A STUDY OF REAL HYPERSURFACES IN COMPLEX SPACE FORMS WITH CONDITION ON THE JACOBI OPERATOR

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ABSTRACT

In this thesis, we first review some basic concepts in Riemannian geometry and then give a detailed survey on real hypersurfaces in a non-flat complex space form. After this, some new results on characterizing real hypersurfaces in a non-flat complex space form in terms of R_ξ are presented.

The main results are proved in the last two chapters. We give a characterization for ruled real hypersurfaces in a non-flat complex space form by the condition $(\nabla_X R_\xi)\xi = 0$, for $X \in \Gamma(D)$. Then we prove the non-existence of real hypersurfaces with D-recurrent R_ξ in a non-flat complex space form. We also give some characterizations for totally η-umbilical real hypersurfaces in a non-flat complex space form in terms of ∇R_ξ. Then we obtain the non-existence of real hypersurfaces in a non-flat complex space form with Codazzi type R_ξ.

ABSTRAK

Tesis ini bermula dengan mengkaji semula konsep-konsep asas dalam geometri Riemannan dan kemudian memberi kaji tinjauan terperinci tentang hiperpermukaan nyata dalam bentuk ruangan kompleks tak-mendatar. Kemudian, beberapa hasil baru yang mencirikan hiperpermukaan nyata dalam bentuk ruangan kompleks tak-mendatar dalam sebutan R_ξ dibentangkan.

Hasil-hasil utama dibuktikan dalam dua bab akhir. Pencirian hiperpermukaan nyata tergaris dalam bentuk ruangan kompleks tak-mendatar melalui syarat $(\nabla_X R_\xi)\xi = 0$, untuk $X \in \Gamma(D)$ diberi. Kemudian dibuktikan tak kewujudan hiperpermukaan nyata dengan R_ξ bertalu-D dalam bentuk ruangan kompleks tak-mendatar. Diberi juga beberapa pencirian bagi hiperpermukaan nyata yang umbilik-η dalam bentuk ruangan kompleks tak-mendatar dalam sebutan ∇R_ξ. Kemudian diperolehi tak kewujudan hiperpermukaan nyata dalam bentuk ruangan kompleks tak-mendatar dengan R_ξ berjenis Codazzi.

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Contents

Chapter 1

Introduction

The study of real hypersurfaces of Kaehler manifolds has been an important subject in the geometry of submanifolds, especially when the ambient space is a complex space form. Complex space forms are one family of the simplest Kaehler manifolds, and they are the complex case analogues of real space forms. The complex structure of the ambient space induces an almost contact structure on its real hypersurfaces. The interaction between the almost contact structure and shape operator results in interesting properties of real hypersurfaces in complex space forms (for instance, see [24, 35, 38, 39, 40, 62]).

The Jacobi operator is an important symmetric tensor field of type $(1, 1)$ defined on a Riemannian manifold. It reflects the curvature of the manifold. For a real hypersurface in a non-flat complex space form, a typical Jacobi operator is the structure Jacobi operator. Classifications of real hypersurfaces in non-flat complex space forms under conditions of struture Jacobi operator is an active topic and a number of results on this topic have been achieved recently (for example, see [10, 17, 42, 50, 51, 52, 53, 55]). We also have obtained some new results on this topic (cf. [29]), which will be presented and proved in the last two chapters as the main aim of this thesis.

It is known that there does not exist any real hypersurface in a non-flat complex space form $M_n(c)$, $n \geq 3$, with parallel structure Jacobi operator (cf. [42]). Hence many conditions weaker than the parallelism of the structure Jacobi operator have been considered.

Recently, the D-parallelism condition was considered and it was proved in [53] that there does not exist any real hypersurface in $\mathbb{C}P^n$, $n \geq 3$, with D-parallel structure Jacobi operator. In [50] and [59], the recurrent condition of R_ξ was considered. The non-existence of real hypersurfaces with recurrent structure Jacobi operator in $M_n(c)$, $n \geq 3$, has been proved in [59]. In [60], real hypersurfaces in $M_2(c)$ with D-recurrent structure Jacobi operator have been studied. Complimentary to these results, we will prove the non-existence of real hypersurfaces with D-recurrent structure Jacobi operator in $M_n(c)$, $n \geq 3$, in Chapter 5. Actually, we will consider a more generalized condition $(\nabla_X R_\xi)\xi = 0$, for any $X \in \Gamma(D)$, and give a characterization for ruled real hypersurfaces. Then we use this result to obtain the non-existence of real hypersurfaces with D-recurrent structure Jacobi operator as a byproduct.

Another way to weaken the parallelism of R_ξ is to consider some conditions on the explicit expressions of the covariant derivative of R_ξ. In [21] and [33], the following conditions for the shape operator A and the Ricci tensor S

$$(\nabla_X A)Y = -c\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\},$$
$$(\nabla_X S)Y = k\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\}$$

have been studied respectively for real hypersurfaces in $\mathbb{C}P^n$. In [12], these two conditions were considered for real hypersurfaces in $\mathbb{C}H^n$. Hence in Chapter 6, it is reasonable to study a similar condition by replacing A and S with R_ξ:

$$(\nabla_X R_\xi)Y = k(\langle \phi X, Y\rangle \xi + \eta(Y)\phi X), \tag{1.1}$$

for any $X, Y \in \Gamma(TM)$.

In [52], the non-existence of real hypersurface in $\mathbb{C}P^n$ with Codazzi type structure Jacobi operator, i.e., $(\nabla_X R_\xi)Y = (\nabla_Y R_\xi)X$, for any $X, Y \in \Gamma(TM)$, has been stated. In Chapter 6, we will genaralize this statement to $M_n(c)$, $n \geq 3$.

Inspired by the Codazzi type condition as well as (1.1), we will study a generalized condition

$$\begin{aligned}\langle(\nabla_X R_\xi)Y-(\nabla_Y R_\xi)X,W\rangle = \;& k(2\eta(W)\langle\phi X,Y\rangle+\eta(Y)\langle\phi X,W\rangle\\ &-\eta(X)\langle\phi Y,W\rangle)\end{aligned}$$

for any $X,Y\in\Gamma(TM)$ in Chapter 6.

To summarize, we list the main results of this thesis as follows.

Chapter 5

- Give a characterization of ruled real hypersurfaces in a non-flat complex space form $M_n(c)$, $n\geq 3$, by the condition $(\nabla_X R_\xi)\xi=0$.
- Obtain the non-existence of real hypersurfaces in $M_n(c)$, $n\geq 3$, with D-recurrent structure Jacobi operator.

Chapter 6

- Give some characterizations in terms of ∇R_ξ for totally η-umbilical real hypersurfaces in $M_n(c)$, $n\geq 3$.
- Prove the non-existence of real hypersurfaces in $M_n(c)$, $n\geq 3$, with Codazzi type structure Jacobi operator.

In this thesis, we also give a biref review on Riemannian geometry with an emphasis on the real hypersurfaces in complex space forms. This review (from Chapter 2 to Chapter 4) is the background and preliminary of our results. In Chapter 2, we introduce our notations in Riemannian geometry and basic structures on Riemannian manifolds. In Chapter 3, we review the geometry of Riemannian submanifolds. In Chapter 4, we review the theory and development of real hypersurfaces in non-flat complex space forms and give a retrospect on classifications and non-existence results of real hypersurfaces in non-flat complex space forms concerning A, S and R_ξ.

Chapter 2

Foundations in Riemannian geometry

Riemannian geometry is an important branch in differential geometry, and it has intimate relations with many other branches in mathematics. In this chapter we give a brief review on the theory of Riemannian geometry.

In Section 2.1, we review some basic notions on Riemannian geometry. In Section 2.2, we discuss Hermitian metrics and Kaehler manifolds. In Section 2.3, we describe the construction of non-flat complex space forms, giving the background of our results in this thesis. In Section 2.4, we discuss Riemannian and semi-Riemannian submersions.

2.1 Basic notions for Riemannian manifolds

In this section, we recall some basic ideas and formulas on a Riemannian manifold. The standard models of real space forms are also discussed.

Definition 2.1.1. (i) An *m-dimensional manifold M* is a Hausdorff topological space for which every point has a neighborhood O such that O is homeomorphic to an open set Ω in $\mathbb{R}^m$.

(ii) Such a homeomorphism

$$x : O \longrightarrow \Omega$$

is called a *chart*. A family of charts $\{O_\tau, x_\tau\}$ is called an *atlas* if O_τ constitute an open covering of M.

(iii) A chart is called *compatible* with an atlas if adding the chart to the atlas yields again an atlas. An atlas is called *maximal* if any chart compatible with it is already contained in it.

(iv) An atlas is called *differentiable (smooth)* if all chart transformations

$$x_{\tau_1} \circ x_{\tau_2}^{-1} : \quad x_{\tau_1}(O_{\tau_1} \cap O_{\tau_2}) \longrightarrow x_{\tau_2}(O_{\tau_1} \cap O_{\tau_2}),$$

for $O_{\tau_1} \cap O_{\tau_2}$ non-empty, are differentiable (smooth).

(v) A *differentiable (smooth) manifold* M is a manifold with a maximal differentiable (smooth) atlas.

Remark 2.1.1. In this thesis, we always assume manifolds to be smooth, connected, paracompact and without boundaries.

Let M be a manifold. and let $C^\infty(M)$ denotes the ring of all smooth functions on M. A *smooth tangent vector field* is a map

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$

satisfying the following conditions:

(1) $X(f + \lambda g) = Xf + \lambda Xg$,

(2) $X(fg) = (Xf)g + f(Xg)$,

for all $f, g \in C^\infty(M)$ and real number λ.

For a vector bundle E over a manifold M, we denote by $\Gamma(E)$ the collection of all smooth cross sections in E. We use TM to denote the tangent bundle of M and use T^*M to denote the cotangent bundle of M. Then X is a smooth tangent vector field on M if and only if $X \in \Gamma(TM)$.

The *Lie-bracket* $[X, Y]$ of $X, Y \in \Gamma(TM)$ is defined as

$$[X, Y]f = X(Yf) - Y(Xf),$$

for any $f \in C^\infty(M)$. Then $[X, Y] \in \Gamma(TM)$. For $X, Y, Z \in \Gamma(TM)$, the Lie-bracket satisfies the following properties

$[X,Y] = -[Y,X]$,

$[\lambda X + \mu Z, Y] = \lambda[X,Y] + \mu[Z,Y]$, for $\lambda, \mu \in \mathbb{R}$,

$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X,Y]$, for $f, g \in C^\infty(M)$,

$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

Theorem 2.1.1. *(page 134-135 of [9], page 116 of [22]) Let M be a manifold. Then there exists a smooth covariant tensor field of rank 2 (i.e., a smooth tensor field of type $(0,2)$), $g : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$ on M, satisfying*

(1) $g(X,Y) = g(Y,X)$ for any $X, Y \in \Gamma(TM)$;

(2) $g(X,X) \geq 0$ for any $X \in \Gamma(TM)$, and equality holds if and only if $X = 0$.

The tensor field g is called a *Riemannian metric* of M. If M is equipped with a Riemannian metric g, then M is called a *Riemannian manifold*. For convenience, we always denote $g(X,Y)$ as $\langle X, Y \rangle$.

If a manifold M is equipped with a smooth covariant tensor field of rank 2, also denoted by g, satisfying the condition (1) in Theorem 2.1.1 and that g is non-degenerate (which is weaker than the condition (2)) on T_pM for each $p \in M$, and the dimension of the negative-definite subspace of g in T_pM is constant for any $p \in M$, then g is called a *semi-Riemannian metric* and M is called a *semi-Riemannian manifold*. This dimension is called the *index* of the semi-Riemannian metric g.

Let M_1 and M_2 be Riemannian (semi-Riemannian) manifolds with Riemannian (semi-Riemannian) metrics g_1 and g_2 respectively. A diffeomorphism

$$f: \; M_1 \longrightarrow M_2$$

is called a *Riemannian (semi-Riemannian) isometry* if for any $p \in M_1$ and $X_p, Y_p \in T_pM_1$,

$$g_2(df(p)X_p, df(p)Y_p) = g_1(X_p, Y_p).$$

In this situation, M_1 and M_2 are said to be *isometric* to each other (cf. page 58 of [41]).

Given a Riemannian manifold M, there exists a unique affine connection ∇ on M such that for $X, Y, Z \in \Gamma(TM)$,

$$[X, Y] = \nabla_X Y - \nabla_Y X, \tag{2.1}$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \tag{2.2}$$

This connection is called the *Levi-Civita connection* of M.

For a tensor field T of type (p, q) on M,

$$T : \underbrace{\Gamma(TM) \times ... \times \Gamma(TM)}_{q \quad times} \longrightarrow \underbrace{\Gamma(TM) \times ... \times \Gamma(TM)}_{p \quad times},$$

we define the *covariant derivative* $\nabla_X T$ of the tensor field T along a vector field $X \in \Gamma(TM)$ by

$$(\nabla_X T)(Y_1, ..., Y_q) := \nabla_X(T(Y_1, ..., Y_q)) - \sum_{k=1}^{q} T(Y_1, ..., \nabla_X Y_k, ..., Y_q)$$

for any $Y_1, ..., Y_q \in \Gamma(TM)$.

The Lie derivative L_X for $X \in \Gamma(TM)$ is given as follows: for $Y \in \Gamma(TM)$, $a \in \Gamma(T^*M)$ and $f \in C^\infty(M)$,

$$\begin{aligned} L_X f &= Xf, \\ L_X Y &= [X, Y], \\ (L_X a)Y &= X(aY) - a[X, Y] \\ &= da(X, Y) + Y(aX), \end{aligned}$$

where $da(X, Y) = X(aY) - Y(aX) - a([X, Y])$. In general, when we regard a tensor field T of type (p, q), $p, q \geq 1$, as a differentiable multilinear map with respect to smooth sections a_1, a_2, ..., $a_p \in \Gamma(T^*M)$ and smooth sections Y_1, Y_2, ..., $Y_q \in \Gamma(TM)$, then

$$\begin{aligned} (L_X T)(a_1, ..., a_p; Y_1, ..., Y_q) : &= X(T(a_1, ..., a_p; Y_1, ..., Y_q)) \\ &- \sum_{k=1}^{p} T(a_1, ..., L_X a_k, ..., a_p; Y_1, ..., Y_q) \\ &- \sum_{k=1}^{q} T(a_1, ..., a_p; Y_1, ..., [X, Y_k], ..., Y_q). \end{aligned}$$

Let $f : M_1 \longrightarrow M_2$ be a smooth map between manifolds M_1 and M_2. Consider the differential $df : T_pM_1 \longrightarrow T_{f(p)}M_2$ for any point $p \in M_1$. Then df induces a map $f^* : \Gamma(T^*M_2) \longrightarrow \Gamma(T^*M_1)$ by $f^*(a)(X) = a(df(X))$ for any $a \in \Gamma(T^*M_2)$ and $X \in \Gamma(TM_1)$. Furthermore, for a tensor field T of type $(0, q)$ on M_2, f induces a covariant tensor field f^*T on M_1 by

$$(f^*T)(X_1, ..., X_q) = T((df)X_1, ..., (df)X_q)$$

for any $X_1, ..., X_q \in \Gamma(TM_1)$. In particular, given an arbitrary Riemannian (semi-Riemannian) metric g_2 on M_2, f induces a Riemannian (semi-Riemannian) metric $g_1 = f^*g_2$ on M_1. In addition, if f is a diffeomorphism, then f is a Riemannian (semi-Riemannian) isometry.

Let M be a Riemannian manifold. We define the *curvature tensor* R as follows:

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$. It can be shown that R satisfies

$$\begin{aligned} \langle R(X, Y)Z, W\rangle &= -\langle R(Y, X)Z, W\rangle \\ = -\langle R(X, Y)W, Z\rangle &= \langle R(Z, W)X, Y\rangle. \end{aligned} \tag{2.4}$$

We also have the *first Bianchi identity*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

and the *second Bianchi identity*

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0.$$

If $\nabla R = 0$, then M is called a *locally symmetric space*.

Let $X, Y \in \Gamma(TM)$. For any point $p \in M$, let $\{e_1, ..., e_m\}$ be an orthonormal basis of T_pM. We define the *Ricci tensor* $Ric(X, Y)$ at the point p as

$$Ric_p(X_p, Y_p) := -\frac{1}{m-1}\sum_{j=1}^{m}\langle R(X_p, e_j)Y_p, e_j\rangle.$$

Since the trace of a bilinear map is invariant under non-degenerate linear transformations, Ric_p is well-defined and does not depend on the choice of basis

$\{e_1, ..., e_m\}$. Hence we can choose $\{e_1, ..., e_m\}$ such that $e_m = X_p$ to obtain

$$Ric_p(X_p, Y_p) = -\frac{1}{m-1}\sum_{j=1}^{m-1}\langle R(X_p, e_j)Y_p, e_j\rangle.$$

It can be shown that Ric is a symmetric smooth covariant tensor field of rank 2 on M. We define the *Ricci operator* S as a tensor field of type (1,1) in the following way:

$$\langle SX, Y\rangle = Ric(X, Y)$$

for $X, Y \in \Gamma(TM)$.

For a Riemannian manifold M with metric g, if there exists a constant k on M such that

$$Ric = kg,$$

then M is called an *Einstein manifold*.

For any point $p \in M$, let $X_p, Y_p \in T_pM$ be orthonormal vectors. The *sectional curvature* $K(X_p, Y_p)$ of M at p with respect to the plane Span$\{X_p, Y_p\}$ is given by

$$K(X_p, Y_p) = -\langle R(X_p, Y_p), X_p.Y_p\rangle.$$

It can be proved that if we change X_p, Y_p to X'_p, Y'_p such that $\{X'_p, Y'_p\}$ is also an orthonormal basis of Span$\{X_p, Y_p\}$, then $K(X'_p, Y'_p) = K(X_p, Y_p)$.

Let I be an interval. A curve $\gamma : I \longrightarrow M$ is called a *geodesic* on M if

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0.$$

Given $p \in M$, it can be proved that there exists $\epsilon(p) > 0$ such that for any unit vector $v \in T_pM$, there exists a unique geodesic

$$\begin{aligned}\gamma_v : \quad (-\epsilon(p), \epsilon(p)) &\longrightarrow M,\\ t &\longmapsto \gamma_v(t)\end{aligned}$$

such that $\gamma(0) = p$, $\partial_t\gamma(0) = v$. Hence it makes sense to define the *exponential map* $\exp_p : B_p(\epsilon(p)) \longrightarrow M$ by

$$\exp_p(tv) = \gamma_v(t)$$

for any $t \in (-\epsilon(p), \epsilon(p))$ and any unit vector $v \in T_pM$. Here $B_p(\epsilon(p))$ denotes the ball of radius $\epsilon(p)$ in T_pM centered at 0. M is called *complete* if for all $p \in M$, the exponential map $\exp_p$ is defined on the whole T_pM, i.e., the above $\epsilon(p)$ can be chosen as any positive number.

We define a subset Ω of TM as

$$\Omega = \{(p, u) \in TM |\ p \in M, |u| < \epsilon(p)\}.$$

Then the exponential map $\exp : \Omega \longrightarrow M$ can be defined by

$$\exp(p, u) = \exp_p u$$

for any $(p, u) \in \Omega$. If M is complete, then $\Omega = TM$.

Let $\gamma :\ I \longrightarrow M$ be a curve in M. A (smooth) map

$$X :\ I \longrightarrow TM$$

is called a *(smooth) vector field along* γ if $\gamma = \Pi \circ X$, where $\Pi : TM \longrightarrow M$ denotes the canonical projection.

For a smooth vector field $X(t)$ along a geodesic $\gamma(t)$, X is called a *Jacobi field* if it satisfies the *Jacobi equation*

$$\partial_t \partial_t X + R(X, \dot{\gamma})\dot{\gamma} = 0. \tag{2.5}$$

Here we write $\partial_t \partial_t X$ to denote $\nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} X(t)$.

Given a vector field $V \in \Gamma(TM)$, the *Jacobi operator* $R_V : \Gamma(TM) \longrightarrow \Gamma(TM)$ is given by

$$R_V(X) = R(X, V)V.$$

Then the Jacobi equation can be reduced to $\partial_t \partial_t X + R_{\dot{\gamma}} X = 0$.

Lemma 2.1.2. *For a symmetric(anti-symmetric) tensor field T of type (1,1) on a Riemannian manifold M and $X \in \Gamma(TM)$, $\nabla_X T$ is also a symmetric(anti-symmetric) tensor field of type (1,1).*

Proof. Case 1. *T is symmetric.*

For $Y, Z \in \Gamma(TM)$,

$$\langle TY, Z\rangle = \langle Y, TZ\rangle.$$

Hence

$$\begin{aligned}\langle(\nabla_X T)Y, Z\rangle &= X\langle TY, Z\rangle - \langle T\nabla_X Y, Z\rangle - \langle TY, \nabla_X Z\rangle \\ &= X\langle Y, TZ\rangle - \langle \nabla_X Y, TZ\rangle - \langle Y, T\nabla_X Z\rangle \\ &= \langle Y, (\nabla_X T)Z\rangle.\end{aligned}$$

Hence $\nabla_X T$ is symmetric.

Case 2. *T is anti-symmetric.*

For $Y, Z \in \Gamma(TM)$,

$$\langle TY, Z\rangle = -\langle Y, TZ\rangle.$$

Hence

$$\begin{aligned}\langle(\nabla_X T)Y, Z\rangle &= -X\langle Y, TZ\rangle + \langle \nabla_X Y, TZ\rangle + \langle Y, T\nabla_X Z\rangle) \\ &= -\langle Y, (\nabla_X T)Z\rangle\end{aligned}$$

Hence $\nabla_X T$ is anti-symmetric. □

This lemma is useful while calculating the covariant derivative of S, R_V and other symmetric and anti-symmetric tensor fields of type (1,1) on M.

We give some examples of Riemannian manifolds.

Example 2.1.1. An Euclidean space $\mathbb{R}^n$ with its canonical inner product is a Riemannian manifold with constant sectional curvature 0.

Example 2.1.2. A sphere imbedded in $\mathbb{R}^{n+1}$

$$S^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + ... + x_{n+1}^2 = 1\}$$

with the Riemannian metric induced from the inner product of $\mathbb{R}^{n+1}$ is a Riemannian manifold with constant sectional curvature 1.

Example 2.1.3. Consider $\mathbb{R}^{n+1}$ equipped with the bilinear form

$$\langle v, w\rangle_1 = \sum_{i=1}^{n} v_i w_i - v_{n+1} w_{n+1}.$$

Identifying each tangent space of $\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$ as described above, we get a Lorentzian metric on $\mathbb{R}^{n+1}$, which is also denoted by $\langle\ ,\ \rangle_1$. We denote the manifold $\mathbb{R}^{n+1}$ equipped with this Lorentzian metric by $\mathbb{R}_1^{n+1}$. A hyperbolic space imbedded in $\mathbb{R}_1^{n+1}$ is given by

$$H^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}_1^{n+1} | \langle x, x\rangle_1 = -1, x_{n+1} > 0\}.$$

This is a connected submanifold of $\mathbb{R}_1^{n+1}$ with time-like unit normal vector field

$$\nu_p = p, \text{ for } p \in H^n.$$

The tangent space T_pH^n consists of all vectors orthogonal to ν_p and hence is a space-like linear subspace of $\mathbb{R}_1^{n+1}$. Thus the Lorentzian metric $\langle\ ,\ \rangle_1$ of $\mathbb{R}_1^{n+1}$ induces a Riemannian metric on H^n. Moreover, equipped with this Riemannian metric, H^n is a Riemannian manifold with constant sectional curvature -1.

A complete and simply connected Riemannian manifold of constant sectional curvature c is called a *real space form*. Without lost of generality, we assume $c = 1$ and $c = -1$ for $c > 0$ and $c < 0$ respectively. It is well-known that a real space form is either $\mathbb{R}^n$, S^n or H^n. All these manifolds are locally symmetric spaces.

2.2 Hermitian metric and Kaehler manifolds

In this section, we recall the definition of Hermitian metrics on complex manifolds and discuss the standard models of complex space forms.

An *almost complex structure* on a differentiable manifold M is a tensor field J of type (1,1) which is, at every point $x \in M$, an endomorphism of T_xM such that $J^2 = -I$, where I denotes the identity transformation of T_xM. A manifold M with an almost complex structure J is called an almost complex manifold. Every almost complex manifold is of even dimension and orientable.

We define the torsion of J to be the tensor field N of type (1,1), called the *Nijenhuis torsion*, given by (cf. page 7-8 of [64])

$$N(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]$$

for any vector fields X and Y. If N vanishes identically, then the almost complex structure is called a *complex structure* and M is called a complex manifold.

A *Hermitian metric* on an almost complex manifold M is a Riemannian metric g such that

$$g(JX, JY) = g(X,Y) \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$.

An almost complex manifold (resp. a complex manifold) with a Hermitian metric is called an *almost Hermitian manifold (resp. a Hermitian manifold).* We notice that every almost complex manifold M with a Riemannian metric g admits a Hermitian metric. Indeed, for any almost complex structure J on M, putting

$$\bar{g}(X,Y) = g(X,Y) + g(JX, JY),$$

we obtain a Hermitian metric $\bar{g}$. A Hermitian manifold M is called a *Kaehler manifold* if the almost complex structure J on M is parallel, that is, $\nabla J = 0$.

For a Kaehler manifold M, $p \in M$ and $X_p \in T_pM$, the *holomorphic sectional curvature* with respect to X_p, denoted by $K_H(X_p)$, is the sectional curvature with respect to the plane Span$\{X_p, JX_p\}$ in T_pM. In addition, if X_p is a unit vector, then

$$K_H(X_p) = -\langle R(X_p, JX_p)X_p, JX_p\rangle.$$

We give some examples of Kaehler manifolds.

Example 2.2.1. A complex Euclidean space $\mathbb{C}^n$ with its canonical Hermitian inner product is a Kaehler manifold with constant holomorphic sectional curvature 0.

Example 2.2.2. A complex projective space $\mathbb{C}P^n$ is a Kaehler manifold with constant holomorphic sectional curvature $4c$ for $c > 0$. All complex projective spaces of the same dimension have same geometric structures except for a scalar multiple of the metric, i.e., they are homothetic to each other. Hence in this thesis, we always assume $c = 1$ for $\mathbb{C}P^n$.

Example 2.2.3. A complex hyperbolic space $\mathbb{C}H^n$ is a Kaehler manifold with constant holomorphic sectional curvature $4c$ for $c < 0$. All complex hyperbolic spaces of the same dimension have same geometric structures except for a scalar multiple of the metric. Hence in this thesis, we always assume $c = -1$ for $\mathbb{C}H^n$.

A *complex space form* is a complete and simply connected Kaehler manifold with constant holomorphic sectional curvature $4c$. Without lost of generality, we assume $c = 1$ for $c > 0$ and $c = -1$ for $c < 0$. It is well-known that a complex space form is either $\mathbb{C}^n$, $\mathbb{C}P^n$ or $\mathbb{C}H^n$. They are typical examples of locally symmetric spaces. The constructions of $\mathbb{C}P^n$ and $\mathbb{C}H^n$ will be discussed in the next section.

2.3 The constructions of $\mathbb{C}P^n$ and $\mathbb{C}H^n$

In this section, we review the constructions of $\mathbb{C}P^n$ and $\mathbb{C}H^n$.

2.3.1 The construction of $\mathbb{C}P^n$

Let

$$\mathbb{C}^{n+1}_* = \mathbb{C}^{n+1} - \{0\}$$

with the Hermitian metric $(,)$ and Euclidean metric $\langle , \rangle$, i.e.,

$$(z, w) := \sum_{k=1}^{n+1} z_k \bar{w}_k,$$

$$\langle z, w \rangle := \mathrm{Re}(z, w)$$

for any $z = (z_1, ..., z_{n+1}), w = (w_1, ..., w_{n+1}) \in \mathbb{C}_*^{n+1}$. Consider the relation $\sim$ on $\mathbb{C}_*^{n+1}$ given by

$$z \sim w \text{ if and only if there exists } \lambda \in \mathbb{C}_* \text{ such that } z = \lambda w$$

and let

$$\mathbb{C}P^n := \mathbb{C}_*^{n+1} / \sim .$$

Then

$$\mathbb{C}P^n = \{[z] | z \in \mathbb{C}_*^{n+1}\},$$

where

$$[z] = \{w \in \mathbb{C}_*^{n+1} | w \sim z\}.$$

Let

$$\begin{aligned} S^{2n+1} : &= \{z \in \mathbb{C}_*^{n+1} | (z, z) = 1\} \\ &= \{z \in \mathbb{C}_*^{n+1} | \langle z, z \rangle = 1\} \end{aligned}$$

and the projection $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$given by

$$\pi(z) = [z].$$

It can be shown that π is a smooth map. It can be proved that $\mathbb{C}P^n$ is a complex manifold and the complex structure J of $\mathbb{C}P^n$ is given by $J \circ d\pi = d\pi \circ i$, where i is the complex structure of C_*^{n+1}.

Let

$$V_z S^{2n+1} = \mathrm{Span}_{\mathbb{R}}\{iz\} = \{aiz | a \in \mathbb{R}\},$$

$$\begin{aligned} H_z S^{2n+1} &= \{V_z S^{2n+1}\}^{\perp} \\ &= \{v \in \mathbb{C}^{n+1} | (v, iz) = 0\} \\ &= \{v \in T_z S^{2n+1} | \langle v, iz \rangle = 0\}. \end{aligned}$$

Then

$$T_z S^{2n+1} = H_z S^{2n+1} \oplus V_z S^{2n+1}.$$

Proposition 2.3.1. *For $z \in S^{2n+1}$, $d\pi$ vanishes on $V_z S^{2n+1}$ and $d\pi$ is a linear isomorphism from $H_z S^{2n+1}$ to $T_{\pi(z)}\mathbb{C}P^n$.*

Proof. For $z, z_1 \in S^{2n+1}$, $z_1 \sim z$ if and only if there exists $\lambda \in \mathbb{C}_*$ with $|\lambda| = 1$ such that $z_1 = \lambda z$. Hence we write $\lambda = e^{i\theta}$. Let a curve $z(t)$ in S^{2n+1} given by

$$z(t) = e^{it} z,$$

then

$$\pi(z(t)) = [z],$$

hence

$$d\pi(z)(\partial_t z(0)) = 0.$$

However

$$\partial_t z(0) = \partial_t (e^{it} z)|_{t=0} = iz,$$

hence by the linearity of $d\pi$, we see that

$$d\pi(z)(V_z S^{2n+1}) = \{0\}.$$

Since π is a surjective smooth map, we see that $d\pi(z)$ is surjective and by virtue of the dimensions of S^{2n+1} and $\mathbb{C}P^n$, we know that $d\pi(z)$ is an isomorphism from $H_z S^{2n+1}$ to $T_{\pi(z)}\mathbb{C}P^n$. □

The map π is called the *Hopf fibration*, and S^{2n+1} is a principal fiber bundle over $\mathbb{C}P^n$ with projection map π and structure group S^1 (cf. [32]). For any $p \in \mathbb{C}P^n$, there exists $z \in S^{2n+1}$ with $\pi(z) = p$. By the above proposition, for any $X_p \in T_p\mathbb{C}P^n$, there exists a unique $X'_z \in H_z S^{2n+1}$ such that

$$d\pi(X'_z) = X_p.$$

X'_z is called the *horizontal lift* of the tangent vector X_p at z.

Proposition 2.3.2. *(page 6-8 of [13]) Let* $z, z_1 \in S^{2n+1}$ *with* $\pi z = \pi z_1 = p$. *We denote* X'_z *and* X'_{z_1} *as the horizontal lifts of* $X_p \in T_p\mathbb{C}P^n$ *at* z *and* z_1 *respectively. Suppose*

$$\begin{aligned} E_t : \quad & S^{2n+1} \longrightarrow S^{2n+1}, \\ & w \longmapsto e^{it}w. \end{aligned}$$

Then E_t *is an isometry on* S^{2n+1} *and*

$$dE_t(z)X'_z = X'_{z_1}.$$

It follows from Proposition 2.3.2 that the Riemannian metric on S^{2n+1} is invariant by the structure group S^1. Hence we are able to define a Riemannian metric on $\mathbb{C}P^n$ by

$$\langle X_p, Y_p \rangle_{\mathbb{C}P^n} = \langle X'_z, Y'_z \rangle_{S^{2n+1}} \tag{2.7}$$

for any $p \in \mathbb{C}P^n$ and $X_p, Y_p \in T_p\mathbb{C}P^n$. Then with this metric, $\mathbb{C}P^n$ can be proved to be a Kaehler manifold with constant holomorphic sectional curvature 4 (page 6 to page 8, [13]; page 273 to page 278, [23]).

2.3.2 The construction of $\mathbb{C}H^n$

We define $(,)_1$ and $\langle , \rangle_1$ on $\mathbb{C}^{n+1}$ as

$$(z, w)_1 = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k,$$

$$\langle z, w \rangle_1 = \mathrm{Re}(z, w)_1$$

for any $z = (z_0, z_1, ..., z_n), w = (w_0, w_1, ..., w_n) \in \mathbb{C}^{n+1}$. Let

$$D^{n+1} = \{z \in \mathbb{C}^{n+1} | (z, z)_1 < 0\},$$

$$H_1^{2n+1} = \{z \in \mathbb{C}^{n+1} | (z, z)_1 = -1\}.$$

Then for any $z \in D^{n+1}$, there exists $a \in \mathbb{R}$ and $z_1 \in H_1^{2n+1}$ such that $z = az_1$. D^{n+1} is an open subset of $\mathbb{C}^{n+1}$ and H_1^{2n+1} is a connected real hypersurface in D^{n+1}, called the *anti-De Sitter space*. It can be seen that $\langle,\rangle_1$ is a semi-Riemannian metric on $\mathbb{C}^{n+1}$. With this metric, it can be proved that H_1^{2n+1} is a connected real hypersurface with Lorentzian metric of index 1 and constant sectional curvature -1.

We consider the relation $\sim$ on D^{n+1}:

$$z \sim w \text{ if and only if there exists } \lambda \in \mathbb{C}_* \text{ such that } z = \lambda w,$$

and let

$$[z] = \{w \in D^{n+1} | w \sim z\}.$$

Let

$$\begin{aligned} \mathbb{C}H^n &:= \{[z] | z \in D^{n+1}\} \\ &= \{[z] | z \in H_1^{2n+1}\}. \end{aligned}$$

Let the projection $\pi : H_1^{2n+1} \longrightarrow \mathbb{C}H^n$ given by $\pi(z) = [z]$. Then π is a smooth map and H_1^{2n+1} is a principal fiber bundle, with the projection map π and structure group S^1.

It can be proved that $\mathbb{C}H^n$ constructed in this way is a complex manifold and its complex structure J is given by $J \circ d\pi = d\pi \circ i$, where i is the complex structure of D^{n+1}.

Let

$$V_z H_1^{2n+1} = \text{Span}\{iz\},$$

$$H_z H_1^{2n+1} = (V_z H_1^{2n+1})^{\perp},$$

for any $z \in M$ and $(V_z H_1^{2n+1})^{\perp}$ is the orthogonal compliment of $(V_z H_1^{2n+1})$ in $T_z H_1^{2n+1}$ with respect to the metric $\langle,\rangle_1$. It can be shown that for any $z \in H_1^{2n+1}$, $d\pi(z)(V_z H_1^{2n+1}) = 0$ and $H_z H_1^{2n+1}$ is linearly isomorphic to $T_{\pi(z)}\mathbb{C}H^n$ via the map $d\pi(z)$. Hence for any $z \in H_1^{2n+1}$ and $X_{\pi(z)} \in T_{\pi(z)}\mathbb{C}H^n$, there exists a unique $X'_z \in H_z H_1^{2n+1}$ such that $d\pi(z)(X'_z) = X_{\pi(z)}$. The vector X'_z is called the *horizontal lift* of $X_{\pi(z)}$.

Proposition 2.3.3. *For any $z \in H_1^{2n+1}$, $\langle , \rangle_1$ is negative definite on $V_z H_1^{2n+1}$.*

Proof. Let $z, w \in H_1^{2n+1}$ and $w \sim z$. Then there exists $\lambda \in \mathbb{C}_*$ such that $w = \lambda z$. Note that

$$\begin{aligned} -1 &= \sum_{k=1}^{n} w_k \bar{w}_k - w_{n+1}\bar{w}_{n+1} \\ &= \sum_{k=1}^{n} \lambda z_k \overline{(\lambda z_k)} - \lambda z_{n+1}\overline{(\lambda z_{n+1})} \\ &= \lambda\bar{\lambda}(\sum_{k=1}^{n} z_k \bar{z}_k - z_{n+1}\bar{z}_{n+1}) \\ &= -\lambda\bar{\lambda}, \end{aligned}$$

Hence we see that $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$. On the other hand,

$$\begin{aligned} \langle iz, iz\rangle_1 &= \mathrm{Re}(iz, iz)_1 \\ &= \mathrm{Re}(z, z)_1 \\ &= \langle z, z\rangle_1 \\ &= -1. \end{aligned}$$

Therefore, $\langle , \rangle_1$ is negative definite on $V_z H_1^{2n+1}$. □

By the above proposition, since $\langle , \rangle_1$ is a Lorentzian metric on H_1^{2n+1} with index 1, we see that the semi-Riemannian metric $\langle , \rangle_1$ is positive-definite on $H_z H_1^{2n+1}$. Hence it induces a Riemannian metric on $\mathbb{C}H^n$:

$$\langle X_p, Y_p\rangle = \langle X'_z, Y'_z\rangle_1, \tag{2.8}$$

where $p \in \mathbb{C}H^n$, $z \in H_1^{2n+1}$, $\pi z = p$, $X_p, Y_p \in T_p\mathbb{C}H^n$, $X'_z, Y'_z \in T_z H_1^{2n+1}$ and $d\pi(z)(X'_z) = X_p$, $d\pi(z)(Y'_z) = Y_p$. It can be proved that the metric on $\mathbb{C}H^n$ defined by (2.8) is independent of the choice of z hence the metric on $\mathbb{C}H^n$ is well defined. Moreover, $\mathbb{C}H^n$ is a Kaehler manifold with constant holomorphic sectional curvature -4 (cf. page 282 to page 285, [23]; page 236 to page 237, [37]).

2.4 Riemannian and semi-Riemannian submersions

In this section, we recall the definitions and some fundamental formulas of Riemannian submersions and semi-Riemannian submersions (cf. page 2-3 of [13], page 212-213 of [41]).

Definition 2.4.1. (i) For manifolds N and M, a smooth surjective map $\pi : N \longrightarrow M$ is called a *submersion* if $d\pi$ is surjective at each point of N.

(ii) Suppose N and M are Riemannian manifolds, and $\pi : N \longrightarrow M$ is a submersion. If $d\pi$ preserves Riemannian metric, i.e.,

$$\langle d\pi(X_a), d\pi(Y_a)\rangle_M = \langle X_a, Y_a\rangle_N$$

for any $a \in N$ and $X_a, Y_a \in T_aN$, then π is called a *Riemannian submersion.*

(iii) Suppose N and M are semi-Riemannian manifolds with indices s' and s respectively, $s \leq s'$. A *semi-Riemannian submersion* is a submersion $\pi : N \longrightarrow M$ which satisfies

(a) the fibers $\pi^{-1}(x)$, for all $x \in M$, are semi-Riemannian submanifolds of N;

(b) $d\pi$ preserves inner products of vectors normal to fibers.

Definition 2.4.2. Suppose $\pi : N \longrightarrow M$ is a Riemannian or semi-Riemannian submersion and a denotes a point of N.

(i) Let $V_aN = \text{Ker}(d\pi(a))$ and $VN = \bigcup_{a\in N} V_aN$. Then V_aN is called the *vertical space* at a and VN is called the *vertical distribution.*

(ii) Let $H_aN = (V_aN)^{\perp}$ and $HN = \bigcup_{a\in N} H_aN$. Then H_zN is called the *horizontal space* at a and HN is called the *horizontal distribution.*

Proposition 2.4.1. *For a Riemannian or semi-Riemannian submersion $\pi : N \longrightarrow M$, H_zN is isometric to $T_{\pi(z)}M$ for any $z \in N$ via the map $d\pi$.*

By this proposition, for any $p \in M$ and $v \in T_pM$, there exists a unique $v' \in H_zN$ such that $d\pi(z)v' = v$, where $z \in N$ and $\pi z = p$. The tangent vector v' is called the *horizontal lift* of v.

It is known that

$$\pi: \ H_1^{2n+1} \longrightarrow \mathbb{C}H^n$$

is a semi-Riemannian submersion with horizontal and vertical distributions

$$HH_1^{2n+1} = \cup_{z \in H_1^{2n+1}} H_z H_1^{2n+1},$$
$$VH_1^{2n+1} = \cup_{z \in H_1^{2n+1}} V_z H_1^{2n+1}$$

respectively. In this case, $s = 1$ and $s' = 0$.

On the other hand,

$$\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$$

is a Riemannian submersion with horizontal and vertical distributions

$$HS^{2n+1} = \cup_{z \in S^{2n+1}} H_z S^{2n+1},$$
$$VS_1^{2n+1} = \cup_{z \in S^{2n+1}} V_z S^{2n+1}$$

respectively.

Now we list some formulas for Riemannian and semi-Riemannian submersions. Let ∇' and ∇ denote the connections of N and M respectively. For simplicity, we use the same symbol $\langle , \rangle$ to denote the Riemannian or semi-riemannian metrics of both M and N. For any $z \in N$ and tangent vector $V \in T_z N$, let $(V)^v$ be the vertical component of V. Let K_M and K_N denote the sectional curvatures of M and N respectively. For any $X, Y \in \Gamma(TM)$, we have

$$[X', Y']^v = [X', Y'] - [X, Y]',$$

$$(\nabla'_{X'} Y')^v = \nabla'_{X'} Y' - (\nabla_X Y)';$$

and for any $\bar{X}, \bar{Y} \in \Gamma(TN)$ spanning non-degenerate planes at each point,

$$K_M(d\pi \bar{X}, d\pi \bar{Y}) = K_N(\bar{X}, \bar{Y}) + \frac{3}{4} \langle [\bar{X}, \bar{Y}]^v, [\bar{X}, \bar{Y}]^v \rangle / (\langle \bar{X}, \bar{X} \rangle \langle \bar{Y}, \bar{Y} \rangle - \langle \bar{X}, \bar{Y} \rangle^2).$$

Chapter 3

Riemannian submanifolds

Geometry of Riemannian submanifolds is one of the main branches in Riemannian geometry, and it contains various topics, for example, hypersurfaces in real and complex space forms, minimal submanifolds, CR-submanifolds, isometric imbeddings of Riemannian manifolds in space forms, etc. In this chapter, we give a survey on certain aspects of Riemannian submanifolds that are related to our work.

In Section 3.1, we briefly review the general theory of Riemannian submanifolds. In Section 3.2, we discuss hypersurfaces and give the definition of geodesic hyperspheres and tubes of a Riemannian manifold. In Section 3.3, we state some basic formulas in the study of hypersurfaces in real space forms and real hypersurfaces in complex space forms.

3.1 General theory of Riemannian submanifolds

In this section, we review some fundamental ideas and formulas in Riemannian submanifolds. We also recall the concepts of totally geodesic, minimal and umbilical submanifolds at the end of this section.

Definition 3.1.1. (page 21 of [9]) Suppose M is an m-dimensional and N is an n-dimensional manifold, $m < n$. If there exists a smooth map

$$\varphi : M \longrightarrow N$$

such that at each point $p \in M$, the tangent map

$$(d\varphi)_p : T_pM \longrightarrow T_{\varphi(p)}N$$

is non-degenerate, then M is called an *immersed submanifold* of N through the *immersion* φ. In addition if the immersion φ is injective, then φ is called an *imbedding* and M is called an *imbedded submanifold* of N.

Remark 3.1.1. For an imbedding $\varphi : M \longrightarrow N$, since φ is injective, the differentiable structure on M can be transported to $\varphi(M)$, making $\varphi : M \longrightarrow \varphi(M)$ a diffeomorphism. On the other hand, being a subset of N, $\varphi(M)$ has an induced topology from N. Generally, $\varphi(M)$ is not necessarily a topological subspace of N. Actually, the topology from M through φ is stronger than the topology induced from N (cf. page 23-page 24 of [9]).

On the other hand, some geometers use a stronger definition for submanifolds. They require a submanifold to be a topological subspace (cf. page 16 of [41]). However, we do not need to take this assumption since Definition 3.1.1 is enough for this thesis.

By using the implicit function theorem, it can be proved that an immersed submanifold is locally an imbedded submanifold. In this thesis, we mainly focus on local properties of submanifolds hence we will always assume submanifolds to be imbedded submanifolds unless otherwise specified.

If M and N are both Riemannian manifolds and the imbedding (immersion) φ preserves the Riemannian metric, i.e., for any $p \in M$ and $X_p, Y_p \in T_pM$,

$$\langle X_p, Y_p \rangle_M = \langle (d\varphi)_p X_p, (d\varphi)_p Y_p \rangle_N,$$

then M is called a *Riemannian submanifold* of N and φ is called a *Riemannian imbedding (immersion)*. Let M_1 and M_2 be Riemannian submanifolds of N. If there exists an isometry $T : N \longrightarrow N$ such that $T(M_1) = M_2$, then M_1 is said to be congruent to M_2.

Let M be a Riemannian submanifold of Riemannian manifold N. Then for any $p \in M$,

$$T_pN = T_pM \oplus T_p^{\perp}M,$$

where $T_p^{\perp}M$ denotes the orthogonal compliment of T_pM in T_pN. Here we do not distinguish M and φM since they are isometric to each other. Let $T^{\perp}M = \bigcup_{p\in M} T_p^{\perp}M$. Then $T^{\perp}M$ is a fibre bundle over M, called the *normal bundle* of the submanifold M in N. A smooth *normal vector field* ν of M is a smooth cross-section of $T^{\perp}M$, i.e., $\nu \in \Gamma(T^{\perp}M)$. Let ∇ and $\bar{\nabla}$ denote the Levi-Civita connections of M and N respectively. Then for $X, Y \in \Gamma(TM)$, we write

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where $h(X, Y) \in \Gamma(T^{\perp}M)$. It can be proved that

$$h(X + Y, Z) = h(X, Z) + h(Y, Z), \tag{3.1}$$

$$h(X, fY) = fh(X, Y), \tag{3.2}$$

$$h(X, Y) = h(Y, X), \tag{3.3}$$

for any $X, Y, Z \in \Gamma(TM)$ and $f \in C^{\infty}(M)$. The map

$$h : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(T^{\perp}M) \tag{3.4}$$

is called the *second fundamental form* of M. For each fixed point $p \in M$, $h :\ T_pM \times T_pM \longrightarrow T_p^{\perp}M$ is a symmetric bilinear map.

Choose an arbitrary normal vector field $\nu \in \Gamma(T^{\perp}M)$. Let

$$\langle A_\nu X, Y\rangle = \langle h(X, Y), \nu\rangle. \tag{3.5}$$

By (3.2), (3.3) and (3.4), we see that A_ν is well-defined and it is a symmetric operator on TM. Hence for any $X, Y \in \Gamma(TM)$,

$$\langle A_\nu X, Y\rangle = \langle X, A_\nu Y\rangle.$$

A_ν is called the *shape operator* of M with respect to the normal vector field ν. We have

$$\langle A_\nu X, Y\rangle = \langle h(X, Y), \nu\rangle = \langle \bar{\nabla}_X Y, \nu\rangle = -\langle \bar{\nabla}_X \nu, Y\rangle,$$

for $X, Y \in \Gamma(TM)$. Hence we can write

$$A_\nu X = -(\bar{\nabla}_X \nu)^{\intercal},$$

$$\bar{\nabla}_X \nu = (\bar{\nabla}_X \nu)^{\intercal} + (\bar{\nabla}_X \nu)^{\perp} = -A_\nu X + (\bar{\nabla}_X \nu)^{\perp}.$$

Here $(\bar{\nabla}_X \nu)^{\intercal}$ and $(\bar{\nabla}_X \nu)^{\perp}$ denote the component of $\bar{\nabla}_X \nu$ in TM and the component in $T^{\perp}M$ respectively. For $X \in \Gamma(TM)$ and $\nu \in \Gamma(T^{\perp}M)$, we write

$$\nabla^{\perp}_X \nu := (\bar{\nabla}_X \nu)^{\perp},$$

then it can be verified that $\nabla^{\perp}$ is a connection on $T^{\perp}M$, i.e., $C^{\infty}(M)$-linear in X, additive in ν and for $f \in C^{\infty}(M)$ (cf. page 135 of [4]),

$$\nabla^{\perp}_X (f\nu) = f\nabla^{\perp}_X \nu + (Xf)\nu.$$

$\nabla^{\perp}$ is called the *normal connection* of the submanifold M. The corresponding *normal curvature tensor* $R^{\perp}$ is given by

$$R^{\perp}(X, Y) = \nabla^{\perp}_X \nabla^{\perp}_Y - \nabla^{\perp}_Y \nabla^{\perp}_X - \nabla^{\perp}_{[X,Y]}. \tag{3.6}$$

Let $\bar{R}$ and R denote the curvature tensors of N and M respectively. By a long but direct calculation, the following formulas can be obtained (cf. page 135-138 of [4]).

The Gauss Equation:

$$\langle R(X,Y)Z, W\rangle = \langle \bar{R}(X,Y)Z, W\rangle + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle; \tag{3.7}$$

The Ricci Equation:

$$\langle \bar{R}(X,Y)\nu, \zeta\rangle - \langle R^{\perp}(X,Y)\nu, \zeta\rangle = \langle [A_\zeta, A_\nu]X, Y\rangle; \tag{3.8}$$

The Codazzi Equation:

$$(\nabla_X h)(Y, Z, \nu) - (\nabla_Y h)(X, Z, \nu) = \langle \bar{R}(X,Y)Z, \nu\rangle. \tag{3.9}$$

Here in the Ricci equation, we have used the notation

$$[A_\nu, A_\zeta] = A_\nu \circ A_\zeta - A_\zeta \circ A_\nu$$

for vector fields $\nu, \zeta \in \Gamma(T^\perp M)$; and in the Codazzi equation, we have used the notation

$$\begin{aligned}(\nabla_X h)(Y, Z, \nu) &= X\langle h(Y,Z), \nu\rangle - \langle h(\nabla_X Y, Z), \nu\rangle \\ &\quad -\langle h(Y, \nabla_X Z), \nu\rangle - \langle h(Y,Z), \nabla^\perp_X \nu\rangle.\end{aligned}$$

Let $\{\nu_1, ..., \nu_{n-m}\}$ be a local orthonormal frame of $T^\perp M$. We define

$$\begin{aligned}H: &= \frac{1}{m}\mathrm{Trace}(h) \\ &= \frac{1}{m}\sum_{j=1}^{n-m}\mathrm{Trace}(A_{\nu_j})\nu_j.\end{aligned}$$

Then H is a normal vector field on M, called the *mean curvature vector field.*

M is called a *minimal submanifold* of N if $H = 0$; and M is called a *totally geodesic submanifold* of N if $h = 0$. A submanifold is totally geodesic if and only if its shape operator satisfies $A_\nu = 0$ for any $\nu \in \Gamma(T^\perp M)$. The class of totally geodesic submanifolds can be generalized to a larger class of submanifolds, so-called totally umbilical submanifolds. If there exists a non-zero normal vector field $\nu \in \Gamma(T^\perp M)$ such that $A_\nu = \lambda I$, where $\lambda \in C^\infty(M)$, then ν is called an *umbilical section* on M, and M is called an *umbilical submanifold* with respect to ν. If M is umbilical with respect to every $\nu \in \Gamma(T^\perp M)$, then M is called a *totally umbilical submanifold.*

3.2 Hypersurfaces and typical examples

In this section we consider the special case that the m-dimensional manifold M is a hypersurface of the n-dimensional manifold N, i.e., $n - m = 1$. Typical examples of hypersurfaces with nice geometric properties are geodesic spheres and tubes.

Suppose M is orientable. We choose a unit normal vector field ν globally defined on M. Hence all umbilical hypersurfaces are totally umbilical. We fix an orientation of M and denote ν as the corresponding unit normal vector field. Then we can simply write $A := A_\nu$. For each $p \in M$, the eigenvalues $\lambda_1(p)$, ...,

$\lambda_m(p)$ of the linear operator $A(p)$ on T_pM are called *principal curvatures*; the corresponding eigenvectors X_1, ..., $X_m \in T_pM$ are called principal vectors. Without lost of generality, it can be assumed

$$\lambda_1(p) \leq ... \leq \lambda_m(p). \tag{3.10}$$

Let Q be the maximal open dense subset of M such that on each connected component of Q, the multiplicities of all principal curvature functions are constant. It can be proved that $\lambda_1, ..., \lambda_m$ are continuous functions on M and smooth functions on Q. The function

$$\hat{h} = \frac{1}{m}\mathrm{Trace}(A)$$

is called the *mean curvature* of M. The function

$$G = \mathrm{Det}(A)$$

is called the *Gauss-Kronecker curvature* of M (cf. page 231 of [63]). In particular, if N is a Euclidean space $\mathbb{R}^n$, then the Gauss-Kronecker curvature of M is also called *Gaussian curvature* (cf. page 96 of [57]).

Since we have assumed that ν is unit, we obtain $\langle \nabla^{\perp}_X \nu, \nu \rangle = 0$, hence

$$\nabla^{\perp}_X \nu = 0.$$

The Ricci equation becomes trivial. On the other hand, we have

$$\begin{aligned}(\nabla_X h)(Y, Z, \nu) &= X\langle AY, Z\rangle - \langle A\nabla_X Y, Z\rangle - \langle \nabla_X Z, AY\rangle \\ &= \langle (\nabla_X A)Y, Z\rangle + \langle A\nabla_X Y, Z\rangle + \langle AY, \nabla_X Z\rangle \\ &\quad -\langle A\nabla_X Y, Z\rangle - \langle \nabla_X Z, AY\rangle \\ &= \langle (\nabla_X A)Y, Z\rangle.\end{aligned}$$

Hence the Codazzi equation can be reduced to

$$\langle \bar{R}(X, Y)Z, \nu\rangle = \langle (\nabla_X A)Y, Z\rangle - \langle (\nabla_Y A)X, Z\rangle.$$

We give some typical examples of hypersurfaces in N.

Example 3.2.1. Geodesic (hyper)spheres

For any point $p \in N$, it can be proved that there exists $\epsilon(p) > 0$ such that the restriction of exp to $B_p(\epsilon)$ is a diffeomorphism onto an open subset of N. Here $B_p(\epsilon)$ denotes the ball in T_pM with radius ϵ origined at $0 \in T_pM$. For $0 < r \leq \epsilon$, let

$$G_p(r) = \{\exp(p, rv) : v \in T_pN, |v| = 1\}.$$

Then $G_p(r)$ is called a *geodesic (hyper)sphere* of radius r in N.

From the well-known Gauss Lemma, we have

Proposition 3.2.1. *(page 69-70 of [4]) $G_p(r)$ is a hypersurface in N, and $G_p(r)$ is diffeomorphic to S^{n-1}. Its unit normal vector field is given by*

$$\nu(exp(p, rv)) = dexp(p, rv)(0, v).$$

Example 3.2.2. Tubes

Let P be a submanifold of N. Suppose there exists a constant $\epsilon > 0$ such that the exponential map can be defined on $T^{\perp}P(\epsilon) = \{(p, u) \in T^{\perp}P : |u| \leq \epsilon\}$, and the restriction of exp to $T^{\perp}P(\epsilon)$ is injective. Suppose $d\exp(p, u)$ is non-degenerate for all $(p, u) \in T^{\perp}P(\epsilon)$. For $0 < r \leq \epsilon$, let

$$T(r) = \{\exp(p, rv) : (p, v) \in T^{\perp}P, |v| = 1\}.$$

Then $T(r)$ is a hypersurface in N, called a *tube* of distance (or radius) r over P.

As a generalization of Gauss Lemma, we have

Proposition 3.2.2. *(page 28-31 of [16]) Let $T(r)$ be a tube over P and a point $q \in T(r)$, $q = exp(p, rv)$, $(p, v) \in T^{\perp}P$ and $|v| = 1$. Then the unit normal vector field ν of $T(r)$ is given by*

$$\nu(q) = dexp(p, rv)(0, v).$$

We may consider a point in N as a 0-dimensional submanifold. In this case, the tube $T(r)$ is a geodesic sphere, and Proposition 3.2.2 will be reduced to Proposition 3.2.1.

3.3 Fundamental equations for hypersurfaces in real space forms and real hypersurfaces in complex space forms

In this section we state the Gauss equations and Codazzi equations of hypersurfaces in $\mathbb{R}^n$, S^n and H^n as well as real hypersurfaces in $\mathbb{C}^n$, $\mathbb{C}P^n$ and $\mathbb{C}H^n$. We state the following well-known theorem without proof.

Theorem 3.3.1. *(page 241-242 of [7]) Let N be a Riemannian manifold. Then for any point $p \in N$, the curvature tensor $\bar{R}$ of N at p is uniquely determined by the sectional curvatures of all the two-dimensional subspaces of the tangent space T_pN.*

Let $N_n(c)$ be $\mathbb{R}^n$ for $c = 0$, S^n for $c = 1$ and H^n for $c = -1$. Let M be an orientable hypersurface in $N_n(c)$, and ν a unit normal vector field. From the above theorem, since $N_n(c)$ has constant sectional curvatures c, it is straight forward that the expression of $\bar{R}$ is given by

$$\langle \bar{R}(X,Y)Z, W\rangle = c\{\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\}. \tag{3.11}$$

By applying (3.11), the Gauss equation and Codazzi equation for hypersurfaces in $N_n(c)$ are given respectively as follows:

$$R(X,Y)Z = c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = 0.$$

For the complex case, we also have a result similar to Theorem 3.3.1.

Theorem 3.3.2. *(page 64-67 of [8]) For an almost complex manifold N with almost complex structure J, if the curvature tensor $\bar{R}$ of N satisfies*

$$\langle \bar{R}(X,Y)Z, W\rangle = \langle \bar{R}(X,Y)JZ, JW\rangle,$$

for any $X, Y, Z, W \in \Gamma(TN)$, then for any point $p \in N$, the curvature tensor $\bar{R}$ at p is uniquely determined by the holomorphic sectional curvatures of all complex lines of the tangent space T_pN.

If N is a Kaehler manifold with Levi-Civita connection $\bar{\nabla}$, then $\bar{\nabla} J = 0$ and so

$$\begin{aligned}\langle \bar{R}(X,Y)JZ, JW\rangle &= \langle \bar{\nabla}_X\bar{\nabla}_Y JZ - \bar{\nabla}_Y\bar{\nabla}_X JZ - \bar{\nabla}_{[X,Y]}JZ, JW\rangle \\ &= \langle J\bar{\nabla}_X\bar{\nabla}_Y Z - J\bar{\nabla}_Y\bar{\nabla}_X Z - J\bar{\nabla}_{[X,Y]}Z, JW\rangle \\ &= \langle \bar{R}(X,Y)Z, W\rangle.\end{aligned}$$

Let $M_n(c)$ be $\mathbb{C}^n$ for $c = 0$, $\mathbb{C}P^n$ for $c = 1$, and $\mathbb{C}H^n$ for $c = -1$. Then we can verify that the curvature tensor $\bar{R}$ of the Kaehler manifold $M_n(c)$ is given by

$$\begin{aligned}\langle \bar{R}(X,Y)Z, W\rangle &= c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle X,JW\rangle\langle Y,JZ\rangle \\ &\quad -\langle X,JZ\rangle\langle Y,JW\rangle - 2\langle X,JY\rangle\langle Z,JW\rangle\}, \qquad (3.12)\end{aligned}$$

where $X, Y, Z, W \in \Gamma(TM_n(c))$. The above expression can be determined by using Theorem 3.3.2.

Let M be an orientable real hypersurface in $M_n(c)$ with unit normal vector field ν. Since $\langle J\nu, \nu\rangle = 0$, $J\nu \in \Gamma(TM)$. $\xi = -J\nu$ is called the *structure vector field* of M. Let D denote the distribution determined by tangent vectors perpendicular to ξ at each point of M. D is called the *holomorphic distribution* on M. We always assume $n \geq 2$ in this thesis since when $n = 1$ the real hypersurface M degenerated to a curve, which is trivial.

Now, we define a tensor field ϕ of type (1,1) and a 1-form η on M by

$$JX = \phi X + \eta(X)\nu, \quad \eta(X) = \langle X, \xi\rangle = \langle JX, \nu\rangle.$$

for $X \in \Gamma(TM)$. Then

$$\eta(\xi) = 1, \quad \eta(X) = 0$$

for $X \in \Gamma(D)$. By $J^2 = -I$ and $\langle JX, Y\rangle = -\langle X, JY\rangle$, we have

$$\begin{aligned}\langle \phi X, \xi\rangle &= \langle JX - \eta(X)\nu, \xi\rangle \\ &= \langle JX, \xi\rangle \\ &= -\langle X, J\xi\rangle \\ &= -\langle X, \nu\rangle \\ &= 0.\end{aligned}$$

Hence for $X \in \Gamma(TM)$, $\phi X \in \Gamma(D)$ and $\eta(\phi X) = 0$. Furthermore,

$$\begin{aligned} \phi^2 X &= \phi(JX - \eta(X)\nu) \\ &= J^2 X - \eta(X) J\nu \\ &= -X + \eta(X)\xi. \end{aligned}$$

For $X, Y \in \Gamma(TM)$,

$$\begin{aligned} \langle \phi X, Y \rangle &= \langle JX - \eta(X)\nu, Y \rangle \\ &= \langle JX, Y \rangle \\ &= -\langle X, JY \rangle \\ &= -\langle X, \phi Y \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \phi X, \phi Y \rangle &= \langle JX - \eta(X)\nu, JY - \eta(Y)\nu \rangle \\ &= \langle JX, JY \rangle - \langle JX, \eta(Y)\nu \rangle - \langle JY, \eta(X)\nu \rangle + \eta(X)\eta(Y) \\ &= \langle JX, JY \rangle - \eta(X)\eta(Y). \end{aligned}$$

From the above observation, the set of tensors $(\phi, \xi, \eta, \langle , \rangle)$ is an *almost contact metric structure* on M.

For $p \in M$, let $\rho(p) : T_p M_n(c) \longrightarrow T_p M$ be the canonical projection. Then by using the parallelism of J in $M_n(c)$, we have

$$\begin{aligned} (\nabla_X \phi) Y|_p &= \nabla_X(\phi Y)|_p - \phi(\nabla_X Y)|_p \\ &= \rho(p)\{\bar{\nabla}_X(\phi Y) - J(\nabla_X Y)\}|_p \\ &= \rho(p)\{\bar{\nabla}_X(JY - \eta(Y)\nu) - J(\nabla_X Y)\}|_p \\ &= \rho(p)\{J(\bar{\nabla}_X Y) - \eta(Y)\bar{\nabla}_X \nu - J(\nabla_X Y)\}|_p \\ &= \rho(p)\{Jh(X,Y)\nu - \eta(Y)\bar{\nabla}_X \nu\}|_p \\ &= \rho(p)\{\langle AX, Y \rangle J\nu + \eta(Y) AX\}|_p \\ &= \{\eta(Y) AX - \langle AX, Y \rangle \xi\}|_p, \end{aligned}$$

i.e.,

$$(\nabla_X \phi) Y = \eta(Y) AX - \langle AX, Y \rangle \xi.$$

Similarly, we have

$$\begin{aligned}
\{\nabla_X \xi\}|_p &= \rho(p)\{-\bar{\nabla}_X(J\nu)\}|_p \\
&= \rho(p)\{-J\bar{\nabla}_X(\nu)\}|_p \\
&= \rho(p)\{JAX\}|_p \\
&= \{\phi AX\}|_p.
\end{aligned}$$

Hence we obtain

$$\nabla_X \xi = \phi AX.$$

By using (3.12), the Gauss equation and Codazzi equation of M in $M_n(c)$ can be written in the following form respectively:

$$\begin{aligned}
R(X,Y)Z &= c\{\langle Y,Z\rangle X - \langle X,Z\rangle\ Y + \langle \phi Y,Z\rangle \phi X - \langle \phi X,Z\rangle \phi Y \\
&\quad -2\langle \phi X,Y\rangle \phi Z\} + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY,
\end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X,Y\rangle \xi\},$$

for $X,Y,Z \in \Gamma(TM)$.

Chapter 4

Theory and history on geometry of hypersurfaces in real and complex space forms

The study of hypersurfaces in space forms has a long history and is one of the most natural topics in Riemannian geometry of submanifolds. It origined from the study of surfaces in $\mathbb{R}^3$, then it has been generalized to hypersurfaces in $\mathbb{R}^n$, S^n and H^n. After that, the study of real hypersurfaces in complex space forms emerged. By now, differential geometers have studied extensively the geometric structures of real hypersurfaces in complex space forms. Many classifications and non-existence results of real hypersurfaces in non-flat complex space forms in terms of various conditions have been obtained.

In Section 4.1, we review briefly the history of Riemannian geometry of submanifolds in real space forms. Some significant theorems will be reviewed in this section. In Section 4.2, we give the foundational theory of real hypersurfaces in non-flat complex space forms. Then some important families as well as examples of real hypersurfaces in non-flat complex space forms will be given. In Section 4.3, we review some classifications and non-existence results of real hypersurfaces in non-flat complex space forms, which motivate our research.

4.1 History and origination of Riemannian submanifolds in space forms

In this section, we review the history and development of Riemannian submanifolds as well as hypersurfaces in real space forms. We state the Gauss Egregium Theorem, Nash's imbedding theorem, and some results on hypersurfaces in real space forms.

Riemannian geometry originated from the study of submanifolds of $\mathbb{R}^n$ in the past several centuries. There are two ways to identify two imbedded submanifolds M_1, M_2 in $\mathbb{R}^n$ as equivalent:

(1) M_1 is congruent to M_2 if there exists an isometry $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $T(M_1) = M_2$.

(2) M_1 and M_2 are isometric if there exists an isometry f between the Riemannian manifolds M_1 and M_2 such that $f(M_1) = M_2$.

Remark 4.1.1. Suppose there exists an isometry $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $T(M_1) = M_2$. Let f be the restriction of T on M_1. Then it can be shown that f is an isometry from M_1 to M_2. Hence (1) implies (2). However, (2) does not imply (1) (cf. page 3B-27 to page 3B-28 of [56]).

Geometers usually use the first way of identification when studying extrinsic properties of submanifolds of $\mathbb{R}^n$. Extrinsic properties of a submanifold M of $\mathbb{R}^n$ are determined by the Riemannian metric of M as well as its second fundamental form. While the second way of identification is often used when studying the intrinsic properties of the submanifolds, and intrinsic properties of M are determined by the Riemannian metric of M, independent from the second fundamental form.

A hypersurface in $\mathbb{R}^3$ is also called a *surface*. According to the classical theory of surfaces (cf. chapter 2 and chapter 3 of [56]), geometers give a surface, in most cases, by expressing its rectangular coordinates x_i as functions of two parameters u, v in a certain interval:

$$x_i = x_i(u, v), \quad u \in (u_1, u_2), \quad v \in (v_1, v_2), \quad i = 1, 2, 3.$$

In this case, only local properties are concerned and a surface is regarded as a chart of a 2-dimensional immersed Riemannian submanifold of $\mathbb{R}^3$.

Before Gauss found the *Gauss Egregium Theorem*, L. Euler and other mathematicians had considered curves and surfaces in $\mathbb{R}^3$ and studied their extrinsic properties. After Gauss found the Gauss Egregium Theorem in 1827, manifolds have become possible to be an independent concept, and intrinsic properties of manifolds have become an important research area. Lots of geometers began to study the intrinsic properties of Riemannian manifolds.

Theorem 4.1.1. (Gauss Egregium Theorem) *(cf. page 3B-27 of [56]) Let M be a 2-dimensional manifold and $f_1, f_2 : M \longrightarrow \mathbb{R}^3$ be two immersions. Then the canonical Riemannian metric of $\mathbb{R}^3$ induces Riemannian metrics g_1 on $f_1(M)$ and g_2 on $f_2(M)$. If $f_1^* g_1 = f_2^* g_2$, then for any point $p \in M$, the Gaussian curvature of $f_1(M) \subset \mathbb{R}^3$ at $f_1(p)$ equals to the Gaussian curvature of $f_2(M)$ at $f_2(p)$.*

Remark 4.1.2. Theorem 4.1.1 can be generalized to hypersurfaces in $\mathbb{R}^{n+1}$ (cf. page 98 of [57]). For a hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, the Gaussian curvature is invariant under isometry if n is even, and invariant up to sign if n is odd.

This theorem shows that the Gaussian curvature is intrinsic. Inspired by this theorem, geometers do not only study submanifolds of $\mathbb{R}^n$, but also study Riemannian manifolds.

A submanifold in $\mathbb{R}^n$ is a Riemannian manifold. At first glance, a Riemannian manifold might not be possible to be imbedded in $\mathbb{R}^n$. Hence it seems that the intrinsic study of Riemannian manifolds is much more general than the study of Riemannian submanifolds of $\mathbb{R}^n$ at some point. However, the fact is totally different. The following theorem, known as the *Nash's Imbedding Theorem*, was proved by John Nash in 1950s.

Theorem 4.1.2. *([36]) Every compact m-dimensional Riemannian manifold can be isometrically imbedded in any small portion of a $l(m)$-dimensional Euclidean space $\mathbb{R}^{l(m)}$ with $l(m) = \frac{1}{2}m(3m + 11)$. Every non-compact m-dimensional Riemannian manifold can be isometrically imbedded in any small* *portion of a $k(m)$-dimensional Euclidean space $\mathbb{R}^{k(m)}$ with $k(m) = \frac{1}{2}m(m + 1)(3m + 11)$.*

Remark 4.1.3. From Theorem 4.1.2, we see that when we study submanifolds of $\mathbb{R}^n$, we are studying all Riemannian manifolds indeed.

There are huge amounts of results on submanifolds of $\mathbb{R}^n$. For example, it is known that a hypersurface in $\mathbb{R}^n$ is totally umbilical if and only if it is locally congruent to $\mathbb{R}^{n-1}$ or S^{n-1}. It is natural to extend the study of submanifolds of $\mathbb{R}^n$ to submanifolds of space forms, i.e., submanifolds of $\mathbb{R}^n$, S^n and H^n. Let $N_n(c)$ be a real space form with constant sectional curvature c, where $c = 0$ or ± 1. Generally, the following theorem concerning totally umbilical submanifolds in $N_n(c)$ has been proved.

Theorem 4.1.3. *(page 50-page 54 of [6]) Let M be an m-dimensional totally umbilical submanifold of $N_n(c)$, $1 \leq m < n$. Then M is locally congruent to one of the following:*

(a) an m-dimensional totally geodesic subspace $N_m(c)$,

(b) a geodesic hypersphere of an $(m + 1)$-dimensional totally geodesic subspace $N_{m+1}(c)$.

A typical example of submanifolds is hypersurfaces, which is not as difficult to investigate as the general case of submanifolds. Because of this reason, many geometers have studied hypersurfaces in real space forms for quite a long time. It can be verified that all Riemannian manifolds of constant sectional curvatures are Einstein manifolds. Hence in $\mathbb{R}^n$, all totally umbilical hypersurfaces are Einstein manifolds. Generally, Einstein hypersurfaces of real space forms were classified through the work of Cartan and Thomas in [61] and Fialkow in [14] (cf. [5]).

4.2 Foundational theory on real hypersurfaces in $M_n(c)$

For non-flat complex space forms, many situations are quite different from real space forms. In this section, we study geometric structures of real hypersurfaces in non-flat complex space forms. We review the concepts of Hopf hypersurfaces, totally η-umbilical real hypersurfaces and ruled real hypersurfaces. We also give some celebrated theorems on these real hypersurfaces, especially for Hopf hypersurfaces. Then we give some examples of Hopf hypersurfaces in non-flat complex space forms.

In the rest of this chapter, let M be an orientable real hypersurface in a non-flat complex space form $M_n(c)$, $n \geq 2$. We denote by $\alpha = \eta(A\xi)$.

Definition 4.2.1. (page 244 of [37]) A real hypersurface M in $M_n(c)$ is called a *Hopf hypersurface* if ξ is principal, i.e., $A\xi = \alpha\xi$ at each point of M.

The theory of real hypersurfaces in complex space forms has been rapidly developed in the past four decades. Many properties and classifications of real hypersurfaces in complex space forms under various conditions have been proved. The next proposition is a critical step of the development of this theory.

Proposition 4.2.1. *For a Hopf hypersurface M in $M_n(c)$ with $A\xi = \alpha\xi$, α is constant.*

This proposition was first proved by M. Okumura in [40], 1975, for the case of $\mathbb{C}P^n$. In Chapter 2 of [37], we can find a proof for the case of $M_n(c)$. With the help of Proposition 4.2.1, since we have assumed M to be orientable, we can replace ν with $-\nu$ if necessary such that $\alpha \geq 0$ on the Hopf hypersurface M.

Proposition 4.2.2. *(page 245 of [37]) Let M be a Hopf hypersurface in $M_n(c)$. Then*

$$A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - c\phi = 0.$$

Let $X \in \Gamma(D)$ be a principal vector field with $AX = \lambda X$ on a Hopf hypersurface M, where λ is a function on M. Then by the above proposition, $(\lambda - \alpha/2)A\phi X - (\alpha\lambda/2 + c)\phi X = 0$. Furthermore, if ϕX is also a principal vector field with $A\phi X = \mu\phi X$, then λ and μ satisfy the relation

$$\lambda\mu - \frac{\alpha}{2}(\lambda + \mu) - c = 0. \tag{4.1}$$

The equation (4.1) is useful in the proof of our results in this thesis. In particular, if $\lambda = \mu$, then

$$\lambda^2 = \alpha\lambda + c. \tag{4.2}$$

In order to study the geometric structure of a Hopf hypersurface M, some differential geometers investigated the following map Φ_r, for $r > 0$:

$$\begin{aligned} \Phi_r : M &\longrightarrow M_n(c), \\ p &\longmapsto \exp(p, r\nu_p), \end{aligned}$$

where exp is the exponential map of $M_n(c)$. In [5] and [34], Hopf hypersurfaces have been studied under the assumption that the rank of Φ_r is constant q, i.e., the image of Φ_r

$$W = \{\Phi_r(p) | p \in M\}$$

is a q-dimensional real submanifold of $M_n(c)$.

Theorem 4.2.3. *([5]) Let M be an orientable Hopf hypersurface of $\mathbb{C}P^n$ with $\alpha = 2\cot 2r$. Suppose Φ_r has constant rank q on M. Then q is even and every point $p \in M$ has a neighborhood V such that $\Phi_r V$ is an imbedded complex $(q/2)$-dimensional submanifold of $\mathbb{C}P^n$, and V lies on a tube of radius r over $\Phi_r V$.*

Theorem 4.2.4. *([34]) Let M be an orientable Hopf hypersurface of $\mathbb{C}H^n$. Suppose Φ_r has constant rank q on M.*

(1) If $\alpha = 2\coth 2r$, then every point $p \in M$ has a neighborhood V such that $\Phi_r V$ is a complex $(q/2)$-dimensional submanifold of $\mathbb{C}H^n$, and V lies on a tube of radius r over $\Phi_r V$.

(2) If $\alpha \neq 2\coth 2r$, then every point $p \in M$ has a neighborhood V such that $\Phi_r V$ is an anti-holomorphic q-dimensional real submanifold of $\mathbb{C}H^n$, and V lies on a tube of radius r over $\Phi_r V$.

The most important family of Hopf hypersurfaces in $M_n(c)$ is Hopf hypersurfaces with constant principal curvatures. This family of Hopf hypersurfaces are well-behaved. Indeed, most of the conditions imposed on the real hypersurfaces lead to characterizations of certain subfamilies of Hopf hypersurfaces with constant principal curvatures. The following two celebrated theorems classify this family of well-behaved Hopf hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ respectively.

Theorem 4.2.5. *([20]) Let M be a Hopf hypersurface with constant principal curvatures in $\mathbb{C}P^n$. Then M is locally congruent to one of the following manifolds:*

(A_1) a geodesic hypersphere of radius r in $\mathbb{C}P^n$, where $0 < r < \pi/2$;

(A_2) a tube of radius r over totally geodesic $\mathbb{C}P^p$ $(0 < p < n-1)$, where $0 < r < \pi/2$;

(B) a tube of radius r over complex quadric Q_{n-1}, where $0 < r < \pi/4$;

(C) a tube of radius r over $\mathbb{C}P^1 \times \mathbb{C}P^{(n-1)/2}$, where $0 < r < \pi/4$ and $n > 4$ is odd;

(D) a tube of radius r over complex Grassmann $G_{2,5}$, where $0 < r < \pi/4$ and $n = 9$;

(E) a tube of radius r over Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.

Remark 4.2.1. In $\mathbb{C}P^n$, a real hypersurface is a geodesic hypersphere of radius r if and only if it is a tube of radius $\frac{\pi}{2} - r$ over $\mathbb{C}P^{n-1}$, where $0 < r < \pi/2$. Hence if we replace the parameter r by $\frac{\pi}{2} - r$, then (A_1) will be written as

(A_1) a tube of radius r over a hyperplane $\mathbb{C}P^{n-1}$, where $0 < r < \pi/2$.

Theorem 4.2.6. *([1]) Let M be a Hopf hypersurface with constant principal curvatures in $\mathbb{C}H^n$. Then M is locally congruent to one of the following:*

(A_0) a horosphere;

(A_1) a geodesic hypersphere or a tube over a hyperplane $\mathbb{C}H^{n-1}$;

(A_2) a tube over a totally geodesic $\mathbb{C}H^p$ $0 < p < n-1$;

(B) a tube over a totally real hyperbolic space $\mathbb{R}H^n$.

Remark 4.2.2. We call the lists in Theorem 4.2.5 and Theorem 4.2.6 as *Takagi's list* and *Montiel's list* respectively.

Now we will study an important subfamily of Hopf hypersurfaces: totally η-umbilical real hypersurfaces. Before the definition of totally η-umbilical real hypersurfaces in $M_n(c)$, the following proposition gives the reason why we study such a class of real hypersurfaces.

Proposition 4.2.7. *There does not exist any totally umbilical real hypersurface in $M_n(c)$.*

Proof. Suppose there exists a totally umbilical real hypersurface M with $\lambda \in C^\infty(M)$ such that $AX = \lambda X$ for all $X \in \Gamma(TM)$. Choose a unit vector field $X \in \Gamma(D)$ and let $Y = \phi X$ in the Codazzi equation,

$$\begin{aligned}(X\lambda)\phi X - (\phi X\lambda)X &= (\nabla_X A)\phi X - (\nabla_{\phi X} A)X \\ &= c\{\eta(X)\phi^2 X - \eta(\phi X)\phi X - 2\langle \phi X, \phi X\rangle \xi\} \\ &= -2c\xi.\end{aligned}$$

The left-hand side of the above equation is a cross-section of D, which cannot equal to $-2c\xi$ for $c \neq 0$. We get a contradiction hence M cannot exist. □

We can weaken the totally umbilical condition and study the so-called *totally η-umbilical real hypersurfaces* in $M_n(c)$. This class of real hypersurfaces do exist in $M_n(c)$.

Definition 4.2.2. For a real hypersurface M in $M_n(c)$, M is said to be totally η-umbilical if there exists a continuous function λ on M such that $AX = \lambda X + (\alpha - \lambda)\eta(X)\xi$.

By the above definition, it can be verified that totally η-umbilical real hypersurfaces in $M_n(c)$ satisfy $A\xi = \alpha\xi$. Hence with the help of Proposition 4.2.1, we have α is constant. It is direct to check that a real hypersurface M in $M_n(c)$ is totally η-umbilical if and only if $AX = \lambda X$ for all $X \in \Gamma(D)$. Furthermore, with the help of (4.2), we see that λ is constant. Hence a totally η-umbilical real hypersurface is one of the spaces in the Takagi's list and Montiel's list.

By investigating the shape operator of each class of real hypersurfaces listed in Theorem 4.2.5 and Theorem 4.2.6, totally η-umbilical real hypersurfaces in $M_n(c)$ can be classified immediately.

Theorem 4.2.8. *([5], [34]) Let M be a real hypersurface in $M_n(c)$. Then M is totally η-umbilical if and only if it is locally congruent to one of the following:*

For $c = 1$,

(a) a geodesic hypersphere in $\mathbb{C}P^n$.

For $c = -1$,

(a) a geodesic hypersphere in $\mathbb{C}H^n$;

(b) a tube around complex hyperbolic hyperplane in $\mathbb{C}H^n$;

(c) a horosphere in $\mathbb{C}H^n$.

The corresponding principal curvatures of the real hypersurfaces in Theorem 4.2.8 are in Table 4.1.

Table 4.1:

	Case	Radius	α	λ
$c > 0$	(a)	r	$2\cot 2r$	$\cot r$
$c < 0$	(a)	r	$2\coth 2r$	$\coth r$
$c < 0$	(b)	r	$2\coth 2r$	$\tanh r$
$c < 0$	(c)	-	2	1

Besides Hopf hypersurfaces, there is another important class of real hypersurfaces in $M_n(c)$: ruled real hypersurfaces.

Ruled real hypersurfaces of complex space forms are characterized by having a one-codimensional foliation whose leaves are complex totally geodesic submanifolds of the ambient space (cf. [30]). Equivalently, we have the following definition for ruled real hypersurfaces.

Definition 4.2.3. A ruled real hypersurface in $M_n(c)$ is a real hypersurface satisfying the condition $\langle AX, Y \rangle = 0$, for all vector fields $X, Y \in \Gamma(D)$.

Proposition 4.2.9. *Let M be a ruled real hypersurface in $M_n(c)$. Then M is non-Hopf.*

Proof. Let M be a ruled real hypersurface. Suppose to the contrary, we assume M to be Hopf. Then we have $A\xi = \alpha\xi$, for some constant α, and $\langle AY, Z \rangle = 0$ for any $Y, Z \in \Gamma(D)$. Hence $AY = 0$ for any $Y \in \Gamma(D)$ and $AX = \eta(X)A\xi = \alpha\eta(X)\xi$ for any $X \in \Gamma(TM)$. Therefore, M is a totally η-umbilical real hypersurface. With the help of Table 4.1, we see that such a totally η-umbilical real hypersurface does not exist. Therefore, M is non-Hopf. □

We can prove that a real hypersurface is ruled if and only if its shape operator could be expressed as $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$, and $AX = 0$ for $X \perp \text{Span}\{\xi, U\}$, where $\langle U, U \rangle = 1$ and β non-vanishing on an open dense subset of M. We also have the following lemma describing the covariant derivative of the shape operator for ruled real hypersurfaces.

Lemma 4.2.10. *([26],[58]) If M is a ruled real hypersurface in $M_n(c)$, $n \geq 3$, then we have*

$$(\nabla_X A)Y = \{-c\langle \phi X, Y \rangle + \eta(AY)\langle X, V \rangle + \eta(AX)\langle Y, V \rangle\}\xi$$

for all vector fields $X, Y \in \Gamma(D)$, where $V = \phi A\xi$.

There are also some theorems which characterize real hypersurfaces in $M_n(c)$ with restrictions on the number of distinct principal curvatures.

Theorem 4.2.11. *([5],[34]) Suppose $n \geq 3$ and M is a real hypersurface in $M_n(c)$ with at most two distinct principal curvatures at each point. Then M is locally congruent to one of the following:*

For $c = 1$,

(a) a geodesic hypersphere.

For $c = -1$,

(a) a horosphere,

(b) a geodesic hypersphere,

(c) a tube over a complex hyperbolic hyperplane,

(d) a tube of radius $\dfrac{\log(2+\sqrt{3})}{2}$ over a totally real hyperbolic space.

For real hypersurfaces in $M_n(c)$ with three distinct principal curvatures at each point, some results have also been obtained (cf. [2], [3], page 267-268 of [37]). We only list one of these results.

Theorem 4.2.12. *(page 267 of [37]) Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 3$. Suppose $\alpha^2 + 4c \neq 0$ and the number of distinct principal curvatures is three at each point. Then M is an open subset of a member of Takagi's list or Montiel's list.*

In the following examples, we give the constructions of type A_0, type A_1 and type A_2 real hypersurfaces listed in Theorem 4.2.5 and Theorem 4.2.6 (cf. [37]). Other real hypersurfaces in non-flat complex space forms can also be constructed by similar method.

Example 4.2.1. Real hypersurfaces of type A_1, type A_2 in $\mathbb{C}P^n$.

Let M' be a hypersurface in S^{2n+1} given by

$$\begin{aligned} M' &= S^{2p+1}(\cos r) \times S^{2q+1}(\sin r) \\ &= \{z = (z_1, z_2) \in \mathbb{C}^{p+1} \times \mathbb{C}^{q+1} |\ \langle z_1, z_1\rangle = \cos^2 r,\ \langle z_2, z_2\rangle = \sin^2 r\}, \end{aligned}$$

where $r \in (0, \pi/2)$ and integers $p \geq 0$, $q \geq 0$, $p + q = n - 1$. Then by a direct computation, a unit normal vector of M' at $z \in M'$ is given by

$$N_z = \tan r z_1 - \cot r z_2,$$

where we write $z_1 = (z_1, 0)$ and $z_2 = (0, z_2)$ for simplicity. Let A' be the shape operator of M' in S^{2n+1} corresponding to N_z at each $z \in M'$. Then it can be computed that

$$\begin{aligned} A'\bar{X}_z &= -\tan r \bar{X}_z, & \bar{X}_z &\in T_z S^{2p+1}(\cos r), \\ A'\bar{X}_z &= \cot r \bar{X}_z, & \bar{X}_z &\in T_z S^{2q+1}(\sin r). \end{aligned}$$

Let π be the projection from S^{2n+1} to $\mathbb{C}P^n$, $M = \pi(M')$, z be a point in M' and $x = \pi(z)$. Then x is a point in M. It follows that

$$\nu_x = d\pi(N_z),$$
$$\xi_x = -d\pi(iN_z),$$

and M is a real hypersurface in $\mathbb{C}P^n$ with unit normal vector ν_x at $x \in M$. Denote the shape operator of M in $\mathbb{C}P^n$ by A. Then by the definition of the shape operator and using the Riemannian submersion formulas at the end of Section 2.4,

$$A'X' = (AX)' + \eta(X)iz,$$
$$A'(iz) = \xi',$$

where we omit the point for simplicity and $(\ )'$ is the horizontal lift of $(\ )$. We let $T_1 = \{X \in D_x | X' \in T_z S^{2p+1}(\cos r)\}$ and $T_2 = \{X \in D_x | X' \in T_z S^{2q+1}(\sin r)\}$. Then

$$T_x M = \mathrm{Span}\{\xi\} \oplus T_1 \oplus T_2.$$

Let $\{e_1, .., e_{2p}; e_{2p+1}, ..., e_{2p+2q}; \xi\}$ be an orthonormal basis of $T_x M$, where $\{e_1, ..., e_{2p}\} \subset T_1$ and $\{e_{2p+1}, ..., e_{2p+2q}\} \subset T_2$. Then $\{e'_1, .., e'_{2p}; e'_{2p+1}, ..., e'_{2p+2q}; \xi', iz\}$ is an orthonormal basis of $T_z M'$. With respect to the basis $\{e'_1, ..., e'_{2p}; e'_{2p+1}, ..., e'_{2p+2q}; \xi, iz\}$, we have

$$A' = \begin{pmatrix} -\tan r I_{2p} & 0 & 0 & 0 \\ 0 & \cot r I_{2q} & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

By the formulas of Riemannian submersion, with respect to the basis $\{e_1, ..., e_{2p}; e_{2p+1}, ..., e_{2p+2q}; \xi\}$, we have

$$A = \begin{pmatrix} -\tan r I_{2p} & 0 & 0 \\ 0 & \cot r I_{2q} & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

Since

$$A'\xi' = (A\xi)' + iz = \alpha\xi' + iz,$$

$$\xi' = -iN_z = -\tan r iz_1 + \cot r iz_2,$$

$$iz_1 \in T_z S^{2p+1}(\cos r), \qquad iz_2 \in T_z S^{2q+1}(\sin r),$$

we have

$$\begin{aligned}
0 &= (A'\xi' - iz) - \alpha\xi' \\
&= -\tan r A'(iz_1) + \cot r A'(iz_2) - iz_1 - iz_2 - \alpha(-\tan r iz_1 + \cot r iz_2) \\
&= (-1 + \tan^2 r) iz_1 + (-1 + \cot^2 r) iz_2 - \alpha(-\tan r iz_1 + \cot r iz_2) \\
&= (\frac{\cos^2 r - \sin^2 r}{\sin r \cos r} - \alpha)(-\tan r iz_1 + \cot r iz_2) \\
&= (2\cot 2r - \alpha)\xi',
\end{aligned}$$

hence $\alpha = 2\cot 2r$.

When $p > 0$ and $q > 0$, M is a Hopf hypersurface of type A_2 in $\mathbb{C}P^n$, i.e., a tube of radius r $(0 < r < \frac{\pi}{2})$ over a complex projective subspace of codimension greater than 1. When $p = 0$ or $q = 0$, M is a Hopf hypersurface of type A_1. In detail, when $p = 0$, M is a geodesic hypersphere of radius r $(0 < r < \frac{\pi}{2})$ in $\mathbb{C}P^n$, which is also a tube of radius $\frac{\pi}{2} - r$ over a complex projective hyperplane. When $q = 0$, M is a geodesic hypersphere of radius $\frac{\pi}{2} - r$ $(0 < r < \frac{\pi}{2})$ in $\mathbb{C}P^n$, which is also a tube of radius r over a complex projective hyperplane.

Example 4.2.2. Real hypersurfaces of type A_1 and type A_2 in $\mathbb{C}H^n$.

Let M' be a hypersurface in H_1^{2n+1} given by

$$\begin{aligned}
M' &= H_1^{2p+1}(\cosh r) \times S^{2q+1}(\sinh r) \\
&= \{z = (z_1, z_2) \in \mathbb{C}_1^{p+1} \times \mathbb{C}^{q+1} |\ \langle z_1, z_1\rangle_1 = -\cosh^2 r,\ \langle z_2, z_2\rangle = \sinh^2 r\},
\end{aligned}$$

where $r > 0$ and integers $p \geq 0$, $q \geq 0$, $p + q = n - 1$. Then by a direct computation, a unit normal vector of M' at $z \in M'$ is given by

$$N_z = -(\tanh r z_1 + \coth r z_2).$$

Let A' be the shape operator of M' in H_1^{2n+1} corresponding to N_z at each $z \in M'$. Then it can be computed that

$$A'\bar{X}_z = \tanh r \bar{X}_z, \qquad \bar{X}_z \in T_z H_1^{2p+1}(\cosh r),$$
$$A'\bar{X}_z = \coth r \bar{X}_z, \qquad \bar{X}_z \in T_z S^{2q+1}(\sinh r).$$

Let π be the projection from H_1^{2n+1} to $\mathbb{C}H^n$, $M = \pi(M')$, z be a point in M' and $x = \pi(z)$. Then x is a point in M. It follows that

$$\nu_x = d\pi(N_z),$$
$$\xi_x = -d\pi(iN_z),$$

and M is a real hypersurface in $\mathbb{C}H^n$ with unit normal vector ν_x at $x \in M$. Denote the shape operator of M in $\mathbb{C}H^n$ by A. Then by the definition of the shape operator and using the semi-Riemannian submersion formulas at the end of Section 2.4,

$$A'X' = (AX)' - \eta(X)iz,$$
$$A'(iz) = \xi',$$

where we omit the point for simplicity and $(\ )'$ is the horizontal lift of $(\ )$. We let $T_1 = \{X \in D_x | X' \in T_z H_1^{2p+1}(\cosh r)\}$ and $T_2 = \{X \in D_x | X' \in T_z S^{2q+1}(\sinh r)\}$. Then

$$T_x M = \mathrm{Span}\{\xi\} \oplus T_1 \oplus T_2.$$

Let $\{e_1, .., e_{2p}; e_{2p+1}, ..., e_{2p+2q}; \xi\}$ be an orthonormal basis of $T_x M$, where $\{e_1, ..., e_{2p}\} \subset T_1$ and $\{e_{2p+1}, ..., e_{2p+2q}\} \subset T_2$. Then $\{e'_1, .., e'_{2p}; e'_{2p+1}, ..., e'_{2p+2q}; \xi', iz\}$ is an orthonormal basis of $T_z M'$. With respect to the basis $\{e'_1, ..., e'_{2p}; e'_{2p+1}, ..., e'_{2p+2q}; \xi, iz\}$, we have

$$A' = \begin{pmatrix} \tanh r I_{2p} & 0 & 0 & 0 \\ 0 & \coth r I_{2q} & 0 & 0 \\ 0 & 0 & \alpha & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

By the formulas of semi-Riemannian submersion, with respect to the basis $\{e_1, ..., e_{2p};\ e_{2p+1}, ..., e_{2p+2q}; \xi\}$, we have

$$A = \begin{pmatrix} \tanh r I_{2p} & 0 & 0 \\ 0 & \coth r I_{2q} & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

Since

$$A'\xi' = (A\xi)' - iz = \alpha\xi' - iz,$$

$$\xi' = -iN_z = \tanh r i z_1 + \coth r i z_2,$$

$$iz_1 \in T_z H_1^{2p+1}(\cosh r), \qquad iz_2 \in T_z S^{2q+1}(\sinh r),$$

we have

$$\begin{aligned} 0 &= A'\xi' + iz - \alpha\xi' \\ &= \tanh r A'(iz_1) + \coth r A'(iz_2) + iz_1 + iz_2 - \alpha(\tanh r iz_1 + \coth r iz_2) \\ &= (1 + \tanh^2 r)iz_1 + (1 + \coth^2 r)iz_2 - \alpha(\tanh r iz_1 + \coth r iz_2) \\ &= (\frac{\sinh^2 r + \cosh^2 r}{\sinh r \cosh r} - \alpha)(\tanh r iz_1 + \coth r iz_2) \\ &= (2\coth 2r - \alpha)\xi', \end{aligned}$$

hence $\alpha = 2\coth 2r$.

When $p > 0$ and $q > 0$, M is a Hopf hypersurface of type A_2 in $\mathbb{C}H^n$, i.e., a tube of radius r over a complex hyperbolic subspace of codimension greater than 1. When $p = 0$ or $q = 0$, M is a Hopf hypersurface of type A_1. In detail, when $p = 0$, M is a geodesic hypersphere of radius r in $\mathbb{C}H^n$. When $q = 0$, M is a tube of radius r over a complex hyperbolic hyperplane.

Example 4.2.3. Real hypersurfaces of type A_0 in $\mathbb{C}H^n$.

We construct a family of hypersurfaces in H_1^{2n+1} as follows:

$$M' = \{z = (z_0, ..., z_{n+1}) \in \mathbb{C}_1^{n+1} |\ \langle z, z\rangle_1 = -1,\ (z_0 - z_1)\overline{(z_0 - z_1)} = t,\ t > 0\}.$$

Then at $z \in M'$, it can be calculated that a unit normal vector on the hypersurface M' in H_1^{2n+1} is given by

$$N_z = \frac{1}{t}(z_0 - z_1, z_0 - z_1, 0, ..., 0) - (z_0, ..., z_{n+1}).$$

Let $M = \pi(M')$, z be a point in M' and $x = \pi(z)$. Then x is a point in M. It follows that

$$\nu_x = d\pi(N_z),$$

$$\xi_x = -d\pi(iN_z),$$

where ν_x is a unit normal vector of M. Let $\{e_1, .., e_{2n-2}; \xi\}$ be an orthonormal basis of T_xM with its corresponding hirozontal lift $\{e'_1, .., e'_{2n-2}; \xi'\}$, which together with iz, forms an orthonormal basis of T_zM'.

It can be proved that under the basis $\{e'_1, ..., e'_{2n-2}; \xi', iz\}$, the shape operator of M' is of the form

$$A' = \begin{pmatrix} I_{2n-2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By applying the formulas of semi-Riemannian submersion, we can obtain that with respect to the basis $\{e_1, .., e_{2n-2}; \xi\}$, the shape operator of M is given by

$$A = \begin{pmatrix} I_{2n-2} & 0 \\ 0 & 2 \end{pmatrix}.$$

In the following, by a real hypersurface of type A, we mean of type A_1, A_2 (resp. of A_0, A_1, A_2) for $c > 0$ (resp. $c < 0$).

4.3 Certain results concerning A, S and R_ξ

In this section, we review some classifications and non-existence results of real hypersurfaces in $M_n(c)$ in terms of A, S and the structure Jacobi operator R_ξ.

Definition 4.3.1. For a tensor field T of type $(1, 1)$, defined on a real hypersurface M in $M_n(c)$,

(1) T is called *parallel* if $\nabla T = 0$.

(2) T is called *recurrent* if there exists a 1-form ω on M such that $\nabla T = T \otimes \omega$, i.e., for any $X, Y \in \Gamma(TM)$, $(\nabla_X T)Y = \omega(X)TY$.

(3) T is called *D-parallel* if for any $X \in \Gamma(D)$, $\nabla_X T = 0$.

(4) T is called *D-recurrent* if there exists a 1-form ω on M such that for any $X \in \Gamma(D)$, $Y \in \Gamma(TM)$, $(\nabla_X T)Y = \omega(X)TY$.

(5) T is called *η-parallel* if for any $X, Y, Z \in \Gamma(D)$, $\langle(\nabla_X T)Y, Z\rangle = 0$.

(6) T is called *η-recurrent* if there exists a 1-form ω on M such that for any $X, Y, Z \in \Gamma(D)$, $\langle(\nabla_X T)Y, Z\rangle = \omega(X)\langle TY, Z\rangle$.

(7) T is called *ξ-parallel* if $\nabla_\xi T = 0$.

(8) T is called *ξ-recurrent* if $\nabla_\xi T = \lambda T$, where $\lambda \in C^\infty(M)$.

(9) T is called *cyclic-parallel* if for any $X, Y, Z \in \Gamma(TM)$,

$$\langle(\nabla_X T)Y, Z\rangle + \langle(\nabla_Y T)Z, X\rangle + \langle(\nabla_Z T)X, Y\rangle = 0.$$

(10) T is called *cyclic η-parallel* if for any $X, Y, Z \in \Gamma(D)$,

$$\langle(\nabla_X T)Y, Z\rangle + \langle(\nabla_Y T)Z, X\rangle + \langle(\nabla_Z T)X, Y\rangle = 0.$$

(11) T is called *cyclic-Ryan parallel* if for any $X, Y, Z \in \Gamma(TM)$,

$$(R(X, Y)T)Z + (R(Y, Z)T)X + (R(Z, X)T)Y = 0,$$

where

$$R(X, Y)T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T.$$

(12) T is called *Lie parallel* if $L_X T = 0$ for any $X \in \Gamma(TM)$.

(13) T is called *Lie ξ-parallel* if $L_\xi T = 0$.

(14) T is called *Lie D-parallel* if $L_X T = 0$ for all $X \in \Gamma(D)$.

(15) T is called of *Codazzi type* if

$$(\nabla_X T)Y = (\nabla_Y T)X \tag{4.3}$$

for all $X, Y \in \Gamma(TM)$.

Many of the above conditions have been studied for A, S and R_ξ respectively. A number of classifications and non-existence theorems have been achieved.

4.3.1 Characterizations of real hypersurfaces concerning A

In this subsection, we review some known results of real hypersurfaces in $M_n(c)$ concerning the shape operator.

Theorem 4.3.1. *(page 243 and page 264 of [37]) For a real hypersurface M in $M_n(c)$, the shape operator satisfies*

$$|\nabla A|^2 \geq 4c^2(n-1).$$

The equality occurs if and only if M is locally congruent to a real hypersurface of type A.

In the proof of Theorem 4.3.1, a tensor field T is constructed as

$$T(X,Y) = (\nabla_X A)Y + c\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\}.$$

It is proved that the equality in Theorem 4.3.1 holds if and only if $T = 0$. We have the following result as a byproduct of Theorem 4.3.1.

Theorem 4.3.2. *(page 262-264 of [37]) Let M be a real hypersurface in $M_n(c)$. Then M is a Hopf hypersurface of type A if and only if*

$$(\nabla_X A)Y = -c\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\} \tag{4.4}$$

for any $X, Y \in \Gamma(TM)$.

From Theorem 4.3.1, we know that there does not exist real hypersurfaces in $M_n(c)$ with parallel shape operator. As a generalization, the following theorem concerning the η-parallel condition on the shape operator has been proved.

Theorem 4.3.3. *([28]) Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Then the shape operator A is η-parallel if and only if M is locally congruent to a ruled real hypersurface, or a Hopf hypersurface of type A or type B.*

Besides conditions related to the parallelism of A, the commutativity of A and ϕ has also been considered.

Theorem 4.3.4. *([35],[40], page 262-264 of [37]) Let M be a real hypersurface in $M_n(c)$. Then the shape operator of M satisfies*

$$A\phi = \phi A,$$

if and only if M is locally congruent to a real hypersurface of type A.

Geometers have also considered a weaker condition than $A\phi = \phi A$ for a real hypersurface in $M_n(c)$.

Theorem 4.3.5. *([25]) Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, satisfying the conditions*

(1) $d\alpha(\xi)$ is nowhere zero in an open dense subset of M,

(2) $\langle(\phi A - A\phi)X, Y\rangle = 0$ for any $X, Y \in \Gamma(D)$.

Then M is locally congruent to a ruled real hypersurface.

The following lemma is useful for the proofs of our main results in Chapter 5 and Chapter 6.

Lemma 4.3.6. *([25]) Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose $\langle(\phi A - A\phi)X, Y\rangle = 0$ for all vector fields X, $Y \in \Gamma(D)$. Let $G_1 = \{x \in M : \| \phi A\phi \|_x \neq 0\}$. Then on G_1, we have $grad\,\alpha = \alpha V - 2AV$, where $V = \phi A\xi$. Furthermore, if we suppose $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\Gamma(D)$, and β is a nonvanishing function on G_1, and $AV = 0$, then we have $grad\,\beta = (c + \beta^2)\phi U$.*

4.3.2 Characterizations of real hypersurfaces concerning S

In this subsection, we review some known results of real hypersurfaces in $M_n(c)$ concerning the Ricci operator.

Theorem 4.3.7. *([27],[31]) Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 3$. If the Ricci operator S is η-recurrent, then M is locally congruent to a real hypersurface of type A or type B. Moreover, there exists no real hypersurfaces in $M_n(c)$, $n \geq 3$, with recurrent Ricci operator.*

From the above theorem we see the non-existence of real hypersurfaces with parallel Ricci operator in $M_n(c)$, $n \geq 3$. Hence geometers have studied conditions weaker than the parallelism of S. In [21], the condition

$$(\nabla_X S)Y = k\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\} \tag{4.5}$$

have been studied for real hypersurfaces in $\mathbb{C}P^n$. In [12], this condition was considered for real hypersurfaces in $\mathbb{C}H^n$.

Theorem 4.3.8. *([12], page 278 of [37]) Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Then there exists a non-zero constant k such that M satisfies (4.5) for any $X, Y \in \Gamma(TM)$ if and only if M is locally congruent to a totally η-umbilical real hypersurface.*

On the other hand, geometers have generalized the study of Einstein real hypersurfaces to *pseudo-Einstein* real hypersurfaces. A real hypersurface M in $M_n(c)$ is said to be pseudo-Einstein if the Ricci tensor Ric satisfies

$$Ric = ag + b\eta \otimes \eta,$$

where a, b are smooth functions on M. A real hypersurface M is pseudo-Einstein if and only if its Ricci operator satisfies

$$SX = aX + b\eta(X)\xi$$

for all $X \in \Gamma(TM)$. It has been proved that a, b must be constant on a pseudo-Einstein real hypersurface in $M_n(c)$ (cf. [5] for $n \geq 3$ and [19] for $n = 2$).

Pseudo-Einstein real hypersurfaces in $M_n(c)$, $n \geq 3$, have been studied in [5], [24] and [34]. The results were summarized in [37]. For the case of $n = 2$, pseudo-Einstein real hypersurfaces have been studied in [19]. For a pseudo-Einstein real hypersurface M in $M_2(c)$, M is proved to be Hopf.

Theorem 4.3.9. *(cf. page 271 of [37]) Let M be a pseudo-Einstein real hypersurface in $M_n(c)$, $n \geq 3$. Then M is locally congruent to one of the following:*

For $c = 1$,

(a) a geodesic hypersphere,

(b) a tube of radius r over a complex projective subspace $\mathbb{C}P^p$, $1 \leq p \leq n-2$, $0 < r < \pi/2$ and $cot^2 r = p/q$, where $p + q = n - 1$,

(c) a tube of radius r over a complex quadric Q^{n-1}, $0 < r < \pi/4$ and $cot^2 2r = n - 2$.

For $c = -1$,

(a) a geodesic hypersphere,

(b) a tube over a complex hyperbolic hyperplane,

(c) a horosphere.

From the results on pseudo-Einstein real hypersurfaces, we know the non-existence of Einstein real hypersurfaces in $M_n(c)$, $n \geq 3$, as a consequence. Therefore, geometers weakened the Einstein condition and have studied various weaker conditions for a real hypersurface in $M_n(c)$ in the last two decades.

For a real hypersurface M in $M_n(c)$, we say that M admits a *Ricci soliton* with the soliton vector field ξ if

$$\frac{1}{2}L_\xi g + Ric - \lambda g = 0$$

for a constant λ. Moreover, we say that M admits an *η-Ricci soliton* with the soliton vector field ξ if

$$\frac{1}{2}L_\xi g + Ric - \lambda g - \mu\eta \otimes \eta = 0$$

for some constants λ and μ.

Theorem 4.3.10. *([11]) A real hypersurface in $M_n(c)$ does not admit a Ricci soliton with the soliton vector field ξ.*

Theorem 4.3.11. *([11]) Let M be a real hypersurface in $M_n(c)$. If M admits an η-Ricci soliton, then M is locally congruent to one of the following:*

For $c = 1$,

(a) a geodesic hypersphere in $\mathbb{C}P^n$,

(b) a tube of radius r over $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for $1 \leq k \leq n-2$ and $cot^2 r = k/(n-k-1)$, $r \in (0, \pi/2)$.

For $c = -1$,

(a) a geodesic hypersphere in $\mathbb{C}H^n$,

(b) a tube over $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$,

(c) a tube over $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for $1 \leq k \leq n-2$,

(d) a horosphere in $\mathbb{C}H^n$.

In [15], the *generalized η-Ricci soliton* has been invented. We say that M admits a generalized η-Ricci soliton if

$$\frac{1}{2}L_\xi g(X, Y) + Ric(X, Y) + \lambda g(X, Y) = 0,$$

for λ constant and all $X, Y \in \Gamma(D)$.

4.3.3 Characterizations of real hypersurfaces concerning R_ξ

In this subsection, we retrospect the development of characterizations of real hypersurfaces in $M_n(c)$ concerning the structure Jacobi operator R_ξ, which offers the background and gives encouragement to our results in the last two chapters of this thesis.

From the Gauss equation, we have

$$\begin{aligned} R_\xi Y &= R(Y, \xi)\xi \\ &= c\{Y - \eta(Y)\xi\} + \alpha AY - \eta(AY)A\xi. \end{aligned} \tag{4.6}$$

It follows that

$$
\begin{aligned}
(\nabla_X R_\xi)Y &= \nabla_X(R_\xi Y) - R_\xi(\nabla_X Y) \\
&= \nabla_X\{c\{Y - \eta(Y)\xi\} + \alpha AY - \eta(AY)A\xi\} \\
&\quad -\{c\{\nabla_X Y - \eta(\nabla_X Y)\xi\} + \alpha A\nabla_X Y - \eta(A\nabla_X Y)A\xi\} \\
&= c\{-\eta(Y)\nabla_X\xi - \langle Y, \nabla_X\xi\rangle\xi\} + (X\alpha)AY + \alpha(\nabla_X A)Y \\
&\quad -\eta((\nabla_X A)Y)A\xi - \langle AY, \nabla_X\xi\rangle A\xi \\
&\quad -\eta(AY)(\nabla_X A)\xi - \eta(AY)A(\nabla_X\xi) \\
&= -c\langle Y, \phi AX\rangle\xi - c\eta(Y)\phi AX + (X\alpha)AY + \alpha(\nabla_X A)Y \\
&\quad -\langle(\nabla_X A)Y, \xi\rangle A\xi - \langle Y, A\phi AX\rangle A\xi \\
&\quad -\eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX, \qquad (4.7)
\end{aligned}
$$

for all vector fields X, Y tangent to M. Hence by using (4.6) and (4.7), the conditions concerning R_ξ in this thesis can be reduced to equations concerning A and ϕ.

In [42], the non-existence of real hypersurfaces in $M_n(c)$, $n \geq 3$, with parallel structure Jacobi operator has been proved.

Theorem 4.3.12. *([42]) There exist no real hypersurfaces M in $M_n(c)$, $n \geq 3$, whose structure Jacobi operator is parallel.*

Corollary 4.3.13. *There does not exist any real hypersurface M in $M_n(c)$, $n \geq 3$, with its structure Jacobi operator satisfying $R_\xi = kI$, where k is a function on M.*

Proof. To the contrary, we assume $R_\xi = kI$. Then we have $0 = R_\xi\xi = k\xi$. Hence $k = 0$ at each point of M. Therefore, $\nabla R_\xi = 0$. This contradicts Theorem 4.3.12 and we finish the proof. □

After Theorem 4.3.12, the study of conditions weaker than the parallelism of R_ξ was started, and a number of results have been obtained. One of these results is the non-existence of real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, with Codazzi type structure Jacobi operator.

Theorem 4.3.14. *([52]) There does not exist any real hypersurface in* $\mathbb{C}P^n$, $n \geq 3$, *with Codazzi type structure Jacobi operator.*

Another way to generalize the parallelism condition of R_ξ is to study the conditions of D-parallel, recurrent and D-recurrent structure Jacobi operator. The D-parallelism of R_ξ was considered with a result obtained as stated in the next theorem.

Theorem 4.3.15. *([53]) There does not exist any real hypersurface in* $\mathbb{C}P^n$, $n \geq 3$, *with D-parallel structure Jacobi operator.*

The recurrent condition of R_ξ was considered in [59] for real hypersurfaces in $M_n(c)$, $n \geq 3$, with a non-existence result obtained.

Theorem 4.3.16. *([59]) There exist no real hypersurfaces in* $M_n(c)$, $n \geq 3$, *with recurrent structure Jacobi operator.*

Moreover, the D-recurrent condition of R_ξ was considered for real hypersurfaces in $M_2(c)$ in [60].

Besides the parallel, D-parallel and recurrent conditions on R_ξ, geometers also have studied other kinds of parallelisms, such as cyclic-parallelism, cyclic-Ryan parallelism, etc. Real hypersurfaces in $M_n(c)$, $n \geq 3$, with cyclic-parallel structure Jacobi operator have been studied in [18]. Real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, with cyclic-Ryan parallel structure Jacobi operator have been studied in [46] and [48].

Besides the conditions concerning the covariant derivative of R_ξ, some other conditions on R_ξ have also been studied.

The study of R_ξ was started in 1990s by J.T. Cho, U.H. Ki, etc. In [10], J.T. Cho and U.H. Ki first studied the structure Jacobi operator of a geodesic hypersphere of $\mathbb{C}P^n$. From (4.6), the structure Jacobi operator of a totally η-umbilical real hypersurface in $M_n(c)$ satisfies

$$R_\xi = (c + \alpha\lambda)(I - \eta \otimes \xi).$$

By virtue of this fact, it is reasonable to consider the classification of real hypersurfaces in $M_n(c)$ under the condition

$$R_\xi = k(I - \eta \otimes \xi), \tag{4.8}$$

where k is a function on M. On the other hand, the condition

$$\phi R_\xi = R_\xi \phi \tag{4.9}$$

has also been considered in [10].

Remark 4.3.1. When J.T. Cho and U.H. Ki studied (4.8), the study of real hypersurfaces in $M_n(c)$ in terms of R_ξ was not fully developed. However, from the updated perspective, the condition (4.8) is reasonable. From Corollary 4.3.13, we see that $R_\xi = kI$ cannot be satisfied by any real hypersurface in $M_n(c)$. Hence it is natural to modify this condition slightly and consider (4.8).

From the non-existence of real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator, it is natural to consider

$$(\nabla_X R_\xi)Y = k\langle \phi AX, Y\rangle \xi \tag{4.10}$$

for any $X, Y \in \Gamma(D)$, where k is a non-zero canstant on M; and as a condition weaker than $\phi R_\xi = R_\xi \phi$, the following condition is reasonable:

$$(R_\xi \phi - \phi R_\xi)X = \omega(X)\xi \tag{4.11}$$

for any $X, Y \in \Gamma(D)$, where ω is a 1-form on the real hypersurface M. In [55], a classification was obtained for real hypersurfaces satisfying both (4.10) and (4.11) simultaneously.

Theorem 4.3.17. *([55]) Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. If the structure Jacobi operator of M satisfies (4.10) and (4.11) at the same time, then $k < 0$ and*

(i) when $k \neq -1$, M is locally congruent to a geodesic hypersphere of radius r such that $\cot^2(r) = -k$;

(ii) when $k = -1$, M is locally congruent to a tube over a complex submanifold of $\mathbb{C}P^n$ of radius $\pi/4$.

Characterizing real hypersurfaces concerning the Lie derivative of R_ξ is also an active research topic. In [47], the Lie parallel condition of R_ξ in $\mathbb{C}P^n$, $n \geq 3$, was considered. Recently, the Lie D-parallel condition was also considered by J.D. Pérez and Y.J. Suh in [54]. Indeed, they obtained a classification result under the Lie D-parallelism and $AR_\xi = R_\xi A$ for real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$. For the 2-dimensional case, real hypersurfaces in $M_2(c)$ with Lie D-parallel structure Jacobi operator were studied in [44]. In addition, a classification of real hypersurfaces in $M_n(c)$, $n \geq 3$, with Lie ξ-parallel R_ξ has been obtained in [51]. Besides the Lie parallelism, Lie D-parallelism and Lie ξ-parallelism of R_ξ, the following condition concerning the Lie derivative as well as the covariant derivative of R_ξ

$$L_\xi R_\xi = \nabla_\xi R_\xi$$

has also been considered for real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, in [49] and for real hypersurfaces in $M_2(c)$ in [45].

As generalizations of some of the theorems in this section, some new results of characterizing real hypersurfaces in terms of conditions on R_ξ will be given in the next two chapters, which is the main aim of this thesis.

Chapter 5

A characterization of ruled real hypersurfaces in $M_n(c)$ concerning the covariant derivative on R_ξ

In this chapter, let $M_n(c)$ be a non-flat complex space form. As in Theorem 4.3.12, there exist no real hypersurfaces with parallel structure Jacobi operator in $M_n(c)$, $n \geq 3$. In [53], J.D. Pérez, F.G. Santos and Y.J. Suh studied the non-existence of real hypersurfaces with D-parallel structure Jacobi operator in a complex projective space (cf. Theorem 4.3.15).

The condition of D-recurrent structure Jacobi operator was first considered in [50]. However, a part of the proof given for the classification of real hypersurfaces in $\mathbb{C}P^n$ with its structure Jacobi operator D-recurrent (paragraph 5, page 221 in [50]) cannot be justified. Actually, there does not exist any real hypersurface in $M_n(c)$ for $n \geq 3$ with D-recurrent structure Jacobi operator as will be shown in this chapter (cf. Theorem 5.3.1).

On the other hand, T. Theofanidis and P.J. Xenos proved in [59] that there does not exist any real hypersurface in $M_n(c)$, $n \geq 3$, with recurrent structure Jacobi operator (cf. Theorem 4.3.16). They also studied real hypersurfaces in $M_2(c)$ with D-recurrent structure Jacobi operator in [60].

The main purpose of this chapter is to study the condition

$$(\nabla_X R_\xi)\xi = 0, \tag{5.1}$$

for any $X \in \Gamma(D)$. We shall first prove that a real hypersurface is ruled if and only if its structure Jacobi operator satisfies (5.1) (cf. Theorem 5.2.1). It is clear that (5.1) is weaker than the parallelism, recurrence, D-parallelism and D-recurrence of R_ξ. Hence we can use this result to prove the non-existence of real hypersurfaces in $M_n(c)$, $n \geq 3$, with D-recurrent structure Jacobi operator (cf. Theorem 5.3.1).

In Section 5.1, we prove some useful lemmas. In Section 5.2, we give the characterization of ruled real hypersurfaces under (5.1). In Section 5.3, we use the result obtained in Section 5.2 to prove the non-existence of real hypersurfaces with D-recurrent as well as D-parallel structure Jacobi operator.

5.1 Some lemmas

In order to prove the main results in Chapter 5 and Chapter 6, we will give some lemmas in this section.

Let $\beta = ||\phi A\xi||$. If M is a non-Hopf real hypersurface in $M_n(c)$ then $\beta > 0$. We can define a unit vector field U in $\Gamma(D)$ by $U = -\dfrac{1}{\beta}\phi^2 A\xi$ and $A\xi = \alpha\xi + \beta U$. Moreover, we can define a distribution D_U by

$$D_U = \{X \in T_xM | X \perp \xi, U, \phi U\}, x \in M.$$

For $n \geq 3$, D_U is non-trivial. We also let $\gamma = \langle AU, U\rangle$ and $\delta = \langle A\phi U, \phi U\rangle$.

The following lemmas are essential for the proofs of the main results in Chapter 5 and Chapter 6.

Lemma 5.1.1. *Let M be a non-Hopf real hypersurface in $M_n(c)$, $n \geq 3$. Suppose there exists a unit vector field $Z \in \Gamma(D_U)$ such that $AZ = \lambda Z$ and $A\phi Z = \lambda\phi Z$.*

(a) If M satisfies

$$A\phi U = \delta\phi U \tag{5.2}$$

then

$$(\lambda - \delta)(\lambda^2 - \alpha\lambda - c) = \beta\phi U\lambda. \tag{5.3}$$

(b) If M satisfies

$$AU = \beta\xi + \gamma U \tag{5.4}$$

then

$$(\lambda - \gamma)(\lambda^2 - \alpha\lambda - c) - \beta^2\lambda = 0. \tag{5.5}$$

(c) If M satisfies both (5.2) and (5.4) then

$$\beta\lambda(\lambda - \delta) - (\lambda - \gamma)\phi U\lambda = 0. \tag{5.6}$$

Proof. Suppose M satisfies (5.2). Taking inner product in the Codazzi equation

$$(\nabla_Z A)\xi - (\nabla_\xi A)Z = -c\phi Z$$

with ϕZ, we obtain

$$\begin{aligned}
-c &= \langle \nabla_Z(A\xi) - A\nabla_Z\xi - \nabla_\xi(AZ) + A\nabla_\xi Z, \phi Z\rangle \\
&= \langle \nabla_Z(\alpha\xi + \beta U) - A\phi AZ - \nabla_\xi(AZ), \phi Z\rangle + \langle \nabla_\xi Z, A\phi Z\rangle \\
&= \langle (Z\alpha)\xi + \alpha\nabla_Z\xi + (Z\beta)U + \beta\nabla_Z U - \lambda^2\phi Z - \nabla_\xi(\lambda Z), \phi Z\rangle \\
&\quad + \langle \nabla_\xi Z, \lambda\phi Z\rangle \\
&= \langle \alpha\lambda\phi Z + \beta\nabla_Z U - \lambda^2\phi Z - \lambda\nabla_\xi Z - (\xi\lambda)Z, \phi Z\rangle + \lambda\langle \nabla_\xi Z, \phi Z\rangle \\
&= \alpha\lambda + \beta\langle \nabla_Z U, \phi Z\rangle - \lambda^2.
\end{aligned}$$

Hence

$$\beta\langle \nabla_Z U, \phi Z\rangle = \lambda^2 - \alpha\lambda - c. \tag{5.7}$$

Taking inner product in the Codazzi equation

$$(\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z = 0$$

with Z, we obtain

$$\begin{aligned}
0 &= \langle \nabla_Z(A\phi U) - A\nabla_Z\phi U - \nabla_{\phi U}(AZ) + A\nabla_{\phi U}Z, Z\rangle \\
&= \langle \nabla_Z(\delta\phi U) - A\nabla_Z\phi U - \nabla_{\phi U}(\lambda Z), Z\rangle + \langle \nabla_{\phi U}Z, \lambda Z\rangle \\
&= \langle (Z\delta)\phi U + \delta\nabla_Z\phi U, Z\rangle - \langle \nabla_Z\phi U, AZ\rangle - \phi U\lambda \\
&= (\delta - \lambda)\langle \nabla_Z\phi U, Z\rangle - \phi U\lambda.
\end{aligned}$$

Hence

$$(\delta - \lambda)\langle \nabla_Z\phi U, Z\rangle = \phi U\lambda.$$

On the other hand, we have

$$\begin{aligned}
\nabla_Z\phi U &= (\nabla_Z\phi)U + \phi\nabla_Z U \\
&= \eta(U)AZ - \langle AZ, U\rangle\xi + \phi\nabla_Z U \\
&= \phi\nabla_Z U.
\end{aligned}$$

Thus we obtain

$$\frac{\lambda^2 - \alpha\lambda - c}{\beta} = \langle \nabla_Z U, \phi Z\rangle = -\langle \phi\nabla_Z U, Z\rangle = -\langle \nabla_Z\phi U, Z\rangle.$$

It follows that

$$\begin{aligned}
\phi U\lambda &= (\delta - \lambda)\langle \nabla_Z\phi U, Z\rangle \\
&= -(\delta - \lambda)\frac{\lambda^2 - \alpha\lambda - c}{\beta}.
\end{aligned}$$

Hence we obtain (5.3).

Next, suppose M satisfies (5.4). Taking inner product in the Codazzi equation $(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z = -2c\xi$ with ξ and U respectively, we have

$$\begin{aligned}
-2c &= \langle(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, \xi\rangle \\
&= \langle \nabla_Z A\phi Z - A\nabla_Z \phi Z - \nabla_{\phi Z} AZ + A\nabla_{\phi Z} Z, \xi\rangle \\
&= \langle \nabla_Z(\lambda\phi Z) - \nabla_{\phi Z}(\lambda Z), \xi\rangle - \langle \nabla_Z \phi Z - \nabla_{\phi Z} Z, A\xi\rangle \\
&= \lambda\langle \nabla_Z \phi Z - \nabla_{\phi Z} Z, \xi\rangle - \langle \nabla_Z \phi Z - \nabla_{\phi Z} Z, \alpha\xi + \beta U\rangle \\
&= \lambda\{-\langle \phi Z, \nabla_Z \xi\rangle + \langle Z, \nabla_{\phi Z}\xi\rangle\} \\
&\quad +\alpha\{\langle \phi Z, \nabla_Z \xi\rangle - \langle Z, \nabla_{\phi Z}\xi\rangle\} + \beta\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle \\
&= \lambda\{-\langle \phi Z, \phi AZ\rangle + \langle Z, \phi A\phi Z\rangle\} \\
&\quad +\alpha\{\langle \phi Z, \phi AZ\rangle - \langle Z, \phi A\phi Z\rangle\} + \beta\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle \\
&= \lambda\{-\lambda + \lambda\langle Z, \phi^2 Z\rangle\} \\
&\quad +\alpha\{\lambda - \lambda\langle Z, \phi^2 Z\rangle\} + \beta\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle \\
&= -2\lambda^2 + 2\alpha\lambda + \beta\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle;
\end{aligned}$$

$$\begin{aligned}
0 &= \langle(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z, U\rangle \\
&= \langle \nabla_Z A\phi Z - A\nabla_Z \phi Z - \nabla_{\phi Z} AZ + A\nabla_{\phi Z} Z, U\rangle \\
&= \langle \nabla_Z(\lambda\phi Z) - \nabla_{\phi Z}(\lambda Z), U\rangle - \langle \nabla_Z \phi Z - \nabla_{\phi Z} Z, AU\rangle \\
&= \lambda\langle \nabla_Z \phi Z - \nabla_{\phi Z} Z, U\rangle - \langle \nabla_Z \phi Z - \nabla_{\phi Z} Z, \beta\xi + \gamma U\rangle \\
&= (\gamma - \lambda)\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle + \beta\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, \xi\rangle \\
&= (\gamma - \lambda)\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle + \beta\{\langle \phi Z, \phi AZ\rangle - \langle Z, \phi A\phi Z\rangle\} \\
&= (\gamma - \lambda)\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle + 2\beta\lambda.
\end{aligned}$$

Hence we obtain

$$\langle \nabla_{\phi Z} Z - \nabla_Z \phi Z, U\rangle = \frac{2(\lambda^2 - \alpha\lambda - c)}{\beta},$$

$$(\lambda - \gamma)(\langle \nabla_{\phi Z} Z, U\rangle - \langle \nabla_Z \phi Z, U\rangle) = 2\beta\lambda.$$

Combining these two equations, we obtain (5.5).

Finally if M satisfies both (5.2) and (5.4) then by using (5.3) and (5.5), we get (5.6). □

It is stated in [42] that there exist no real hypersurfaces M in $\mathbb{C}P^n$, $n \geq 3$, with shape operator given by $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$, $AX = -X$ for all $X \perp \xi, U$. However, some crucial steps in the proof were not written in detail (cf. page 1610 of [42]). We shall generalize this statement to $M_n(c)$ in the following lemma and give an alternative proof. This lemma enables us to solve several problems arising when proving our main results.

Lemma 5.1.2. *Let M be a non-Hopf real hypersurface in $M_n(c)$. Suppose M satisfies $A\xi = \epsilon c\xi + \beta U$, $AU = \beta\xi + \epsilon(\beta^2 - c)U$, $A\phi U = -\epsilon c\phi U$, where $\epsilon = \pm 1$. Then $c > 0$. Furthermore if $n \geq 3$, then there exists a vector field $X \in \Gamma(D_U)$ such that $AX \neq -\epsilon X$.*

Proof. Suppose M is such a real hypersurface. Taking inner product in the Codazzi equation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2c\xi$ with U and ξ respectively, we have

$$\begin{aligned}
0 &= \langle (\nabla_U A)\phi U - (\nabla_{\phi U} A)U, U\rangle \\
&= \langle \nabla_U A\phi U - A\nabla_U \phi U - \nabla_{\phi U} AU + A\nabla_{\phi U} U, U\rangle \\
&= \langle \nabla_U(-\epsilon c\phi U) - \nabla_{\phi U}(\beta\xi + \epsilon(\beta^2 - c)U), U\rangle \\
&\quad -\langle \nabla_U \phi U - \nabla_{\phi U} U, \beta\xi + \epsilon(\beta^2 - c)U\rangle \\
&= -\epsilon c\langle \nabla_U \phi U, U\rangle - \beta\langle \nabla_{\phi U}\xi, U\rangle - 2\epsilon\beta(\phi U\beta) \\
&\quad +\beta\langle \phi U, \nabla_U \xi\rangle - \epsilon(\beta^2 - c)\langle \nabla_U \phi U, U\rangle - \beta\langle U, \nabla_{\phi U}\xi\rangle \\
&= -\epsilon\beta^2\langle \nabla_U \phi U, U\rangle - \beta\langle \phi A\phi U, U\rangle - 2\epsilon\beta(\phi U\beta) \\
&\quad +\beta\langle \phi U, \phi AU\rangle - \beta\langle U, \phi A\phi U\rangle \\
&= -\epsilon\beta^2\langle \nabla_U \phi U, U\rangle - \beta\epsilon c - 2\epsilon\beta(\phi U\beta) + \beta\epsilon(\beta^2 - c) - \beta\epsilon c \\
&= \epsilon\beta\{-\beta\langle \nabla_U \phi U, U\rangle + \beta^2 - 3c - 2\phi U\beta\};
\end{aligned}$$

$$
\begin{aligned}
-2c &= \langle(\nabla_U A)\phi U - (\nabla_{\phi U} A)U, \xi\rangle \\
&= \langle\nabla_U(-\epsilon c\phi U) - \nabla_{\phi U}(\beta\xi + \epsilon(\beta^2 - c)U), \xi\rangle \\
&\quad -\langle\nabla_U \phi U - \nabla_{\phi U} U, \epsilon c\xi + \beta U\rangle \\
&= (-\epsilon c - \epsilon c)\langle\nabla_U \phi U, \xi\rangle - \phi U\beta \\
&\quad -\{\epsilon(\beta^2 - c) - \epsilon c\}\langle\nabla_{\phi U} U, \xi\rangle - \beta\langle\nabla_U \phi U, U\rangle \\
&= 2\epsilon c\langle\nabla_U \xi, \phi U\rangle - \phi U\beta \\
&\quad +(\epsilon\beta^2 - 2\epsilon c)\langle\nabla_{\phi U}\xi, U\rangle - \beta\langle\nabla_U \phi U, U\rangle \\
&= 2c(\beta^2 - c) - \phi U\beta \\
&\quad +(\epsilon\beta^2 - 2\epsilon c)\epsilon c - \beta\langle\nabla_U \phi U, U\rangle.
\end{aligned}
$$

Hence we obtain

$$
-\beta\langle\nabla_U \phi U, U\rangle + \beta^2 - 3c - 2\phi U\beta = 0, \tag{5.8}
$$

$$
-\beta\langle\nabla_U \phi U, U\rangle + 3c\beta^2 - 4c^2 + 2c - \phi U\beta = 0.
$$

From these two equations, we obtain

$$
\beta^2 - 3c\beta^2 + 4c^2 - 5c - \phi U\beta = 0. \tag{5.9}
$$

Taking inner product in the Codazzi equation

$$
(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = cU \tag{5.10}
$$

with U, ξ, ϕU respectively, we have

$$
\begin{aligned}
&(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U \\
&= \nabla_{\phi U} A\xi - A\nabla_{\phi U}\xi - \nabla_\xi A\phi U + A\nabla_\xi \phi U \\
&= \nabla_{\phi U}(c\epsilon\xi + \beta U) - A\phi A\phi U + \epsilon c\nabla_\xi \phi U + A\nabla_\xi \phi U \\
&= c\epsilon\phi A\phi U + (\phi U\beta)U + \beta\nabla_{\phi U} U \\
&\quad -\epsilon c(\beta\xi + \epsilon(\beta^2 - c))U + \epsilon c\nabla_\xi \phi U + A\nabla_\xi \phi U \\
&= (c^2 + c(c - \beta^2) + \phi U\beta)U + \beta\nabla_{\phi U} U + \epsilon c\nabla_\xi \phi U \\
&\quad +A\nabla_\xi \phi U - \epsilon c\beta\xi,
\end{aligned}
$$

$$
\begin{aligned}
c &= \langle (\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U, U\rangle \\
&= c^2 + c(c-\beta^2) + \phi U\beta + \epsilon c\langle \nabla_\xi \phi U, U\rangle + \langle \nabla_\xi \phi U, AU\rangle \\
&= c^2 + c(c-\beta^2) + \phi U\beta + \epsilon c\langle \nabla_\xi \phi U, U\rangle - \beta\langle \phi U, \nabla_\xi \xi\rangle \\
&\quad +\epsilon(\beta^2 - c)\langle \nabla_\xi \phi U, U\rangle \\
&= c^2 + c(c-\beta^2) + \phi U\beta - \beta^2 + \epsilon\beta^2\langle \nabla_\xi \phi U, U\rangle,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \langle (\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U, \xi\rangle \\
&= -\epsilon c\beta + \beta\langle \nabla_{\phi U} U, \xi\rangle + \epsilon c\langle \nabla_\xi \phi U, \xi\rangle + \langle \nabla_\xi \phi U, A\xi\rangle \\
&= -\epsilon c\beta - \beta\langle \nabla_{\phi U} \xi, U\rangle - \epsilon c\langle \nabla_\xi \xi, \phi U\rangle \\
&\quad +\langle \nabla_\xi \phi U, \epsilon c\xi + \beta U\rangle \\
&= -\epsilon c\beta - \beta\langle \nabla_{\phi U} \xi, U\rangle - 2\epsilon c\langle \nabla_\xi \xi, \phi U\rangle + \beta\langle \nabla_\xi \phi U, U\rangle \\
&= -4c\epsilon\beta + \beta\langle \nabla_\xi \phi U, U\rangle,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \langle (\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U, \phi U\rangle \\
&= \beta\langle \nabla_{\phi U} U, \phi U\rangle + \langle \nabla_\xi \phi U, A\phi U\rangle \\
&= \beta\langle \nabla_{\phi U} U, \phi U\rangle.
\end{aligned}
$$

Hence we obtain

$$\epsilon\beta^2\langle \nabla_\xi \phi U, U\rangle + 2c^2 - c\beta^2 - \beta^2 - c + \phi U\beta = 0, \tag{5.11}$$

$$\langle \nabla_\xi \phi U, U\rangle - 4\epsilon c = 0, \tag{5.12}$$

$$\langle \nabla_{\phi U} U, \phi U\rangle = 0. \tag{5.13}$$

From (5.11) and (5.12), we obtain

$$3c\beta^2 + 2c^2 - \beta^2 - c + \phi U\beta = 0. \tag{5.14}$$

By summing up (5.9) and (5.14), we obtain

$$c(c-1) = 0,$$

which cannot happen when $c = -1$. Hence $c = 1$ and (5.9) becomes

$$\phi U\beta = -2\beta^2 - 1. \tag{5.15}$$

By substituting (5.15) into (5.8), we have

$$\beta\langle\nabla_U\phi U, U\rangle = 5\beta^2 - 1. \tag{5.16}$$

From (5.12), we have

$$\langle\nabla_\xi\phi U, U\rangle = 4\epsilon. \tag{5.17}$$

In order to prove the second assertion, we shall suppose to the contrary that for any $Z \in \Gamma(D_U)$, $AZ = -\epsilon Z$ with $n \geq 3$. Take inner product in the Codazzi equation $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$ with U, ϕU, ξ respectively, by using $A\phi Z = -\epsilon\phi Z$ and $A\phi AZ = \phi Z$, we have

$$\begin{aligned}(\nabla_Z A)\xi - (\nabla_\xi A)Z &= \nabla_Z A\xi - A\nabla_Z\xi - \nabla_\xi AZ + A\nabla_\xi Z \\ &= \epsilon c\nabla_Z\xi + (Z\beta)U + \beta\nabla_Z U - A\phi AZ + \epsilon\nabla_\xi Z + A\nabla_\xi Z \\ &= -c\phi Z + (Z\beta)U + \beta\nabla_Z U - \phi Z + \epsilon\nabla_\xi Z + A\nabla_\xi Z,\end{aligned}$$

$$\begin{aligned}0 &= \langle(\nabla_Z A)\xi - (\nabla_\xi A)Z, U\rangle \\ &= Z\beta + \langle\epsilon\nabla_\xi Z + A\nabla_\xi Z, U\rangle \\ &= Z\beta + \epsilon\langle\nabla_\xi Z, U\rangle + \langle AU, \nabla_\xi Z\rangle \\ &= Z\beta + \epsilon\langle\nabla_\xi Z, U\rangle + \langle\beta\xi + \epsilon(\beta^2 - c)U, \nabla_\xi Z\rangle \\ &= Z\beta + \epsilon\langle\nabla_\xi Z, U\rangle + \langle\epsilon(\beta^2 - 1)U, \nabla_\xi Z\rangle,\end{aligned}$$

$$\begin{aligned}0 &= \langle(\nabla_Z A)\xi - (\nabla_\xi A)Z, \phi U\rangle \\ &= \beta\langle\nabla_Z U, \phi U\rangle + \epsilon\langle\nabla_\xi Z, \phi U\rangle + \langle\nabla_\xi Z, A\phi U\rangle \\ &= \beta\langle\nabla_Z U, \phi U\rangle,\end{aligned}$$

$$\begin{aligned}0 &= \langle(\nabla_Z A)\xi - (\nabla_\xi A)Z, \xi\rangle \\ &= \beta\langle\nabla_Z U, \xi\rangle + \epsilon\langle\nabla_\xi Z, \xi\rangle + \langle\nabla_\xi Z, A\xi\rangle \\ &= \langle\nabla_\xi Z, \epsilon\xi + \beta U\rangle \\ &= \beta\langle\nabla_\xi Z, U\rangle.\end{aligned}$$

Hence we have

$$\langle\nabla_\xi Z, U\rangle = 0$$

for any $Z \in \Gamma(D_U)$. By replacing Z with ϕZ in the above equation, we have

$$\langle \phi Z, \nabla_\xi U \rangle = -\langle \nabla_\xi \phi Z, U \rangle = 0,$$

and so

$$\begin{aligned}
\langle \nabla_\xi Z, \phi U \rangle &= -\langle Z, \nabla_\xi \phi U \rangle \\
&= -\langle Z, (\nabla_\xi \phi) U + \phi(\nabla_\xi U) \rangle \\
&= -\langle Z, \eta(U) A\xi - \langle A\xi, U \rangle \xi + \phi(\nabla_\xi U) \rangle \\
&= \langle \phi Z, \nabla_\xi U \rangle \\
&= 0.
\end{aligned}$$

By collecting all these above equations, we obtain the following:

$$Z\beta + \epsilon\beta^2 \langle \nabla_\xi Z, U \rangle = 0, \tag{5.18}$$

$$\langle \nabla_Z U, \phi U \rangle = 0, \tag{5.19}$$

$$\langle \nabla_\xi Z, U \rangle = 0, \tag{5.20}$$

$$Z\beta = 0. \tag{5.21}$$

Taking inner product in the Codazzi equation $(\nabla_Z A)U - (\nabla_U A)Z = 0$ with U and with the help of $Z\beta = 0$, we have

$$\begin{aligned}
0 &= \langle (\nabla_Z A)U - (\nabla_U A)Z, U \rangle \\
&= \langle \nabla_Z(\beta\xi + \epsilon(\beta^2 - 1)U) - A\nabla_Z U + \epsilon \nabla_U Z + A\nabla_U Z, U \rangle \\
&= \langle \epsilon \nabla_U Z, U \rangle + \langle -\nabla_Z U + \nabla_U Z, \beta\xi + \epsilon(\beta^2 - 1)U \rangle \\
&= \epsilon\beta^2 \langle \nabla_U Z, U \rangle.
\end{aligned}$$

Hence

$$\langle \nabla_U Z, U \rangle = 0. \tag{5.22}$$

Taking inner product with Z in (5.10), we have

$$\begin{aligned}
0 &= \langle \beta \nabla_{\phi U} U + \epsilon c \nabla_\xi \phi U + A \nabla_\xi \phi U, Z \rangle \\
&= \beta \langle \nabla_{\phi U} U, Z \rangle + \epsilon \langle \nabla_\xi \phi U, Z \rangle - \epsilon \langle \nabla_\xi \phi U, Z \rangle \\
&= \beta \langle \nabla_{\phi U} U, Z \rangle.
\end{aligned}$$

Hence

$$\langle \nabla_{\phi U} U, Z \rangle = 0. \tag{5.23}$$

On the other hand,

$$\begin{aligned} \langle \nabla_{\phi U} U, \xi \rangle &= -\langle U, \nabla_{\phi U} \xi \rangle \\ &= -\langle U, \phi A \phi U \rangle \\ &= -\langle U, (-\epsilon c)\phi^2 U \rangle \\ &= -\epsilon. \end{aligned}$$

From (5.13), (5.23) and the above equation, we obtain

$$\nabla_{\phi U} U = -\epsilon \xi. \tag{5.24}$$

Hence

$$\nabla_{\phi U} \phi U = (\nabla_{\phi U} \phi) U + \phi \nabla_{\phi U} U = 0. \tag{5.25}$$

From (5.16), (5.22) and $\langle \nabla_U U, \xi \rangle = 0$, we obtain

$$\nabla_U U = \frac{1 - 5\beta^2}{\beta} \phi U. \tag{5.26}$$

Hence

$$\nabla_U \phi U = \epsilon(1 - \beta^2)\xi + \frac{5\beta^2 - 1}{\beta} U. \tag{5.27}$$

From (5.17), (5.20) and $\langle \nabla_\xi U, \xi \rangle = 0$, we obtain

$$\nabla_\xi U = -4\epsilon \phi U. \tag{5.28}$$

Finally, we also have

$$\nabla_{\phi U} \xi = \phi A \phi U = \epsilon U. \tag{5.29}$$

Let $X = U$, $Y = \phi U$ and $Z = U$ in the Gauss equation, we have

$$\begin{aligned} R(U, \phi U)U &= -\phi U - \phi U - 2\phi U + \langle A\phi U, U \rangle AU - \langle AU, U \rangle A\phi U \\ &= (\beta^2 - 5)\phi U. \end{aligned} \tag{5.30}$$

On the other hand, it follows from (5.15), (5.24)–(5.29) and

$$R(U, \phi U)U = \nabla_U \nabla_{\phi U} U - \nabla_{\phi U} \nabla_U U - \nabla_{[U, \phi U]} U$$

that

$$
\begin{aligned}
R(U,\phi U)U &= -\epsilon\nabla_U\xi - \nabla_{\phi U}(\frac{(1-5\beta^2)\phi U}{\beta}) - \nabla_{\epsilon(1-\beta^2)\xi+(5\beta^2-1)U/\beta+\epsilon\xi}U \\
&= (1-\beta^2)\phi U - \phi U(\frac{1-5\beta^2}{\beta})\phi U - \frac{1-5\beta^2}{\beta}\nabla_{\phi U}\phi U \\
&\quad +\epsilon(\beta^2-2)\nabla_\xi U + \frac{1-5\beta^2}{\beta}\nabla_U U \\
&= (1-\beta^2)\phi U - \frac{(2\beta^2+1)(5\beta^2+1)}{\beta^2}\phi U \\
&\quad +4(2-\beta^2)\phi U + \frac{(1-5\beta^2)^2}{\beta^2}\phi U \\
&= (10\beta^2-8)\phi U. \qquad (5.31)
\end{aligned}
$$

From (5.30) and (5.31), we see that

$$10\beta^2 - 8 = \beta^2 - 5$$

at each point of M. Hence β is a constant on M. This contradicts (5.15). □

5.2 A characterization of ruled real hypersurfaces in $M_n(c)$ in terms of ∇R_ξ

Our aim in this section is to prove the following theorem.

Theorem 5.2.1. *([29]) Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, satisfying (5.1) for all vector fields $X \in \Gamma(D)$. Then M is locally congruent to a ruled real hypersurface.*

We need some preparations before the proof of Theorem 5.2.1.

Let M be a real hypersurface in $M_n(c)$. Note that by (4.7), we have

$$
\begin{aligned}
(\nabla_X R_\xi)\xi &= -c\phi AX + (X\alpha)A\xi + \alpha(\nabla_X A)\xi - \langle(\nabla_X A)\xi,\xi\rangle A\xi \\
&\quad -\langle\xi, A\phi AX\rangle A\xi - \alpha(\nabla_X A)\xi - \alpha A\phi AX \\
&= -c\phi AX + (X\alpha)A\xi - \{(X\alpha)A\xi - \langle A\nabla_X\xi,\xi\rangle A\xi - \langle A\xi, \nabla_X\xi\rangle A\xi\} \\
&\quad -\langle\xi, A\phi AX\rangle A\xi - \alpha A\phi AX \\
&= -c\phi AX + \langle A\phi AX,\xi\rangle A\xi - \alpha A\phi AX.
\end{aligned}
$$

Hence the condition (5.1) is equivalent to

$$c\phi AX + \alpha A\phi AX - \langle \phi AX, A\xi\rangle A\xi = 0, \tag{5.32}$$

for any tangent vector field $X \in \Gamma(D)$.

In the rest of this section, we further suppose that M satisfies (5.1), which is equivalent to (5.32), and $n \geq 3$. It follows from (5.32) that for any vector field $X, Y \in \Gamma(D)$,

$$\begin{aligned} 0 &= \langle c\phi AX + \alpha A\phi AX - \langle \phi AX, A\xi\rangle A\xi, Y\rangle \\ &\quad + \langle X, c\phi AY + \alpha A\phi AY - \langle \phi AY, A\xi\rangle A\xi\rangle, \end{aligned}$$

i.e.,

$$c\langle(\phi A - A\phi)X, Y\rangle = \langle \phi AX, A\xi\rangle\langle A\xi, Y\rangle + \langle X, A\xi\rangle\langle \phi AY, A\xi\rangle. \tag{5.33}$$

Proposition 5.2.2. *There does not exist Hopf hypersurfaces in $M_n(c)$, $n \geq 3$, satisfying the condition (5.1).*

Proof. Suppose M is such a Hopf hypersurface. The equation (5.33) becomes

$$\langle(\phi A - A\phi)X, Y\rangle = 0 \tag{5.34}$$

for all vector fields $X, Y \in \Gamma(TM)$. Hence $A\phi = \phi A$. Pointwisely, we get that $D_\lambda = \{X \in D : AX = \lambda X\}$ is ϕ-invariant. Hence by (4.2), we obtain

$$\lambda^2 = \alpha\lambda + c. \tag{5.35}$$

Let X be a unit principal vector field in D such that $AX = \lambda X$. Then by (5.32), we obtain

$$\lambda(c + \alpha\lambda) = 0.$$

From the above two equations, we get $\lambda^3 = 0$. Hence $\lambda = 0$ and this contradicts (5.35). □

By the above proposition, we see that M is non-Hopf. It follows from (5.33) that for any vector fields $X, Y \in \Gamma(D)$,

$$\begin{aligned} & c\langle(\phi A - A\phi)X, Y\rangle \\ = & \langle \phi AX, \alpha\xi + \beta U\rangle\langle \beta U, Y\rangle + \langle X, \beta U\rangle\langle \phi AY, \alpha\xi + \beta U\rangle \\ = & -\beta^2\{\langle Y, U\rangle\langle \phi U, AX\rangle + \langle X, U\rangle\langle \phi U, AY\rangle\}. \end{aligned} \tag{5.36}$$

Proposition 5.2.3. *For a non-Hopf real hypersurface M in $M_n(c)$, $n \geq 3$, satisfying condition (5.1), we have*

(a) $A\phi U = \delta\phi U$, *where δ is a function on M,*

(b) $AU = \beta\xi + \left(1 - \frac{\beta^2}{c}\right)\delta U$,

(c) $(\beta^2 - c)(c + \alpha\delta)\delta = 0$.

Proof. Let $X = Y = \phi U$ in (5.36). Then we have

$$\begin{aligned} 0 &= \langle c(\phi A - A\phi)\phi U, \phi U\rangle \\ &= c\langle \phi A\phi U, \phi U\rangle - c\langle A\phi^2 U, \phi U\rangle \\ &= c\langle A\phi U, U\rangle + c\langle AU, \phi U\rangle \\ &= 2c\langle AU, \phi U\rangle, \end{aligned}$$

i.e.,

$$\langle AU, \phi U\rangle = 0. \tag{5.37}$$

If we let $X = U$ and Y an arbitrary vector field in $\Gamma(D)$ in (5.36), then

$$c\langle \phi AU - A\phi U, Y\rangle = -\beta^2\langle A\phi U, Y\rangle. \tag{5.38}$$

On the other hand, we have

$$\begin{aligned} c\langle \phi AU - A\phi U, \xi\rangle &= -c\langle A\phi U, \xi\rangle \\ &= -c\langle \phi U, \alpha\xi + \beta U\rangle \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} -\beta^2\langle A\phi U, \xi\rangle &= -\beta^2\langle \phi U, A\xi\rangle \\ &= 0. \end{aligned}$$

Hence we see that (5.38) also holds for $Y = \xi$. This implies that (5.38) holds for all $Y \in \Gamma(TM)$ and we obtain

$$cA\phi U - c\phi AU = \beta^2 A\phi U. \tag{5.39}$$

By putting $X = \phi U$ and replacing Y with ϕY in (5.36), we have

$$c\langle(\phi A - A\phi)\phi U, \phi Y\rangle = -\beta^2\langle\phi Y, U\rangle\langle\phi U, A\phi U\rangle,$$

i.e.,

$$c\langle A\phi U - \phi AU, Y\rangle = \beta^2\langle A\phi U, \phi U\rangle\langle\phi U, Y\rangle, \tag{5.40}$$

for any vector field $Y \in \Gamma(D)$. On the other hand, since

$$c\langle A\phi U - \phi AU, \xi\rangle = c\langle\phi U, A\xi\rangle = 0,$$
$$\beta^2\langle A\phi U, \phi U\rangle\langle\phi U, \xi\rangle = 0,$$

we see that (5.40) also holds for $Y = \xi$. Therefore, (5.40) holds for all $Y \in \Gamma(TM)$ and we obtain

$$cA\phi U - c\phi AU = \beta^2\langle A\phi U, \phi U\rangle\phi U. \tag{5.41}$$

Putting $X = \phi U$ in (5.32) and taking inner product with U, we get

$$\begin{aligned} 0 &= \langle c\phi A\phi U + \alpha A\phi A\phi U - \langle\phi A\phi U, A\xi\rangle A\xi, U\rangle \\ &= (c - \beta^2)\langle\phi A\phi U, U\rangle + \alpha\langle A\phi A\phi U, U\rangle. \end{aligned} \tag{5.42}$$

From (5.39) and (5.41), we have

$$\begin{aligned} \beta^2 A\phi U &= cA\phi U - c\phi AU \\ &= \beta^2\langle A\phi U, \phi U\rangle\phi U, \end{aligned}$$

hence we get Proposition 5.2.3 (a). Next by using Proposition 5.2.3 (a) and (5.41), we obtain

$$\begin{aligned} c\phi AU &= cA\phi U - \beta^2 A\phi U \\ &= (c - \beta^2)\delta\phi U. \end{aligned}$$

Hence we get (b). Furthermore from (a), (b) and (5.42), we get (c). □

Proof of Theorem 5.2.1. From this proposition we know that D_U is invariant under A. In particular, for any vector field $X, Y \in \Gamma(D_U)$, (5.36) becomes

$$\langle(\phi A - A\phi)X, Y\rangle = 0. \tag{5.43}$$

Let $D_\lambda = \{X \in D_U : AX = \lambda X\}$ denote a pointwise subspace of D_U. Then D_λ is ϕ-invariant.

Let $Y \in \Gamma(D_U)$ be a unit vector field satisfying $AY = \lambda Y$ at each point, where λ is a continuous function on M. Then $A\phi Y = \lambda \phi Y$ at each point. From (5.32) we have

$$\lambda(c + \alpha\lambda) = 0. \tag{5.44}$$

From (5.44), we consider the following two cases.

Case 1. *$A = 0$ on D_U.*

Hence $D_U = D_0$ at each point of M, i.e., $AY = 0$ for any vector field $Y \in \Gamma(D_U)$. From (5.3), we have $\delta = 0$. Therefore, by Proposition 5.2.3, M satisfies $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$, $AX = 0$, for all vector fields X perpendicular to ξ and U at each point. This means that M is a ruled real hypersurface.

Case 2. *$A \neq 0$ on D_U.*

In this case, there exists a unit vector field $Y \in \Gamma(D_U)$, such that $AY = \lambda_1 Y$, where $\lambda_1 \neq 0$ on an open subset of M. We identify this subset with M. By (5.44), $\alpha \neq 0$ and $\lambda_1 = -c/\alpha$. From (5.5) in Lemma 5.1.1, we get

$$0 = \left(-\frac{c}{\alpha} - \delta\left(1 - \frac{\beta^2}{c}\right)\right)\frac{c^2}{\alpha^2} + \frac{\beta^2 c}{\alpha},$$

i.e.,

$$0 = \alpha^2\beta^2 - c^2 - \alpha\delta c + \alpha\beta^2\delta,$$

or

$$\alpha^2\beta^2 = \alpha(c - \beta^2)\delta + c^2. \tag{5.45}$$

By applying (5.45) to Proposition 5.2.3 (c),

$$\begin{aligned} 0 &= (\beta^2 - c)(c + \alpha\delta)\delta \\ &= \beta^2 c\delta - c^2\delta + (\beta^2 - c)\alpha\delta^2 \\ &= \beta^2 c\delta - c^2\delta + \delta(c^2 - \alpha^2\beta^2) \\ &= \beta^2 c\delta - \delta\alpha^2\beta^2 \\ &= (c - \alpha^2)\delta\beta^2. \end{aligned}$$

Hence we get

$$\delta(\alpha^2 - c) = 0. \tag{5.46}$$

We shall consider the following two subcases.

SUBCASE 2-A. *$\delta \neq 0$ at some point of M.*

By the continuity of δ, there exists an open subset G of M such that $\delta \neq 0$ at each point of G. From (5.46), $\alpha^2 = c$ on G. If necessary, we replace the normal vector field N by $-N$, so that $\alpha = c = 1$. Then $\lambda_1 = -1$ on G. Putting $\lambda = -1$ in (5.3), we obtain $\delta = -1$. From (5.44), for any principal unit vector field $Y \in \Gamma(D_U)$ such that $AY = \lambda Y$, $\lambda(\lambda+1) = 0$. Hence by the continuity, λ is constantly 0 or -1. By using (5.3), we see that $\lambda = -1$. This subcase cannot happen according to Lemma 5.1.2 for the case $\epsilon = 1$ and Proposition 5.2.3.

SUBCASE 2-B. *$\delta = 0$ at every point of M.*

By using $\delta = 0$ and (5.36), we obtain $\langle(\phi A - A\phi)X, Y\rangle = 0$ for all X, $Y \in \Gamma(D)$. We use the same notation G_1 as in Lemma 4.3.6. By the continuity of the norm, G_1 is an open subset of M. On G_1, by using Lemma 4.3.6, we have

$$\phi U\alpha = \alpha\beta \tag{5.47}$$

and

$$\phi U\beta = \beta^2 + c. \tag{5.48}$$

From (5.45), we have $\alpha^2\beta^2 = c^2$; then take the covariant derivative of this equation in the direction of ϕU,

$$\beta(\phi U\alpha) + \alpha(\phi U\beta) = 0. \tag{5.49}$$

Putting (5.47), (5.48) into (5.49), with the help of $\alpha \neq 0$, we get

$$2\beta^2 + c = 0.$$

Hence β is constant and by (5.48), we have $\beta^2 + c = 0$. This is a contradiction if G_1 is non-empty.

From the above argument we have G_1 must be empty and $\phi A\phi = 0$ must hold everywhere on M, hence M is a ruled real hypersurface. But this contradicts $D_{-c/\alpha}\neq 0$, which holds in the whole CASE 2. So SUBCASE 2-B is impossible.

From the above observation, the only possibility for M is that it is a ruled real hypersurface. Conversely, it is easy to check that ruled hypersurfaces satisfy (5.32). So we have completed the proof of Theorem 5.2.1.

□

5.3 Non-existence of real hypersurfaces in $M_n(c)$ with D-recurrent structure Jacobi operator

Theorem 5.2.1 in the previous section will lead to the proof of non-existence of real hypersurfaces with D-parallel or D-recurrent structure Jacobi operator in $M_n(c)$, as will be shown in this section.

Theorem 5.3.1. *([29]) There does not exist any real hypersurface M in $M_n(c)$, $n \geq 3$, with its structure Jacobi operator D-recurrent: $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$, for all vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(TM)$. Here ω denotes a 1-form on M.*

Proof. By virtue of Theorem 5.2.1, we only need to verify that the structure Jacobi operator R_ξ of ruled real hypersurfaces cannot be D-recurrent. Suppose there exists a ruled real hypersurface with its structure Jacobi operator D-recurrent. Then its shape operator satisfies $\langle AX, Y\rangle = 0$, for vector fields $X, Y \in \Gamma(D)$. From Lemma 4.2.10, it also satisfies $\langle(\nabla_X A)Y, Z\rangle = 0$ for all vector fields $X, Y, Z \in \Gamma(D)$.

We consider X, $Y \in \Gamma(D)$ for (4.7). First, taking inner product on both sides of (4.7) with a unit tangent vector Z in $\Gamma(D_U)$, and then applying (4.6), we obtain

$$\begin{aligned} -\eta(AY)\langle(\nabla_X A)Z,\xi\rangle &= \langle(\nabla_X R_\xi)Y,Z\rangle \\ &= \omega(X)\langle R_\xi Y,Z\rangle \\ &= c\omega(X)\langle Y,Z\rangle. \end{aligned}$$

It follows from Lemma 4.2.10 that this equation becomes

$$c\eta(AY)\langle\phi X,Z\rangle = c\omega(X)\langle Y,Z\rangle.$$

By putting $Y = U$ and $X = \phi Z$ in the above equation, we obtain $\beta = 0$, which is a contradiction. Hence such a ruled real hypersurface cannot exist. □

From the proof of Theorem 5.3.1, we get the following result:

Corollary 5.3.2. *There does not exist any ruled real hypersurface in $M_n(c)$, $n \geq 3$, with its structure Jacobi operator η-recurrent, i.e., $\langle(\nabla_X R_\xi)Y,Z\rangle = \omega(X)\langle R_\xi Y,Z\rangle$ for all vector fields $X,Y,Z \in \Gamma(D)$.*

From Theorem 5.3.1, we have the following result:

Corollary 5.3.3. *There does not exist any real hypersurface M in $M_n(c)$, $n \geq 3$, with its structure Jacobi operator D-parallel: $(\nabla_X R_\xi)Y = 0$, for all vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(TM)$.*

Remark 5.3.1. Theorem 4.3.16 can also be obtained immediately from Theorem 5.3.1.

Chapter 6

Characterizations for totally η-umbilical real hypersurfaces in $M_n(c)$

In this chapter, we assume that $M_n(c)$ is a non-flat complex space form with $n \geq 3$. We characterize totally η-umbilical real hypersurfaces in $M_n(c)$. We also obtain the non-existence of real hypersurfaces in $M_n(c)$ with Codazzi type structure Jacobi operator.

Since the shape operator A and the Ricci operator S cannot be parallel, the following conditions have been studied (cf. Theorem 4.3.2 and Theorem 4.3.8):

$$(\nabla_X A)Y = -c\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\},$$
$$(\nabla_X S)Y = k\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\}$$

for any $X, Y \in \Gamma(TM)$, where k is a constant. Also, it is known that R_ξ cannot be parallel (cf. Theorem 4.3.12). It is natural to consider a condition on ∇R_ξ that is similar to the above two conditions, i.e.,

$$(\nabla_X R_\xi)Y = k(\langle \phi X, Y\rangle \xi + \eta(Y)\phi X) \tag{6.1}$$

for all $X, Y \in \Gamma(TM)$.

On the other hand, the Codazzi type condition (4.3) is weaker than parallelism, and it is natural since for a totally geodesic hypersurface in a Riemannian manifold (if it exists), the shape operator is of Codazzi type. A

Riemannian manifold is said to have harmonic curvature if its Ricci operator is of Codazzi type. This cannot happen for a Hopf hypersurface in $M_n(c)$ (cf. [37, page 279]). In [52], the non-existence of real hypersurfaces in $\mathbb{C}P^n$ with Codazzi type structure Jacobi operator has been obtained. We will generalize this statement to $M_n(c)$. In fact, we shall first consider a condition weaker than the Codazzi type condition and (6.1), i.e.,

$$\begin{aligned} \langle(\nabla_X R_\xi)Y - (\nabla_Y R_\xi)X, W\rangle = \quad & k(2\eta(W)\langle\phi X, Y\rangle + \eta(Y)\langle\phi X, W\rangle \\ & -\eta(X)\langle\phi Y, W\rangle) \end{aligned} \tag{6.2}$$

for any $X, Y, W \in \Gamma(TM)$. Indeed, we prove the following theorem.

Theorem 6.0.4. *A real hypersurface M in $M_n(c)$ satisfies (6.2) for all $X, Y, W \in \Gamma(TM)$ if and only if M is locally congruent to a totally η-umbilical real hypersurface, or locally congruent to an arbitrary Hopf hypersurface with $\alpha = k = 1$, $c = -1$. Furthermore, we have $k \neq 0$.*

From Theorem 6.0.4, we can then characterize totally η-umbilical real hypersurfaces in $M_n(c)$ under the condition (6.1) as stated in the next theorem.

Theorem 6.0.5. *Let M be a real hypersurface in $M_n(c)$. Then M satisfies (6.1) for all $X, Y \in \Gamma(TM)$ if and only if M is locally congruent to a totally η-umbilical real hypersurface, i.e., one of the following real hypersurfaces:*

For $c > 0$,

(a) geodesic hyperspheres in $\mathbb{C}P^n$.

For $c < 0$,

(a) geodesic hyperspheres in $\mathbb{C}H^n$;

(b) tubes around complex hyperbolic hyperplane in $\mathbb{C}H^n$;

(c) horospheres in $\mathbb{C}H^n$.

Furthermore, we have $k \neq 0$.

We also obtain the non-existence of real hypersurfaces in $M_n(c)$ with Codazzi type structure Jacobi operator from Theorem 6.0.4.

Theorem 6.0.6. *There does not exist real hypersurface M in $M_n(c)$ with its structure Jacobi operator of Codazzi type.*

Finally, Theorem 6.0.4 and Theorem 6.0.5 give two equivalent characterizations for totally η-umbilical real hypersurfaces in $\mathbb{C}P^n$, as stated in the following corollary.

Corollary 6.0.7. *For a real hypersurface M in $\mathbb{C}P^n$, the following conditions are equivalent:*

(a) (6.1) holds for all $X, Y, W \in \Gamma(TM)$;

(b) (6.2) holds for all $X, Y \in \Gamma(TM)$;

(c) M is totally η-umbilical.

6.1 Proofs of the theorems

We will prove the theorems stated above in this section. We will first prove some propositions in preparation for the proof of our theorems.

Proposition 6.1.1. *Let M be a real hypersurface in $M_n(c)$. If M is totally η-umbilical then M satisfies (6.1). Furthermore, $k \neq 0$.*

Proof. For a totally η-umbilical real hypersurface M, we have

$$AX = \lambda X + (\alpha - \lambda)\eta(X)\xi \tag{6.3}$$

for any $X \in \Gamma(TM)$, where λ and α are two constants. Hence by (6.3), we have for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} \eta(AX) &= \lambda\eta(X) + (\alpha - \lambda)\eta(X)\eta(\xi) \\ &= \alpha\eta(X), \end{aligned} \tag{6.4}$$

$$\begin{aligned} A\phi AX &= A\phi(\lambda X + (\alpha - \lambda)\eta(X)\xi) \\ &= \lambda^2\phi X, \end{aligned} \tag{6.5}$$

$$\begin{aligned}(\nabla_X A)Y &= \nabla_X(AY) - A\nabla_X Y\\ &= \nabla_X(\lambda Y + (\alpha-\lambda)\eta(Y)\xi) - \lambda\nabla_X Y - (\alpha-\lambda)\eta(\nabla_X Y)\xi\\ &= (\alpha-\lambda)\{\langle Y, \nabla_X\xi\rangle\xi + \eta(Y)\nabla_X\xi\}\\ &= (\alpha-\lambda)\{\langle Y, \phi AX\rangle\xi + \eta(Y)\phi AX\}\\ &= \lambda(\alpha-\lambda)\{\langle Y, \phi X\rangle\xi + \eta(Y)\phi X\}. \qquad (6.6)\end{aligned}$$

Applying (6.3), (6.4), (6.5) and (6.6) to (4.7), we have

$$\begin{aligned}(\nabla_X R_\xi)Y &= -c\lambda\langle Y, \phi X\rangle\xi - c\lambda\eta(Y)\phi X\\ &\quad +\alpha\lambda(\alpha-\lambda)\{\langle Y, \phi X\rangle\xi + \eta(Y)\phi X\}\\ &\quad -\alpha\lambda(\alpha-\lambda)\langle Y, \phi X\rangle\xi - \alpha\lambda^2\langle Y, \phi X\rangle\xi\\ &\quad -\alpha\eta(Y)\lambda(\alpha-\lambda)\phi X - \alpha\eta(Y)\lambda^2\phi X\\ &= -\lambda(c+\alpha\lambda)\{\langle Y, \phi X\rangle\xi + \eta(Y)\phi X\}. \qquad (6.7)\end{aligned}$$

Hence M satisfies (6.1). From (4.2), we have $\lambda^2 = \alpha\lambda + c$, hence $\lambda \neq 0$. Then from the right-hand side of (6.7), we see that $k = -\lambda(c+\alpha\lambda) = -\lambda^3 \neq 0$. □

Proposition 6.1.2. *Let M be a Hopf hypersurface in $\mathbb{C}H^n$ with $\alpha = 1$. Then M satisfies (6.2) for $k = 1$ and does not satisfy (6.1) for $k = 1$.*

Proof. We suppose $k = 1$. Then (4.7) reduces to

$$\begin{aligned}(\nabla_X R_\xi)Y &= \langle Y, \phi AX\rangle\xi + \eta(Y)\phi AX + (\nabla_X A)Y - \langle Y, (\nabla_X A)\xi\rangle\xi\\ &\quad -\langle Y, A\phi AX\rangle\xi - \eta(Y)(\nabla_X A)\xi - \eta(Y)A\phi AX. \qquad (6.8)\end{aligned}$$

By a direct computation, we have $(\nabla_X A)\xi = \phi AX - A\phi AX$. Substituting this equation into (6.8), we obtain

$$(\nabla_X R_\xi)Y = (\nabla_X A)Y. \qquad (6.9)$$

By using (6.9) and the Codazzi equation, we see that M satisfies (6.2) for $k = 1$.

Next, suppose there exists a Hopf hypersurface M satisfying (6.1) for $k = 1$. Then by (6.9), (6.1) becomes

$$(\nabla_X A)Y = \langle \phi X, Y\rangle\xi + \eta(Y)\phi X. \qquad (6.10)$$

By Theorem 4.3.2, M is locally congruent to a Hopf hypersurface of type A with $\alpha = 1$. According to [37, page 254-257, 260], $\alpha \neq 1$ for real hypersurfaces of type A. This is a contradiction. Hence we conclude that M does not satisfy (6.1) for $k = 1$. □

Proposition 6.1.3. *Let M be a real hypersurface in $M_n(c)$ with its structure Jacobi operator satisfying (6.2). Then for any $X, Y, W \in \Gamma(TM)$,*

$$\begin{aligned} 0 &= \langle(\alpha c + k)(\eta(X)\phi Y - \eta(Y)\phi X - 2\langle\phi X, Y\rangle\xi) + 2\langle AX, \phi AY\rangle A\xi \\ &\quad + c(\langle X, (A\phi + \phi A)Y\rangle\xi + 2\langle\phi X, Y\rangle A\xi - \eta(Y)\phi AX + \eta(X)\phi AY) \\ &\quad + (X\alpha)AY - (Y\alpha)AX + \eta(AX)(\nabla_Y A)\xi - \eta(AY)(\nabla_X A)\xi \\ &\quad + \eta(AX)A\phi AY - \eta(AY)A\phi AX, W\rangle. \end{aligned} \tag{6.11}$$

Proof. Let M be a real hypersurface in $M_n(c)$ satisfying (6.2). Then by applying (4.7), the equation (6.2) becomes

$$\begin{aligned} 0 &= \langle(\nabla_X R_\xi)Y - (\nabla_Y R_\xi)X, W\rangle \\ &\quad - k(2\eta(W)\langle\phi X, Y\rangle + \eta(Y)\langle\phi X, W\rangle - \eta(X)\langle\phi Y, W\rangle) \\ &= \langle -c\langle Y, \phi AX\rangle\xi - c\eta(Y)\phi AX + (X\alpha)AY + \alpha(\nabla_X A)Y \\ &\quad - \langle(\nabla_X A)Y, \xi\rangle A\xi - \langle Y, A\phi AX\rangle A\xi \\ &\quad - \eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX \\ &\quad + c\langle X, \phi AY\rangle\xi + c\eta(X)\phi AY - (Y\alpha)AX - \alpha(\nabla_Y A)X \\ &\quad + \langle(\nabla_Y A)X, \xi\rangle A\xi + \langle X, A\phi AY\rangle A\xi \\ &\quad + \eta(AX)(\nabla_Y A)\xi + \eta(AX)A\phi AY, W\rangle \\ &\quad - k(2\eta(W)\langle\phi X, Y\rangle + \eta(Y)\langle\phi X, W\rangle - \eta(X)\langle\phi Y, W\rangle) \end{aligned}$$

for any $X, Y, W \in \Gamma(TM)$. With the help of the Codazzi equation, the above equation can be reduced to (6.11). □

Proposition 6.1.4. *Let M be a Hopf hypersurface in $M_n(c)$ satisfying (6.2). Then one of the following holds:*

(1) M is a totally η-umbilical real hypersurface;

(2) $\alpha = k = 1$, $c = -1$.

Proof. Recall that α is a constant when M is Hopf. Let $X = \xi$, $Y, W \in \Gamma(D)$ in (6.11). Then we have

$$(\alpha c + k)\langle \phi Y, W\rangle + (\alpha^2 + c)\langle \phi AY, W\rangle = 0. \tag{6.12}$$

Let $Y \in \Gamma(D)$ be a unit principal vector field with $AY = \lambda Y$ and let $W = \phi Y$ in (6.12). Then

$$(\alpha^2 + c)\lambda + (\alpha c + k) = 0. \tag{6.13}$$

We consider two cases: $\alpha^2 + c \neq 0$, $\alpha^2 + c = 0$.

Case-I. $\alpha^2 + c \neq 0$.

Since Y is an arbitrary unit principal vector field in $\Gamma(D)$, we have $AX = \lambda X$ for all $X \in \Gamma(D)$, where

$$\lambda = -\frac{\alpha c + k}{\alpha^2 + c}.$$

Therefore, M is totally η-umbilical.

Case-II. $\alpha^2 + c = 0$.

Then $c = -1$, and by replacing the unit normal vector field N with $-N$ if necessary, we have $\alpha = 1$. By (6.13), we have $k = \alpha = 1$. □

For the next step, we will first prove the non-existence of non-Hopf real hypersurfaces satisfying condition (6.1) or (6.2), and then prove the theorems.

Proposition 6.1.5. *There does not exist any non-Hopf real hypersurface M in $M_n(c)$ satisfying (6.2).*

Proof. Suppose M is a non-Hopf real hypersurface satisfying (6.2) with $A\xi = \alpha\xi + \beta U$, β a non-vanishing function and $U \in \Gamma(D)$ a unit vector field.

Let $W = \xi$ in (6.11). Then we have

$$\begin{aligned} 2k\langle \phi X, Y\rangle + 2\alpha\langle A\phi AX, Y\rangle + c\langle(\phi A + A\phi)X, Y\rangle \\ -\eta(AY)\eta(A\phi AX) + \eta(AX)\eta(A\phi AY) = 0 \end{aligned} \tag{6.14}$$

for any $X, Y \in \Gamma(TM)$. From (6.14), we have

$$\begin{aligned} 2k\phi X + 2\alpha A\phi AX + c(\phi A + A\phi)X \\ -\eta(A\phi AX)A\xi - \eta(AX)A\phi A\xi = 0 \end{aligned} \tag{6.15}$$

for any $X \in \Gamma(TM)$.

Let $X = \xi$ in (6.15). Then we have

$$\begin{aligned} 0 &= 2\alpha A\phi A\xi + c\phi A\xi - \eta(A\phi A\xi)A\xi \\ &= \alpha\beta A\phi U + c\beta\phi U. \end{aligned}$$

Hence $\alpha \neq 0$ and

$$A\phi U = -\frac{c}{\alpha}\phi U. \tag{6.16}$$

Let $X = \phi U$ in (6.15). Then with the help of (6.16), we have

$$\begin{aligned} 0 &= -2kU + 2\alpha A\phi A\phi U + c\phi A\phi U - cAU - \eta(A\phi A\phi U)A\xi \\ &\quad -\eta(A\phi U)A\phi A\xi \\ &= -2kU + 2cAU + \frac{c^2}{\alpha}U - cAU - \frac{c}{\alpha}\eta(AU)(\alpha\xi + \beta U) \\ &= cAU + (\frac{c^2}{\alpha} - 2k - \frac{c\beta^2}{\alpha})U - c\beta\xi. \end{aligned}$$

This means that

$$AU = \beta\xi + (\frac{2k}{c} + \frac{\beta^2 - c}{\alpha})U. \tag{6.17}$$

Hence D_U is A-invariant.

Let $X = \phi U$, $Y \in \Gamma(D_U)$, $W \in \Gamma(TM)$ in (6.11). Then we have

$$(\phi U\alpha)AY + \frac{c}{\alpha}(Y\alpha)\phi U = 0. \tag{6.18}$$

Taking inner product with ϕU in (6.18), we have $Y\alpha = 0$.

Let $X = U$, $Y \in \Gamma(D_U)$, $W \in \Gamma(TM)$ in (6.11). Then with the help of $Y\alpha = 0$, we have

$$(U\alpha)AY + \beta(\nabla_Y A)\xi + \beta A\phi AY = 0. \tag{6.19}$$

Let $X = \xi$, $Y \in \Gamma(D_U)$, $W \in \Gamma(TM)$ in (6.11). Then by the fact that $Y\alpha = 0$, we have

$$(\xi\alpha)AY + c\phi AY + (\alpha c + k)\phi Y + \alpha(\nabla_Y A)\xi + \alpha A\phi AY = 0. \tag{6.20}$$

From (6.19) and (6.20), we have

$$(\xi\alpha - \frac{\alpha}{\beta}U\alpha)AY + c\phi AY + (\alpha c + k)\phi Y = 0. \tag{6.21}$$

Let Y be a unit vector field with $AY = \lambda Y$ and then taking inner product with ϕY in (6.21), we obtain

$$c\lambda + \alpha c + k = 0. \tag{6.22}$$

Therefore,

$$AX = \lambda X, (\lambda = -\frac{\alpha c + k}{c}) \tag{6.23}$$

for any $X \in \Gamma(D_U)$. On the other hand if we put $X = Y$ in (6.15), then we have

$$\alpha\lambda^2 + c\lambda + k = 0. \tag{6.24}$$

From (6.22) and (6.24) we have

$$\lambda^2 = c. \tag{6.25}$$

Hence $c = 1$ and $\lambda = \pm 1$. It follows that $\phi U\lambda = 0$. So by (5.6) in Lemma 5.1.1, (6.16), (6.24) and (6.25), we obtain $\alpha = -\lambda(= \mp 1)$ and $k = 0$. By substituting all these quantities into (6.16), (6.17) and (6.23), we see that M satisfies the hypothesis of Lemma 5.1.2 but $AX = \pm X$ for any $X \in \Gamma(D_U)$. This is a contradiction and the proof is completed. □

Proof of Theorem 6.0.4. ($\Leftarrow$): It follows directly from Proposition 6.1.1 and Proposition 6.1.2.

($\Rightarrow$): Suppose M satisfies (6.2). By Proposition 6.1.4 and Proposition 6.1.5, M is either totally η-umbilical or a Hopf hypersurface with $\alpha = k = 1$, $c = -1$. Furthermore, by Proposition 6.1.1, we can see that $k \neq 0$. □

Proof of Theorem 6.0.5. ($\Leftarrow$): It has been proved in Proposition 6.1.1.

($\Rightarrow$): Suppose M is a real hypersurface satisfying (6.1). Then M also satisfies (6.2). Hence from Theorem 6.0.4, M is either totally η-umbilical or a Hopf hypersurface with $\alpha = k = 1$, $c = -1$. However, by Proposition 6.1.2, the latter case cannot occur. We conclude that M is totally η-umbilical, so it is locally congruent to one of the real hypersurfaces listed in Theorem 4.2.8. □

Proof of Theorem 6.0.6. Suppose the Jacobi operator is of Codazzi type. Then M satisfies (6.2) with $k = 0$. This contradicts the fact that $k \neq 0$ as stated in Theorem 6.0.4. The proof is completed. $\square$

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