining them, and the vertices u and v are then *incident* with the edge uv. Similarly, two distinct edges e and f are *adjacent* if they are incident to a common vertex, otherwise they are *non-adjacent*. Here, the two distinct adjacent vertices are *neighbours* to each other. A set of pairwise non-adjacent vertices is called an *independent set of vertices*.

A *loop* is an edge that joins a vertex to itself. Two or more edges joining the same pair of vertices are called *multiple edges*. As opposed to graphs with multiple edges, a *simple graph* is a graph having neither loops nor multiple edges.

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ if and only if $uv \in E(G)$. If there is an isomorphism from G to H, we say that G and H are *isomorphic*, written as $G \cong H$.

If $v_i \in V(G)$, a *walk* in G is a finite sequence of edges of the form v_0v_1, v_1v_2, ..., $v_{m-1}v_m$, denoted as $v_0v_1v_2...v_{m-1}v_m$, in which any two consecutive edges are adjacent or identical. The number of edges in a walk is called its *length*. In this case, the length of this walk is m. A walk in which all the edges are distinct is a *trail*. If, in addition, all the vertices in a walk are disctinct, then the trail is a *path*. A path with n vertices is denoted as P_n. A walk $v_0v_1...v_m$ is closed if $v_0 = v_m$. A *closed path* containing at least one edge is a *cycle*. Note that any loop or pair of multiple edges is a cycle. A cycle of length k is known as k-cycle and denoted as C_k. A 3-cycle is often called a *triangle*.

A graph G is said to be *connected* if for any two vertices u and v in G, there exists a path from u to v; if there is no such path, then G is said to be *disconnected*. Further, if a graph is disconnected, then it is the disjoint union of several connected graphs called the *connected components* of the graph. Therefore, a graph is connected if and only if it has only one component; it is disconnected if and only if it has more than one component. A graph G is said to be *k-connected* (or *k-vertex-connected*) if it has more than k vertices and the result of deleting any (perhaps empty) set of fewer than k vertices is a connected graph.

The *crossing number* of a graph G, denoted as $cr(G)$, is the minimum number of crossings (of its edges) among the drawings of G in the plane. Clearly, a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

A graph G is *planar* if can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are both incident. Any such drawing is called a *plane graph* of G. A planar graph G is *maximal planar* if adding an edge between any two non-adjacent vertices in G violates its planarity.

The *skewness* of a graph G, denoted as $sk(G)$, is the minimum number of edges in G whose deletion results in a planar graph. The regions defined by a plane graph are called the $faces$ of the graph. A face bounded by n edges is also known as an n-$face$. In a maximal planar graph, each face is bounded by three edges. The *girth* of a graph is the length of a shortest cycle contained in the graph.

1.3 New Graphs from Old

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$and edge set $\{uv : uv \notin E(G), u, v \in V(G)\}$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

If H is isomorphic to a subgraph of G, we say that H is a subgraph of G and write $H \subseteq G$. Subgraphs of G can be obtained by deleting a vertex or an edge. If $v \in V(G)$ and $|V(G)| \geq 2$, then $G - v$ denotes the subgraph obtained by deleting vertex v together with all edges incident with v. Moreover, if $e \in E(G)$, then $G - e$ is the subgraph having vertex set $V(G)$ and edge set $E(G) - \{e\}$.

Further, if a subgraph H of a graph G has the same order as G, then H is called a *spanning subgraph* of G. If V_1 is a non-empty subset of the vertex set of G, then the subgraph *induced* by V_1 is the graph having vertex set V_1 in which two vertices are adjacent if and only if they are adjacent in G.

An *elementary subdivision* of a non-empty graph G is a graph obtained from G by removing some edges $e = uv$ and adding a new vertex w and two new edges uw and vw. A *subdivision* of G is a graph obtained from G by a succession of elementary subdivisions (including the possibility of none). Two graphs G and H are *homeomorphic* if they have isomorphic subdivisions.

A *complete graph* is a simple graph whose vertices are pairwise adjacent. It is denoted as K_n if it has n vertices. The complement of the complete graph with n vertices is denoted as $\overline{K_n}$ where $E(\overline{K_n}) = \emptyset$.

A *bipartite graph* is a graph whose vertex set can be partitioned into two independent sets called *partite sets* V_1 and V_2, such that each edge joins a vertex of V_1 to a vertex of V_2. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different

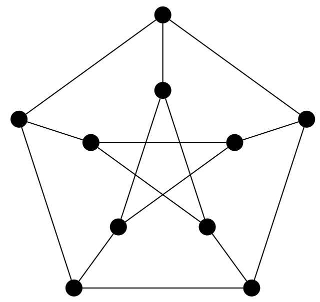


Figure 1.1: The Petersen graph $P(5,2)$

partite sets. When the two sets have m and n vertices respectively, then the complete bipartite graph is denoted as $K_{m,n}$.

A graph G is *k-partite*, $k \geq 1$, if $V(G)$ can be partitioned into k partite sets $V_1, V_2, ..., V_k$ such that each element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. A *complete k-partite graph* is a simple graph with partite sets $V_1, V_2, ..., V_k$ such that for any two vertices $u \in V_i$ and $v \in V_j$, vertex u is adjacent to vertex v if and only if $i \neq j$. If $|V_i| = n_i$, then this graph is denoted as $K_{n_1,n_2,...,n_k}$.

Let G_1 and G_2 be two graphs. By the *join* of G_1 and G_2, denoted as G_1+G_2, we mean the graph with $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1), y \in V(G_2)\}$.

Let n and k be two integers such that $1 \leq k \leq n-1$. The *generalized Petersen graph* $P(n,k)$ is the graph with vertex-set $\{u_i, v_i : i = 0, 1, ..., n-1\}$ and edge-set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : i = 0, 1, ..., n-1$ with subscripts reduced modulo $n\}$. The classical Petersen graph $P(5,2)$ is depicted in Figure 1.1.

Chapter 2

Crossing Number

2.1 Introduction

The study of crossing numbers began during the Second World War with Paul Turán's *brick factory problem* that dates back as far as 1944. During the Second World War, Turán was forced to work in a brick factory, using hand-pulled carts that ran on tracks to move bricks from kilns to stores. When tracks crossed, several bricks fell from the carts and had to be replaced by hand.

So, a question arises, see [39]: what are the minimum number of crossings of these tracks from kilns to stores? The problem, when formulated into graph theory, is to find the crossing number of the complete bipartite graph $K_{m,n}$.

In October 1952, Turán communicated the brick factory problem to other mathematicians during his first visit to Poland. Solutions were then published by topologist Zarankiewicz in 1954, see [44]. Years later, in 1968, Guy [12] pointed out an error in Zarankiewicz's proof and hence his formula yields only an upper bound for the minimum number of crossings in complete bipartite graphs. Many mathematicians have investigated Zarankiewicz's conjecture on the crossing number of complete bipartite graph. To date, in general, the long standing Zarankiewicz's conjecture has not been proved or disproved.

2.2 Crossing Number of Some Families of Graphs

The investigation on the crossing number of graphs is a very difficult problem. Despite much effort by many people, exact formulas of crossing numbers are known only for relatively few graphs. To date, mathematicians still do not know the exact value of the crossing numbers for basic graphs such as the complete graph and the complete bipartite graph. In fact, finding the crossing number of a given graph is a difficult task. In [10], Garey and Johnson have shown that the problem of determining the crossing numbers of graphs is NP-complete.

This section highlights some recent developments of crossing numbers of graphs. In addition, some known good drawings of these families of graphs are also given.

2.2.1 Complete Bipartite Graphs

In [44], Zarankiewicz illustrated a drawing of complete bipartite graph $K_{m,n}$ with $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings. The description of such drawing is quite simple:

Divide the m vertices into two sets of equal (or nearly equal) sizes and place the two sets equally spaced on the x-axis on either side of the origin. Do the same for the n vertices, placing them on the y-axis, and then join the appropriate pairs of vertices by straight-line segments.

Figure 2.1 illustrates a drawing for $K_{4,5}$ with 8 crossings.

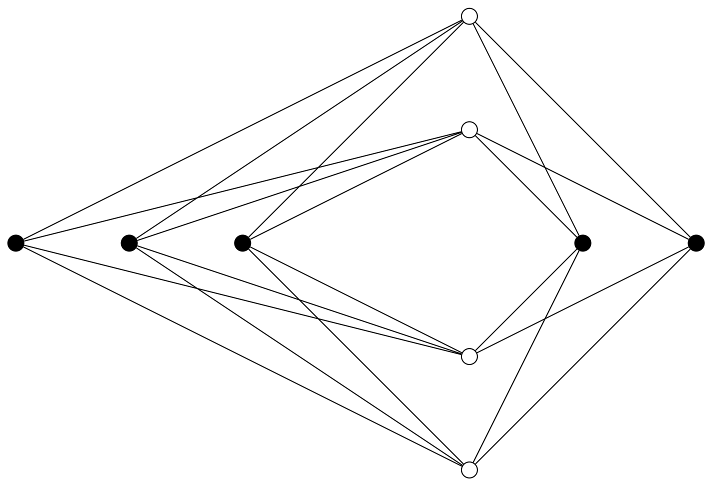


Figure 2.1: A drawing of $K_{4,5}$ with 8 crossings

Zarankiewicz [44] claimed that

$$cr(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor = Z(m,n) \qquad (2.1)$$

In 1968, Guy [12] showed that the proof given by Zarankiewicz was invalid but equation (2.1) remains undecided. Thus, equation (2.1) is known as

Zarankiewicz's Conjecture. However, some partial results are known. To date, it has been proved that the conjecture is true for $\min\{m,n\} \leq 6$, by Kleitman [20], and for the special cases $2 \leq m \leq 8$, and $7 \leq n \leq 10$, by Woodall [42].

Kleitman [20] also obtained the following lower bound of $cr(K_{m,n})$ for $m \geq 5$.

$$cr(K_{m,n}) \geq \frac{1}{5}m(m-1)\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor \tag{2.2}$$

In 2003, Nahas [32] improved the above lower bound for sufficiently large m and n, and stated the new bound as follows:

$$cr(K_{m,n}) \geq \frac{1}{5}m(m-1)\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 9.9 \times 10^{-6}m^2n^2 \tag{2.3}$$

In 2006, Klerk, Maharry, Pasechnik, Richter and Salazar, see [21], showed that, for $m \geq 9$, $\lim_{n\to\infty} \frac{cr(K_{m,n})}{Z(m,n)} \geq \alpha \frac{m}{m-1}$ where $\alpha = 0.83$. The authors in [22] then strengthened the result by improving α to 0.8594.

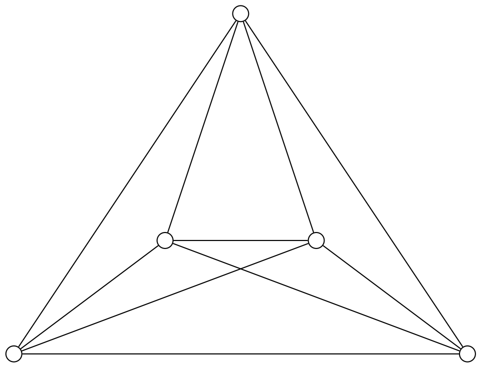
2.2.2 Complete Graphs

In 1960, Guy [11] obtained an upper bound on the crossing number for the complete graph K_n.

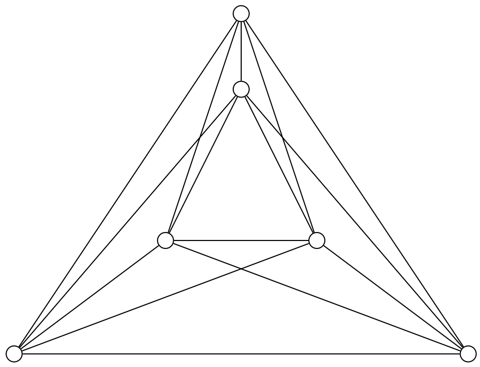
$$cr\,(K_n) \leq \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor \tag{2.4}$$

.

Figure 2.2 shows drawings of complete graphs K_5 and K_6 with one and three crossings respectively.



$(a)\ K_5$



$(b)\ K_6$

Figure 2.2: Complete graphs on 5 and 6 vertices

The inequality (2.4) may be written as:

$$cr(K_n) \leq \tfrac{1}{64}n(n-2)^2(n-4), \qquad \text{for } n \text{ even;}$$

$$cr(K_n) \leq \tfrac{1}{64}(n-1)^2(n-3)^2, \qquad \text{for } n \text{ odd.}$$

He then conjectured that inequality (2.4) is true for all natural numbers n. Later in 1972, Guy [13] proved that the conjecture is true for $n \leq 10$. No improved upper bound on crossing number of complete graph in general has been published to date.

In 1997, Richter and Thomassen investigated the relationship between the crossing number of complete graphs and complete bipartite graphs, see [35]. Recently, in [33], Pan and Richter proved that Guy's conjecture is true for $n = 11, 12$, that is $cr(K_{11}) = 100$ and $cr(K_{12}) = 150$.

2.2.3 Complete Tripartite Graphs

As for the crossing number of the complete tripartite graph $K_{l,m,n}$, to date, there are only a few results for the case where both l and m are small values.

In the pioneering work, see [1], in 1986, Asano proved that $cr(K_{1,3,n}) = Z(4,n) + \lfloor \frac{n}{2} \rfloor$, and $cr(K_{2,3,n}) = Z(5,n) + n$. Figure 2.3 shows a drawing of complete tripartite graph $K_{1,3,5}$ with 10 crossings which contains a subgraph of $K_{4,5}$. Several years later, by assuming that Zarankiewicz's conjecture is true, it has been shown that $cr(K_{1,4,n}) = n(n-1)$ in [18], $cr(K_{1,6,n}) = Z(7,n) + 6\lfloor \frac{n}{2} \rfloor$ in [16] and that $cr(K_{1,8,n}) = Z(9,n) + 12\lfloor \frac{n}{2} \rfloor$ in [17].

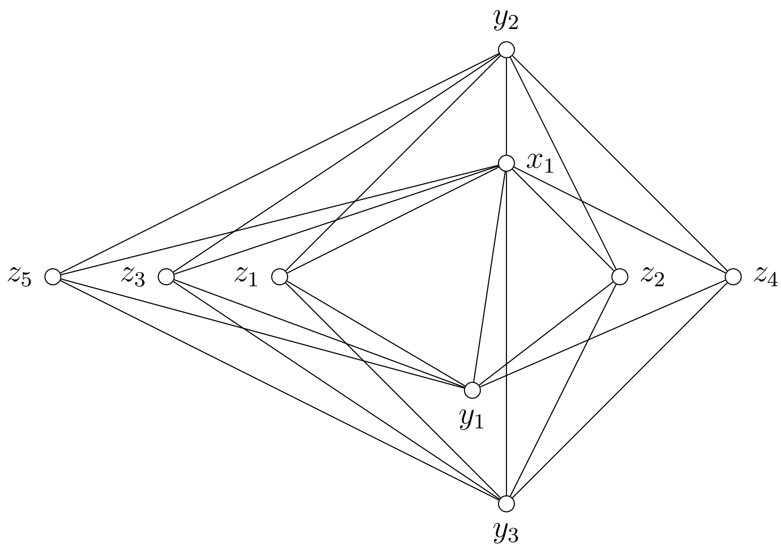


Figure 2.3: Complete tripartite graph $K_{1,3,5}$ with 10 crossings

In 2008, Zheng, Lin and Yang (see [45] and [46]) proved that, for $min\{m,n\} \geq$ 2,

$$cr(K_{1,m,n}) \leq Z(m+1,n+1) - \lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{n}{2} \rfloor;$$

$$cr(K_{2,m,n}) \leq Z(m+2, n+2) - mn.$$

The equalities hold for $min\{m, n\} \leq 3$ in both cases. Zheng, Lin and Yang in [45] then conjectured that it is true in general.

Later, in 2008, Ho [15] obtained a lower bound of the crossing number of complete tripartite graph, which is $cr(K_{1,m,n}) \geq cr(K_{m+1,n+1}) - \lfloor \frac{n}{m} \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \rfloor$ by constructing drawings of $K_{m+1,n+1}$ which preserve all the optimality from a good drawing of $K_{1,m,n}$. Ho [15] showed that the equality holds for $m = 4$.

In fact, due to the complexity of complete tripartite graphs, especially with bigger value of the order of each partite set, no further results on crossing number of complete tripartite graphs has been obtained.

2.2.4 Join of Graphs

In 2007, Kleš̃c̃ gave the exact values of crossing numbers for the join of two paths, the join of two cycles, and for the join of path and cycle, see [23].

Theorem 2.1 *[23]*

(a) $cr(P_m + P_n) = Z(m, n)$ *for min* $\{m, n\} \leq 6$;

(b) $cr(P_m + C_n) = Z(m, n) + 1$ *for any* $m \geq 2$, $n \geq 3$ *with* $min\{m, n\} \leq 6$;

(c) $cr(C_m + C_n) = Z(m, n) + 2$ *for any* $m \geq 3$, $n \geq 3$ *with* $min\{m, n\} \leq 6$.

Figure 2.4 shows an optimal drawing of $C_m + C_n$.

For the join of path and cycle with graphs of order 3, see Theorem 2.2. Note that $K_3 + P_n$ and $K_3 + C_n$ are isomorphic with $C_3 + P_n$ and $C_3 + C_n$

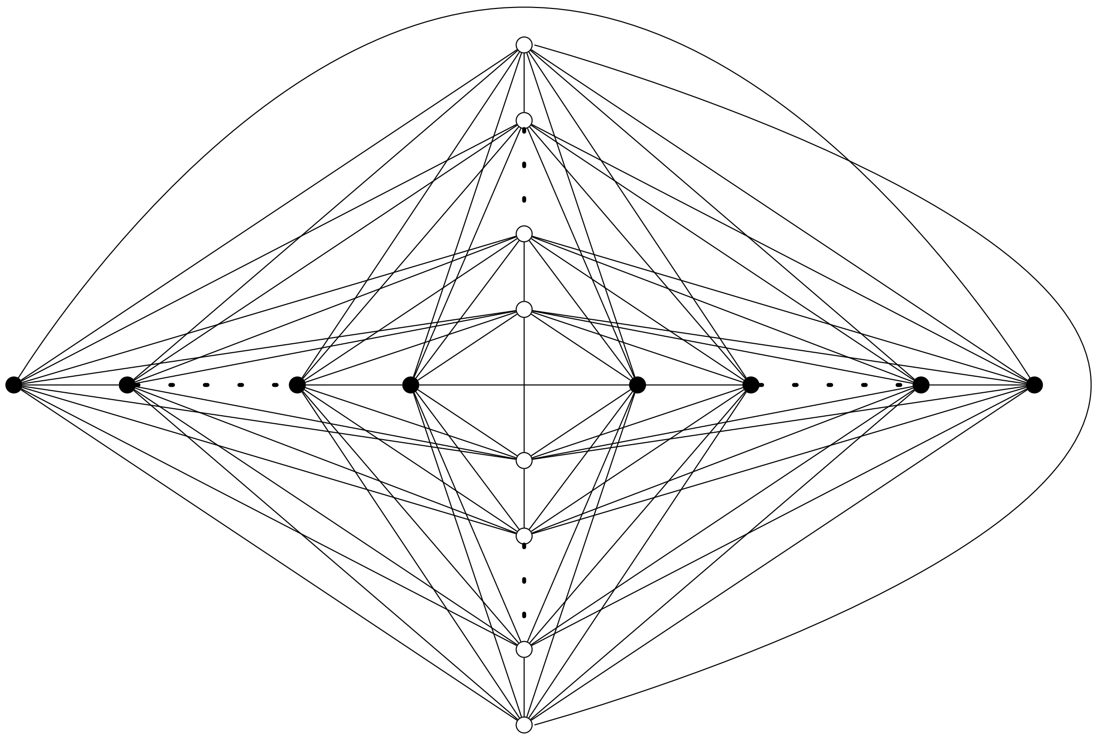


Figure 2.4: The graph $C_m + C_n$

respectively. Thus the crossing numbers of these graphs follows from Theorem 2.1.

Theorem 2.2 *[23] For* $n \geq 3, cr(3K_1 + P_n) = cr((K_1 \cup P_2) + P_n) = cr(3K_1 + C_n) = cr((K_1 \cup P_2) + C_n) = Z(3, n)$.

In [23], Kleš̃č extended the results in Theorem 2.1 to the join products $G + P_n$ and $G + C_n$ where G is a graph with order at most 4. Table 2.1 shows a summary of the crossing numbers for the join products of path, cycle and n isolated vertices with all graphs G of order four, see [23] and [25]. Both connected and disconnected graphs of G are taken into account. Figure 2.5

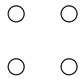
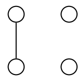
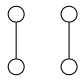
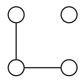
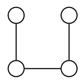
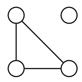
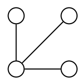
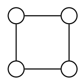
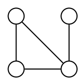
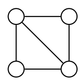
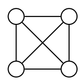
G	$cr(G + nK_1)$		$cr(G + P_n)$		$cr(G + C_n)$	
	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 3$
	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 3$
	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 3$
	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$	$Z(4, n) + 1$	$n \geq 3$
	$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$	$Z(4, n) + 1$	$n \geq 4$
	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor + 2$	$n \geq 3$
	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor + 2$	$n \geq 3$
	$Z(4, n)$	$n \geq 1$	$Z(4, n) + 1$	$n \geq 2$	$Z(4, n) + 2$	$n \geq 3$
	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor + 2$	$n \geq 3$
	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor + 1$	$n \geq 2$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor + 3$	$n \geq 3$
	$Z(4, n) + n$	$n \geq 1$	$Z(4, n) + n + 1$	$n \geq 2$	$Z(4, n) + n + 4$	$n \geq 3$

Table 2.1: Summary of crossing numbers for $G + nK_1$, $G + P_n$ and $G + C_n$.

shows a good drawing of $K_4 + P_n$ with $Z(4, n) + n + 1$ crossings.

There were only known exact value of crossing number for join of several particular graphs of order 5 and order 6 with n isolated vertices as well as with P_n and C_n, see [24], [26] and [27].

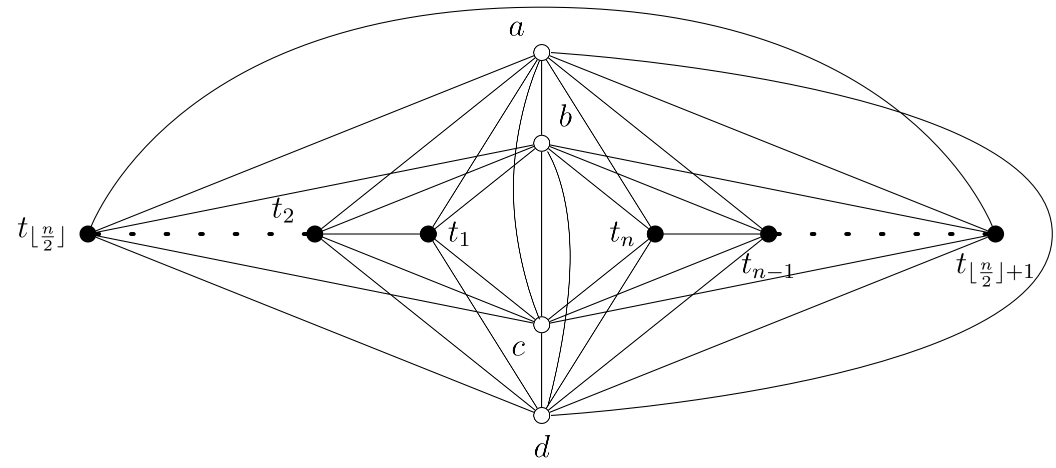


Figure 2.5: The graph $K_4 + P_n$

2.2.5 Generalized Petersen Graphs

The determination of the crossing number for the generalized Petersen graph was first studied in 1981.

Theorem 2.3 *[7]*

$$cr(P(m,2)) = \begin{cases} 0 & if\ m = 3\ or\ m\ is\ even \\ 2 & if\ m = 5 \\ 3 & if\ m \geq 7\ is\ odd. \end{cases}$$

Note that the graphs $P(2n+1,n)$, $P(2n+1,n+1)$ and $P(2n+1,2n-1)$ are all isomorphic to $P(2n+1,2)$.

Earlier in 1967, Guy and Harary [14] have established the crossing number of all Möbius ladders to be 1 and since $P(2k,k)$ is a subdivision of Möbius ladder M_{2k}, it follows that for $k \geq 3$, $cr(P(2k,k)) = 1$.

In 1986, Fiorini [8] proved that $P(8,3)$ has crossing number 4 and claimed that $P(10,3)$ also has crossing number 4. Years later, in [31], McQuillan and Richter gave a short proof of the first claim and showed that the second claim is false. Then, again, Richter and Sazalar [34] found a gap in some of Fiorini's proofs which invalidated his principal results about $cr(P(n,3))$. However, Fiorini has shown the following.

Theorem 2.4 *[8]*

(i) $cr(P(8,3)) = 4$

(ii) $cr(P(9,3)) = 2$

(iii) $cr(P(12,3)) = 4$

In 1997, Saražin [37] proved the following.

Theorem 2.5 *[37]* $cr(P(10,4)) = 4$.

In 2002, Richter and Sazalar [34] determined the crossing number of $P(3k+h,3)$.

Theorem 2.6 *[34]*

$$cr(P(3k+h,3)) = \begin{cases} k+h & if\ h \in \{0,2\} \\ k+3 & if\ h = 1. \end{cases}$$

for each $k \geq 3$*, with the single exception of* $P(9,3)$ *where* $cr(P(9,3)) = 2$*.*

In 2003, Fiorini and Gausi [9] determined the crossing number of $P(3k, k)$ (see [8] and [34]) and showed that $cr(P(3k, k)) = k$. However, Chia and Lee [4] pointed out a flaw in the proof in [9], and provided a correct proof.

In 2013, Yang, Zheng and Xu [43] proved that $cr(P(10, 3)) = 6$.

To date, the crossing numbers of other generalized Petersen graphs remain unknown.

2.3 Planarity Criterion

Planarity being such a fundamental property, the problem of deciding whether a given graph is planar is remarkably important. There is a simple formula relating the number of vertices, edges, and faces in a connected plane graph. It was first established for polyhedral graphs by Euler (1752), and is known as *Euler's Formula*.

Theorem 2.7 *(Euler's Formula) Let G be a connected plane graph with p vertices, q edges and f faces. Then*

$$p - q + f = 2.$$

Euler's formula can easily be extended to disconnected graph. For plane graphs in general, we have the following.

Corollary 2.8 *Let G be a plane graph with p vertices, q edges, f faces and k components. Then*

$$p - q + f = 1 + k.$$

Note that in any graph, the sum of the vertex degrees is equal to twice the number of edges. This is called the *Hand-shaking Lemma*.

An immediate consequence is the following.

Corollary 2.9 *Let G be a planar graph with q edges, f faces and girth g. Then*

$$2q \geq gf.$$

By Theorem 2.7 and Collorary 2.9, we have the following.

Corollary 2.10 *Let G be a simple connected planar graph with p vertices and q edges, $p \geq 3$. Then $q \leq 3p - 6$. Furthermore, $q = 3p - 6$ if and only if G is a maximal planar graph.*

For the case where a plane graph G contains no triangle, the girth of G is at least 4. Hence by Theorem 2.7 and Collorary 2.9 again, we have the following theorem.

Theorem 2.11 *Suppose G is a planar graph containing no triangle, having p vertices and q edges, $p \geq 3$. Then*

$$q \leq 2p - 4.$$

An important consequence of Colloraries 2.9 and 2.10 is the following.

Theorem 2.12 *$K_{3,3}$ and K_5 are non-planar.*

$K_{3,3}$ and K_5 are important in determining the planarity of a graph. The most well known characterization of planar graphs is the Kuratowski's Theorem.

Theorem 2.13 *(Kuratowski's Theorem). A graph is planar if and only if it contains no subgraph which is a subdividion of K_5 or $K_{3,3}$.*

2.4 Crossing-critical Graphs

2.4.1 Definitions

A graph G is said to be *crossing-critical* if deleting any edge decreases its crossing number on the plane, that is $cr(G-e) < cr(G)$ for every edge e of G.

Let $F_n(cr)$ denote the family of crossing-critical graphs G, where $cr(G) > n$ and $cr(G-e) \leq n$ for every edge e of G and no vertex of G has degree two. For every graph H, $cr(H) \leq n$ if and only if H does not contain a subgraph homeomorphic to a member of $F_n(cr)$. Clearly, $F_n(cr)$ is a family of some "forbidden subgraphs" of graph H with $cr(H) \leq n$.

2.4.2 Some Ideas and Examples

From Theorem 2.13, we conclude that the members of $F_0(cr)$ are K_5 and $K_{3,3}$. Now, a question arises: For every $n \geq 1$, is $F_n(cr)$ a finite set?

Note that there exists infinitely many 2-connected graphs of $F_n(cr)$ for every $n \geq 1$, see Figure 2.6. Then, the following question is of some interest:

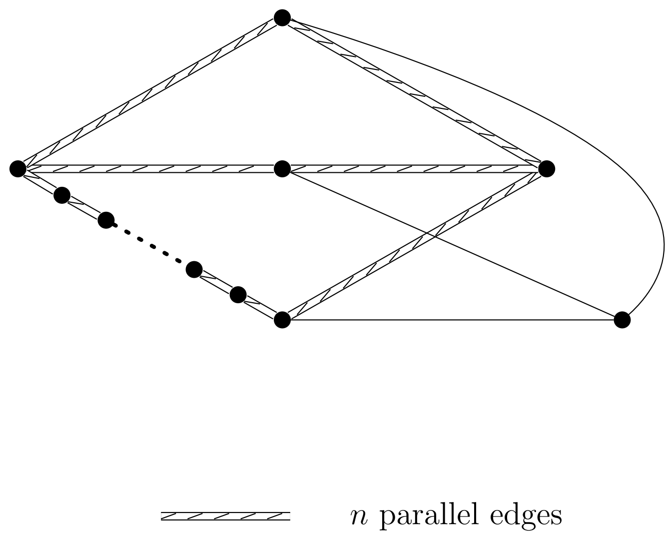


Figure 2.6:

Has $F_n(cr)$ infinitely many 3-connected members?

The problem was first studied by Širáň [38] in 1984.

Theorem 2.14 *[38] For any $n \geq 3$, there is an infinite family of 3-connected crossing-critical graphs with crossing number n.*

Further, Širáň conjectured that, there are at most finitely many 3-connected graphs in $F_1(cr)$ where $cr(G) = 2$. Few years later, in contrast to this conjecture, Kochol [28] presented a construction of an infinite family of such graphs and extended the result to other integers.

Theorem 2.15 *[28] For any $n \geq 2$ there is an infinite family of 3-connected crossing-critical simple graphs with crossing number n.*

Chapter 3

Skewness of Graphs

3.1 Introduction

Similar to crossing number, skewness of graphs could be thought of as a measure of how non-planar a graph is. Graph operations such as vertex deletion, vertex splitting and edge deletion have been widely studied (see [29]) in making a non-planar graph into a planar one. While it is hard to find the crossing numbers for the complete graphs, the complete bipartite graphs and the join of graphs, the determination of skewness of these graphs turns out to be not too difficult (see [6] and [29]).

In this section, we are more concerned with the edge deletion operation, a topic that has been studied intensively and that has many applications in VSLI circuit routing. It is also associated with the Maximum Planar Subgraph Problem (see [29] for more details).

In [29], Liebers suggested the following definition on skewness of graphs.

Definition 3.1 *(maximum planar subgraph, skewness) If a graph $G' = (V, E')$ is a planar subgraph of a graph $G = (V, E)$ such that there is no planar subgraph $G'' = (V, E'')$ of G with $|E''| > |E'|$, then G' is called a maximum planar subgraph of G, and the number of deleted edges, $|E| - |E'|$ is called the skewness of G.*

In short, the skewness of a graph G, denoted $sk(G)$, is the minimum number of edges in G whose deletion results in a planar graph. It is clear that $sk(G) \leq cr(G)$. [30] shows that the problem of determining the skewness of a graph is NP-complete.

3.2 $\pi - skewness$

In this section, we provide some elementary facts about skewness of graphs and introduce the definition of $\pi - skew$ graph.

Remark 3.2 *(i) $sk(G) = 0$ if and only if G is a planar graph.*

(ii) If two graphs G_1 and G_2 are homeomorphic, then $sk(G_1) = sk(G_2)$.

(iii) If H is a subgraph of a graph G, then $sk(H) \leq sk(G)$.

(iv) If there is a planar graph H which can be obtained by deleting r edges from the graph G, then $sk(G) \leq r$.

(v) $sk(G) = 1$ if $cr(G) = 1$.

Recalling that $cr(K_5) = 1$ and $cr(K_{3,3}) = 1$. By Remark 3.2 (v), we can conclude that $sk(K_5) = 1$ and $sk(K_{3,3}) = 1$.

Euler's formula (Theorem 2.7) together with the Hand-shaking Lemma (Corollary 2.9) yield the following.

Theorem 3.3 *Let G be a graph on p vertices, q edges and having girth g. If G is planar, then*

$$q \leq \tfrac{g}{g-2}(p-2).$$

Hence, we have the following.

Theorem 3.4 *Let G be a graph on p vertices, q edges and having girth g. Then*

$$sk(G) \geq q - \tfrac{g}{g-2}(p-2).$$

One natural thing to ask is: *When does equality holds?*

Definition 3.5 *Let G be a graph on p vertices, q edges and having girth g. Define*

$$\pi(G) = \lceil q - \tfrac{g}{g-2}(p-2) \rceil.$$

Note that the general form of this combinatorial invariant appeared in [19].

We say that the graph G is $\pi - skew$ if $sk(G) = \pi(G)$.

Theorem 3.6 *For any connected graph G, we have $sk(G) \geq \pi(G)$.*

3.3 Known Results

We now turn our attention to some known results on the skewness of some families of graphs.

Proposition 3.7 *Suppose G is the complete graph K_n. Then $sk(G) = \pi(G) = \binom{n-3}{2}$.*

Proposition 3.8 *[4] Let $m, n \geq 2$. Then $sk(K_{m,n}) = \pi(K_{m,n}) = (m-2)(n-2)$.*

Now, we give a short proof for $sk(K_{m,n}) = \pi(K_{m,n})$. By Theorem 3.6, we have $sk(K_{m,n}) \geq \pi(K_{m,n})$. It remains to prove that there exists a planar graph by removing $(m-2)(n-2)$ (which is equal to $\pi(K_{m,n})$) edges from $K_{m,n}$. Figure 3.1 shows a planar subgraph of $K_{m,n}$ after deleting $(m-2)(n-2)$ edges. Hence, $K_{m,n}$ is $\pi - skew$. Note that a complete bipartite graph is of girth 4.

In [4], Chia and Lee gave some partial results on skewness of complete tripartite graph $K_{m,n,t}$.

Proposition 3.9 *[4] Suppose $n \leq t$ and G is the graph $K_{1,n,t}$. Then $sk(G) = \pi(G) + t - 1$.*

Proposition 3.10 *[4] Suppose m, n and t are integers such that $2 \leq m \leq n \leq t$ where $t \leq m + n - 2$. Let G be the complete tripartite graph $K_{m,n,t}$. Then $sk(G) = \pi(G)$.*

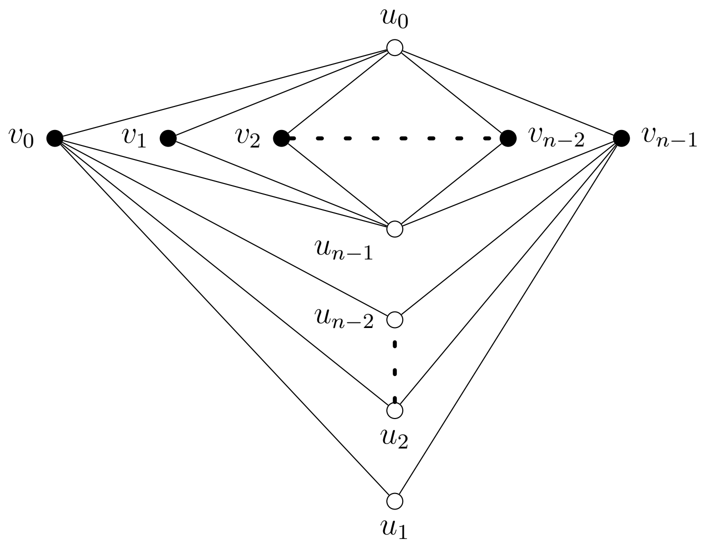


Figure 3.1: A planar graph obtained by removing $(m-2)(n-2)$ edges from $K_{m,n}$.

Also, Chia and Lee conjectured that $sk(K_{m,n,m+n-1}) = \pi(K_{m,n,m+n-1}) + 1$ for $2 \leq m \leq n$. Later, in Chapter 4, we shall determine completely the skewness of complete tripartite graph.

Corollary 3.11 *[4] Suppose G is the complete $n-$partite graph $K_{2,2,\ldots,2}$ where $n \geq 2$. Then $sk(G) = \pi(G)$.*

A wealth of literature have gone into finding the crossing number of some generalized Petersen graphs. However, there is not much research on the skewness of the generalized Petersen graphs.

Some results concerning the skewness of generalized Petersen graphs are given below.

Theorem 3.12 *(a) [3] $sk(P(3k,k)) = \lceil \frac{k}{2} \rceil + 1$ if $k \geq 4$.*

(b) [4] $sk(P(2k,k)) = 1$ where $k \geq 3$.

(c) [4] $sk(P(k,2)) = \begin{cases} 2 & if\ k \geq 5\ is\ odd \\ 0 & otherwise. \end{cases}$

(d) [5] $sk(P(4k,k)) = k+1$ if $k \geq 4$ is even.

(e) [5] $sk(P(12,3)) = 3$.

(f) [5] $sk(P(4k,k)) \leq k+2$ if $k \geq 5$ is odd.

It was conjectured in [5] that $sk(P(4k,k)) = k+2$ if $k \geq 5$ is odd and $cr(P(4k,k)) = 2k+1$ if $k \geq 4$.

3.4 Skewness-critical Graphs

A graph G is $skewness-critical$ if deleting any edge decreases its skewness on the plane, that is $sk(G-e) < sk(G)$ for every edge e of G.

We ask: For each natural number $n \geq 1$, does there exist a 3-connected skewness-critical graph G such that $sk(G) = n$?

Theorem 3.13 *For each natural number $n \geq 1$, there exists a 3-connected skewness-critical graph G such that $sk(G) = n$.*

Proof: Let G be the complete bipartite graph $K_{3,m}$ with $m \geq 3$. Note that $K_{3,m}$ is edge-transitive. Since $sk(K_{3,m}) = m-2$, the result follows. $\square$

Chapter 4

Skewness of the Join of Graphs

4.1 Introduction

Suppose G is a graph with p vertices and having girth g. If G is π-skew, then G contains a planar subgraph H with p vertices and $\lceil \frac{g}{g-2}(p-2) \rceil$ edges. In particular, if G is π-skew and is of girth 3, then H is a spanning maximal planar subgraph of G. Besides complete graph and complete bipartite graph, another example of a π-skew graph is given by the $n-$cube whose skewness has been determined in [6].

Theorem 4.1 *Suppose that G is a complete graph or a complete bipartite graph. Then G is π-skew.*

In the rest of the sections, we determine completely the skewness of the complete $k-$partite graphs for $k = 3, 4$. This is done by first establishing some results concerning the skewness for the join of two graphs and then applying the results on the complete multipartite graphs.

4.2 Join of Graphs

Lemma 4.2 *For each $i = 1, 2$, let G_i be a connected graph on p_i vertices. Then*

$$sk(G_1 + G_2) \geq sk(G_1) + sk(G_2) + (p_1 - 2)(p_2 - 2).$$

Proof: Note that $G_1 + G_2$ contains a subgraph K_{p_1,p_2} whose skewness is $\pi(K_{p_1,p_2}) = (p_1 - 2)(p_2 - 2)$ by Theorem 4.1. In order to obtain a spanning planar subgraph of $G_1 + G_2$, we need to delete at least $sk(G_1)$ edges from G_1, at least $sk(G_2)$ edges from G_2, and at least $(p_1 - 2)(p_2 - 2)$ from K_{p_1,p_2}. But this means that $sk(G_1 + G_2) \geq sk(G_1) + sk(G_2) + (p_1 - 2)(p_2 - 2)$, and the lemma follows. □

A direct consequence is the following. Let $\overline{K_n}$ denote the complement of the complete graph with n vertices.

Corollary 4.3 *Let G be a connected graph on p vertices. Then*

$$sk(G + \overline{K_s}) \geq sk(G) + (p - 2)(s - 2).$$

We shall now introduce two techniques, which we call *vertex triangulation* of a face and *building faces* on a given edge. These techniques will be used throughout the rest of the sections.

Definition 4.4 *Let H denote a plane graph. (i) Let F denote a face of H. By a vertex triangulation of F, we mean inserting a new vertex x inside F and joining x to each vertex of F. (ii) Let uv be an edge of H. By building a fetch on uv with a new vertex x we mean joining x to u and v.*

Theorem 4.5 *Let G be a connected graph on p vertices with girth 3. Suppose $sk(G) = \pi(G)$. Then for any positive integer s,*

$$sk(G + \overline{K_s}) = \begin{cases} \pi(G + \overline{K_s}) & if\ s \leq 2p - 4 \\ \\ \pi(G + \overline{K_s}) + s - 2p + 4 & otherwise. \end{cases}$$

Proof: Since G is of girth 3, $\pi(G) = |E(G)| - 3(p-2)$. Because $sk(G) = \pi(G)$, we have $|E(G)| - sk(G) = 3(p-2)$, and this means that there exists a maximal planar subgraph H of G with p vertices and $3(p-2)$ edges.

Draw H on the plane with no crossings. Note that H has $2p-4$ faces, and that each face is a triangle.

If $s \leq 2p-4$, we can add s new vertices, one inside each face of H, and join each new vertex v to the three vertices on the face containing v. The result is a maximal planar subgraph of $G + \overline{K_s}$ on $p + s$ vertices. This proves that $sk(G + \overline{K_s}) = \pi(G + \overline{K_s})$.

Hence assume that $s \geq 2p - 4$, and write $t = s - 2(p-2)$. We shall show that $sk(G + \overline{K_s}) = \pi(G + \overline{K_s}) + t$.

By Corollary 4.3, $sk(G + \overline{K_s}) \geq sk(G) + (p-2)(s-2)$, and it is routine to check that $sk(G) + (p-2)(s-2) = \pi(G + \overline{K_s}) + t$.

To show that $sk(G+\overline{K_s}) \leq \pi(G+\overline{K_s})+t$, we do the following.

Consider the plane subgraph H of G (from the second paragraph) and vertex-triangulate each of the $2p-4$ faces of H. We then build fetches on a fixed edge of H with each of the remaining $t = s-2p+4$ vertices. The resulting graph J is planar, and clearly has $p+s$ vertices and $3(2p-4)+3(p-2)+2t = 9(p-2)+2t = 5(p-2)+2s$ edges, which means that $sk(G+\overline{K_s}) \leq \pi(G+\overline{K_s})+t$.

This completes the proof. □

Theorem 4.5 can be used to generate infinitely many $\pi-$skew graphs recursively. Since $G+\overline{K_s}$ is of girth 3 if $s \leq 2p-4$, it follows from Theorem 4.5 that $(G+\overline{K_s})+\overline{K_r}$ is $\pi-$skew if $r \leq 2(p+s)-4$. This process can be repeated to generate $\pi-$skew graphs of the form $G+K_{m_1,m_2,\ldots,m_r}$ where $m_1 \leq 2p-4$ and $m_i \leq 2(p+m_1+\ldots+m_{i-1})-4$ for each $i = 2,\ldots,r$.

One may ask: *To what extent can the conditions on G in Theorem 4.5 be relaxed and we still get $\pi-$skewness in $G+\overline{K_s}$?* We do not know the answer in general. Some special cases are considered in the next section.

4.3 Complete Tripartite Graphs

If a graph G with p vertices has girth 4, then it is not true in general that $sk(G) = \pi(G)$ implies that $sk(G+\overline{K_s}) = \pi(G+\overline{K_s})$ if $s \leq p-2$. For example, take G to be the 4-cycle and $s = 1$; then $sk(G+\overline{K_s}) = 0$, but $\pi(G+\overline{K_s}) < 0$.

However, for the special case when G is the complete bipartite graph $K_{m,n}$, where $2 \leq m \leq n$, it has been shown in [4] that $sk(G+\overline{K_s}) = \pi(G+\overline{K_s})$ if

$n \leq s \leq m+n-2$. Note that $K_{m,n} + \overline{K_s} = K_{m,n,s}$. Also, it has been shown in [4] that, if $H \cong K_{m,n,m+n-1}$, then $\pi(H) \leq sk(H) \leq \pi(H)+1$. Also, it was believed that $sk(H) = \pi(H)+1$. We shall now determine completely the skewness of $K_{m,n,r}$ with the use of Theorem 4.5. Here, a much easier proof (than the one given in [4]) for the case $n \leq s \leq m+n-2$ is also provided.

Theorem 4.6 *Let G be the complete tripartite graph $K_{m,n,r}$, where $m \leq n \leq r$. Then*

$$sk(G) = \begin{cases} 0 & if \quad m = 1 = n \\ \pi(G)+r-1 & if \quad m=1, n \geq 2 \\ \pi(G) & if \quad 2 \leq m \leq n \leq r \leq m+n-2 \\ \pi(G)+r+2-m-n & if \quad 2 \leq m \leq n \leq r \text{ and } r > m+n-2. \end{cases}$$

Proof: If $m = 1 = n$, then G is a planar graph, and hence $sk(G) = 0$.

It has been shown in [4] that $sk(K_{1,n,r}) = (n-1)(r-2)$. Hence $sk(K_{1,n,r}) = \pi(K_{1,n,r}) + r - 1$.

Hence, assume that $m \geq 2$. Recall that G is isomorphic to $K_{m,n} + \overline{K_r}$ and that $sk(K_{m,n}) = (m-2)(n-2)$. Hence there is a planar spanning subgraph H of $K_{m,n}$ obtained by deleting $(m-2)(n-2)$ edges from $K_{m,n}$.

Since H has $m+n$ vertices and $2(m+n-2)$ edges, it has $m+n-2$ faces, where each face is of length 4. Draw H on the plane with $m+n-2$ faces.

Assume first that $r > m+n-2$, and write $r = m+n-2+t$ for some integer $t > 0$. Then vertex-triangulate each face of H and follow this by building t fetches on a fixed edge of H. The resulting planar graph J is a spanning subgraph of G, and $|E(J)| = 2(m+n-2) + 4(m+n-2) + 2(r-m-n+2) = 4(m+n-2) + 2r$. This means that $sk(G) \leq |E(G)| - |E(J)|$. Since $\pi(G) = |E(G)| - 3(m+n+r-2)$, we have $sk(G) \leq \pi(G) + r + 2 - m - n$.

On the other hand, by Corollary 4.3, $sk(G) \geq (m-2)(n-2) + (m+n-2)(r-2)$. Clearly, $(m-2)(n-2)+(m+n-2)(r-2) = \pi(G)+r+2-m-n$. This proves that $sk(G) = \pi(G) + r + 2 - m - n$ in this case.

Next, assume that $r \leq m+n-2$. To show that $sk(G) = \pi(G)$, we just need to show that there is a spanning maximal planar subgraph M of G. To do this, we shall construct a plane graph $H_{m,n,r}$ on $m+n$ vertices having r faces so that $H_{m,n,r}$ is a subgraph of $K_{m,n}$. Then, by vertex-triangulating every face of $H_{m,n,r}$, we obtain the required planar graph M.

For this purpose, let $V(H_{m,n,r}) = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$, where $s = n-m$ and $t = r-n$ for some non-negative integers s and t. The edges of $H_{m,n,r}$ are as follows.

(i) $u_m v_i$ is an edge for all $i = 1, 2, ..., n$.

(ii) $v_n u_i$ is an edge for all $i = 1, 2, ..., m$.

(iii) $u_i v_i$ is an edge for all $i = 1, 2, ..., m-1$.

(iv) $u_{m-1} v_i$ is an edge for every $m-1 \leq i \leq m-1+s$.

(v) $u_i v_{i-1}$ is an edge for every $m-t \leq i \leq m-1$ if $t \geq 1$.

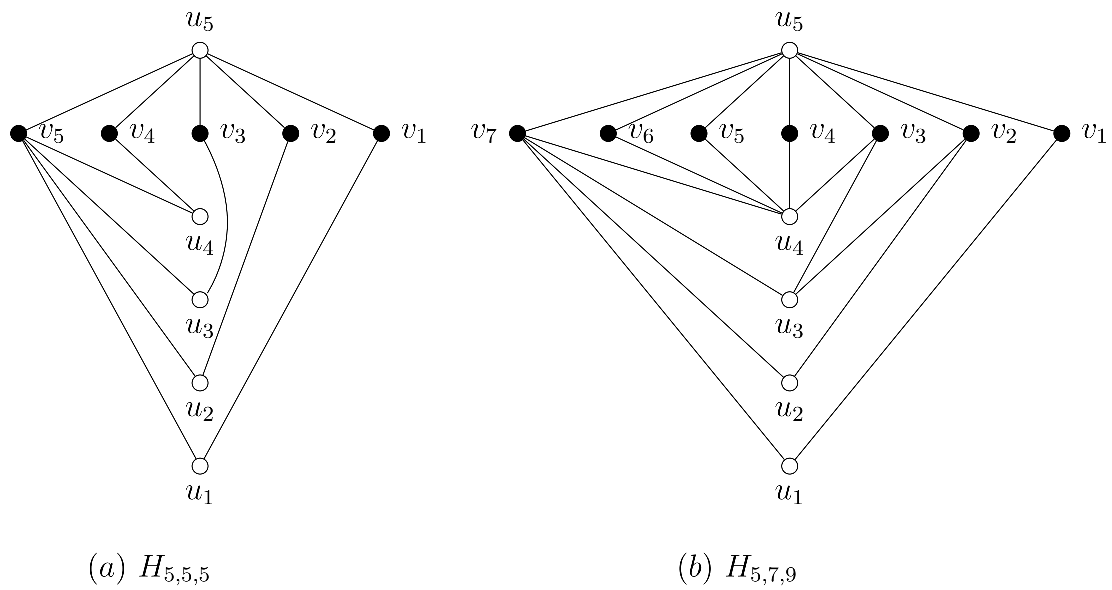


Figure 4.1: $H_{m,n,r}$ with r faces

The graphs $H_{5,5,5}$ and $H_{5,7,9}$ are depicted in Figure 4.1 (a) and (b), respectively. Note that the numbers of vertices and edges in $H_{m,n,r}$ are $m+n$ and $m+n+r-2$, respectively. Hence the number of faces in $H_{m,n,r}$ is r (by Euler's polyhedron formula). This completes the proof. $\square$

4.4 Complete 4-partite Graphs and More

We now apply the results developed in the previous sections to determine completely the skewness of complete 4-partite graphs plus some other more general results concerning $\pi-$skew graphs. Let G denote the complete $r-$partite graph $K_{m_1,m_2,\ldots,m_r}$. Then $G+\overline{K_s}$ is the complete $(r+1)-$partite graph $K_{m_1,m_2,\ldots,m_r,m_s}$. An immediate consequence of Theorem 4.5 is the following.

Corollary 4.7 *Suppose that the complete r-partite graph* $K_{m_1,m_2,\ldots,m_r}$ *is* $\pi-$ *skew, where* $r \geq 3$*. Then so is the complete* $(r+1)$*-partite graph* $K_{m_1,m_2,\ldots,m_r,m}$ *for any positive integer* $m \leq 2(m_1 + m_2 + \ldots + m_r) - 4$*.*

In [4], it was shown that, if G is the complete $r-$partite graph $K_{2,2,\ldots,2}$, where $r \geq 2$, then G is $\pi-$skew. The following result generalizes this.

Corollary 4.8 *Suppose* G *is the complete* $r-$*partite graph* $K_{n,n,\ldots,n}$ *where* $n \geq 1$ *and* $r \geq 2$*. Then* G *is* $\pi-$*skew.*

Proof: When $n = 1$, G is the complete graph K_r, and hence is $\pi-$skew. So we may assume that $n \geq 2$.

When $r = 2$, $sk(G) = \pi(G)$ by Theorem 4.1. So we may assume that $r \geq 3$.

When $n = 2$ and $r = 3$, we see that G is a maximal planar graph, and hence is $\pi-$skew. By Corollary 4.7 and by induction on r, it follows that, if G is the complete $r-$partite graph $K_{2,2,\ldots,2}$, then $sk(G) = \pi(G)$.

So we may assume that $n \geq 3$. By Theorems 4.1 and 4.6, $K_{n,n}$ and $K_{n,n,n}$ are $\pi-$skew. By Corollary 4.7 and by induction on r, it follows that $K_{n,n,\ldots,n}$ is $\pi-$skew. $\square$

By Corollary 4.8 and Theorem 4.5, we have the following.

Corollary 4.9 *Suppose that* G *is the complete* $(r+1)-$*partite graph* $K_{n,n,\ldots,n,m}$ *where* $n \geq 1$ *and* $r \geq 3$*. Then*

$$sk(G)=\begin{cases} \pi(G) & if\ m \le 2nr-4 \\ \\ \pi(G)+m-2nr+4 & otherwise. \end{cases}$$

Therefore it follows directly from Corollary 4.9 that $sk(K_{1,1,1,m})$ is equal to 0 if $m \le 2$ and is equal to $m-2$ otherwise.

Proposition 4.10 *Let G be the complete 4-partite graph $K_{1,1,m,n}$ where $2 \le m \le n$. Then*

$$sk(G) = \begin{cases} \pi(G) & if\ n \le m+1 \\ \\ \pi(G)+n-m-1 & otherwise. \end{cases}$$

Proof: Note that $G \cong K_{1,1,m} + \overline{K_n}$ and that $K_{1,1,m}$ can be drawn on the plane with $m+1$ faces, where two of them are 3-faces and $m-1$ of them are 4-faces. If we do a vertex-triangulation on each face followed by building fetches on some fixed edge, the result follows. □

A natural question arises. What is the skewness of the complete $(r+2)-$partite graph $K_{1,\dots,1,m,n}$ (which can be written as $K_r + K_{m,n}$), where $r \ge 3$? A more general result is given below.

Theorem 4.11 *Let G be a connected graph on p vertices and with girth 3. Suppose $sk(G) = \pi(G)$ and $m \le n$.*

(i) Suppose $m \le 2p-4$. Then

$$sk(G+K_{m,n})=\begin{cases} \pi(G+K_{m,n}) & if\ n \leq 2(p+m-2) \\ \\ \pi(G+K_{m,n})+n-2(p+m-2) & otherwise. \end{cases}$$

(ii) Suppose $m > 2p-4$. Then

$$sk(G+K_{m,n})=\begin{cases} \pi(G+K_{m,n}) & if\ n \leq 4(p-2)+m \\ \\ \pi(G+K_{m,n})+n-m-4p+8 & otherwise. \end{cases}$$

Proof: (i) This follows directly from Theorem 4.5, since $G+K_{m,n} \cong (G+\overline{K_m})+\overline{K_n}$ and $G+\overline{K_m}$ is of girth 3 and is $\pi-$skew if $m \leq 2p-4$.

(ii) Suppose that $m > 2p-4$. By Theorem 4.5, $sk(G+\overline{K_m}) = \pi(G+\overline{K_m})+m-2p+4$. From the proof of Theorem 4.5, we know that there is a plane subgraph J of $G+\overline{K_m}$ with $p+m$ vertices and $5(p-2)+2m$ edges. Moreover, J has $4(p-2)+m$ faces, where $m-2p+4$ of them are 4-faces and the remaining $6p-12$ of them are triangles.

If $n \leq 4(p-2)+m$, then we vertex-triangulate each of the 4-faces of J first and then the triangles (if necessary) to get a maximal planar spanning subgraph of $G+K_{m,n}$. (Note that this is possible because $m-2(p-2) \leq m \leq n$.) This shows that $sk(G+K_{m,n}) = \pi(G+K_{m,n})$ in this case.

If $n > 4(p-2)+m$, then, to the resulting maximal planar subgraph obtained above, build fetches on a particular edge to get a planar subgraph with $p+m+n$ vertices and $7p+4m+2n-14$ edges. This shows that $sk(G+K_{m,n}) \leq \pi(G+K_{m,n})+n-m-4p+8$.

On the other hand, since $G + K_{m,n} \cong G + \overline{K_m} + \overline{K_n}$, by Corollary 4.3, $sk(G + K_{m,n}) \geq sk(G + \overline{K_m}) + (p + m - 2)(n - 2)$. Since $m > 2p - 4$, by Theorem 4.5, $sk(G + K_{m,n}) \geq \pi(G + \overline{K_m}) + m - 2p + 4 + (p + m - 2)(n - 2)$, and hence $sk(G + K_{m,n}) \geq \pi(G + K_{m,n}) + n - m - 4p + 8$.

This completes the proof. $\square$

Corollary 4.12 *Let G be the complete 4-partite graph $K_{1,m,n,r}$ where $2 \leq m \leq n \leq r$. Then*

$$sk(G) = \begin{cases} \pi(G) & \text{if } n \leq r \leq 2m + n - 1 \\ \\ \pi(G) + r - 2m - n + 1 & \text{otherwise.} \end{cases}$$

Proof: If $n \leq r \leq m + n - 2$, then $K_{m,n,r}$ is $\pi-$skew by Theorem 4.6. As such, by Theorem 4.5, $sk(K_{1,m,n,r}) = sk(K_{m,n,r} + \overline{K_1}) = \pi(K_{1,m,n,r})$.

Suppose that $r > m + n - 2$. Then $K_{1,m,n,r} = K_{1,m,n} + \overline{K_r}$. By Theorem 4.6, we have $sk(K_{1,m,n}) = (m - 1)(n - 2)$. In proving that $sk(K_{1,m,n}) = (m - 1)(n - 2)$, the authors in [4] showed (by construction) that there is a spanning planar subgraph J_0 of $K_{1,m,n}$ obtained by deleting $(m - 1)(n - 2)$ edges from $K_{1,m,n}$. It turns out that J_0 has $2m+n-1$ faces, where $2m$ of them are 3-faces and $n-1$ of them are 4-faces. For ease of reference, the graph J_0 is described here. Take a vertex x and join it to the vertices $v_1, v_3, ..., v_{2m-1}$ of the path $v_1v_2...v_{2m-1}$. Then join a new vertex y to all the vertices $v_1, v_2, ..., v_{2m-1}, x$. Now build fetches on the edge xy with $n - m$ vertices to get the graph J_0.

If $r \leq 2m+n-1$, then we may first use $n-1$ new vertices to vertex-triangulate each 4-face of J_0, and then use the remaining $r-n+1$ vertices to vertex-triangulate each 3-face of J_0. The resulting graph is a maximal planar graph on $1+m+n+r$ vertices. Hence $sk(G) = \pi(G)$ in this case.

Now consider the case $r > 2m+n-1$. First, we construct a maximal planar graph by vertex-triangulating each face of J_0 (using $2m+n-1$ new vertices), and then we build fetches on two adjacent vertices of J_0 using the remaining $r-2m-n+1$ new vertices. The resulting graph is planar, and clearly has $1+m+n+r$ vertices and $5m+4n+2r-4$ edges. This means that $sk(K_{1,m,n,r}) \leq |E(K_{1,m,n,r})| - (5m+4n+2r-4) = \pi(K_{1,m,n,r}) + r - 2m - n + 1$.

On the other hand, by Corollary 4.3 and Theorem 4.6, we have $sk(K_{1,m,n,r}) \geq sk(K_{1,m,n}) + (m+n-1)(r-2) = (m-1)(n-2) + (m+n-1)(r-2)$, and hence $sk(K_{1,m,n,r}) \geq \pi(K_{1,m,n,r}) + r - 2m - n + 1$.

This completes the proof. $\square$

We are now left with the last family of complete 4-partite graphs to consider.

Corollary 4.13 *Let G be the complete 4-partite graph $K_{m,n,r,s}$ where $2 \leq m \leq n \leq r \leq s$.*

(i) Suppose $r \leq m+n-2$. Then

$$sk(G) = \begin{cases} \pi(G) & if\ s \leq 2(m+n+r-2) \\ & \\ \pi(G) + s - 2(m+n+r-2) & if\ s > 2(m+n+r-2). \end{cases}$$

(ii) Suppose $r > m+n-2$. Then

$$sk(G) = \begin{cases} \pi(G) & if\ s \leq 3(m+n-2)+r \\ & \\ \pi(G) + s - 3(m+n-2) - r & if\ s > 3(m+n-2)+r. \end{cases}$$

Proof: (i) If $r \leq m+n-2$, then $sk(K_{m,n,r}) = \pi(K_{m,n,r})$, by Theorem 4.6. The result then follows as a consequence of Theorem 4.5.

(ii) Since $r > m+n-2$, we have $sk(K_{m,n,r}) = \pi(K_{m,n,r}) + r + 2 - m - n$ according to Theorem 4.6. We first assume that $s \leq 3(m+n-2)+r$.

From the proof of Theorem 4.6, we know that there is a spanning subgraph J of $K_{m,n,r}$ obtained by deleting $sk(K_{m,n,r})$ edges from it. Note that J has precisely $r-m-n+2$ 4-faces and $4(m+n-2)$ 3-faces. Since $m+n-2 < r \leq s \leq 3(m+n-2)+r$, we can vertex-triangulate the 4-faces (and then the 3-faces if necessary) to obtain a spanning maximal planar subgraph H_1 of $K_{m,n,r,s}$. This proves that $sk(G) = \pi(G)$ in this case.

Now assume that $s > 3(m+n-2)+r$. Then, to the maximal planar graph H_1, we build fetches on two adjacent vertices of H_1 with $s-3(m+n-2)-r$ new vertices. This means that $sk(K_{m,n,r,s}) \leq \pi(K_{m,n,r,s}) + s - 3(m+n-2) - r$.

On the other hand, by Corollary 4.3 and Theorem 4.6, we have $sk(K_{m,n,r,s}) \geq \pi(K_{m,n,r,s}) + s - 3(m+n-2) - r$. $\square$

4.5 Conclusion

We have determined completely the skewness of the complete $k-$partite graph for $k = 2, 3, 4$ and hence characterized all such graphs which are $\pi-$skew. The same techniques can probably be used to determine the skewness of complete $k-$partite graphs for $k \geq 5$. However when k gets larger, one might need to develop further techniques in order to reduce the number of cases that need to be considered and to overcome the large amount of computations involved.

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