ABSTRACT

The *clique graph* of a graph G is the graph obtained by taking the cliques of G as vertices, and two vertices are adjacent if and only if the corresponding cliques have non-empty intersection. A graph G is *self-clique* if it is isomorphic to its clique graph. Clique graphs have been studied for some time. However, not much is known about self-clique graphs. Self-clique graphs were first introduced and studied by Escalante [*Abh. Math. Sem. Univ. Hamburg* **39** (1973) 59-68]. Since then, self-clique graphs have been characterized for some classes of graphs. Chia [*Discrete Math.* **212** (2000) 185-189] gave a characterization of connected self-clique graphs in which all cliques have size two, except for precisely one clique.

The main objective in this thesis is to characterize all connected self-clique graphs with given clique sizes. Some known results on the characterizations of clique graphs and self-clique graphs are presented. We obtain a characterization for the set of all connected self-clique graphs having all cliques but two of size 2. We also give several results on connected self-clique graphs in which each clique has the same size k for $k = 2$ and $k = 3$.

ABSTRAK

Graf klik bagi sesuatu graf G diperolehi dengan mengambil klik-klik bagi G sebagai bucu-bucunya dan dua bucu adalah bersebelahan jika dan hanya jika klik-klik yang bersepadan dengan bucu-bucu tersebut mempunyai persilangan tak kosong. Graf G adalah *swa-klik* jika ia berisomorfisma dengan graf kliknya. Graf klik telah dikaji untuk sesuatu masa, tetapi tidak banyak yang diketahui mengenai graf swa-klik. Graf swa-klik mula diperkenalkan dan dikajikan oleh Escalante [*Abh. Math. Sem. Univ. Hamburg* **39** (1973) 59-68]. Sejak itu, graf swa-klik telah dicirikan untuk beberapa kelas graf. Chia [*Discrete Math.* **212** (2000) 185-189] memberikan pencirian bagi semua graf swa-klik terkait dengan semua klik mempunyai saiz 2 kecuali satu kliknya yang mempunyai saiz lebih daripada 2.

Tujuan utama tesis ini adalah mencirikan semua graf swa-klik terkait dengan saiz klik yang tertentu. Beberapa hasil kajian tentang pencirian graf klik dan swa-klik yang dikenali dipersembahkan. Pencirian bagi semua graf swa-klik terkait dengan semua klik mempunyai saiz 2 kecuali dua kliknya yang mempunyai saiz lebih daripada 2 diperolehi. Beberapa hasil berkenaan dengan graf swa-klik terkait dengan setiap kliknya mempunuyai saiz yang sama dengan k, di mana $k = 2$ dan $k = 3$ juga diberikan.

ACKNOWLEDGEMENTS

When I started out as a graduate student years ago, I had no idea what the future would bring. During the course of this work I have experienced happiness but also sorrow and despair. Had I known, I would probably have chosen to do something completely different, but fortunately predictions are difficult, especially if they concern the future. Over the years, this work has given me the chance to get to know many people, who have inspired me by their wisdom, helpfulness and good humor and altogether made the time spent on this project worthwhile.

First I wish to send a thought of thankfulness and utmost gratitude to my supervisor, Professor Chia Gek Ling for introducing me into this field. With his profound understanding of mathematic science, he served as a great inspiration to me. Without his countless hours of review and encouragement, this work would never have been completed. I also wish to thank the Head, academic and general staffs of the Institute of Mathematical Sciences, University of Malaya, for their assistance in various capacities. I would also like to thank University of Malaya for their financial support for this PhD.

I wish to express my gratitude and affection to my parents for always being there when I need them. Without their never ending support and encouragement this work would never have been possible. You bring sunshine into my life! Several other persons have some way or the other, contributed to this thesis without being mentioned below. Please accept my apologies, your contribution and support is also acknowledged.

Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. (See [24] for an introduction on intersection graphs.) Intersection graphs have received much attention in graph theory for some time and an early result showed that any undirected graph may be represented as an intersection graph [34]. Recently, intersection graphs have received much attention in the study of algorithmic graph theory and their applications [11, 20, 35]. On the other hand, many important families of graphs can be described as intersection graphs of more restricted types of set families. Some well-known special classes of intersection graphs include interval graphs, chordal graphs, circular-arc graphs, line graphs, clique graphs, string graphs and so on. There are several papers written on this subject, dated some sixty years ago. Besides that, two books, [11] and [35], appeared recently where intersection graphs play a central role.

There is an intersection graph associated with every graph which depends on its complete subgraphs. The clique graph of a given graph G is the intersection graph of the family of maximal complete subgraphs of G. By definition, a graph G is self-clique if G is isomorphic with its clique graph. The study of clique graphs begun over 30 years ago [23, 17, 25, 38], but very little is known about self-clique graphs. Clique graphs were also included in the books [11], [35] and [36]. The main objective in Chapter 2 is to study some characterizations of clique graphs.

Self-clique graphs are important in the study of clique graphs and the problem of characterizing self-clique graphs is one of the main unsolved problems in graph theory. Indeed, no good characterization of this class of graphs is known. Self-clique graphs were discovered and studied by Escalante [17] in 1973. Lately, self-clique graphs have received much attention ([4], [8], [9], [13], [15], [16], [28], [29], [30], [31]). We will discuss some of these results in Chapter 3.

The idea of constructing self-clique graphs which contain given lists of clique sizes was first introduced by Escalante [17], who characterized self-clique graphs in which all cliques have size two. This idea was extended by Chia [13], who characterized self-clique graphs all of whose cliques have size two, except for precisely one clique. Recently, Larrión et al. gave a characterization of the self-clique graphs such that at most 3 cliques have more than two vertices in [30]. The main objective in this thesis is to give some new characterizations of connected self-clique graphs with given clique sizes.

In Chapter 4, we present a characterization of connected self-clique graphs with exactly two cliques of size greater than two. We divide this class of graphs into six subclasses. The characterization seems to lead to an efficient recognition algorithm for this class of graphs.

Let $\mathcal{G}(k)$ denote the set of all connected self-clique graphs where each clique is of size k. In the last chapter, we determine (i) all graphs in $\mathcal{G}(2)$, (ii) all 4-regular graphs of $\mathcal{G}(3)$ and (iii) all those in $\mathcal{G}(3)$ in which the degree of any vertex is at most 4. We also show the existences of 5-regular graphs and 6-regular graphs in $\mathcal{G}(3)$. Further, we examine the existence of graphs in $\mathcal{G}(3)$ whose vertices are of mixed degrees.

In the following section of this chapter, we will present some basic definitions which will be frequently referred to throughout this thesis. For those terms and definitions not included here, the reader is referred to [7].

1.2 Basic Definitions

By a *graph* $G = (V, E)$, we mean a finite non-empty set $V = V(G)$, called the set of *vertices* of G, and a finite set $E = E(G)$, called the set of *edges* of G, consisting of unordered pairs of distinct elements of $V(G)$. $V(G)$ is also called the *vertex-set* of G and $E(G)$ is also called the *edge-set* of G.

Let u and v be two vertices of G. Any edge e of the form uv is said to *join* u and v; the vertices u and v are called the *ends* of e. In this case, u and v are said to be *adjacent* and e is said to be *incident* to u and v. Two distinct edges of G are *adjacent* if they have at least one vertex in common.

A graph G whose edge-set $E(G)$ is empty is called a *null graph* and G is called a *complete graph* if every two vertices are adjacent in G. A complete graph with n vertices is denoted by K_n. A *triangle* is a complete graph with exactly three vertices.

A *bipartite graph* is a graph whose vertex-set can be split into two sets X and Y in such a way that each edge of the graph joins a vertex in X to a vertex in Y. A *complete bipartite graph* is a bipartite graph in which each vertex in X is joined to each vertex in Y by exactly one edge. We denote the complete bipartite graph by $K_{r,s}$ where $|X| = r$ and $|Y| = s$.

A *subgraph* of G is a graph $S = (V(S), E(S))$ such that $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$. If $V(S) = V(G)$, then S is called a *spanning subgraph* of G. If $W \subseteq V(G)$, then the *subgraph induced by* W (denoted by $G[W]$) is the subgraph of G whose vertex-set is W and whose edge-set is the set of those edges of G that have both ends in W. If S is a complete graph, then S is called a *complete subgraph* of G.

Two or more edges joining the same pair of vertices are called *multiple edges*, and an edge joining a vertex to itself is called a *loop*. A graph with neither loops nor multiple edges is called a *simple graph*. Throughout this thesis, we are mainly concerned with simple graphs.

The *degree* of a vertex $v \in V(G)$ is the number of edges incident to v, and is denoted by $d(v)$. A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is called an *end vertex*. We say that a simple graph G is *regular* if all the vertices of G have the same degree. In particular, if the degree of each vertex is r, then G is *regular of degree r*.

Two graphs G and H are said to be *isomorphic*, written $G \cong H$, if there is a one-to-one mapping ϕ from $V(G)$ onto $V(H)$ such that $xy \in E(G)$ if and only if $\phi(x)\phi(y) \in E(H)$.

A *walk* of a graph G, $W = v_0e_1v_1e_2v_2 \ldots e_kv_k$, is an alternating sequence of vertices and edges such that the ends of e_i are v_{i-1} and v_i for $1 \leq i \leq k$. It is *closed* if $v_0 = v_k$ and is *open* otherwise. W is called a *path* if the vertices $v_0, v_1, \ldots, v_k$ are distinct. A *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. Let P_n and C_n denote a path and a cycle on n vertices respectively.

A graph G is *connected* if every two vertices are joined by a path in G and it is *disconnected* otherwise. Any disconnected graph G can be expressed as a union of a finite number of connected subgraphs. Each of these connected subgraphs is called a *component* of G.

Chapter 2

Clique Graphs

2.1 Introduction

Let Q be a complete subgraph of G. If Q is not properly contained in any other complete subgraph of G, then Q is called a *maximal complete subgraph* of G. A *clique* of G is a maximal complete subgraph of G. Let $\mathcal{K}(G)$ denote the set of all cliques of G. Then the *clique graph* of G, denoted by $K(G)$, is the graph whose vertex-set is $\mathcal{K}(G)$ and two vertices are adjacent in $K(G)$ if and only if the corresponding cliques have a non-empty intersection. Throughout this thesis, $\mathcal{K}(G)$ always stands for the set of all cliques of G and $K(G)$ to be the clique graph of G. Henceforth we refrain from mentioning it. On the other hand, we call G a *clique inverse* graph of $K(G)$.

Example 2.1 Let G and H be the graphs shown in Figure 2.1. There are four cliques in the graph G. That is, $\mathcal{K}(G) = \{Q_1, Q_2, Q_3, Q_4\}$. So $V(K(G))$ $= \mathcal{K}(G)$ and $E(K(G)) = \{Q_i\, Q_j \mid Q_i \cap Q_j \neq \emptyset,\ i \neq j\}$. Hence the clique graph of G is the graph H which is K_4, and G is a clique inverse of the graph H.

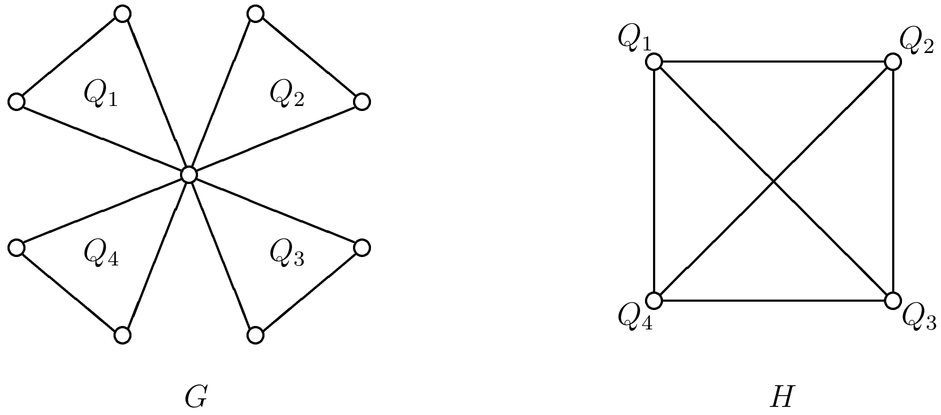


Figure 2.1: H is the clique graph of G and G is a clique inverse graph of H

Clique graphs have been considered in a number of articles [23, 17, 25, 38]. A problem of interest, in the context of intersection graph theory and especially in the study of clique graphs, is to characterize clique graphs of special classes of graphs. Many questions about clique graphs are still open. Probably one of the important questions is related to the characterization of clique graphs.

This chapter is a survey of results concerning the characterization and recognition of some classes of clique graphs.

2.2 A General Characterization

A graph G is said to have the *Helly property* (or is *Helly*) if for every subset $\{Q_i \mid i \in I\} \subseteq \mathcal{K}(G)$ with $Q_i \cap Q_j \neq \emptyset$ for all $i, j \in I$, then the total intersection is non-empty, that is, $\bigcap_{i \in I} Q_i \neq \emptyset$. We call such a graph a *clique-Helly graph*. Clique-Helly graphs have been introduced in [23] and they have played a central role in the study of clique graphs.

Example 2.2 Let G be the graph shown in Figure 2.1. Then $\mathcal{K}(G) = \{Q_1, Q_2, Q_3, Q_4\}$. We see that $Q_i \cap Q_j \neq \emptyset$ for all $1 \leq i, j \leq 4$, and

$Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \neq \emptyset$. Thus the graph G satisfies the Helly property and so G is a clique-Helly graph.

Example 2.3 The graph Z in Figure 2.2 is called the *Hajós graph*. We see that there are four cliques Q_1, Q_2, Q_3 and Q_4 in the Hajós graph and $Q_i \cap Q_j \neq \emptyset$ for all $i \neq j$. But $Q_1 \cap Q_2 \cap Q_3 \cap Q_4 = \emptyset$. Hence the Hajós graph is not a clique-Helly graph.

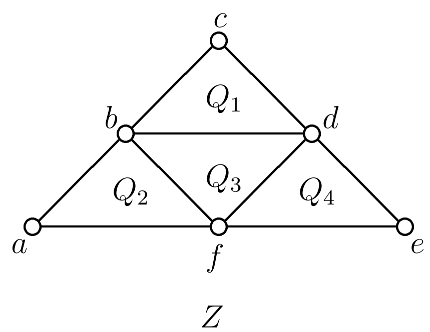


Figure 2.2: The Hajós Graph

In [23], Hamelink posed the following question:

Question 2.1 *Given a graph H, is it the clique graph of some graph G? If it is, how do we find one such graph G?*

In the same paper [23], Hamelink found an interesting sufficient condition for this question. He showed that if H is clique-Helly, then it is the clique graph of some graph G.

Theorem 2.1 *[23] If H is clique-Helly, then H is the clique graph of some graph G.*

Example 2.4 Let X_1, X_2 and X_3 be the graphs shown in Figure 2.3. We see that X_1, X_2 and X_3 all satisfy the Helly property. X_1 is the clique graph of X_2 and X_2 is the clique graph of X_3.

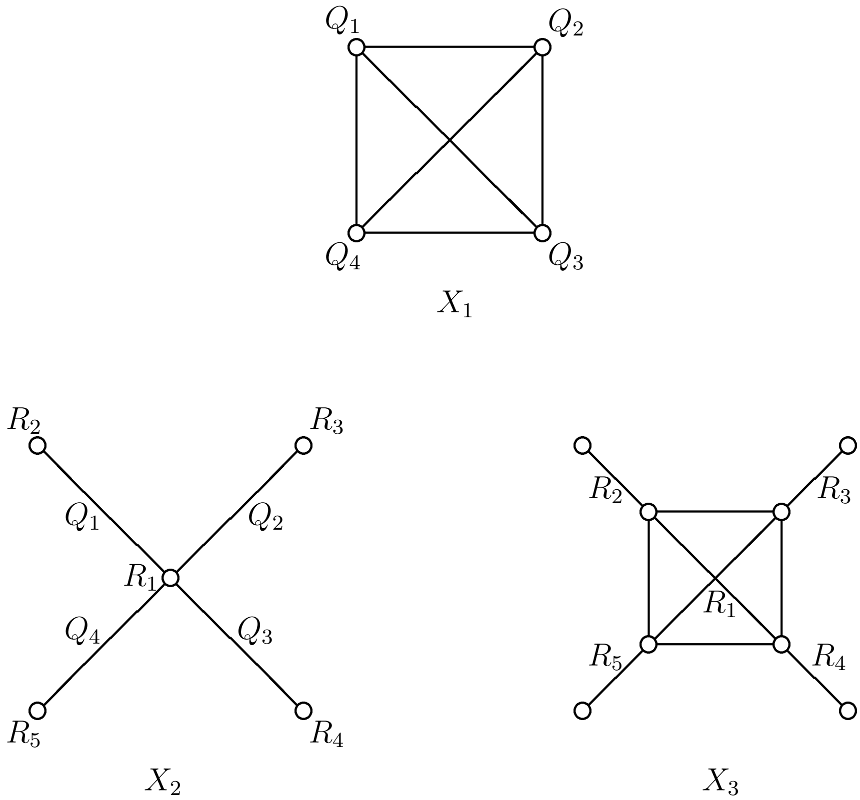


Figure 2.3: $K(X_2) = X_1$ and $K(X_3) = X_2$

However, the converse of Theorem 2.1 is not true as shown in the following example.

Example 2.5 *[38]* Let G and H be the graphs shown in Figure 2.4. Then $H = K(G)$.

In the graph H, take $\{L_1, L_2, L_3\}$ to be a set of cliques of H, where

$$\begin{aligned} L_1 &= \{Q_6, Q_7, Q_8, Q_9\}, \\ L_2 &= \{Q_1, Q_2, Q_3, Q_6\}, \\ \text{and} \qquad L_3 &= \{Q_3, Q_4, Q_5, Q_8\}. \end{aligned}$$

Then

$$L_1 \cap L_2 = \{Q_6\}, \quad L_1 \cap L_3 = \{Q_8\}, \quad L_2 \cap L_3 = \{Q_3\}.$$

But $L_1 \cap L_2 \cap L_3 = \emptyset$. Hence H does not satisfy the Helly property and so H is not a clique-Helly graph.

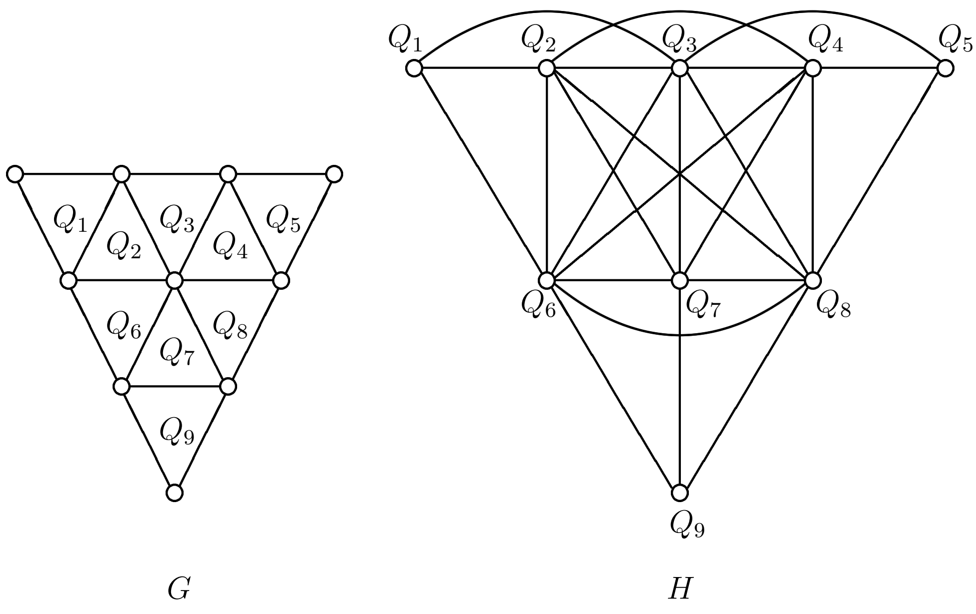


Figure 2.4: $H = K(G)$ but H is not clique-Helly.

In [23], Hamelink gave a method for constructing a graph G having H as its clique graph, provided the graph H is clique-Helly. We shall call this construction Method I.

Method I:

Let $V(G) = V(H) \cup \mathcal{K}(H)$, where $\mathcal{K}(H) = \{Q_1, \ldots, Q_n\}$.

1. If $h \in V(H)$, then $hQ_i \in E(G) \Leftrightarrow h \in Q_i$,

2. $Q_iQ_j \in E(G) \Leftrightarrow i \neq j$ and $Q_i \cap Q_j \neq \emptyset$,

3. If $h, h' \in V(H)$ then $hh' \notin E(G)$.

We illustrate this by giving the following example.

Example 2.6 Let H be the graph shown in Figure 2.5. Then $\mathcal{K}(H) = \{Q_1, Q_2, Q_3, Q_4\}$. Moreover $Q_i \cap Q_j \neq \emptyset$ for all $1 \leq i, j \leq 4$, and $Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \neq \emptyset$. Hence H is a clique-Helly graph. Let $V(G) = \{a, b, c, d, e, f, g, h, Q_1, Q_2, Q_3, Q_4\}$. By using Method 1, we construct the graph G as shown in Figure 2.5. It is easy to show that the clique graph of G is H.

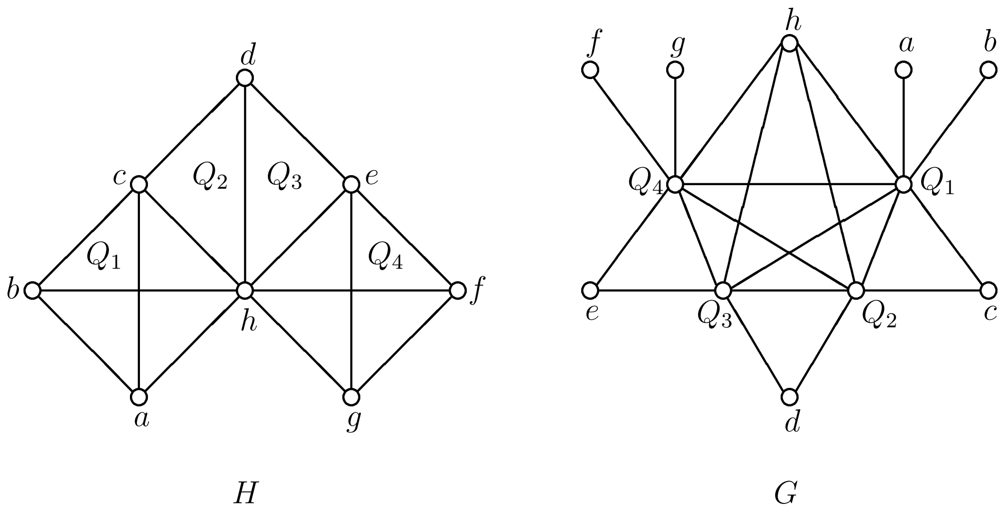


Figure 2.5: H is the clique graph of G

On the other hand, Hamelink found the following result which shows the existence of graphs that are not clique graphs. Some examples are presented in Example 2.7.

Theorem 2.2 *[23]* *Any graph H containing a clique Q on 3 vertices $\{a, b, c\}$ and 3 other cliques A, B and C such that*

$$\begin{aligned} V(Q) \cap V(A) &= \{a, b\}, \\ V(Q) \cap V(B) &= \{b, c\}, \\ \text{and} \qquad V(Q) \cap V(C) &= \{a, c\}, \end{aligned}$$

is not the clique graph of any graph.

Example 2.7 The graphs Z_2, Z_3 and Z_4 in Figure 2.6 are called *extended Hajós graphs*. For $i = 1, 2, 3, 4$, let Q_i, A_i, B_i and C_i be four cliques of the graphs Z_i and let a_i, b_i and c_i be the vertices of the clique Q_i, as shown in Figure 2.6. Then

$$\begin{aligned} V(Q_i) \cap V(A_i) &= \{a_i, b_i\}, \\ V(Q_i) \cap V(B_i) &= \{b_i, c_i\}, \\ \text{and} \qquad V(Q_i) \cap V(C_i) &= \{a_i, c_i\}. \end{aligned}$$

Thus none of these graphs are clique graphs by Theorem 2.2.

Remark : The Hajós graph is the smallest graph which is not clique-Helly and has the property given in Theorem 2.2.

A *complete edge cover* of a graph G is a collection $\mathcal{C}$ of complete subgraphs of G such that if $xy \in E(G)$ then $\{x, y\}$ is contained in some element of $\mathcal{C}$. A *Helly complete edge cover* of G is an complete edge cover of G satisfying the Helly property.

A necessary and sufficient condition for a graph H to be a clique graph of some graph G was found by Roberts and Spencer in [38]. The following theorem is a well-known characterization of clique graphs.

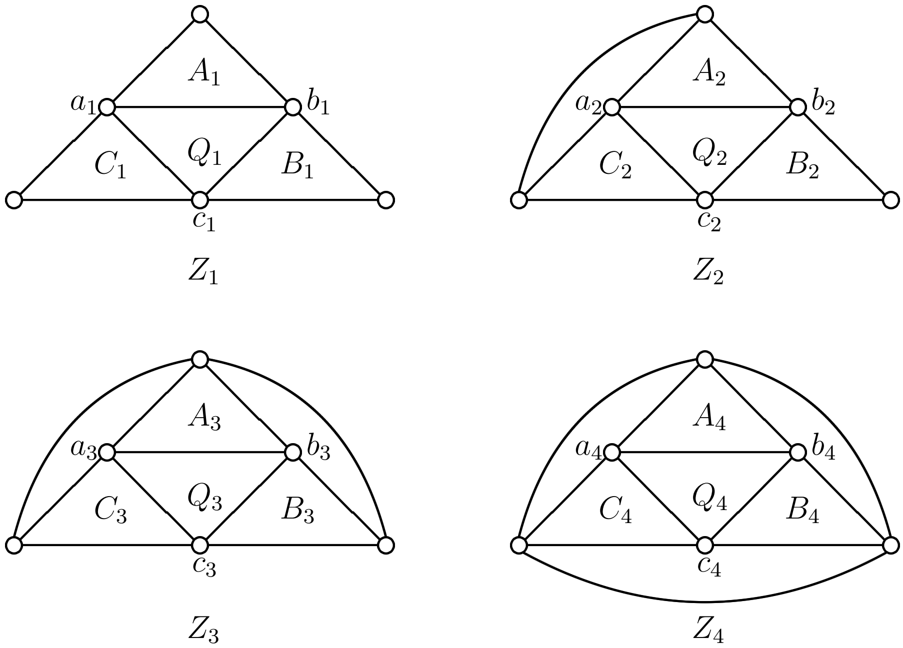


Figure 2.6: Extended Hajós Graphs

Theorem 2.3 *[38] A graph H is a clique graph if and only if there exists a Helly complete edge cover of H.*

For a graph H that satisfies the condition in Theorem 2.3, we may also construct a graph G having H as its clique graph by using the following method which has been restated by Roberts and Spencer in [38] and also by Szwarcfiter in [40].

Method II:

Let $L = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be a Helly complete edge cover of a given clique graph H and $h \in V(H)$. Then construct a graph G, with vertex-set $V(G) = V(H) \cup L$, as follows:

1. $h\mathcal{C}_i \in E(G) \Leftrightarrow h \in \mathcal{C}_i$,

2. $\mathcal{C}_i\mathcal{C}_j \in E(G) \Leftrightarrow i \neq j$ and $\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset$,

3. the graph G contains no other edges.

Example 2.8 Let X be the graph shown in Figure 2.7. It is easy to check that X is isomorphic to the graph H in Figure 2.4. Let

$$\begin{aligned}
\mathcal{C}_1 &= \{h_1, h_2, h_3\},\\
\mathcal{C}_2 &= \{h_1, h_2, h_6\},\\
\mathcal{C}_3 &= \{h_3, h_4, h_5\},\\
\mathcal{C}_4 &= \{h_4, h_5, h_8\},\\
\mathcal{C}_5 &= \{h_6, h_7, h_9\},\\
\mathcal{C}_6 &= \{h_7, h_8, h_9\},\\
\mathcal{C}_7 &= \{h_2, h_3, h_4, h_6, h_7, h_8\},\\
\text{and} \quad L &= \{\mathcal{C}_1, \ldots, \mathcal{C}_7\}
\end{aligned}$$

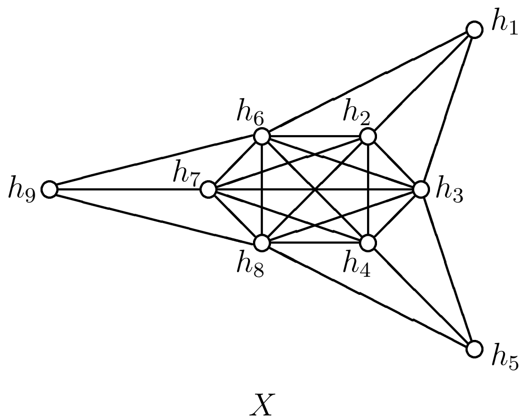
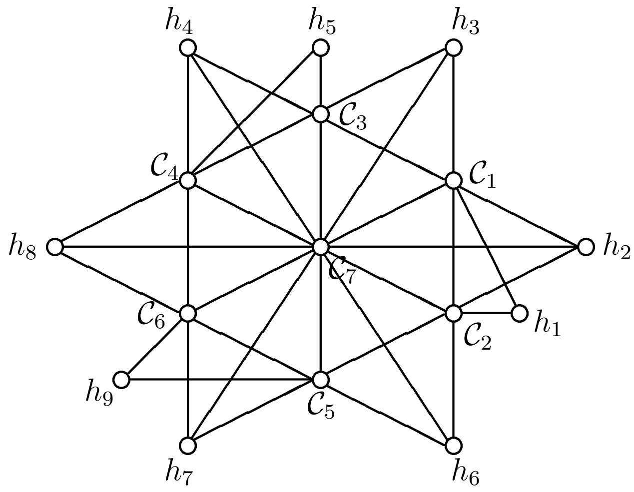


Figure 2.7: X is isomorphic to H in Figure 2.4

Let $I = \{1, \ldots, 7\}$. Clearly, L is a complete edge cover of X. We see that $\bigcap_{i \in I} \mathcal{C}_i = \emptyset$ and there exist some $i, j \in I$ such that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$. Therefore, L is a Helly complete edge cover of X. Hence, X is a clique graph by Theorem 2.3.

Now let a clique inverse graph of X be Y and we construct it by using Method II. Let $V(Y) = V(X) \cup L$. Then for $h_i \in V(X)$ and $\mathcal{C}_j \in L$,

$h_i\mathcal{C}_j \in E(Y)$ precisely when $h_i \in \mathcal{C}_j$, and for $\mathcal{C}_i, \mathcal{C}_j \in L$, $\mathcal{C}_i\mathcal{C}_j \in E(Y)$ when $i \neq j$ and $\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset$. The graph Y is shown in Figure 2.8.



Y

Figure 2.8: Y is a clique inverse graph of X in Figure 2.7

A consequence of Theorem 2.3 is that a K_4-free graph is a clique graph if and only if it is clique-Helly. The following theorem might be useful in recognizing clique graphs.

Theorem 2.4 *[22]* *Let H be a clique graph. If H contains an induced subgraph H' isomorphic to an extended Hajós graph, then H' must be contained in a (not necessarily induced) subgraph of H isomorphic to the graphs A or B of Figure 2.9.*

Recently, Alcón and Gutierrez [3] proved that deciding whether a given graph is a clique graph is an NP-complete problem.

Theorem 2.5 *[3]* *Clique graph recognition is NP-complete.*

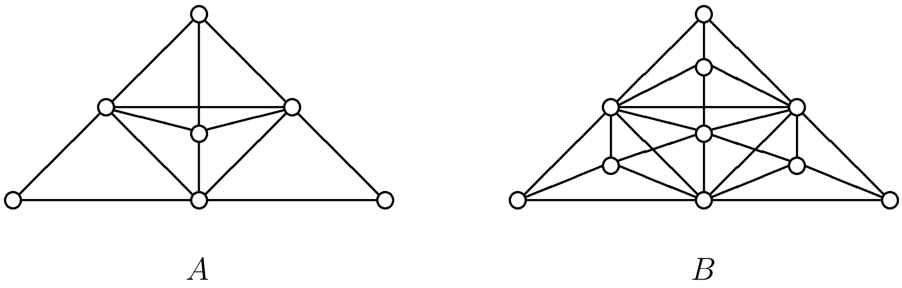


Figure 2.9: Graphs A and B

Let G be a graph and T a triangle of G. The *extended triangle* of G, relative to T, is the subgraph of G induced by the vertices which form a triangle with at least one edge of T. Let H be a subgraph of G. A vertex $v \in V(H)$ is *universal* in H whenever v is adjacent to every other vertex of H. The problem of whether or not clique-Helly graphs can be recognized in polynomial time has been mentioned as an open problem by Brandstädt [11] and Prisner [37]. In [39], Szwarcfiter described a characterization of clique-Helly graphs and it led to a polynomial time algorithm for recognizing graphs of this class.

Theorem 2.6 *[39] A graph G is a clique-Helly graph if and only if every extended triangle of G has a universal vertex.*

The recognition algorithm for clique-Helly graphs follows directly from Theorem 2.6. Given a connected graph G, for each of its triangles T, find the extended triangle T' relative to T, and verify if T' has a universal vertex. G is clique-Helly if and only if a universal vertex exists for each extended triangle T'.

Example 2.9 Let H be the graph shown in Figure 2.10. Let $T = \{x_2, x_4, x_6\}$ be one of the triangles in the graph H. Denote an extended triangle of H

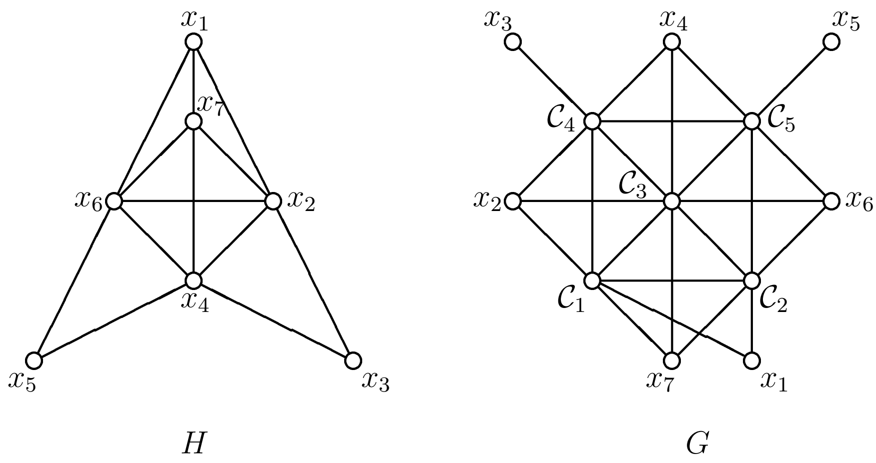


Figure 2.10: H is not a clique-Helly graph and $K(G) = H$

relative to T by T' where T' is the subgraph obtained from H by deleting the edge x_1x_7 from H. It is easy to check that the subgraph T' does not contain any vertex v such that v is adjacent to all the vertices in T'. This means that H has an extended triangle T' which does not contain any universal vertices. Thus by Theorem 2.6, H is not a clique-Helly graph.

Since clique-Helly graphs are always clique graphs and they have been characterized by looking at its triangles, Alcón and Gutierrez [1, 2] presented a generalized notion of extended triangle which allows a blending of the techniques of Roberts and Spencer [38] and Szwarcfiter [39] as follows.

Lemma 2.1 *[1, 2]* *Let $\mathcal{F}$ be a complete edge cover of a graph H and T be a triangle of H. Let $\mathcal{F}_T = \{\mathcal{C}_i \,|\, \mathcal{C}_i \in \mathcal{F}$ and $\mathcal{C}_i \cap T$ contains at least two vertices of $T\}$. A graph H is a clique graph if and only if there exists a complete edge cover $\mathcal{F}$ of H such that for every triangle T of H, $\cap \mathcal{C}_i \neq \emptyset$ for all $\mathcal{C}_i \in \mathcal{F}_T$.*

Example 2.10 Let H be the graph shown in Figure 2.10. Note that there are 9 triangles in the graph H. For $i = 1, \ldots, 9$, denote the triangles in H by

T_i as follows:

$$\begin{aligned} T_1 &= \{x_1, x_2, x_6\} \\ T_2 &= \{x_1, x_2, x_7\} \\ T_3 &= \{x_1, x_6, x_7\} \\ T_4 &= \{x_2, x_3, x_4\} \\ T_5 &= \{x_2, x_4, x_6\} \\ T_6 &= \{x_2, x_4, x_7\} \\ T_7 &= \{x_2, x_6, x_7\} \\ T_8 &= \{x_4, x_5, x_6\} \\ \text{and} \quad T_9 &= \{x_4, x_6, x_7\} \end{aligned}$$

Take

$$\begin{aligned} \mathcal{C}_1 &= \{x_1, x_2, x_7\} \\ \mathcal{C}_2 &= \{x_1, x_6, x_7\} \\ \mathcal{C}_3 &= \{x_2, x_4, x_6, x_7\} \\ \mathcal{C}_4 &= \{x_2, x_3, x_4\} \\ \text{and} \quad \mathcal{C}_5 &= \{x_4, x_5, x_6\} \end{aligned}$$

Then $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5\}$ is a complete edge cover of the graph H.

Now

$$\begin{aligned} F_{T_1} &= F_{T_2} = F_{T_3} = F_{T_7} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\} \\ F_{T_4} &= \{\mathcal{C}_3, \mathcal{C}_4\} \\ F_{T_5} &= \{\mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5\} \end{aligned}$$

$$
\begin{aligned}
F_{T_6} &= \{\mathcal{C}_1, \mathcal{C}_3, \mathcal{C}_4\} \\
F_{T_8} &= \{\mathcal{C}_3, \mathcal{C}_5\} \\
\text{and} \quad F_{T_9} &= \{\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_5\}
\end{aligned}
$$

We see that $\cap \mathcal{C}_i \neq \emptyset$ for all $\mathcal{C}_i \in F_{T_j}$, $1 \leq j \leq 9$. Thus by Lemma 2.1, H is a clique graph of some graph G. By taking $L = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5\}$ and using Method II, we construct the graph G which is shown in Figure 2.10.

Besides that, Alcón and Gutierrez [1] also obtained a new characterization of clique graphs which does not depend on the property of the Helly complete edge cover. For a detailed discussion of their results, the reader may refer to [1] and [2].

Chapter 3

Self-clique Graphs

3.1 Iterated Clique Graphs

Let G be a graph. Define $K^s(G)$ recursively by $K^0(G) = G$ and $K^s(G) = K(K^{s-1}(G))$, $s > 0$. We call $K^s(G)$ the *s-th iterated clique graph of* G. The value of the smallest s such that $K^s(G) \cong G$, for some $s > 0$, is called the *period* of G, and the graph G is called a *periodic graph* of period s. In particular, a graph G of period 1 is called a *self-clique* graph, that is $K(G) \cong G$.

Example 3.1 Let G denote the graph of Figure 3.1(a). The clique graph $K(G)$ of G is shown in Figure 3.1(b). Note that the cliques of G are denoted by P_i and they become the vertices of $K(G)$. On the other hand, the cliques of $K(G)$ are denoted by Q_i. We see that Q_i's form the vertices of the graph G, that is the clique graph of $K(G)$ is G. Hence we have $K^2(G) \cong G$ and so the period of G is 2.

Once again, clique-Helly graphs play a central role in the study of iterated clique graphs. The following theorem is fundamental and its proof provides

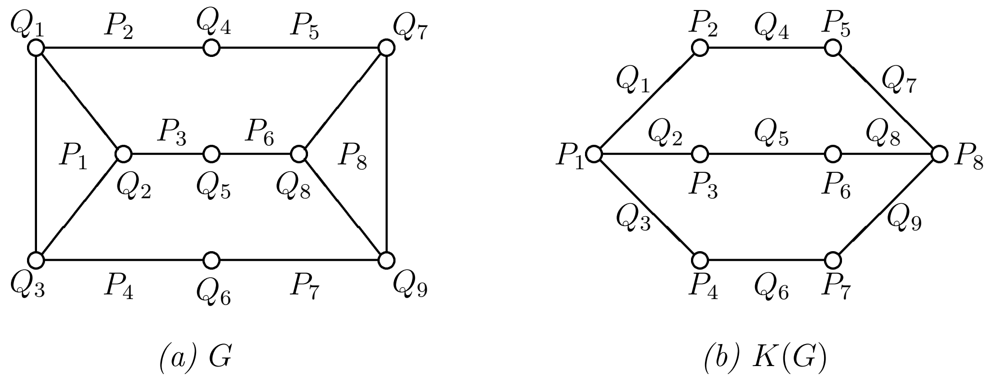


Figure 3.1: $K^2(G) \cong G$

a simple and complete description of the second iterated clique graph of the clique-Helly graph. For the proof of this theorem, the reader may refer to [17] and [40].

Theorem 3.1 *[17]* *Let G be a clique-Helly graph. Then $K^2(G)$ is an induced subgraph of G.*

The following theorem was found by Hedetniemi and Slater [25].

Theorem 3.2 *[25]* *If G is a triangle-free connected graph with at least three vertices, then $K^2(G) \cong G - \{v \in V(G) \,|\, d(v) = 1\}$.*

Let G^* denote the graph obtained by contracting each component of G which is a complete graph to an isolated vertex. A graph G is said to have the T_1 *property* if for every two distinct vertices u, $v \in V(G)$ with $d(u) \geq 2$ and $d(v) \geq 2$, there exist $Q_1, Q_2 \in \mathcal{K}(G)$ such that $u \in Q_1$, $v \notin Q_1$ and $u \notin Q_2$, $v \in Q_2$. Theorem 3.2 was then generalized by Lim [32] to a wider class of graphs.

Theorem 3.3 *[32]* *If G is a graph which satisfies the Helly property and the T_1 property, then $K^2(G) \cong G^* - \{v \in V(G^*) \,|\, d(v) = 1\}$.*

In [32], Lim proved that if G is a triangle-free connected graph, then G satisfies the Helly property and the T_1 property. Since G is a connected graph, $G \cong G^*$. Thus we have $K^2(G) \cong G - \{v \in V(G) \,|\, d(v) = 1\}$ by Theorem 3.3. Moreover, if G is a disconnected graph such that each component of G is a triangle-free graph with at least three vertices, then each component has the Helly property and the T_1 property. Thus G has the Helly property and this means that Theorem 3.2 is also true if connectedness of G is dropped. So Theorem 3.3 is a generalization of Theorem 3.2.

3.2 Self-clique Graphs

Clique graphs have been studied for some time. However, not much is known about self-clique graphs. Self-clique graphs were first introduced and studied by Escalante [17] in 1973; among others, he proved the existence of both clique-Helly and non-clique-Helly self-clique graphs, and also that every graph is an induced subgraph of a clique-Helly self-clique graph. Besides that, Balconi [5] also has some related results. In [27], Hedman asked if such graphs can be characterized. Self-clique graphs have also been studied by Escalante and Toft in [18]. The problem of characterizing self-clique graphs remains open. However, self-clique graphs have been characterized for some classes of graphs, like triangle-free graphs [17], graphs with all cliques but one of size 2 [13], clique-Helly graphs [9, 17, 30] and hereditary clique-Helly graphs [31].

The *neighbourhood* of a vertex $u \in V(G)$ is the set $N(u)$ consisting of all vertices $v \in V(G)$ which are adjacent to u. The *closed neighbourhood* is $N[u] = N(u) \cup \{u\}$. If v and w are vertices of G satisfying $N[v] = N[w]$ then v, w are *twins*. If $N[v] \subseteq N[w]$ then v is *dominated* by w. We say that a vertex v is *dominated* only if v is dominated by a different vertex $w \neq v$.

Theorem 3.4 *[17]* *Let G be a clique-Helly graph. Then G is self-clique if and only if G has no dominated vertices.*

The following is a direct consequence of Theorem 3.4.

Theorem 3.5 *[17]* *Let G be a connected graph whose cliques are all of size* 2. *Then G is self-clique if and only if G is a cycle of length at least* 4.

This result was extended by Chia [13], who characterized self-clique graphs all of whose cliques have size two, except for precisely one clique. Moreover, a construction of self-clique graphs under certain condition was also given by Chia in [14]. We'll discuss Chia's results in Section 3.3.

Let $v_1, v_2, \ldots, v_n$ be the vertices of a graph G and let $\mathcal{K}(G) = \{Q_1,\ Q_2,\ \ldots, Q_m\}$. The $(0,1)$-matrix $A = (a_{ij})$, where $a_{ij} = 1$ if and only if vertex v_i belongs to clique Q_j is called a *clique matrix* of G. A matrix $A' = (a'_{ij})$ derived from A by permuting the columns of A is called a *permuted matrix* of A. A matrix is *quasi-symmetric* if its family of row vectors and family of column vectors are identical; in particular, every symmetric matrix is quasi-symmetric.

Let G be a graph with the same number of vertices and cliques. A *vertex-clique duality* of G is a bijection ϕ which maps vertices to cliques and cliques to vertices in such a way that, for $v \in V(G)$ and $Q \in \mathcal{K}(G)$, $v \in Q$ if and only if $\phi(Q) \in \phi(v)$. This notion was introduced under the name *self-duality* by Balconi [5].

In [9], Bondy et al. characterized self-clique graphs which are clique-Helly in terms of permuted matrices of their clique matrices, and also in terms of the concept of vertex-clique duality. Their elegant proofs made use of the spectra of the clique matrices of graphs.

Theorem 3.6 *[9] A graph is clique-Helly and self-clique if and only if it has a quasi-symmetric clique matrix.*

Theorem 3.7 *[9] A graph is clique-Helly and self-clique if and only if it admits a vertex-clique duality.*

In view of Theorem 3.6, one may ask whether every clique-Helly self-clique graph has a symmetric clique matrix. Bondy et al. [9] showed that the answer to this question is no. They showed that the graph in Figure 3.2 is a clique-Helly and self-clique graph, but it has no symmetric clique matrix by an exhaustive computer check. Moreover, it is a smallest graph with these properties.

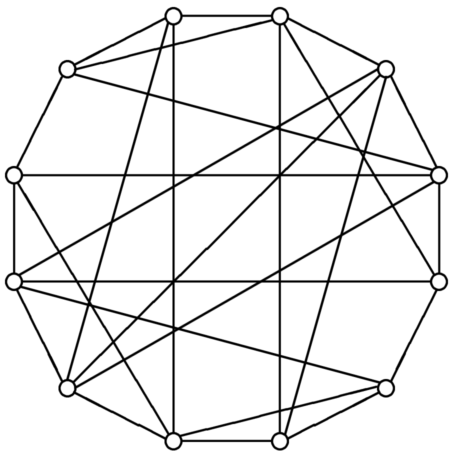


Figure 3.2: A clique-Helly self-clique graph with no symmetric clique matrix

3.3 Constructions for Self-clique Graphs

The characterization of self-clique graphs does not seem to be an easy problem although in certain special cases, this may be manageable. In this section, we will discuss some constructions for self-clique graphs given by Balakrishnan and Paulraja [4], Chia [13, 14], Bondy et al. [9] and Larrión et al. [28, 29].

3.3.1 Balakrishnan and Paulraja's Construction

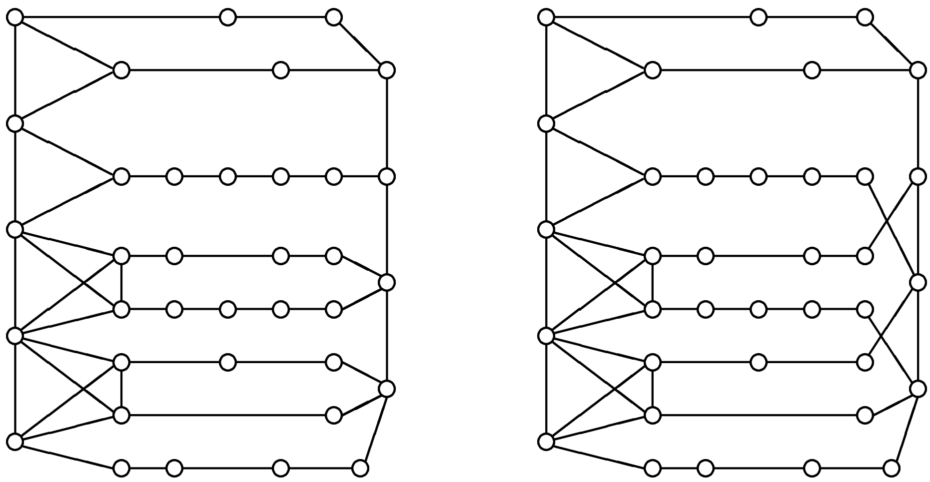
A construction for self-clique graphs was given by Balakrishnan and Paulraja in [4]. We will present their method in this section.

A vertex in a connected graph is a *cut-vertex* if its removal leaves a disconnected graph. A *nonseparable* graph is a connected, nontrivial graph that has no cut-vertices. A *block* of a graph is a maximal nonseparable subgraph. Let $\mathcal{B}(G)$ denote the set of all blocks of G. Then the *block graph* of G, denoted by $B(G)$, is the graph whose vertex-set is $\mathcal{B}(G)$ and two vertices are adjacent in $B(G)$ if and only if the corresponding blocks have non-empty intersection. A vertex that belongs to a unique clique is called *unicliqual vertex*.

Let H be a block graph such that each vertex belongs to at most two blocks and let $\mathcal{K}(H) = \{Q_1, \ldots, Q_t\}$ be the set of all cliques of H. Construct a self-clique graph G as follows:

For each i, $1 \le i \le t$, if $Q_i \cong K_s$ for some s, then let $T_i = K_{1,s}$. For all $1 \le i \ne j \le t$, T_i and T_j have a unique edge in common if and only if $Q_i \cap Q_j \ne \emptyset$ and let T be the union of T_i thus constructed.

For $1 \le i \le t$, let $\{u_1, \ldots, u_{n_i}\}$ and $\{v_1, \ldots, v_{n_i}\}$ denote the sets of unicliqual vertices in Q_i and end vertices of T_i respectively in some ordering. Now we may



(a) G_1 (b) G_2

Figure 3.3: G_1 and G_2 are self-clique graphs

join the vertices in H to the vertices in T by using either of the following ways.

(1) For $1 \leq k \leq n_i$, join u_k to v_k by a path of length at least 1 with its internal vertices not belonging to $H \cup T$ and let this path be P_i. Require further that the paths P_i, $1 \leq i \leq t$ are mutually disjoint. Let $P = \bigcup_{i=1}^{t} P_i$. Then $G = H \cup P \cup T$ is a self-clique graph.

(2) Instead of joining the unicliqual vertices of Q_i with the end vertices of T_i by paths (as in (1)), one may as well join a unicliqual vertex of Q_i to an end vertex of T_j for $i \neq j$, by a path P_{ij} of length at least 1. In this case, one must join a unicliqual vertex of Q_j to an end vertex of T_i by a path P_{ji} of length equal to that of P_{ij}. The resulting graphs would also be self-clique graphs.

Figure 3.3(a) and (b) each shows an example of a graph that can be constructed by using method (1) and (2) respectively.

3.3.2 Chia Graphs

In 2000, Chia [13] gave a characterization of the first family of self-clique graphs in which all cliques have size two, except for precisely one clique. Such graphs are called *Chia graphs* (see [28]).

Suppose a_1, a_2, $\ldots$, a_p are positive integers. Then the sequence (a_1, a_2, $\ldots$, a_p) is called the *clique size sequence* of a graph G if the p elements in $\mathcal{K}(G)$ can be arranged in such a way that the i-th element of $\mathcal{K}(G)$ is of size a_i, $i = 1, 2, \ldots, p$. Let K_n denote a complete graph on n vertices and let its vertices be denoted by v_1, v_2, $\ldots$, v_n. Chia [13] described two families of graphs whose clique size sequence is $(2, \ldots, 2, n)$ where $n \geq 3$ as follows:

Family $\mathcal{G}_{r,s}$:

Let $r \geq 0$ and $s \geq 1$ be two integers such that $2r + s = n$. Join a new vertex z to the vertices v_1, $\ldots$, v_s of K_n and subdivide the new edges zv_1, $\ldots$, zv_s such that at most one of them is not subdivided. If $r \geq 1$, overlap r cycles C_1, $\ldots$, C_r each of length at least 4 all at the vertex z. Also, for each $i = 1$, $\ldots$, r, overlap a duplicate of C_i with K_n at the edge $v_{s+2i-1}v_{s+2i}$. If $r = 0$, then the latter two operations are omitted.

Let $\mathcal{G}_{r,s}$ denote the set of all graphs obtained in this manner.

Family $\mathcal{H}_{t,q}$:

Let t and q be two non-negative integers such that $2t + q = n - 1$. If $q = 0$, then overlap t cycles C_1, $\ldots$, C_t each of length at least 4 all at the same vertex v_1 of K_n. Also, for each $i = 1$, $\ldots$, t, overlap a duplicate of C_i with K_n at the edge $v_{2i}v_{2i+1}$. Note that in this case, n is odd.

If $q \geq 1$, let C_1^*, $\ldots$, C_q^* be q cycles each of length at least 4. For each $i = 1$, $\ldots$, q, overlap C_i^* with K_n at the edge v_1v_{i+1}. If $t \geq 1$, then overlap t cycles C_1, $\ldots$, C_t each of length at least 4 all at the same vertex v_1 of K_n. Also, for each $i = 1$, $\ldots$, t, overlap a duplicate of C_i with K_n at the edge $v_{q+2i}v_{q+2i+1}$.

If $t = 0$, then the latter two operations are omitted.

Let $\mathcal{H}_{t,q}$ denote the set of all graphs obtained in this manner.

The general drawing of the graphs $\mathcal{G}_{r,s}$ and $\mathcal{H}_{t,q}$ are shown in Appendix A. Figure 3.4 depicts some examples of self-clique graphs with clique size sequence $(2, \ldots, 2, 6)$. Note that, if we contract all edges in the path joining z and x in $G_1 \in \mathcal{G}_{0,6}$ (respectively $G_2 \in \mathcal{G}_{1,4}$ and $G_3 \in \mathcal{G}_{2,2}$), so that z and x coincide to become a single vertex $z = x$, then this produces the graph $G_4 \in \mathcal{H}_{0,5}$ (respectively $G_5 \in \mathcal{H}_{1,3}$ and $G_6 \in \mathcal{H}_{2,1}$).

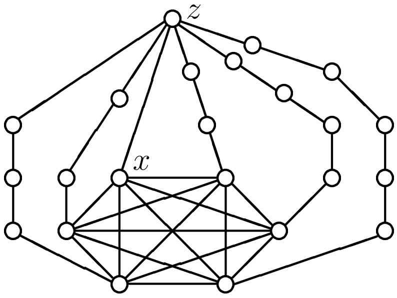
Theorem 3.8 *[13] Let G be a connected graph whose clique size sequence is $(2, \ldots, 2, n)$, $n \geq 3$. Then G is self-clique if and only if $G \in \mathcal{G}_{r,s} \cup \mathcal{H}_{t,q}$ where $r, t, q \geq 0$ and $s \geq 1$ are integers such that $2r + s = n$ and $2t + q = n - 1$.*

Remark : If G is a graph in $\mathcal{G}_{r,s}$ such that the paths joining the vertex z to the vertices of K_n have length at least 3, then contracting all the edges of a path joining z to some v_i, $i \leq s$, so that z and v_i eventually coincide will result in a graph which is a member of the family $\mathcal{H}_{r,s-1}$.

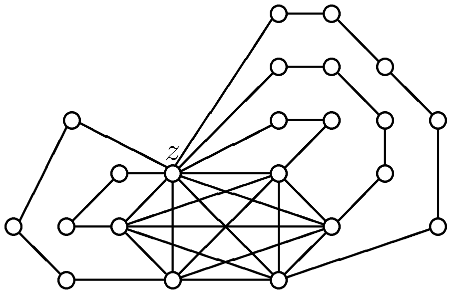
3.3.3 Chia-type Graphs

Shortly after publication of [13], a lot of works has been done by many people in finding the characterization of self-clique graphs. One of these works that has been done is to generalize Chia graphs [13]. More recently, Larrión et al. have generalized the concept of Chia graphs in [30] and [28].

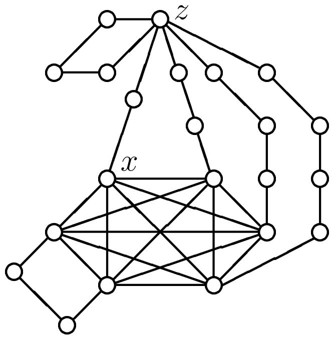
An *automorphism* of a graph G is an isomorphism of G onto itself. The collection of all automorphisms of G forms a group under the composition of



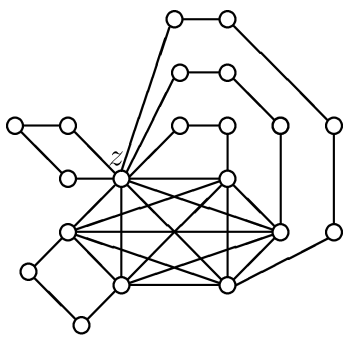
$G_1 \in \mathcal{G}_{0,6}$



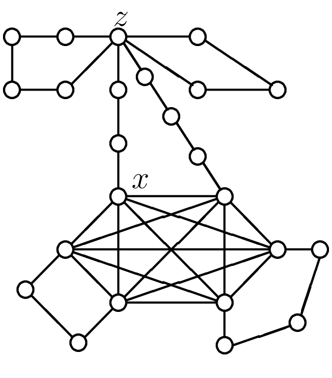
$G_4 \in \mathcal{H}_{0,5}$



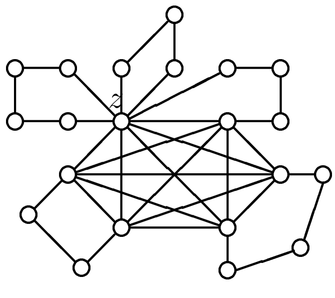
$G_2 \in \mathcal{G}_{1,4}$



$G_5 \in \mathcal{H}_{1,3}$



$G_3 \in \mathcal{G}_{2,2}$



$G_6 \in \mathcal{H}_{2,1}$

Figure 3.4: Self-clique graphs with clique size sequence $(2, \ldots, 2, 6)$

mappings. This automorphism group is called the *automorphism group* of G and is denoted by *Aut(G)*.

The *vertex-clique bipartite graph* of a graph G is the graph $BK(G)$ with vertex-set $V(G) \cup V(K(G))$ and edge-set $\{vQ \mid v \in V(G),\ Q \in V(K(G))$ and $v \in Q\}$. If (X, Y) is a bipartition of $BK(G)$, then $X = V(G)$ and $Y = V(K(G))$.

A bipartition (X, Y) of a bipartite graph B is said to be *self-dual* if there exists an automorphism σ of B which transforms (X, Y) into its dual (Y, X) i.e. $\sigma(X) = Y$ and $\sigma(Y) = X$. Such a σ is called a *self-duality* or a *part-switching automorphism* of B.

Theorem 3.9 *[30] If G is a connected graph, then the following are equivalent:*

(1) G is Helly and self-clique.

(2) $BK(G)$ is self-dual.

A connected graph G is said to be *involutive* if $BK(G)$ has a part switching *involution* (automorphism of order 2), i.e. a part-switching automorphism $\pi \in \mathrm{Aut}(BK(G))$ such that π^2 is an identity. Therefore, if G is involutive, then $BK(G)$ is self-dual and by Theorem 3.9, G is a Helly self-clique graph.

A *large clique* is a clique with more than two vertices, and a *Chia-Type* graph is a self-clique graph with at most 3 large cliques (see [30]).

Theorem 3.10 *[30] Let G be a connected graph such that at most 3 cliques of G have more than 2 vertices. Then G is self-clique if and only if G is involutive.*

The *k-th power* of a graph G is the graph G^k, whose vertices are those of G, two of them being adjacent whenever their distance in G is at most k. A graph

G is called *N-Sperner* whenever $N(v) \subseteq N(w) \Rightarrow v = w$. Let $\delta(G)$ denote the minimum vertex degree of a graph G. The following result characterizes Chia-type graphs.

Theorem 3.11 *[30] Assume that at most 3 cliques of a connected non-trivial graph G have more than 2 vertices. Then G is self-clique if and only if $G \cong B^2[X]$ for some connected bipartite graph B satisfying:*

(1) B is N-Sperner and has a part switching involution.

(2) $\delta(B) \geq 2$ and at most 6 vertices have degree greater than 2.

(3) Every hexagon of B has a chord.

By attaching a *p-leg* to a graph means attaching a path of length $p \geq 0$ to one of the vertices and putting a loop at the free end of the new path (attaching a 0-leg is just attaching a loop). A *lobster graph* is any graph G with $\delta(G) = 2$ which is constructed starting with a single vertex x and attaching to x any number of legs (at least one leg) and any number of cycles all sharing precisely the vertex x. By a cycle here we mean one with at least 3 vertices, but there can be a loop at x (at most one) if a 0-leg was attached to it. The *strict square* of H is the (necessarily loopless) graph $H^{[2]}$ with $V(H^{[2]}) = V(H)$ and $uv \in E(H^{[2]})$ if and only if the distance between u and v in H is 2.

In [30], Larrión et al. gave an alternative characterization of Chia graphs as follows.

Theorem 3.12 *[30] A graph G is a Chia graph if and only if $G \cong H^{[2]}$ for some lobster graph H satisfying:*

(1) Every cycle in H has even length at least 8.

(2) If H has a 0-leg, it does not have a 1-leg nor a 2-leg.

(3) If H has a 1-leg, it is unique.

Figure 3.5 shows an example of Chia graph that can be constructed by using Theorem 3.12.

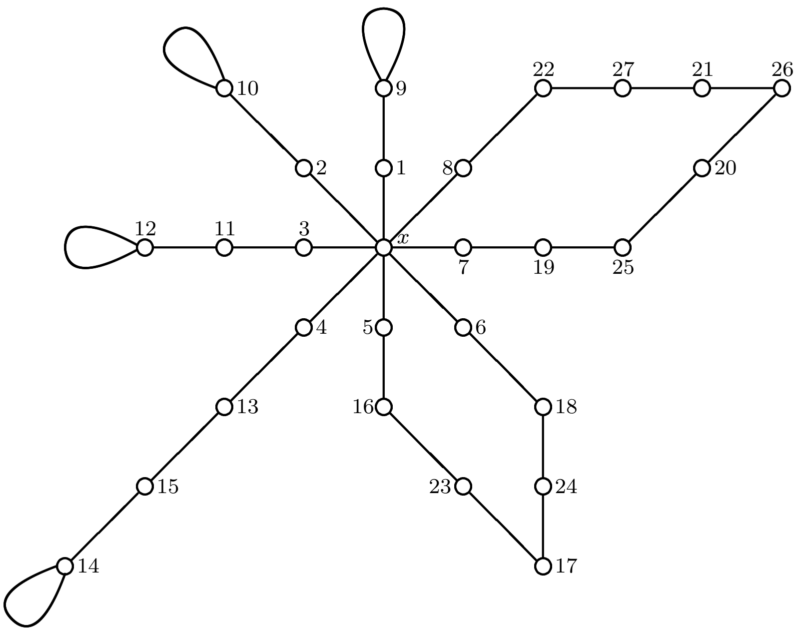
The *Cartesian Product*, $G_1 \times G_2$, is the graph where $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and for $v_1, w_1 \in V(G_1)$ and $v_2, w_2 \in V(G_2)$, $(v_1, v_2)(w_1, w_2) \in E(G_1 \times G_2)$ if and only if $v_1w_1 \in E(G_1)$ and $v_2w_2 \in E(G_2)$. A graph H is *good* if $K_2 \times H \cong BK(G)$ for some involutive self-clique graph G. A *possibly loopy* graph G is a graph which is allowed to have at most one loop at each vertex. A graph G is called *N-Helly* if the family $\{N(v) \,|\, v \in V(G)\}$ has the Helly property.

Theorem 3.13 *[28] A graph H is good if and only if it is N-Helly and N-Sperner.*

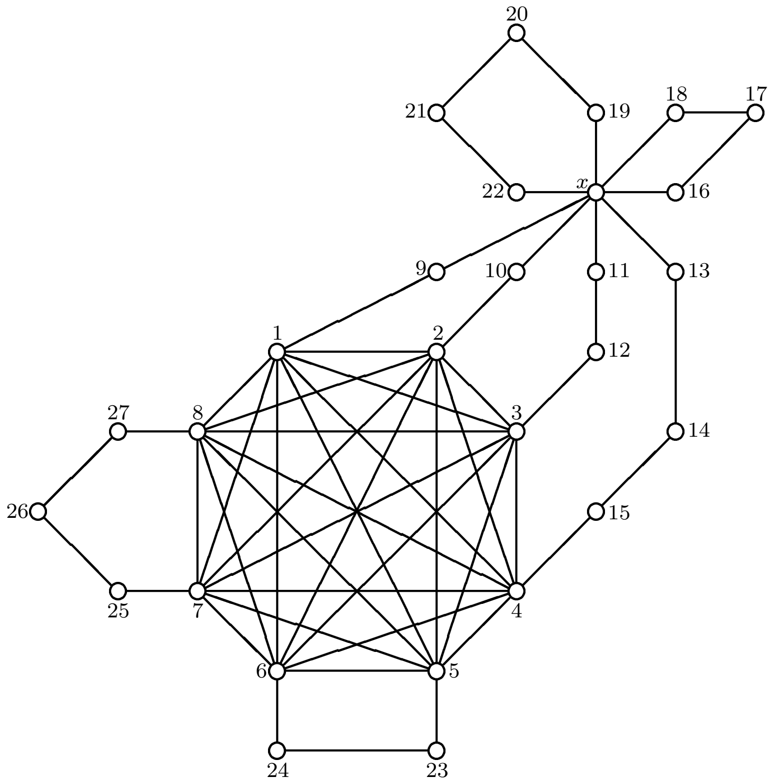
Theorem 3.14 *[30] A graph G is involutive if and only if $G \cong H^{[2]}$ for some possibly loopy, good, connected, non-bipartite graph H. In this case, $BK(G) \cong K_2 \times H$.*

Define the *clique-size set* of a graph G, to be the set of clique sizes in G, disregarding multiplicities. For example, a Chia graph has clique-size set $\{2, p\}$, for some $p \geq 3$. In [28], Larrión et al. generalized the concept of Chia graphs by giving an interpolation type construction, which gives a self-clique graph with clique-size set $\{2, 3, 4, \ldots, p\}$ for any $p \geq 3$.

Consider the graph $K_{1,p}$ and identify each leaf i in $K_{1,p}$ with a vertex in a given graph H_i and call the resulting graph $star(H_1, \ldots, H_p)$. Now define



(a) A lobster graph H

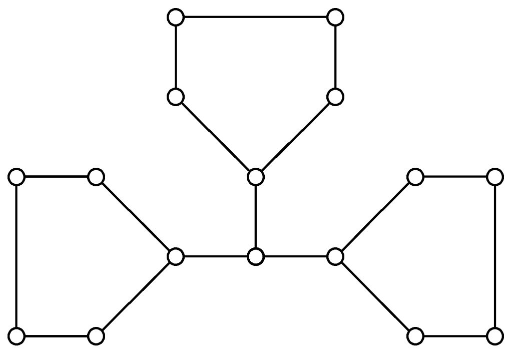


(b) $G \cong H^{[2]}$

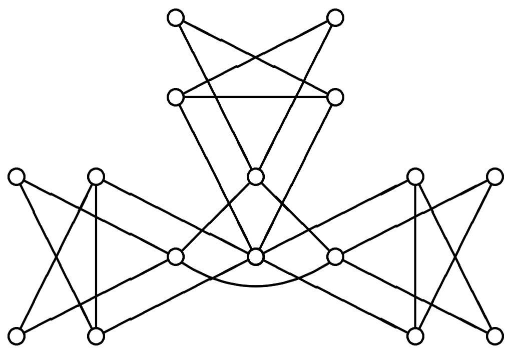
Figure 3.5: G is a Chia graph

$S(3) = star(C_5, C_5, C_5)$ (see Figure 3.6(a)), and recursively generate a sequence of star graphs as follows:

$$\begin{aligned} S(3) &= star(C_5, C_5, C_5), \\ S(n+1) &= star(\underbrace{S(n), S(n), \ldots, S(n)}_{(n+1)-\text{times}}), \qquad n \geq 3 \end{aligned}$$



(a) The star graph $S(3)$



(b) $S(3)^{[2]}$

Figure 3.6: $S(3)^{[2]}$ is self-clique with clique-size set $\{2, 3\}$

In $S(n+1)$, each leaf vertex in the underlying graph $K_{1,n+1}$ is identified with the center vertex in $S(n)$. Larrión et al. [28] showed that the star graphs $S(p)$, $p \geq 3$ are N-Helly and N-Sperner and so the strict square graph $S(p)^{[2]}$ is an involutive self-clique graph. The self-clique graph $S(3)^{[2]}$ is shown in Figure 3.6(b). In general, they gave the following result.

Theorem 3.15 *[28]* *If $H_1, \ldots, H_p$ is a family of loopless good graphs, with $\delta(H_i) \geq 2$, then $star(H_1, \ldots, H_p)$ is good.*

Figure 3.7 shows an example of a self-clique graph $H^{[2]}$ with clique size set $\{2, 3, 4\}$, where $H = star(C_5, C_7, S(3))$.

We conclude this subsection with the following two theorems which were found by Larrión et al. in [28].

Theorem 3.16 *[28]* *For any finite set $X = \{2, x_1, \ldots, x_p\}$ of distinct integers, there exists a self-clique graph with clique-size set X.*

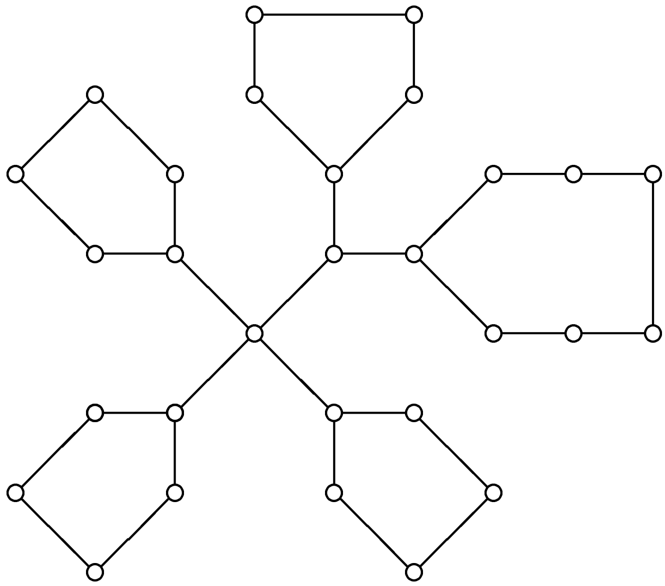
Theorem 3.17 *[28]* *For any set $Y = \{l_1, \ldots, l_p\}$ with $l_i \geq 2$, there exist infinitely many self-clique graphs with clique-size set Y.*

3.3.4 Other Constructions

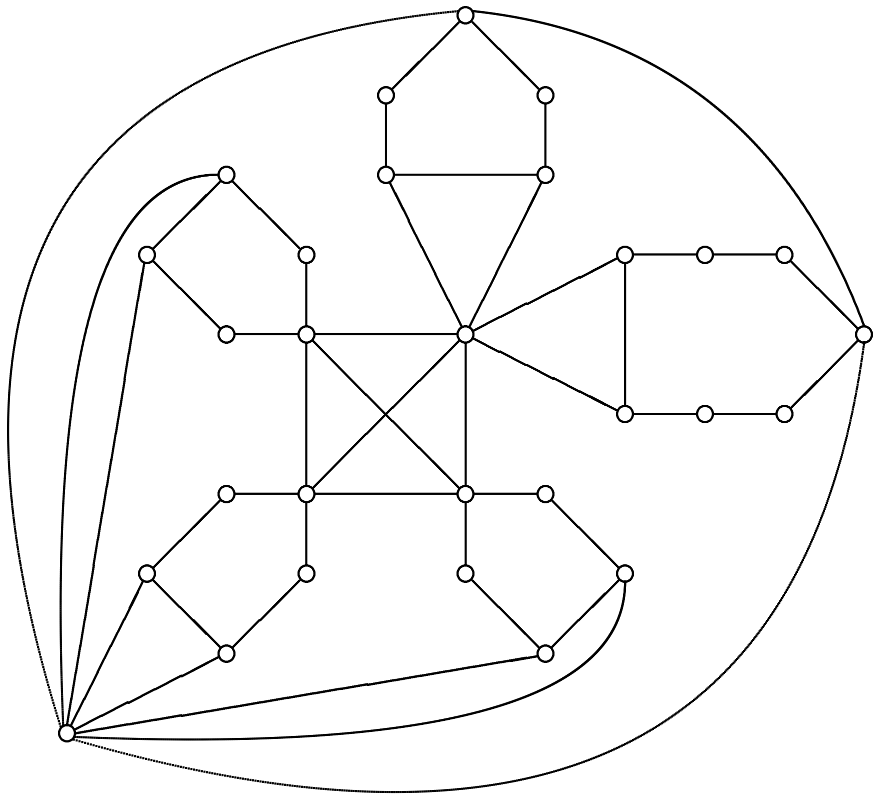
In [14], Chia also gave a construction for disconnected self-clique graphs, and another construction of self-clique graphs which makes use of those graphs that have the Helly property and the T_1 property.

(I) Suppose H is a graph such that $K^n(H) \cong H$, where $n \geq 2$, then it is easy to prove that the graph $G = H \cup K(H) \cup \cdots \cup K^{n-1}(H)$ is a self-clique graph. In this case, we see that G is a disconnected graph.

(II) Let H be a connected graph without end vertices, and H has the Helly property and the T_1 property. Then by Theorem 3.3, it is not difficult to see that $K^2(H) \cong H$. Let w be a vertex in H and let $K(w)$ denote the induced subgraph of $K(H)$ whose vertex-set is the set of all cliques of H containing w. Let $G_1(H)$ denote the graph obtained by joining a new vertex z to w and to all



(a) The star graph $H = star(C_5, C_7, S(3))$



(b) $H^{[2]}$

Figure 3.7: $H^{[2]}$ is self-clique with clique-size set $\{2, 3, 4\}$

vertices of $K(w)$ of $K(H)$. This construction is depicted in Figure 3.8 and the clique graph of $G_1(H)$ is shown in Figure 3.9.

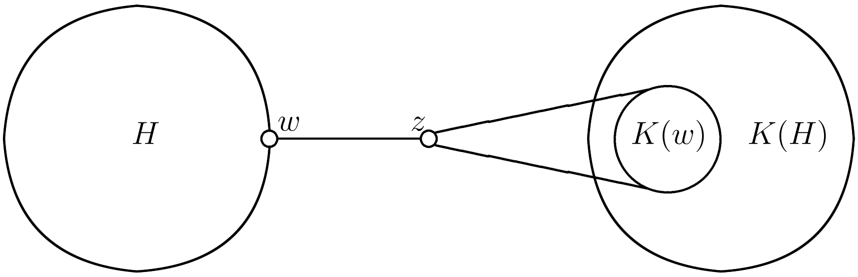


Figure 3.8: The graph $G_1(H)$

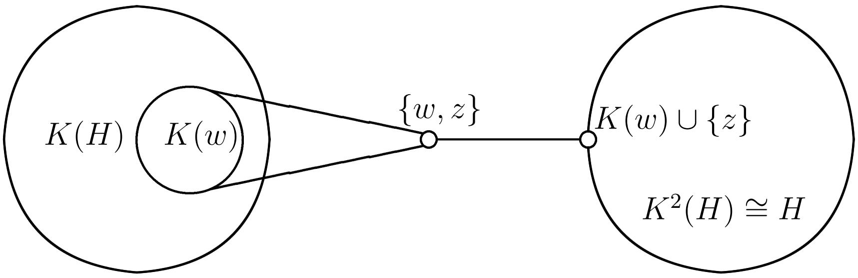
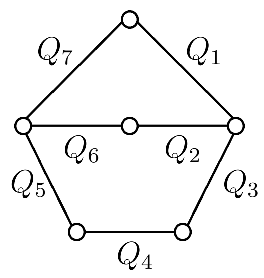


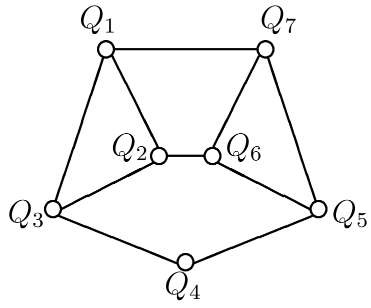
Figure 3.9: The graph $K(G_1(H))$

Theorem 3.18 *[14]* *Let H be a connected graph without end vertices. Suppose H has the Helly property and the T_1 property. Then $G_1(H)$ is a connected self-clique graph.*

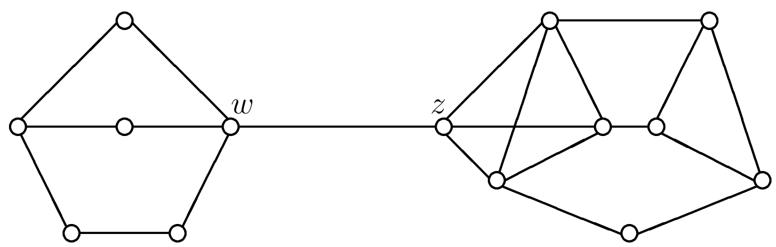
Example 3.2 Let H denote the graph of Figure 3.10(a). Since H is triangle-free, it has the Helly property and the T_1 property. Hence we have $K^2(H) \cong H$. The clique graph $K(H)$ of H is shown in Figure 3.10(b). Note that the cliques of H are denoted by Q_i and they become the vertices of $K(H)$. Now, the graph $G_1(H)$ is shown in Figure 3.10(c). It is easy to verify that this graph is self-clique.



(a) H



(b) $K(H)$



(c) $G_1(H)$

Figure 3.10: $K^2(H) \cong H$ and $G_1(H)$ is self-clique

In [14], Chia posed the following question which remains open.

Question 3.1 *Suppose H is a connected graph such that $K^n(H) \cong H$ for some $n \geq 3$ and that $K^i(H) \ncong H$ for $1 \leq i < n$. How could one join up the n graphs H, $K(H)$, $\ldots, K^{n-1}(H)$ so as to obtain a connected self-clique graph?*

Let $g(G)$ denote the *girth* of a graph G, that is, the length of the shortest cycle of G, if any. Examples of connected self-clique graphs which are not clique-Helly were given by Escalante [17], while in [9], Bondy et al. have described two new classes of self-clique graphs which are clique-Helly. The following theorem was found by Bondy et al. which was also independently found by Larrión et al..

Theorem 3.19 *[9, 28, 30]* *Let G be a graph, where $\delta(G) \geq 2$ and $g(G) \geq 6k+1$, $k \geq 1$. Then G^{2k} is a clique-Helly self-clique graph.*

The authors claimed that Theorem 3.19 is best possible in the following sense: if either $\delta(G) < 2$ or $g(G) < 6k+1$, then G^{2k} is not necessarily a self-clique graph. A graph G formed by a cycle graph C_7 on 7 vertices, together with an additional vertex adjacent exactly to one vertex of the cycle is an example where the degree condition of Theorem 3.19 fails, while the girth condition is satisfied for $k = 1$. However G^2 is not self-clique. On the other hand, a cycle graph C_6 on 6 vertices is an example of a graph where the degree condition is satisfied for $k = 1$, the girth condition fails and C_6^2 is not a self-clique graph. Figure 3.11 shows two examples of self-clique graphs that satisfy the conditions of Theorem 3.19.

The second class of self-clique graphs of Bondy et al. [9] is introduced based on the following expansion operation.

Let G be a graph and S a set of vertices of G. The *expansion of G relative to S* is the graph H obtained from G by iteratively replacing each vertex $v \in S$ by a clique $Q(v)$ of size $d_G(v)$ and making each $w \in Q(v)$ adjacent in H to exactly one neighbour of v in G, these neighbours being distinct (see Figure 3.12) . $Q(v)$ is called the *expanded clique* at v.

Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. The *reduced adjacency matrix* of G is the $m \times n$ $(0,1)$-matrix $B = (b_{ij})$, where $b_{ij} = 1$ if and only if $x_i y_j \in E(G)$.

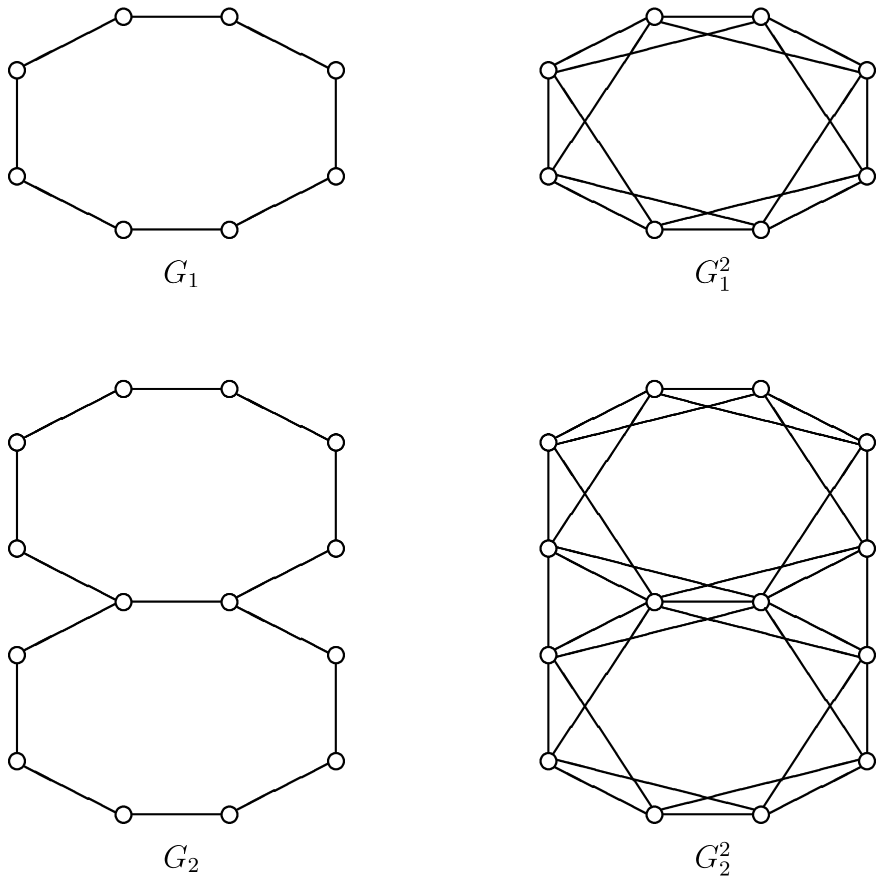


Figure 3.11: G_1^2 and G_2^2 are self-clique graphs

Theorem 3.20 *[9]* *Let G be a bipartite graph with bipartition (X, Y), admitting a symmetric reduced adjacency matrix. Then the expansion of G relative to Y is a clique-Helly self-clique graph.*

Example 3.3 *[9]* Let G denote the graph of Figure 3.12(a). Let $Y = \{y_1, y_2, y_3\}$ be a set of vertices of G. Replace vertices y_1, y_2 and y_3 by cliques $Q(y_1) = \{y_{11}, y_{12}\}$, $Q(y_2) = \{y_{21}, y_{22}, y_{23}\}$ and $Q(y_3) = \{y_{31}, y_{32}\}$ respectively. For each $w \in Q(y_i)$, w is adjacent to exactly one neighbour of y_i for $i = 1, 2, 3$, as shown in Figure 3.12(b), where H is the expansion graph of G relative to Y.

We see that the reduced adjacency matrix of G is $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ which satisfies the hypothesis of Theorem 3.20. Hence the graph H is a clique-Helly self-clique graph.

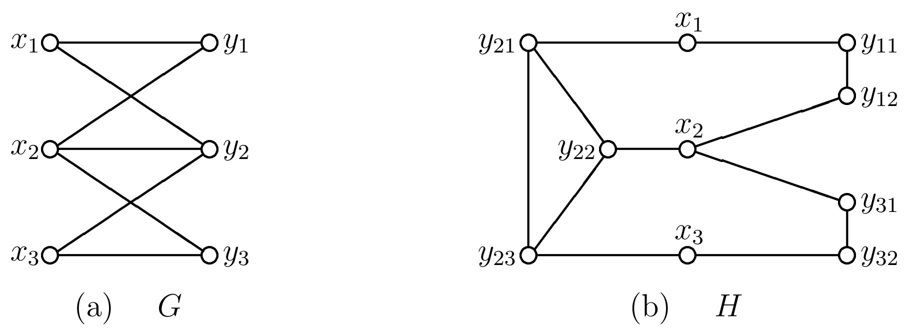


Figure 3.12: H is the expansion of G and H is a clique-Helly self-clique graph.

Let G be a graph. We *subdivide* a graph G by inserting a new vertex of degree 2 into each edge of G and let S be a set of all inserted new vertices. If $\tilde{G}$ is the subdivision of G, then $\tilde{G}$ is bipartite and has a natural bipartition $(X, Y) = (V(G), S)$. Note that, every vertex in S has degree 2.

Let G_1 and G_2 be any two disjoint graphs. Take G_1 and add three extra vertices $\{x_1, y_1, z_1\}$, make x_1 adjacent to every vertex in $G_1 \cup \{y_1, z_1\}$ and make y_1 adjacent to every vertex in $G_1 \cup \{x_1, z_1\}$. Call the resulting graph G_1'. Now subdivide G_1' to obtain G_1''. Do the same to G_2 with three other extra vertices $\{x_2, y_2, z_2\}$ to obtain G_2' and then subdivide it to get G_2''. Then G_1'' and G_2'' are connected, triangleless (therefore Helly) and without dominated vertices. The only maximal-degree vertices in G_i'' are the extra x_i and y_i, so any isomorphism $G_1'' \to G_2''$ induces an isomorphism $G_1' \to G_2'$ and so $G_1 \cong G_2$.

Define a new graph G_{12} by $V(G_{12}) = V(G_1^{''}) \cup V(K(G_2^{''}))$ and $E(G_{12}) = E(G_1^{''}) \cup E(K(G_2^{''})) \cup \{z_1 Q \mid Q \in V(K(G_2^{''}))$ and $z_2 \in Q\}$. This is just the disjoint union of $G_1^{''}$ and $K(G_2^{''})$ plus 2 specific edges. Larrión et al. [29] proved that the graph G_{12} obtained in this manner is self-clique if $G_1 \cong G_2$.

Theorem 3.21 *[29] Given any two graphs G_1 and G_2, construct G_{12} as above. If G_1 and G_2 are isomorphic, then G_{12} is self-clique.*

Example 3.4 Let G_1 and G_2 be the complete graphs K_2. Without loss of generality, let $G \cong G_1 \cong G_2$ be the graph shown in Figure 3.13(a). Construct the graph $G^{'}$ by adding three extra vertices $\{x, y, z\}$ to the graph G. Further, let x be adjacent to a, b, y, z and y be adjacent to a, b, x, z (Figure 3.13(b)). Now replace each edge of $G^{'}$ by a new path of length 2 and denote the graph obtained in this manner by $G^{''}$ (Figure 3.13(c)). The cliques of $G^{''}$ are denoted by Q_i. The clique graph of $G^{''}$, $K(G^{''})$ is shown in Figure 3.13(d). Construct the graph G_{12} by $V(G_{12}) = V(G^{''}) \cup V(K(G^{''}))$ and $E(G_{12}) = E(G^{''}) \cup E(K(G^{''})) \cup zQ_{15} \cup zQ_{16}$. Hence, by Theorem 3.21, G_{12} is self-clique.

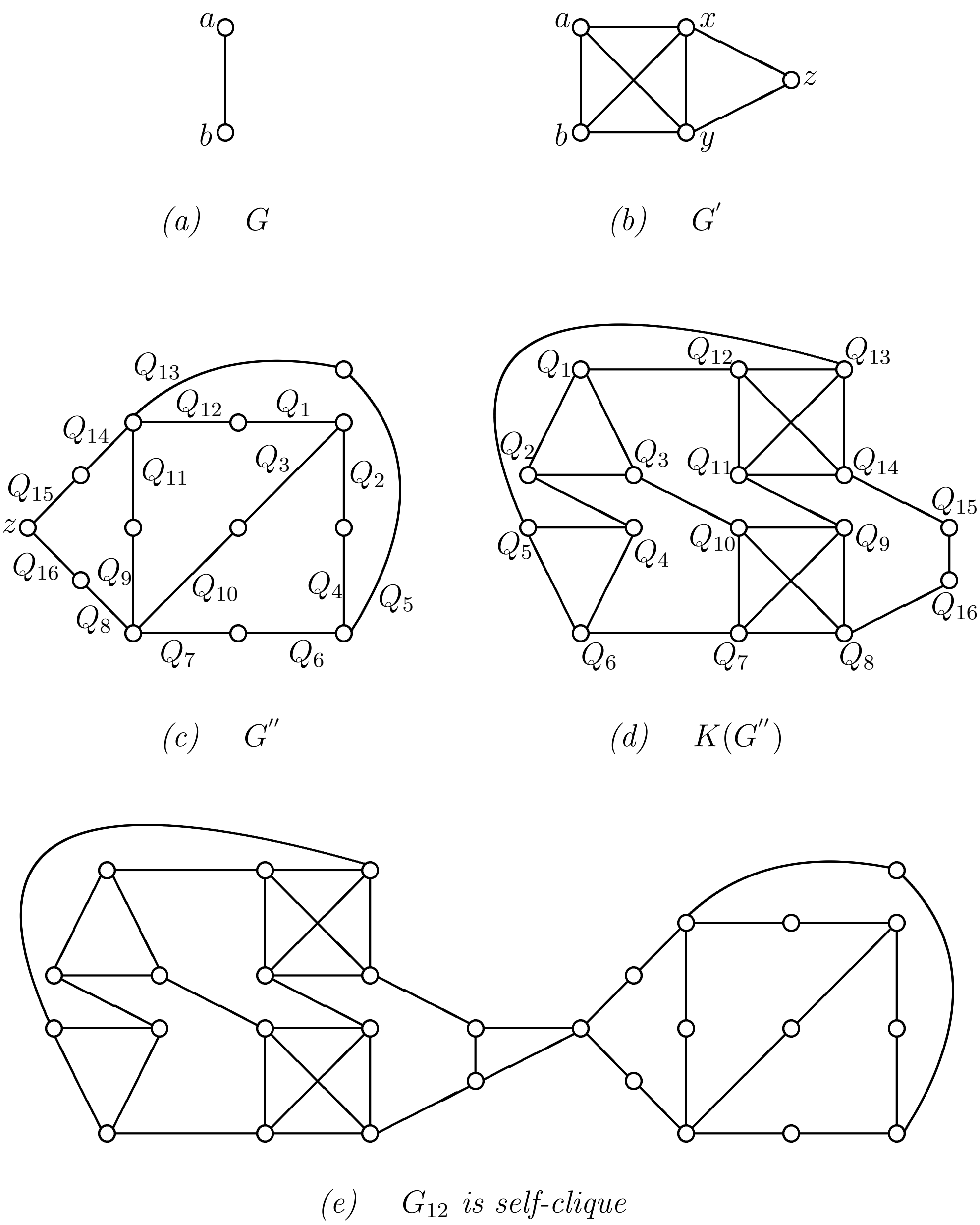


Figure 3.13: A construction of G_{12} from $G_1 \cong G_2 \cong G$

Chapter 4

On Self-clique Graphs With Given Clique Sizes

4.1 Introduction

Let G be a connected graph and let Q be a clique in G. Recall that Q is called a *large clique* if it is of size at least 3, otherwise it is called a *small clique*. Any clique in G which has a non-empty intersection with Q is called a *neighboring clique* of Q.

The set of all connected self-clique graphs with only one large clique has been determined in [13]. The set of all connected self-clique graphs having at most three large cliques has been determined in [30]. In this chapter, we present a characterization of the set of all connected self-clique graphs having precisely two large cliques. The result is stated in Theorem 4.1 while the proof is given in Section 4.3. The approach adopted here tends to be more elementary, describing directly in terms of graphs all the self-clique graphs with only two large cliques, and giving a way to construct them in a systematic way. It is known that the problem of recognizing self-clique graphs is isomorphism-complete (see [9] and [29]). However, the characterization presented in this chapter seems to lead to an efficient algorithm for recognizing self-clique graphs with only two large cliques. Perhaps it should also be reminded that, although this family of graphs is a subfamily of the family described in [30], it has the advantage that the characterization says precisely the subset with exactly two large cliques.

It might be possible to simplify the proofs using some previously known results (such as the matrix characterizations of [9] or some known results in [30]) but we feel that it is more appropriate to do this elsewhere since it involves an entirely different approach.

Alternatively, we may describe this in a different way. Let $\mathcal{G}(m,n)$ denote the family of all connected self-clique graphs whose clique-size sequence is $(2,\ldots,2,m,n)$ where $m,n \geq 3$. The purpose of this chapter is to determine $\mathcal{G}(m,n)$.

4.2 The Theorem

In this section, we shall describe six families of connected self-clique graphs whose clique size sequence is $(2,\ldots,2,m,n)$ where $m,n \geq 3$.

Let G be a graph. By a *pure path* in G, we mean a path in G whose internal vertices (if any) are of degree 2. A pure path in G is said to be *broken* if one of its ends is an end vertex.

Let K_m and K_n denote the complete graphs on m and n vertices respectively and $m,n \geq 3$. Let the vertices of K_m and K_n be $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$

respectively.

F(1) Let $q, r, s_1, s_2, t_1, t_2 \geq 0$ be some integers such that $q+r+s_1+2t_1 = m$ and $q + r + s_2 + 2t_2 = n$ where $q + r \geq 1$.

(a) If $s_1 \geq 1$ (respectively $s_2 \geq 1$), join a new vertex x (respectively y) to the vertices $x_1, \ldots, x_{s_1}$ of K_m (respectively $y_1, \ldots, y_{s_2}$ of K_n) and subdivide the new edges $xx_1, \ldots, xx_{s_1}$ (respectively $yy_1, \ldots, yy_{s_2}$) such that at most one of them is not subdivided.

If $s_1 = 0$ (respectively $s_2 = 0$), then the operation (a) is omitted.

(b) If $q \geq 1$, join x to y with q multiple edges and subdivide these q edges such that at most one of them is not subdivided and obtain q pure paths from x to y. In the event that one of these pure paths is not subdivided, then all the other $q - 1$ pure paths from x to y are of length at least 3. If $q = 1$, then such restriction is omitted. Let these pure paths (from x to y) be denoted $F_1, \ldots, F_q$.

For every pure path F_i of length f_i from x to y obtained in this way, we add a pure path F_i' of length $f_i - 1$ joining the vertex x_{i+s_1} to the vertex $y_{q+1-i+s_2}$ for $i = 1, 2, \ldots, q$. Note that there is at most one i for which $f_i = 1$. In such case, let $i = 1$ and identify the two vertices x_{1+s_1} and y_{q+s_2} into a single vertex.

If $q = 0$, then the operation (b) is omitted.

(c) If $t_1 \geq 1$ (respectively $t_2 \geq 1$), overlap t_1 cycles $C_1, \ldots, C_{t_1}$ (respectively t_2 cycles $D_1, \ldots, D_{t_2}$) each of length at least 4 all at the vertex x (respectively y). Also for each $i = 1, 2, \ldots, t_1$ (respectively $i = 1, 2, \ldots, t_2$), overlap a duplicate of C_i (respectively D_i) with K_m (respectively K_n) at the edge $x_{q+s_1+2i-1}x_{q+s_1+2i}$ (respectively $y_{q+s_2+2i-1}y_{q+s_2+2i}$).

If $t_1 = 0$ (respectively $t_2 = 0$), then the operation (c) is omitted.

(d) If $r \geq 1$, join x (respectively y) to $y_{q+1+s_2+2t_2}, \ldots, y_{q+r+s_2+2t_2}$ (respectively $x_{q+1+s_1+2t_1}, \ldots, x_{q+r+s_1+2t_1}$) and subdivide the edges $xy_{q+1+s_2+2t_2}$, $\ldots, xy_{q+r+s_2+2t_2}$ (respectively $yx_{q+1+s_1+2t_1}, \ldots, yx_{q+r+s_1+2t_1}$) such that at most one of them is not subdivided. We further require that the length of the pure path from x to $y_{q+i+s_2+2t_2}$ to be equal to the length of the pure path from y to $x_{q+i+s_1+2t_1}$ for $i = 1, 2, \ldots, r$.

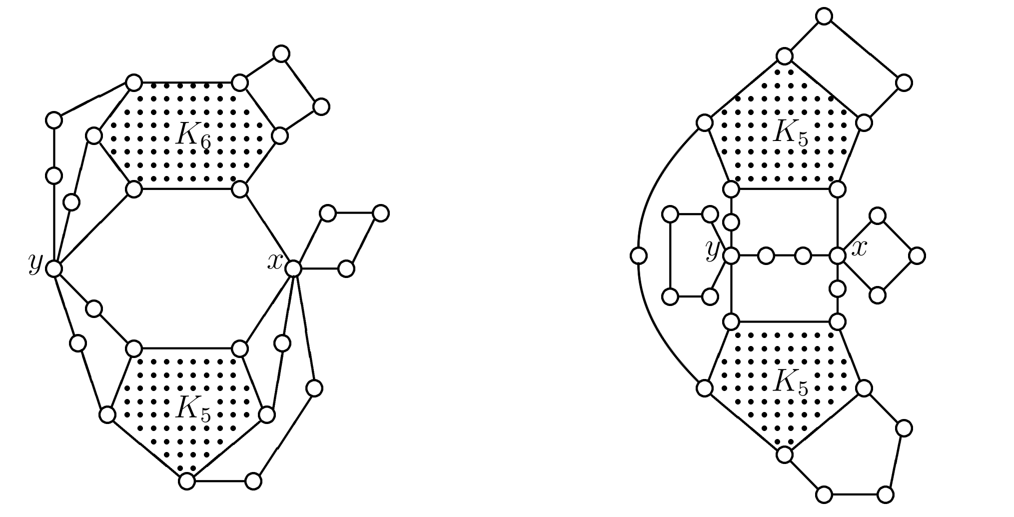
If $r = 0$, then the operation (d) is omitted.

Write $s = (s_1, s_2)$ and $t = (t_1, t_2)$ and let $\mathcal{G}_1(q, r; s, t)$ denote the set of all graphs obtained in this manner. In the event that $s_1 + s_2 = 0$, then we require that $q \geq 1$ and $r \geq 1$. Figure 4.1 depicts some graphs in $\mathcal{G}_1(q, r; s, t)$.

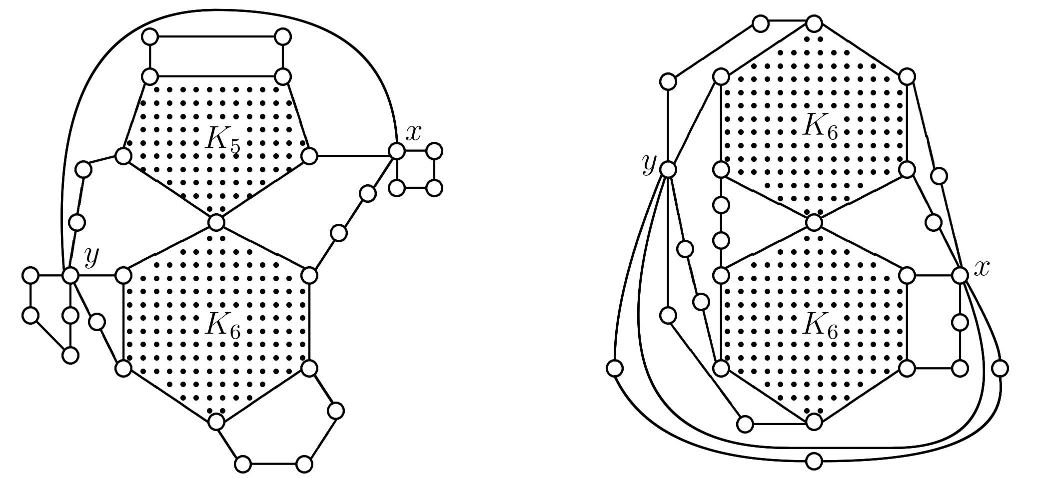
F(2) Let $G_1 \in \mathcal{G}_1(q, r; s, t)$ and suppose $q, r, s = (s_1, s_2)$ and $t = (t_1, t_2)$ are such that $q + r + s_1 + 2t_1 = m$, $q + r + s_2 + 2t_2 = n$, $q + r \geq 1$ and $s_1 \geq 1$.

If $s_1 = 1$, contract all edges in the pure path joining x and x_1 so that x and x_1 coincide to become a single vertex $x = x_1$. If $s_1 \geq 2$, assume further that the pure path from x to x_i has length at least 3 for $i = 2, \ldots, s_1$. Again contract all edges in the pure path joining x and x_1 so that x and x_1 coincide to become a single vertex $x = x_1$. We further require that if there is a pure path F_j from x to y of length 1, then all the pure paths from $x_{q+i+s_1+2t_1}$ to y and from $y_{q+i+s_2+2t_2}$ to x, $i = 1, \ldots, r$ must be of length at least 2. Let G_2 denote the resulting graph. Note that in such edge contraction, the pure path from x to x_i in G_1 gives rise to a cycle C_i^* in G_2 which overlaps with K_m at the edge x_1x_i, $i = 2, \ldots, s_1$.

Let $\mathcal{G}_2(q, r; s', t)$ denote the set of all graphs obtained in this manner. Here, notice that $s' = (s'_1, s'_2)$, where $s'_1 = s_1 - 1 \geq 0, s'_2 = s_2 \geq 0$ and $t = (t_1, t_2)$ are



$q = 0, r = 3, s = (1, 2), t = (1, 0)$ $\qquad$ $q = 1, r = 1, s = (1, 1), t = (1, 1)$



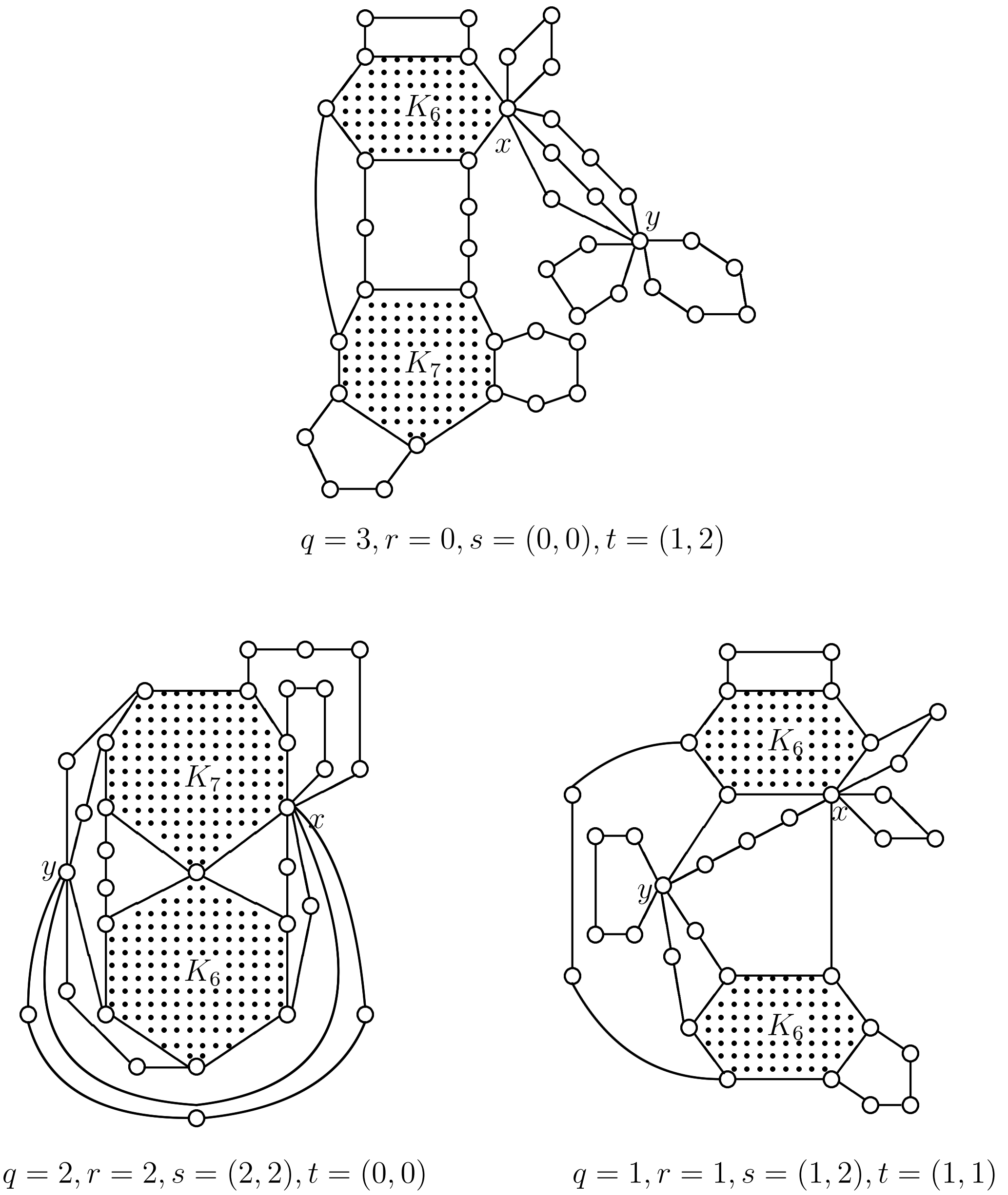
$q = 1, r = 1, s = (1, 2), t = (1, 1)$ $\qquad$ $q = 2, r = 2, s = (2, 2), t = (0, 0)$

Figure 4.1: Some graphs in $\mathcal{G}_1(q, r; s, t)$

such that $q + r + s'_1 + 2t_1 = m - 1$, $q + r + s'_2 + 2t_2 = n$, $q + r \geq 1$. Figure 4.2 depicts some graphs in $\mathcal{G}_2(q, r; s, t)$.

F(3) Let $G_3 \in \mathcal{G}_2(q, r; s', t)$ where $q, r, s' = (s'_1, s'_2)$ and $t = (t_1, t_2)$ are such that $q + r + s'_1 + 2t_1 = m - 1$, $q + r + s'_2 + 2t_2 = n$, $r \geq 1$.

If $r = 1$, contract all edges in the pure path joining x and $y_{q+1+s_2+2t_2}$ (respectively y and $x_{q+1+s_1+2t_1}$) so that they coincide to become a single vertex $x = y_{q+1+s_2+2t_2}$ (respectively $y = x_{q+1+s_1+2t_1}$). If $r \geq 2$, assume further that the



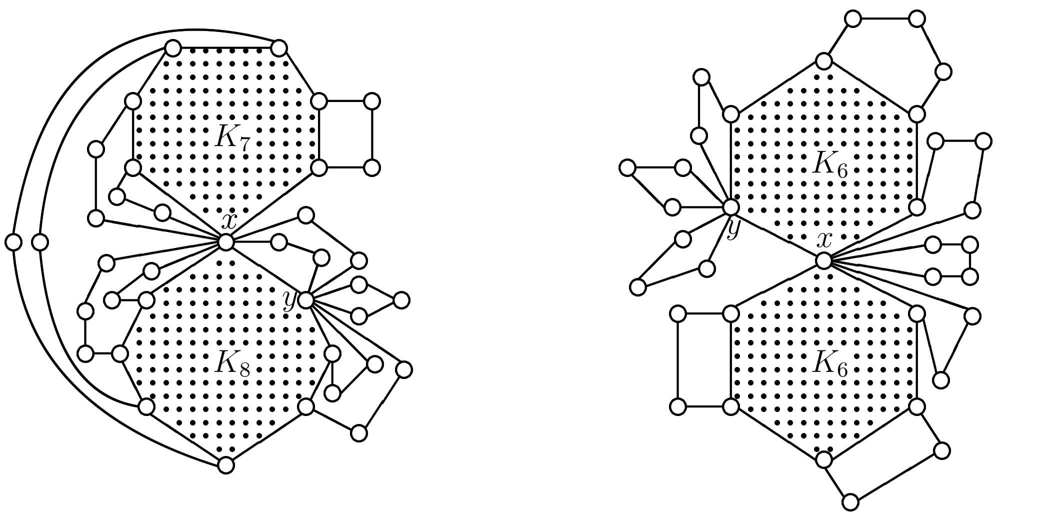
$q = 3, r = 0, s = (0, 0), t = (1, 2)$

$q = 2, r = 2, s = (2, 2), t = (0, 0)$ $q = 1, r = 1, s = (1, 2), t = (1, 1)$

Figure 4.2: Some graphs in $\mathcal{G}_2(q, r; s, t)$

pure path from x to $y_{q+i+s_2+2t_2}$ (respectively y to $x_{q+i+s_1+2t_1}$) has length at least 3 for $i = 1, \ldots, r-1$. Again contract all edges in the pure path joining x and $y_{q+r+s_2+2t_2}$ (respectively y and $x_{q+r+s_1+2t_1}$) so that they coincide to become a single vertex $x = y_{q+r+s_2+2t_2}$ (respectively $y = x_{q+r+s_1+2t_1}$). We further require that all the pure paths F_i from x to y must be of length at least 3 (and as such all the pure paths F_i' must be of length at least 2). In addition, all the pure paths from y to y_j, $j = 1, \ldots, s_2$ must be of length at least 2. Let G_4 denote the resulting graph. Note that the pure path from x to $y_{q+i+s_2+2t_2}$ (respectively y to $x_{q+i+s_1+2t_1}$) in G_3 gives rise to a cycle E_i in G_4 which overlap K_n (respectively K_m) at the edge $xy_{q+i+s_2+2t_2}$ (respectively $yx_{q+i+s_1+2t_1}$) for $i = 1, \ldots, r-1$.

Let $\mathcal{G}_3(q, r'; s', t)$ denote the set of all graphs obtained in this manner. Notice that $r' = r - 1 \geq 0$, $s' = (s'_1, s'_2)$, $s'_1, s'_2 \geq 0$ and $t = (t_1, t_2)$ are such that $q + r' + s'_1 + 2t_1 = m - 2, q + r' + s'_2 + 2t_2 = n - 1$. Figure 4.3 depicts some graphs in $\mathcal{G}_3(q, r; s, t)$.



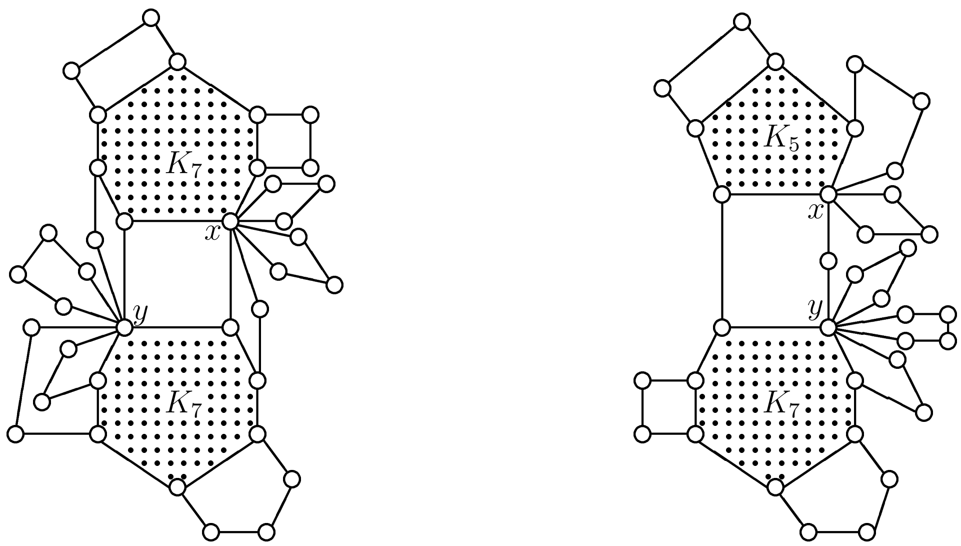
$q = 2, r = 2, s = (0, 2), t = (1, 0)$ $\qquad$ $q = 0, r = 1, s = (1, 0), t = (1, 2)$

Figure 4.3: Some graphs in $\mathcal{G}_3(q, r; s, t)$

F(4) Let $G_1 \in \mathcal{G}_1(q, r; s, t)$ and assume that $s_1, s_2 \geq 1$. Contract the edges in the pure path joining x and x_1 (respectively y and y_1) so that x and x_1 (respectively y and y_1) coincide to become a single vertex $x = x_1$ (respectively $y = y_1$). In the event that $s_1 \geq 2$ (respectively $s_2 \geq 2$), then assume further that the pure path from x to x_i (respectively y to y_j) has length at least 3 for $i = 2, \ldots, s_1$ (respectively $j = 2, \ldots, s_2$). We further require that the pure path

F_j from x to y must be of length at least 2. Let G_5 denote the resulting graph. As in case F(2), note that the pure path from x to x_i (respectively y to y_j) in G_1, becomes a cycle C_i^* (respectively D_j^*) in G_5 which overlaps with K_m (respectively K_n) at the edge $x_1x_i, i = 2, \ldots, s_1$ (respectively $y_1y_j, j = 2, \ldots, s_2$).

Let $\mathcal{G}_4(q, r; s', t)$ denote the set of all graphs obtained in this manner. Notice that $s' = (s_1', s_2')$ where $s_1' = s_1 - 1 \geq 0, s_2' = s_2 - 1 \geq 0$ and $t = (t_1, t_2)$ are such that $q + r + s_1' + 2t_1 = m - 1, q + r + s_2' + 2t_2 = n - 1, q + r \geq 1$. Figure 4.4 depicts some graphs in $\mathcal{G}_4(q, r; s, t)$.



$q = 0, r = 2, s = (0, 2), t = (2, 1)$ $\qquad$ $q = 1, r = 0, s = (1, 1), t = (1, 2)$

Figure 4.4: Some graphs in $\mathcal{G}_4(q, r; s, t)$

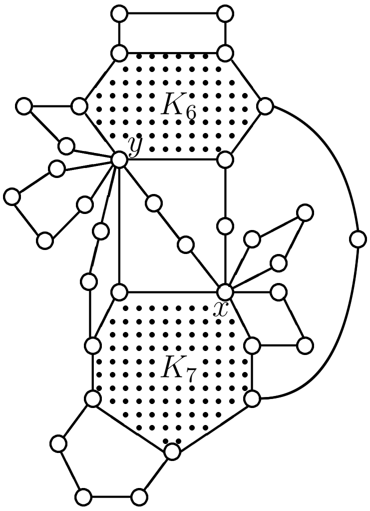
F(5) Let $G_1 \in \mathcal{G}_1(q, r; s, t)$ be such that $r \geq 1$. Impose the same condition set in F(1)(d) to the pure paths joining x to $y_{q+i+s_2+2t_2}$ and joining y to $x_{q+i+s_1+2t_1}$, $i = 1, 2, \ldots, r$. Contract the edges in the pure path joining x and $y_{q+r+s_2+2t_2}$ (respectively y and $x_{q+r+s_1+2t_1}$) so that x and $y_{q+r+s_2+2t_2}$ (respectively y and $x_{q+r+s_1+2t_1}$) coincide to become a single vertex $x = y_{q+r+s_2+2t_2}$

(respectively $y = x_{q+r+s_1+2t_1}$). In the event that $r \geq 2$, then assume further that the pure path from x to $y_{q+i+s_2+2t_2}$ (respectively y to $x_{q+i+s_1+2t_1}$) has length at least 3 for $i = 1, \ldots, r-1$. We further require that the pure path F_j from x to y must be of length at least 2. Let G_6 denote the resulting graph. Note that the pure path from x to $y_{q+i+s_2+2t_2}$ (respectively y to $x_{q+i+s_1+2t_1}$) in G_1, becomes a cycle E_i in G_6 which overlaps with K_n (respectively K_m) at the edge $y_{q+r+s_2+2t_2}y_{q+i+s_2+2t_2}$ (respectively $x_{q+r+s_1+2t_1}x_{q+i+s_1+2t_1}$), $i = 1, \ldots, r-1$.

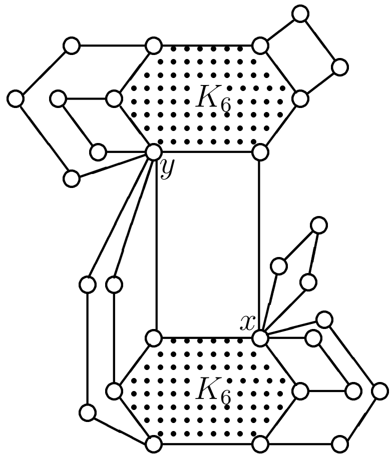
Let $\mathcal{G}_5(q, r'; s, t)$ denote the set of all graphs obtained in this manner. Notice that $r' = r - 1 \geq 0$, $s = (s_1, s_2)$, $s_1, s_2 \geq 0$ and $t = (t_1, t_2)$ are such that $q + r' + s_1 + 2t_1 = m - 1, q + r' + s_2 + 2t_2 = n - 1$. In the event that $s_1 + s_2 = 0$, then we require that $q \geq 1$. Figure 4.5 depicts some graphs in $\mathcal{G}_5(q, r; s, t)$.

F(6) Let $G_7 \in \mathcal{G}_4(q, r; s', t)$ where $q, r, s' = (s'_1, s'_2)$ and $t = (t_1, t_2)$ are such that $q + r + s'_1 + 2t_1 = m - 1$, $q + r + s'_2 + 2t_2 = n - 1$, $r \geq 1$ and $s'_1, s'_2 \geq 0$.

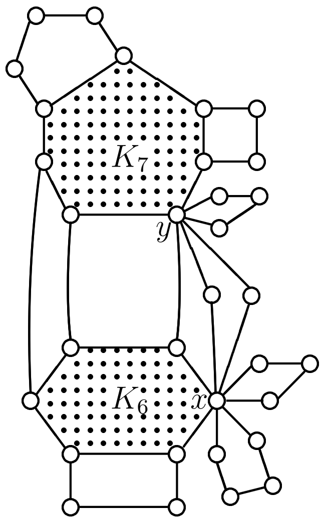
Contract all edges in the pure path joining x and $y_{q+r+s_2+2t_2}$ (respectively y and $x_{q+r+s_1+2t_1}$) so that they coincide to become a single vertex $x = y_{q+r+s_2+2t_2}$ (respectively $y = x_{q+r+s_1+2t_1}$). In the event that $r \geq 2$, then assume that the pure path from x to $y_{q+i+s_2+2t_2}$ (respectively y to $x_{q+i+s_1+2t_1}$) has length at least 3 for $i = 1, \ldots, r-1$. Let G_8 denote the resulting graph. Note that the pure path from x to $y_{q+i+s_2+2t_2}$ (respectively y to $x_{q+i+s_1+2t_1}$) in G_7 gives rise to a cycle E_i in G_8 which overlaps with K_n (respectively K_m) at the edge $xy_{q+i+s_2+2t_2}$ (respectively $yx_{q+i+s_1+2t_1}$) for $i = 1, \ldots, r-1$. Also, we assume that the pure path F_i from x to y has length at least 3 for each $i = 1, \ldots, q$. As such each pure path F'_i has length at least 2.



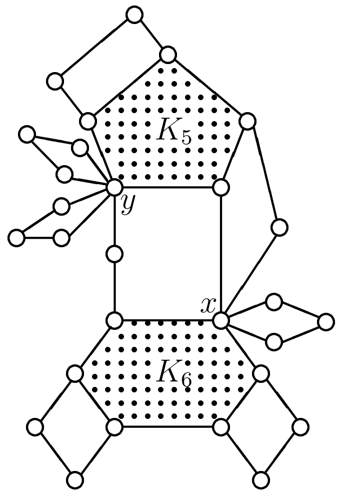
$q = 1, r = 1, s = (1, 2), t = (1, 1)$



$q = 0, r = 2, s = (1, 3), t = (1, 0)$



$q = 2, r = 0, s = (0, 1), t = (2, 1)$

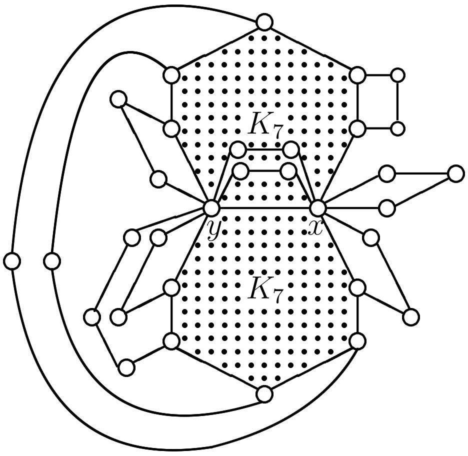


$q = 0, r = 0, s = (2, 1), t = (1, 2)$

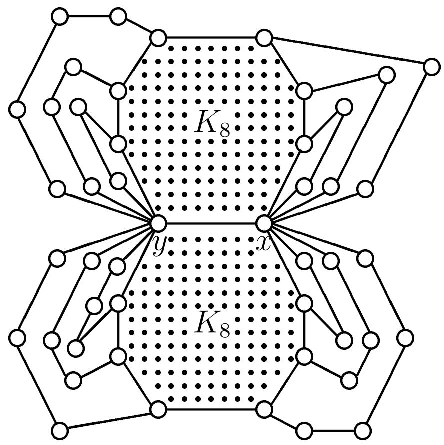
Figure 4.5: Some graphs in $\mathcal{G}_5(q, r; s, t)$

Let $\mathcal{G}_6(q, r'; s', t)$ denote the set of all graphs obtained in this manner. Notice that $r' = r - 1 \geq 0$, $s' = (s'_1, s'_2)$, $s'_1, s'_2 \geq 0$ and $t = (t_1, t_2)$ are such that $q + r' + s'_1 + 2t_1 = m - 2$ and $q + r' + s'_2 + 2t_2 = n - 2$. Figure 4.6 depicts some graphs in $\mathcal{G}_6(q, r; s, t)$.

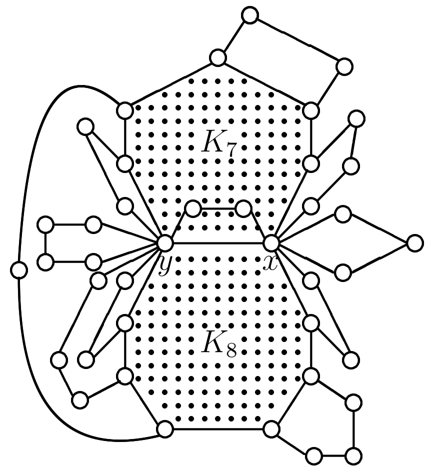
The general drawings of the graphs $\mathcal{G}_i(q, r; s, t)$ for $1 \leq i \leq 6$ are shown in Appendix A.



$q = 2, r = 1, s = (0, 2), t = (1, 0)$



$q = 0, r = 3, s = (3, 3), t = (0, 0)$



$q = 1, r = 1, s = (1, 2), t = (1, 1)$

Figure 4.6: Some graphs in $\mathcal{G}_6(q, r; s, t)$

Theorem 4.1 *Let G be a connected graph whose clique size sequence is* $(2, \dots, 2, m, n)$ *where $m, n \geq 3$. Then G is self-clique if and only if $G \in \mathcal{G}_i(q, r; s, t)$ for some $1 \leq i \leq 6$.*

4.3 Proof of Theorem 1

Let G denote a graph. Throughout this section, let φ be a mapping from $\mathcal{K}(G)$ onto $V(K(G))$ such that for any two cliques $Q_1, Q_2 \in \mathcal{K}(G)$, $\varphi(Q_1)\varphi(Q_2) \in$

$E(K(G))$ if and only if $Q_1 \cap Q_2 \neq \emptyset$. By the action of φ on $\mathcal{K}(G)$, it means the operation of forming the clique graph $K(G)$ of G.

Sufficiency

This is by direct verification. To help go through this verification, some observations are in order.

Let $G \in \mathcal{G}_1(q, r; s, t)$.

(i) The vertex x (respectively y) in G is such that there are precisely m (respectively n) cliques in G each of size 2 which contain the vertex x (respectively y). Under the action of φ, these m (respectively n) cliques are mapped to m (respectively n) vertices which form a clique K_m^* of size m (respectively K_n^* of size n) in $K(G)$. The two cliques K_m^* and K_n^* in $K(G)$ correspond to the two cliques K_m and K_n in G.

(ii) On the other hand, the clique K_m (respectively K_n) in G is such that each of its vertices, with at most one exception, is adjacent to precisely one other distinct vertex of G giving precisely m (respectively n) cliques all but at most one of size 2. Under the action of φ, the clique K_m (respectively K_n) is mapped to a vertex, called X (respectively Y), in $K(G)$ and those m (respectively n) cliques adjacent to K_m (respectively K_n) are mapped to m (respectively n) vertices in $K(G)$ each adjacent to X (respectively Y). The vertex X (respectively Y) in $K(G)$ together with its neighbors correspond to the vertex x (respectively y) in G.

(iii) Suppose $t_1 \geq 1$ (respectively $t_2 \geq 1$). Consider those cycles C_i, $i = 1, \ldots, t_1$ (respectively D_j, $j = 1, \ldots, t_2$) which overlap at the vertex x (respectively y) in G. Under the action of φ, each C_i (respectively D_j) is mapped to a cycle of the same length in $K(G)$ which overlaps with K_m^* (respectively K_n^*) at an edge.

As for those duplicate cycles C_i, $i = 1, \ldots, t_1$ (respectively D_j, $j = 1, \ldots, t_2$) which overlap with K_m (respectively K_n) at an edge in G, the action of φ maps each of these C_i (respectively D_j) to a cycle of the same length in $K(G)$ which overlap at the vertex X (respectively Y).

(iv) Suppose $q \geq 1$. The action of φ maps the pure path F_i (from x to y in G) of length f_i, $i = 1, \ldots, q$ to a pure path (from a vertex in K_m^* to a vertex in K_n^* in $K(G)$) of length $f_i - 1$. On the other hand, the action of φ maps the cliques K_m and K_n and the pure path $F_i^{'}$ (from a vertex in K_m to a vertex in K_n in G) of length $f_i - 1$ to a pure path (from X to Y in $K(G)$) of length f_i.

(v) Suppose $r \geq 1$. Recall that, in the graph G, for each pure path P_1 from x to a vertex in K_n, there is a pure path P_2 (having the same length as P_1) from y to a vertex in K_m and conversely. Under the action of φ, P_1 (respectively P_2) is mapped to P_2^* (respectively P_1^*) in $K(G)$ where P_2^* (respectively P_1^*) is a pure path from a vertex in K_m^* to the vertex Y (respectively from a vertex in K_n^* to the vertex X) and having the same length as P_1 (respectively P_2).

(vi) Suppose $s_1 \geq 1$ (respectively $s_2 \geq 1$). The action of φ maps any pure path from x to a vertex in K_m (respectively from y to a vertex in K_n) in G to a pure path of the same length from a vertex in K_m^* to the vertex X (respectively from a vertex in K_n^* to the vertex Y) in $K(G)$.

Putting all the observations (i) - (vi) together, it follows that $G \in \mathcal{G}_1(q, r; s, t)$ is self-clique with clique size sequence $(2, \ldots, 2, m, n)$ for some $m, n \geq 3$.

The case $G \in \mathcal{G}_i(q, r; s, t)$, $i \geq 2$ is treated in a similar way.

Necessity

Since $G \in \mathcal{G}(m,n)$, G contains two cliques K_m and K_n. Moreover, since $G \cong K(G)$, there exist cliques $Q_1, \ldots, Q_m$ (respectively $R_1, \ldots, R_n$) in G which form the vertices of a clique K_m^* of size m (respectively K_n^* of size n) in $K(G)$. As such, $Q_i \cap Q_j \neq \emptyset$ (respectively $R_i \cap R_j \neq \emptyset$) for every $i \neq j$.

In fact, since all cliques in G other than K_m and K_n are of size 2, we see that

$$\bigcap_{i=1}^{m} Q_i = \{x\} \quad \text{and} \quad \bigcap_{j=1}^{n} R_j = \{y\} \tag{4.1}$$

for some vertices x and y in G and $x \neq y$.

Furthermore, since $G \cong K(G)$, there exist m cliques $Q_1^*, \ldots, Q_m^*$ and n cliques $R_1^*, \ldots, R_n^*$ in $K(G)$ such that

$$\bigcap_{i=1}^{m} Q_i^* = \{x^*\} \quad \text{and} \quad \bigcap_{j=1}^{n} R_j^* = \{y^*\} \tag{4.2}$$

where x^* and y^* are two vertices in $K(G)$ such that $x^* \neq y^*$.

We shall show that the two large cliques in G have at most two vertices in common.

(L1) *Suppose $|K_m \cap K_n| = k$. Then $k \leq 2$.*

Since $G \cong K(G)$, we have, in $K(G)$, $|K_m^* \cap K_n^*| = k$ so that $K_m^* \cap K_n^*$ contains a complete subgraph on k vertices. In view of this, we may assume (without loss of generality) that $Q_i = R_i$ for $i = 1, \ldots, k$.

If the above statement is not true, then $k \geq 3$. But then this implies that there exists $1 \leq i \leq k$ such that $Q_i = K_2 = R_i$ and we have $\cap_{j=1}^{k} Q_j = \{x\}$ and $\cap_{j=1}^{k} R_j = \{x\}$, a contradiction to (4.1). $\square$

(L2) *Let z be a vertex in K_m (respectively K_n) such that z is neither in $\{x, y\}$ nor in $K_m \cap K_n$. Then z is adjacent to at most one vertex in $G - (K_m \cup K_n)$.*

If there are two vertices z_1 and z_2 in $G - K_m$ (respectively $G - K_n$) that are both adjacent to z, then z_1 and z_2 are not adjacent in G because G contains no triangle other than those in $K_m \cup K_n$. Under the action of φ, K_m, $\{z_1, z\}$ and $\{z_2, z\}$ give rise to a triangle in $K(G)$. However, this is impossible because any triangle in $K(G)$ must be contained in $K_m^* \cup K_n^*$. Hence any vertex in K_m (respectively K_n) is adjacent to at most one vertex in $G - (K_m \cup K_n)$. $\square$

Suppose v is a vertex in G (respectively $K(G)$). Let $\beta(v)$ (respectively $\beta^*(v)$) denote the number of small cliques containing v in G (respectively $K(G)$). Since there are at most two large cliques containing x (respectively y), we see that $m - 2 \leq \beta(x) \leq m$ (respectively $n - 2 \leq \beta(y) \leq n$).

(L3) *Let Q be a clique in G. Then $\varphi(Q) = x^*$ (respectively $\varphi(Q) = y^*$) only if Q is K_m (respectively K_n) unless $m = n$ (in which case we have either $\varphi(K_m) = x^*$ and $\varphi(K_n) = y^*$ or else $\varphi(K_m) = y^*$ and $\varphi(K_n) = x^*$).*

To see that the above statement is true, suppose $\varphi(Q) = x^*$ and $\varphi(R) = y^*$. Suppose $\beta(x) = m - t_1$ and $\beta(y) = n - t_2$ where $0 \leq t_1, t_2 \leq 2$ and $m \neq n$. Since $\beta^*(x^*) = \beta(x)$ and $\beta^*(y^*) = \beta(y)$, we see that x^* (respectively y^*) is contained in $m - t_1$ (respectively $n - t_2$) small cliques in $K(G)$.

Note that

$(O1)$ x (respectively y) is contained in $K_m \cup K_n$ if and only if $t_1 \geq 1$ (respectively $t_2 \geq 1$). Equivalently, x^* (respectively y^*) is contained in $K_m^* \cup K_n^*$ if and only if $t_1 \geq 1$ (respectively $t_2 \geq 1$).

Note also that

$(O2)$ $\varphi(Q_i)$ is in K_m^* and $\varphi(R_j)$ is in K_n^*.

In view of $(O1)$ and $(O2)$, it follows that if $t_1 = 0$, then $\varphi(Q) = x^*$ implies that Q is either K_m or K_n. Likewise, if $t_2 = 0$, then $\varphi(R) = y^*$ implies that R is either K_m or K_n.

Similarly, if $t_1 \geq 1$ (respectively $t_2 \geq 1$), then $\varphi(Q) = x^*$ implies that Q (respectively R) is K_m, K_n or the Q_i's (respectively R_j's).

Now we shall show that neither Q nor R can be a small clique.

Assume on the contrary that $Q = Q_r$ for some $1 \leq r \leq m$ and that Q_r is a small clique. Now, this implies that x^* is a vertex on the clique K_m^* in $K(G)$.

Let Q' be a neighboring clique of Q_r and Q' is not any of the Q_i's. Then clearly, $\varphi(Q')$ is adjacent to x^*. There are two cases to consider. In each case, we shall show that $\beta^*(x^*) \leq 1$ which yields a contradiction (since $\beta^*(x^*) = \beta(x) = m - t_1$) unless $m = 3$ and $t_1 = 2$.

Case (i): Q' is a small clique R_j for some $1 \leq j \leq n$

In this case, $\varphi(Q')$ is a vertex on K_n^*. If x^* is not adjacent to other vertices of K_n^* in $K(G)$, then $\beta^*(x^*) = 1$. If x^* is adjacent to another vertex in K_n^*, then x^* is a vertex in K_n^*, in which case, $\beta^*(x^*) = 0$.

Case (ii): $Q' \neq R_j$ for any $1 \leq j \leq n$

In this case, since K_m^* and K_n^* are the only two large cliques in $K(G)$, we see that $\varphi(Q')$ is the only vertex, other than those in K_m^*, that is adjacent to x^* in $K(G)$. Consequently, we have $\beta^*(x^*) = 1$.

Similarly, R is not a small clique R_r, for any $1 \leq r \leq n$, unless $n = 3$ and $t_2 = 2$.

Now, consider the case $m = 3 = n$ and $t_1 = 2 = t_2$. In this case, both x and y are contained in the two large cliques K_m and K_n. Consequently, $K_m \cap K_n = \{x, y\}$ by (L1). Hence we may assume that $Q_i = R_i$ where $i = 1, 2$. Then, under the action of φ, we see that $\varphi(Q_1), \varphi(Q_2), \varphi(Q_3)$ and $\varphi(Q_1), \varphi(Q_2), \varphi(R_3)$ each form a large clique of size 3 in $K(G)$. Clearly these two large cliques share a common edge $\varphi(Q_1)\varphi(Q_2)$. Hence neither $\varphi(Q_3)$ nor $\varphi(R_3)$ can be x^*. Therefore $\varphi(K_m) = x^*$ and $\varphi(K_n) = y^*$ (or equivalently $\varphi(K_m) = y^*$ and $\varphi(K_n) = x^*$) in this case.

Hence, we have $\{Q, R\} = \{K_m, K_n\}$.

Suppose on the contrary that $\varphi(K_n) = x^*$. Then $\varphi(K_m) = y^*$. Since $m \neq n$, we may assume that $m < n$.

Recall that $\beta^*(x^*) = m - t_1$ and $\beta^*(y^*) = n - t_2$. Moreover $t_1 \geq 1$ (respectively $t_2 \geq 1$) if and only if x (respectively y) is in $K_m \cup K_n$ (by $(O1)$).

We assert that $t_1 = 0 = t_2$.

To see this, we need to show that neither x nor y is in $K_m \cup K_n$.

Suppose $z \in \{x, y\}$. Because $\varphi(K_m) = y^*$ and $\varphi(K_n) = x^*$, we see that

$(O3)$ z in K_m implies that y^* is in K_m^* if z is x (respectively in K_n^* if z is y), and

$(O4)$ z in K_n implies that x^* is in K_m^* if z is x (respectively in K_n^* if z is y).

(i) Suppose y is in K_m. Then by $(O3)$, y^* is in K_n^* and this implies that y is in K_n (because $K(G) \cong G$). Hence y is in $K_m \cap K_n$. By $(O4)$, x^* is in K_n^* and hence x is in K_n (because $K(G) \cong G$). But then, by $(O4)$ again, x^* is in K_m^* and hence x is in $K_m \cap K_n$. Therefore $t_1 = 2 = t_2$. Since $\varphi(K_m) = y^*$, we see that K_m has $n - t_2$ neighboring small cliques, other than those containing x and y, a contradiction to (L2) (because $m < n$).

(ii) Suppose y is in K_n. Then by $(O4)$, x^* is in K_n^* and this implies that x is in K_n (because $K(G) \cong G$). By $(O4)$ again, x^* is in K_m^* and hence x is in $K_m \cap K_n$. By $(O3)$, y^* is in K_m^* and hence y is in $K_m \cap K_n$. Therefore $t_1 = 2 = t_2$. As in (i) above, we see that K_m has $n - t_2$ neighboring small cliques, other than those containing x and y, a contradiction to (L2) (because $m < n$).

If x is in $K_m \cup K_n$, then argue in a similar way as in (i) and (ii) above, we will obtain a contradiction. This proves the assertion.

Since $t_1 = 0 = t_2$ and $\varphi(K_m) = y^*$, we see that K_m has n neighboring small cliques, a contradiction to (L2).

This completes the proof. □

In the rest of the arguments, by $K(G) \cong G$ we mean φ is an isomorphism between G and $K(G)$.

(L4) *Suppose $\beta(x) = m - 2$. Then $\beta(y) \leq n - 1$. Likewise if $\beta(y) = n - 2$, then $\beta(x) \leq m - 1$.*

To see this, suppose on the contrary that $\beta(x) = m - 2$ and $\beta(y) = n$. Then x is in $K_m \cap K_n$ and y is not in $K_m \cup K_n$. Since $\varphi(K_m) = x^*$ and $\varphi(K_n) = y^*$ (by

(L3)), we see that x^* and y^* are contained in K_m^*. But this is a contradiction because $K(G) \cong G$. Hence $\beta(y) \leq n-1$. Similarly, if $\beta(y) = n-2$, then $\beta(x) \leq m-1$. $\square$

(L5) *Let z be a vertex in K_m (respectively K_n) such that z is neither in $\{x, y\}$ nor in $K_m \cap K_n$.*

(i) If $K_m \cap K_n = \emptyset$, then z is adjacent to precisely one vertex in $G - K_m$ (respectively $G - K_n$).

(ii) If $K_m \cap K_n \neq \emptyset$, then z is adjacent to precisely one vertex in $G - (K_m \cup K_n)$.

By (L2), any vertex in K_m (respectively K_n) is adjacent to at most one vertex in $G - (K_m \cup K_n)$.

Suppose $\beta(x) = m - t_1$ and $\beta(y) = n - t_2$ where $0 \leq t_1, t_2 \leq 2$.

Suppose $t_1 = 0$. Then $t_2 \leq 1$ by (L4). Moreover y is not in K_m (otherwise, by (L3), x^* is in K_n^*, a contradiction). Since $\beta^*(x^*) = \beta(x)$ and $\beta^*(y^*) = \beta(y)$, by (L3), we see that K_m has m neighboring cliques and that K_n has $n - t_2$ neighboring cliques other than those (small cliques) containing y if y is in K_n.

Suppose $t_1 = 1$. Then $0 \leq t_2 \leq 2$. The case $t_2 = 0$ is similar to the preceding case. So assume that $1 \leq t_2 \leq 2$. If $t_2 = 1$, then x and y are in $K_m \cup K_n$. However, x and y must be in different large cliques (otherwise $K_m^* \cap K_n^*$ contains x^* or y^*, a contradiction because $K(G) \cong G$). Since $\beta^*(x^*) = \beta(x)$ and $\beta^*(y^*) = \beta(y)$, by (L3), we see that K_m (respectively K_n) has $m-1$ (respectively $n-1$) neighboring cliques other than those (small cliques) containing z where $z \in \{x, y\}$.

If $t_2 = 2$, then y is in $K_m \cap K_n$. By (L3), x^* and y^* are both contained in K_n^*. It follows that x is in K_n. Since $\beta^*(x^*) = \beta(x)$ and $\beta^*(y^*) = \beta(y)$, by (L3), we see that K_m has $m-1$ neighboring cliques other than those (small cliques) containing y while K_n has $n-2$ neighboring cliques other than those (small cliques) containing x or y.

Suppose $t_1 = 2$. Then $1 \leq t_2 \leq 2$ by (L4). If $t_2 = 1$, the situation is similar to the preceding case. So assume that $t_2 = 2$. Then $K_m \cap K_n = \{x, y\}$. Consequently, $\beta(x) = m-2 = \beta^*(x^*)$ and $\beta(y) = n-2 = \beta^*(y^*)$ and K_m (respectively K_n) has $m-2$ (respectively $n-2$) neighboring small cliques other those (small cliques) containing x and y.

This completes the proof. $\square$

(L6) *Let w be a vertex of G and w is not in $K_m \cup K_n \cup \{x, y\}$. Then w is of degree 2 in G.*

Suppose w is of degree l in G. Then there are l small cliques containing w. These l cliques give rise to a clique of size l in $K(G)$. Since $G \in \mathcal{G}(m, n)$, it must be the case that $l \leq 2$. Clearly $l \neq 0$ because G is connected.

If $l = 1$, then there is a broken pure path P of length $\alpha \geq 1$ connecting w and a vertex z in $K_m \cup K_n \cup \{x, y\}$.

(i) If $z = x$ (respectively $z = y$), then the action of φ on G sends the path P to a broken pure path P^* of length $\alpha - 1$ in $K(G)$ with one of its ends in K_m^* (respectively K_n^*). Now, $G \cong K(G)$ implies that there is also a broken pure path P_1 of length $\alpha - 1$ in G with one of its ends in K_m (respectively K_n).

(ii) If z is some vertex on K_m (respectively K_n), then the action of φ on G sends the path P to a broken pure path P^* of length α in $K(G)$ where one of its ends is x^* (respectively y^*) by (L3). Again, $G \cong K(G)$ implies that there is also a broken pure path P_2 of length α in G where one of its ends is x (respectively y).

Repeat the same argument to P_1 and P_2 (by combining the observations (i) and (ii)), in a finite number of steps, we arrive at a contradiction: either $G \ncong K(G)$ or else $K_m \cup K_n$ contains a vertex not adjacent to any vertex in $G - (K_m \cup K_n)$, which by (L5), is impossible. $\square$

Let the vertices of K_m and K_n be denoted $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ respectively. Also, let $V(Q_i) = \{x, u_i\}$ where $i = 1, \ldots, \beta(x)$ and $V(R_j) = \{y, v_j\}$ where $j = 1, \ldots, \beta(y)$. Note that u_i and u_j (respectively v_i and v_j) are not adjacent if $i \neq j$. We have the following observations.

(C1) If there is a pure path in G connecting u_i and u_j (respectively v_i and v_j), $i \neq j$, then this pure path yields a cycle C of length at least 4 overlapping at x (respectively y). Under the action of φ, the cycle C is sent to the cycle C^* of the same length (in $K(G)$) overlapping with K_m^* (respectively K_n^*) at the edge Q_iQ_j (respectively R_iR_j). Since $G \cong K(G)$, this implies that in G, there is also a cycle C' of the same length overlapping with K_m (respectively K_n) at the edge x_hx_l (respectively y_hy_l) for some $h \neq l$.

Conversely, if there is a pure path in G connecting x_h and x_l (respectively y_h and y_l), $h \neq l$, then this pure path yields a cycle C' of length at least 4 overlapping with K_m (respectively K_n) at x_hx_l (respectively y_hy_l). The action of φ on C' shows that there is also a cycle C of the same length overlapping at the vertex x (respectively y) (by (L3)).

Suppose there are t_1 (respectively t_2) such cycles overlapping at x (respectively y). Then $t_1, t_2 \geq 0$. Let $t = (t_1, t_2)$.

(C2) If there is a pure path in G connecting u_i and v_j, then this pure path yields a pure path F with x and y as its ends and passing through u_i and v_j. Suppose the length of F is f. Under the action of φ, F is sent to a pure path F^* of length $f-1$ in $K(G)$ with Q_i and R_j as its ends. Since $G \cong K(G)$, this implies that there is also a pure path F' of length $f-1$ in G with x_h and y_l as its ends for some some h and l.

Conversely, if there is a pure path F' in G of length f' with x_h and y_l as its ends for some h and l, then the action of φ on F' shows that there is also a pure path F in G of length $f'+1$ with x and y as its ends and passing through the vertices u_i and v_j for some i and j. (See (L5).)

Suppose there are q such pure paths F. Then $q \geq 0$. Note that if some such pure path has length one, then all other such paths must have length at least 3.

(C3) If there is a pure path in G connecting u_i and y_j (respectively v_i and x_j), then this yields a pure path P of length f (≥ 1) with x and y_j (respectively y and x_j) as its ends and passing through u_i (respectively v_i). Under the action of φ, P is sent to a pure path P^* of length f in $K(G)$ with Q_i and $\varphi(K_n)$ (which is y^* by (L3)) (respectively R_i and $\varphi(K_m)$ (which is x^* by (L3))) as its ends. Since $G \cong K(G)$, this implies that there is also a pure path P' of length f in G with y and x_h (respectively x and y_h) as its ends and passing through v_l (respectively u_l) for some h and l. (See (L5)).

Suppose there are r such pure paths. Then $r \geq 0$.

(C4) If there is a pure path in G connecting u_i and x_j (respectively v_i and y_j), then this yields a pure path P_1 of length f_1 with x and x_j (respectively P_2 of length f_2 with y and y_j) as its ends and passing through u_i (respectively v_i). Under the action φ, P_1 (respectively P_2) is sent to a pure path P_1^* of length f_1 (respectively P_2^* of length f_2) with Q_i and $\varphi(K_m)$ (which is x^* (by (L3))) (respectively R_i and $\varphi(K_n)$ (which is y^* (by (L3))) as its ends. (See (L5).)

Suppose there are s_1 (respectively s_2) such pure paths. Then $s_1, s_2 \geq 0$. Note that, for each s_i, there is at most one such pure path having length equal to one. Let $s = (s_1, s_2)$.

We shall now finish the proof for Theorem 1 by considering all the possible cases on $\beta(x)$ and $\beta(y)$ where $m - 2 \leq \beta(x) \leq m$ and $n - 2 \leq \beta(y) \leq n$.

Note that, by (L6) and since G is connected, every vertex not in $K_m \cup K_n \cup \{x, y\}$ belongs to a pure path of one of the kinds considered above.

Note that, by (L4), if $\beta(x) = m-2$, then $\beta(y) \leq n-1$ and that if $\beta(y) = n-2$, then $\beta(x) \leq m - 1$.

In each of the following cases, we make use of (L4), (L5) and the observations (C1) – (C4) to draw the conclusion of Theorem 1.

(I) $\beta(x) = m - 2$ and $\beta(y) = n - 2$.

In this case, x and y are both in $K_m \cap K_n$. Consequently, $K_m \cap K_n = \{x, y\}$. Moreover, any pure path connecting x and y, x and the x_i's, x and the y_i's, y and the x_i's, y and the y_i's must be of length at least 3. Also, any pure path connecting the x_i's and the y_j's must be of length at least 2 (see (C2)). It follows that $G \in \mathcal{G}_6(q, r; s, t)$ where $s = (s_1, s_2)$, $t = (t_1, t_2)$, $q + r + s_1 + 2t_1 = m - 2$

and $q + r + s_2 + 2t_2 = n - 2$.

(II) $\beta(x) = m - 2$ and $\beta(y) = n - 1$.

Then x is in $K_m \cap K_n$ and y is in $K_m \cup K_n$. Since $\varphi(K_m) = x^*$ and $\varphi(K_n) = y^*$ (by (L3)), x^* and y^* are both contained in K_m^*. It follows that y is in K_m. Hence, any pure path connecting x and y, x and the x_i's, x and the y_i's, y and the x_i's must be of length at least 3. Also, any pure path connecting the x_i's and the y_j's must be of length at least 2 (see (C2)). Moreover, any pure path connecting y and the y_j's must be of length at least 2. It follows that $G \in \mathcal{G}_3(q, r; s, t)$ where $s = (s_1, s_2)$, $t = (t_1, t_2)$, $q + r + s_1 + 2t_1 = m - 2$ and $q + r + s_2 + 2t_2 = n - 1$.

In the event that $\beta(x) = m - 1$ and $\beta(y) = n - 2$, then y is in $K_m \cap K_n$ and it follows in a similar argument that x is in K_n. This would eventually give rise to a graph G which is isomorphic to one in $\mathcal{G}_3(q, r; s, t)$.

(III) $\beta(x) = m - 1$ and $\beta(y) = n - 1$.

Then x and y are in $K_m \cup K_n$. Suppose x and y are contained only in the same large clique, say K_m. (The case that x and y are in K_n is similar.) By (L3), x^* is in $K_m^* \cap K_n^*$, a contradiction because $K(G) \cong G$. Hence either (i) x is in K_m and y is in K_n or else (ii) x is in K_n and y is in K_m. If there is a pure path of length one connecting x and y, then by (C2), some vertex x_i coincides with some vertex y_j (to become a single vertex x_i say). But then xyx_i is a triangle in G which is contained neither in K_m nor in K_n, a contradiction. Hence any pure path connecting x and y must be of length at least 2 in either case.

(i) In this subcase, any pure path connecting x and the x_i's, y and the y_i's must be of length at least 3. Moreover, if there is a pure path connecting x and some vertex y_j of length one (and hence one that connects y and some vertex x_j by (C3)), then all other such pure paths (if they exist) must be of length at least 2. It follows that $G \in \mathcal{G}_4(q, r; s, t)$ where $s = (s_1, s_2)$, $t = (t_1, t_2)$, $q + r + s_1 + 2t_1 = m - 1$, $q + r + s_2 + 2t_2 = n - 1$ and $q + r \geq 1$ (since G is connected).

(ii) In this subcase, any pure path connecting x and the y_i's, y and the x_i's must be of length at least 3. Moreover, if there is a pure path connecting x and some vertex x_j of length one (respectively y and some vertex y_j), then all other such pure paths (if they exist) must be of length at least 2. It follows that $G \in \mathcal{G}_5(q, r; s, t)$ where $s = (s_1, s_2)$, $t = (t_1, t_2)$, $q + r + s_1 + 2t_1 = m - 1$, $q + r + s_2 + 2t_2 = n - 1$ and $q + r \geq 1$ (since G is connected).

(IV) $\beta(x) = m - 1$ and $\beta(y) = n$.

Then y is not in $K_m \cup K_n$ and x is in K_m (otherwise x is in K_n and by (L3), y^* is contained in K_m^*, a contradiction because $K(G) \cong G$). As such, any pure path connecting x and the x_i's must be of length at least 3. Also, if there are pure paths connecting y and the y_i's, then at most one of them is of length one.

If there is a pure path connecting x and y of length one, then all other pure paths connecting x and y must be of length at least 3. Moreover, since some vertex from K_m coincides with some vertex from K_n in this case, any pure path connecting y and the x_i's (and hence any pure path connecting x and the y_i's by (C3)) must be of length at least 2.

If any pure path connecting x and y is of length at least 2, then any pure path connecting y and the x_i's (and hence any pure path connecting x and the y_i's by (C3)) must be of length at least 2 with at most one exception.

It follows that $G \in \mathcal{G}_2(q, r; s, t)$ where $s = (s_1, s_2)$, $t = (t_1, t_2)$, $q + r + s_1 + 2t_1 = m - 1$, $q + r + s_2 + 2t_2 = n$ and $q + r \geq 1$ (since G is connected).

In the event that $\beta(x) = m$ and $\beta(y) = n - 1$, then x is not in $K_m \cup K_n$ and it follows in a similar argument that y is in K_n. This would eventually give rise to a graph G which is isomorphic to one in $\mathcal{G}_2(q, r; s, t)$.

(V) $\beta(x) = m$ and $\beta(y) = n$.

Then neither x nor y is in $K_m \cup K_n$. If there is a pure path connecting x and y of length one, then all other such pure paths must be of length at least 3. Note also that, any pure path connecting x and the x_i's (respectively y and the y_i's) must be of length at least 2 with at most one exception. Also, any pure path connecting x and the y_i's (and hence that connecting y and the x_i's) must be of length at least 2 with at most one exception. It follows that $G \in \mathcal{G}_1(q, r; s, t)$ where $s = (s_1, s_2)$, $t = (t_1, t_2)$, $q + r + s_1 + 2t_1 = m$, $q + r + s_2 + 2t_2 = n$ and $q + r \geq 1$ if $s_1 + s_2 \neq 0$ (since G is connected). In the event that $s_1 = 0 = s_2$, then $q, r \geq 1$.

This completes the proof. $\square$

Chapter 5

On Self-clique Graphs All of Whose Cliques Have Equal Size

5.1 Introduction

This chapter follows in the similar vein of thought by confining the attention on those self-clique graphs whose clique sizes are uniform.

Let $\mathcal{G}(k)$ denote the set of all connected self-clique graphs where each clique is of size k. In the present section, we determine all graphs in $\mathcal{G}(2)$ (Theorem 5.1). In the next section, while unable to determine all graphs in $\mathcal{G}(3)$, we turn to determine all those in $\mathcal{G}(3)$ which are 4-regular (Corollary 5.3) and all those in which the degree of any vertex is at most 4 (Theorem 5.3). In the subsequent sections, we show the existences of 5-regular graphs and 6-regular graphs in $\mathcal{G}(3)$ by constructions (Propositions 5.2 and 5.3). In the final section, we examine the existence of a graph in $\mathcal{G}(3)$ whose set of vertices admits two degrees r and s where $2 \leq r < s \leq 6$. It is shown that, with the exceptions of

$s = 6$ and $r \in \{2, 5\}$, such graphs do not exist in $\mathcal{G}(3)$ unless $r = 4$ and $s = 5$ (Propositions 5.4 to 5.7).

Recall that K_n, C_n and P_n are a complete graph, a cycle and a path on n vertices respectively.

Suppose $G \in \mathcal{G}(2)$. In [17], Escalante showed that, if G is finite, then G is the cycle C_n for some $n \geq 4$. Theorem 5.1 includes the case where G is infinite.

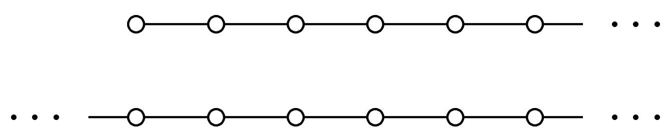


Figure 5.1: Two infinite self-clique graphs

Theorem 5.1 *A graph is in $\mathcal{G}(2)$ if and only if it is either the cycle C_n with $n \geq 4$ or else one of the infinite paths of Figure 5.1.*

Proof: The sufficiency is clear. We prove the necessity part.

Suppose $G \in \mathcal{G}(2)$. Let $x \in V(G)$. Since G is self-clique, $d(x) \leq 2$; otherwise $K(G)$ contains a clique of size larger than 2. Moreover, if $d(x) = 2$, we note that the subgraph induced by the neighbors of x consists of two isolated vertices (because every clique in G is of size 2).

Suppose every vertex in G is of degree 2. Then it follows that, in this case, either G is finite and G is C_n for some $n \geq 4$, or else G is infinite and G is the both way infinite path of Figure 5.1.

Now, suppose G contains an end vertex v. Then G is infinite for otherwise $|V(K(G))| < |V(G)|$ which is impossible because G is self-clique. This implies that G is the one way infinite path of Figure 5.1 .

This completes the proof. $\square$

5.2 4-regular graphs in $\mathcal{G}(3)$ and beyond

Lemma 5.1 *Let $G \in \mathcal{G}(3)$. Then any edge in G is contained in at most two cliques of G.*

Proof: If there is an edge uv of G which is contained in s cliques of G, then these s cliques will give rise to a K_s in $K(G)$. Since $G \in \mathcal{G}(3)$, we must have $s \leq 3$.

Suppose $s = 3$. Then these three cliques give rise to an induced subgraph H in G consisting of three triangles overlapping at the common edge uv. Since G is self-clique, $K(G)$ must also contain an induced subgraph isomorphic to H.

Let U, V and Q_i, $i = 1, 2, 3$ be some cliques of G which form an induced subgraph H^* of $K(G)$ isomorphic to H. Assume further that $U \cup V \cup Q_i$, $i = 1, 2, 3$ are three cliques in $K(G)$ that have UV as the common edge in the subgraph H^*.

Since $U \cap Q_i \neq \emptyset$, and the Q_i's are pairwise disjoint in G, we may assume that $U = \{x_1, x_2, x_3\}$ and that $Q_i = \{x_i, w_i, z_i\}$, $i = 1, 2, 3$. Now assume that $V = \{y_1, y_2, y_3\}$. There are two cases to consider.

Case (i): $|U \cap V| = 2$

In this case, assume that $x_1 = y_1$ and $x_2 = y_2$. Since $Q_3 \cap V \neq \emptyset$, we see that either $y_3 = w_3$ or $y_3 = z_3$ (because $x_3 \notin V$ and $x_1, x_2 \notin Q_3$). In either case, we have y_3 adjacent to x_3 which means that $\{x_1, x_2, x_3, y_3\}$ is a K_4 in G, a contradiction.

Case (ii): $|U \cap V| = 1$

In this case, assume that $x_1 = y_1$. Since $Q_i \cap V \neq \emptyset$, for $i = 2, 3$, we may assume that $y_i = w_i$. But this means that w_2 and w_3 are both adjacent to x_1 in G so that $R_i = \{x_1, x_i, w_i\}$, $i = 2, 3$, U and V are four cliques in G all with the common vertex x_1. This yields a K_4 in $K(G)$, a contradiction because $K(G) \cong G$.

This completes the proof. $\square$

Proposition 5.1 *Suppose $G \in \mathcal{G}(k)$ where $k \geq 2$. Then for any vertex $x \in V(G)$, we have $k - 1 \leq d(x) \leq k(k-1)$.*

Proof: It is clear that $d(x) \geq k - 1$ since each clique in G is of size k.

Since G is self-clique, at each vertex x, there are at most k cliques containing x. Hence the degree of x is at most $k(k-1)$. $\square$

Theorem 5.2 *Suppose $G \in \mathcal{G}(3)$ and let x be a vertex of degree r in G. Then $G[N(x)]$ is*

(i) P_r if $2 \leq r \leq 4$,

(ii) $P_2 \cup P_3$ if $r = 5$ and

(iii) $3P_2$ if $r = 6$.

Proof: Let $Q_1, \ldots, Q_t$ denote the set of cliques in G containing the vertex x. Then clearly, $1 \leq t \leq 3$ because these t cliques form a complete subgraph K_t in $K(G)$. Consequently, we have

$(O1)$ $G[N(x)]$ contains at most 3 edges because each edge in $G[N(x)]$, together with the vertex x, induce a clique of size 3 in G.

Also, since G contains neither cliques of size 2 nor cliques of size 4, we have

$(O2)$ $G[N(x)]$ contains neither isolated vertices nor triangles.

These two observations immediately imply that $G[N(x)]$ is P_r if $2 \leq r \leq 3$.

Suppose $r = 4$. If $G[N(x)]$ is disconnected, then $G[N(x)] \cong 2P_2$ by $(O2)$. In this case, $t = 2$ and Q_1 and Q_2 are such that Q_1Q_2 forms a clique of size 2 in $K(G)$ which is impossible because $K(G) \cong G$. Hence $G[N(x)]$ is connected.

By $(O1)$ and $(O2)$, $G[N(x)]$ is a tree on 4 vertices. If $G[N(x)]$ contains a vertex v of degree 3, then the edge xv is contained in the three cliques Q_1, Q_2 and Q_3, a contradiction to Lemma 5.1. Hence $G[N(x)] \cong P_4$. This proves (i).

Applying observations $(O1)$ and $(O2)$ to the cases $r = 5$ and $r = 6$ lead to the conclusions (ii) and (iii). $\square$

An immediate consequence to the above theorem is the following.

Corollary 5.1 *Suppose $G \in \mathcal{G}(3)$ and G is 4-regular. Then for any vertex $x \in V(G)$, $G[N(x)]$ is the path on 4 vertices.*

Corollary 5.2 *If there is an r-regular graph in $\mathcal{G}(3)$, then $r \geq 4$.*

Proof: Let G be an r-regular graph in $\mathcal{G}(3)$. Clearly, $r \geq 3$.

Suppose $r = 3$. Let x be a vertex of degree 3 in G. By Theorem 5.2(i), we may assume that $x_1x_2x_3$ is the path on 3 vertices in $G[N(x)]$. By Theorem 5.2(i), we may assume that $G[N(x_1)]$ is the path xx_2y for some vertex $y \in V(G)$ where $y \neq \{x, x_1, x_2, x_3\}$. But then this means that $d(x_2) \geq 4$, a contradiction. Hence $r \geq 4$. $\square$

Let G be a graph. Recall that G^k is the k-th power of G . Let $Z_n = \{0, 1, \ldots, n-1\}$ and let Z denote the set of all integers.

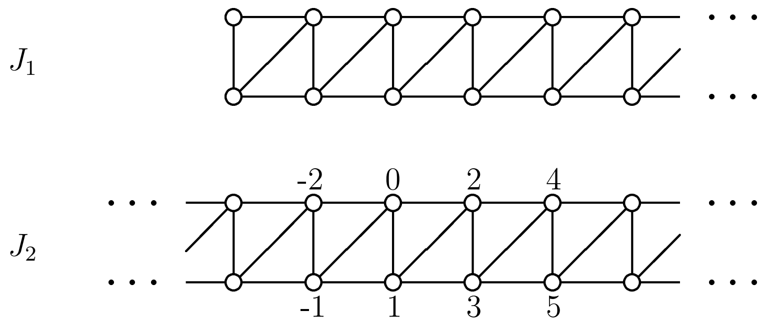


Figure 5.2: Two more infinite self-clique graphs

Theorem 5.3 *Let G be a graph with no vertices of degree 5 or 6. Then $G \in \mathcal{G}(3)$ if and only if G is either the graph C_n^2 for some $n \geq 7$ or else one of the infinite graphs J_1 or J_2 of Figure 5.2.*

Proof: The sufficiency is by direct verification. We now prove the necessity part.

First, we consider the case where G is 4-regular.

Note that the graph C_n^2 has vertex-set given by Z_n and edge-set given by $\{i(i+1), i(i+2) \mid i \in Z_n\}$ with operations reduced modulo n.

Also, note that the both way infinite graph J_2 of Figure 5.2 has vertex-set given by Z, and edge-set given by $\{i(i+1), i(i+2) \mid i \in Z\}$.

For any vertex x in G, let $N[x] := N(x) \cup \{x\}$. With this notation, we note that, for any vertex i in the graph C_n^2 or J_2,

(i) $N[i] = \{i-2, i-1, i, i+1, i+2\}$ and that

(ii) $G[N(i)]$ is the path $(i-2)(i-1)(i+1)(i+2)$ on 4 vertices.

To finish the proof for this case, we just need to show that for any 4-regular graph $G \in \mathcal{G}(3)$, the vertex-set of G can be identified with that of C_n^2 or that of J_2 such that, for any vertex i in G, $N[i]$ is the same set given as in (i) above and $G[N(i)]$ is the path on 4 vertices given as in (ii) above.

Now, by Corollary 5.1, for any vertex x in G, $G[N(x)]$ is a path on 4 vertices. We may label the vertex x as j, for some integer j, and all its neighbors as $j-2, j-1, j+1, j+2$ so that $(j-2)(j-1)(j+1)(j+2)$ is a path on 4 vertices.

Consider the neighbors for $j+1$. Since G is 4-regular, we see that $N(j+1) = \{j-1, j, j+2, u\}$ for some vertex $u \in V(G)$. By Corollary 5.1, $G[N(j+1)]$ is a path on 4 vertices and hence u is adjacent to $j-1$ or $j+2$ in G. If u is adjacent to $j-1$, then $\{j-1, j+1, u\}$ is a clique of G. This clique, together with $G[N[j]]$, form a K_4 in $K(G)$. This is a contradiction because $K(G) \cong G$. Hence u is adjacent to $j+2$. We may label the vertex u as $j+3$ so that $N[j+1] = \{j-1, j, j+1, j+2, j+3\}$ and that $G[N(j+1)]$ is the path $(j-1)(j)(j+2)(j+3)$ on 4 vertices which take the forms (i) and (ii) above respectively.

The above process can be continued. If G is finite, then we should eventually arrive at $G \cong C_n^2$ for some integer $n \geq 7$. If G is not finite, repeating the above process at both the vertices $j-k$ and $j+k$ for each integer k, we should eventually arrive at $G \cong J_2$.

Now, consider the case where G is not 4-regular. By Proposition 5.1, G contains vertices of degree $2, 3$ or 4.

Suppose G contains a vertex x_1 of degree 2 and let $N(x_1) = \{x_2, x_3\}$. By Theorem 5.2(i), $Q_1 = \{x_1, x_2, x_3\}$ is a clique of G, and further, $d(x_2) \geq 3$ and $d(x_3) \geq 3$.

If $d(x_2) = 3 = d(x_3)$, then, by Theorem 5.2(i) again, G is isomorphic to the graph obtained from K_4 by deleting an edge. But this means that $K(G) \ncong G$, a contradiction.

Hence assume that $d(x_3) = 4$. Further, let $N(x_3) = \{x_1, x_2, x_4, x_5\}$ so that $x_1x_2x_4x_5$ is a path of length 4 and that $Q_i = \{x_i, x_{i+1}, x_{i+2}\}$ is a clique of G for each $i = 2, 3$, by Theorem 5.2(i).

If $d(x_2) = 4$, then by Theorem 5.2(i), $G[N(x_2)]$ is the path $x_1x_3x_4z$, for some vertex $z \in V(G)$, $z \neq x_i$, $i = 1, 2, 3, 4$ and $\{x_2, x_4, z\}$ is a clique of G. But, on taking the clique graph of G, we see that $K(G)$ contains a K_4 which is absurd since $K(G) \cong G$. Hence $d(x_2) = 3$.

Now, if $d(x_4) = 3$, then $d(x_5) = 2$ and we have a contradiction because $K(G) \cong K_3 \ncong G$. Hence $d(x_4) = 4$.

By Theorem 5.2(i), $G[N(x_4)]$ is the path $x_2x_3x_5x_6$ for some vertex $x_6 \in V(G)$ and $x_6 \neq x_i$, $i = 1, 2, \ldots, 5$ so that $Q_4 = \{x_4, x_5, x_6\}$ is a clique of G.

Now, if $d(x_5) = 3$, then $d(x_6) = 2$ and we have a contradiction because $K(G) \ncong G$. Hence $d(x_5) = 4$.

Repeat the similar argument as before to the vertex x_k successively, for each $k \geq 5$ where x_k is adjacent to x_{k-1} and x_{k+1} (and by noting that $G[N(x_k)]$ is a path on 4 vetices in G), we see that G is an infinite graph isomorphic to the graph J_1 (shown in Figure 5.2).

Suppose G contains a vertex of degree 3. The preceding discussion implies that if G contains a vertex of degree 2, then $G \cong J_1$. Hence we may assume that G contains no vertices of degree 2. In this case, G contains only vertices of degree 3 or 4. By Proposition 5.6, G is not in $\mathcal{G}(3)$.

This completes the proof. $\square$

Corollary 5.3 *Suppose G is a 4-regular graph. Then $G \in \mathcal{G}(3)$ if and only if G is either the graph C_n^2 for some $n \geq 7$ or else the infinite graph J_2 of Figure 5.2.*

5.3 5-regular graphs in $\mathcal{G}(3)$

In this section, we show the existence of 5-regular graphs in $\mathcal{G}(3)$ by construction.

Definition 5.1 *Let $m,\ n \geq 2$ be two integers. Let $L(m,n)$ denote the graph whose vertex-set is the set of ordered pairs (i,j) where $i \equiv j \pmod 2$, $i \in Z_{4m}$ and $j \in Z_{4n}$ and whose edge-set is $E_1 \cup E_2$ where $E_1 = \{(i,j)(k,l) \mid i,k \in Z_{4m},\ j,l \in Z_{4n},\ |i-k| = 1 = |j-l|\}$ and $E_2 = \{(2i,2j)(2i+2,2j),\ (2i+1,2j+1)(2i+1,2j+3) \mid i,\ k \in Z_{2m},\ j,\ l \in Z_{2n}\ i+j \equiv 1 \pmod 2\}$. Here, the operations on the first (respectively second) index are reduced modulo $4m$ (respectively modulo $4n$).*

The above definition gives a graph whose set of vertices is finite. We may allow the second index to be any integer and obtain an infinite graph.

Definition 5.2 *Let $m \geq 2$ be an integer. Let $L(m)$ denote the graph whose vertex-set is the set of ordered pairs (i,j) where $i \equiv j \pmod 2$, $i \in Z_{4m}$ and $j \in Z$ and whose edge-set is $E_1 \cup E_2$ where $E_1 = \{(i,j)(k,l) \mid i,k \in Z_{4m},\ j,l \in Z,\ |i-k| = 1 = |j-l|\}$ and $E_2 = \{(2i,2j)(2i+2,2j),\ (2i+1,2j+1)(2i+1,2j+3) \mid i \in Z_{2m},\ j \in Z,\ i+j \equiv 1 \pmod 2\}$. Here, the operations on the first index are reduced modulo $4m$.*

Figure 5.3 shows a toroidal drawing of the graph $L(2,2)$. It is routine to verify that $L(2,2)$ is a 5-regular self-clique graph all of whose cliques have size equal to 3. More generally, we have the following result.

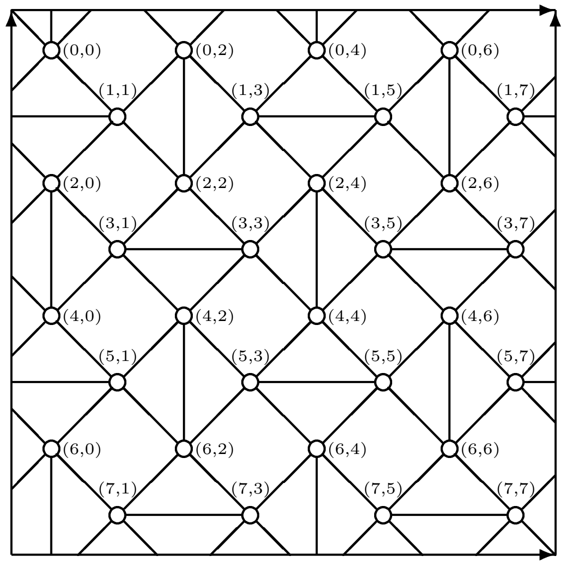


Figure 5.3: The graph $L(2,2)$ drawn on the torus

Proposition 5.2 *For each $m, n \geq 2$, the graphs $L(m,n)$ and $L(m)$ are 5-regular and are both in $\mathcal{G}(3)$.*

Proof: Let G be the graph $L(m,n)$ or $L(m)$.

Let Q be a clique in G. Then Q is one of the following four types.

(i) $(a+1,b-1)(a,b)(a+1,b+1)$, a even,

(ii) $(a-1,b-1)(a,b)(a-1,b+1)$, a even,

(iii) $(a-1,b+1)(a,b)(a+1,b+1)$, a odd, and

(iv) $(a-1,b-1)(a,b)(a+1,b-1)$, a odd.

Let φ be a mapping from $V(K(G))$ to $V(G)$ defined by

$$\varphi(Q) = (a+2, b).$$

Then it is readily checked that φ is an isomorphism from $K(G)$ onto G. $\square$

By allowing the first index of the vertex-set of $L(m)$ to include any integers, we obtain another infinite 5-regular graph which is in $\mathcal{G}(3)$. Denote this graph by L. Its formal definition is given below.

Definition 5.3 *Let L denote the graph whose vertex-set is the set of ordered pairs (i,j) where $i \equiv j \pmod 2$, and $i,\ j \in Z$ and whose edge-set is $E_1 \cup E_2$ where $E_1 = \{(i,j)(k,l) \mid i,j,k,l \in Z,\ |i-k| = 1 = |k-l|\}$ and $E_2 = \{(2i,2j)(2i+2,2j), (2i+1,2j+1)(2i+1,2j+3) \mid i,j,k,l \in Z,\ i+j \equiv 1 \pmod 2\}$.*

5.4 6-regular graphs in $\mathcal{G}(3)$

In this section, we show the existence of 6-regular graphs in $\mathcal{G}(3)$ by construction. Let $A_n = \{i \in Z_n \mid i \not\equiv 0 \pmod 4\}$.

Definition 5.4 *Let $n \geq 4$ be an integer. Let $M(n)$ denote the graph whose vertex-set is $V_1 \cup V_2$ and whose edge-set is $E_1 \cup E_2 \cup E_3 \cup E_4$ where*

(i) $V_1 = \{(i,j) \mid i \in A_{12},\ j \in Z_{2n},\ i \equiv j \pmod 2\}$

$V_2 = \{x_{2i+1} \mid i \in Z_n\}$.

(ii) $E_1 = \{(i,j)(k,l) \mid i,k \in A_{12},\ j,l \in Z_{2n},\ |i-k| = 1 = |j-l|\}$

$E_2 = \{(i,j)(i,j+2), (i,j)x_j \mid i \in A_{12},\ j \in Z_{2n},\ i,j \equiv 1 \pmod 2\}$

$$E_3 = \{(i,j)(i+2,j) \mid i \in A_{12},\ j \in Z_{2n},\ i \equiv 3 \pmod 4,\ j \equiv 1 \pmod 2\}$$

$$E_4 = \{(i,j)(i+4,j) \mid i \in A_{12},\ j \in Z_{2n},\ i,j \equiv 0 \pmod 2\}$$

Here, the operations on the first (respectively second) index are reduced modulo 12 *(respectively modulo* $2n$*).*

The above definition gives a graph whose set of vertices is finite. We may allow the second index to be any integer and obtain an infinite graph.

Definition 5.5 *Let* M *denote the graph whose vertex-set is* $V_1 \cup V_2$ *and whose edge-set is* $E_1 \cup E_2 \cup E_3 \cup E_4$ *where*

(i) $V_1 = \{(i,j) \mid i \in A_{12},\ j \in Z,\ i \equiv j \pmod 2\}$

$V_2 = \{x_{2i+1} \mid i \in Z\}$.

(ii) $E_1 = \{(i,j)(k,l) \mid i,k \in A_{12},\ j,l \in Z,\ |i-k| = 1 = |j-l|\}$

$E_2 = \{(i,j)(i,j+2), (i,j)x_j \mid i \in A_{12},\ j \in Z,\ i,j \equiv 1 \pmod 2\}$

$E_3 = \{(i,j)(i+2,j) \mid i \in A_{12},\ j \in Z,\ i \equiv 3 \pmod 4,\ j \equiv 1 \pmod 2\}$

$E_4 = \{(i,j)(i+4,j) \mid i \in A_{12},\ j \in Z,\ i,j \equiv 0 \pmod 2\}$

Here, the operations on the first index are reduced modulo 12.

Figure 5.4 shows a toroidal drawing of the graph $M(4)$. It is routine to verify that $M(4)$ is a 6-regular self-clique graph all of whose cliques have size equal to 3. More generally, we have the following result.

Proposition 5.3 *For each* $n \geq 4$*, the graphs* $M(n)$ *and* M *are* 6*-regular and are both in* $\mathcal{G}(3)$*.*

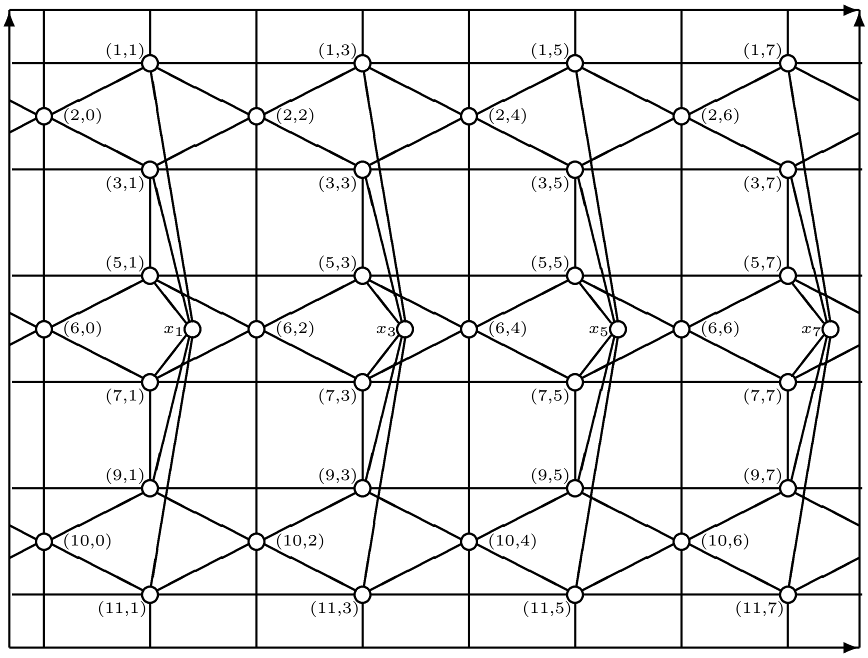


Figure 5.4: The graph $M(4)$ drawn on the torus

Proof: Let G be the graph $M(n)$ or M.

Let Q be a clique in G. Then Q is one of the following four types.

(i) $(a-1,b-1)(a,b)(a-1,b+1)$,

(ii) $(a+1,b-1)(a,b)(a+1,b+1)$,

(iii) $(a-4,b)(a,b)(a+4,b)$,

(iv) $(a,b)(a+2,b)x_b$,

where $a \equiv 2 \pmod 4$, $b \equiv 0 \pmod 2$ for (i), (ii), (iii) and $a \equiv 3 \pmod 4$, $b \equiv 1 \pmod 2$ for (iv).

Let φ be a mapping from $V(K(G))$ to $V(G)$ defined by

$$\varphi(Q) = \begin{cases} (a+1,b+1) & \text{if } Q \text{ is type (i)} \\ (a+3,b+1) & \text{if } Q \text{ is type (ii)} \\ x_{b+1} & \text{if } Q \text{ is type (iii)} \\ (a+3,b+1) & \text{if } Q \text{ is type (iv)} \end{cases}$$

Then it is readily checked that φ is an isomorphism from $K(G)$ onto G. $\square$

5.5 (r, s)-regular graphs in $\mathcal{G}(3)$

In this section, we shall investigate the existence of graphs in $\mathcal{G}(3)$ whose vertices are of mixed degrees. By Proposition 5.1, if $G \in \mathcal{G}(3)$, then $2 \leq d(x) \leq 6$ for any vertex x in G. The more interesting case seems to be those graphs in $\mathcal{G}(3)$ which are almost regular in the sense that, for every vertex x in G, $d(x) = r$ or $d(x) = s$ for some given r and s with $2 \leq r < s \leq 6$.

Definition 5.6 *Let $S \subseteq Z_n$ and let G be a graph. Then G is said to be of type-S if for every $r \in S$, there is a vertex x such that $d(x) = r$. In the event that $S = \{r\}$, then G is called an r-regular graph. If $S = \{r, s\}$, then G is called an (r, s)-regular graph.*

Of course, one could also investigate those graphs in $\mathcal{G}(3)$ which are of type-S where $3 \leq |S| \leq 6$ but we feel that this could be done elsewhere.

Lemma 5.2 *Let x be a vertex of degree 2 in a $(2, s)$-regular graph $G \in \mathcal{G}(3)$, where $3 \leq s \leq 6$. Then x is adjacent to another vertex of degree 2 in G.*

Proof: Let $N(x) = \{x_1, x_2\}$. By Theorem 5.2(i), $Q = \{x, x_1, x_2\}$ is a clique of G.

Suppose $d(x_1) = s = d(x_2)$ and let $N(x_1) = \{x, x_2, y_1, \ldots, y_{s-2}\}$ and let $N(x_2) = \{x, x_1, z_1, \ldots, z_{s-2}\}$.

Suppose $3 \leq s \leq 4$. Since $G[N(x_1)]$ and $G[N(x_2)]$ are both a path on s vertices, by Theorem 5.2(i), we may assume that $y_1 = z_1$ so that $\{x_1, x_2, y_1\}$ is a clique of G, and in addition, if $s = 4$, then so are $\{x_1, y_1, y_{s-2}\}$ and $\{x_2, y_1, z_{s-2}\}$. Now, by taking the clique graph of G, we see that, if $s = 3$, then Q is a vertex of degree 1 in $K(G)$, while if $s = 4$, then $K(G)$ contains a K_4. Either case is a contradiction since $K(G) \cong G$.

Suppose $5 \leq s \leq 6$. By Theorem 5.2(ii) and (iii), each of $G[N(x_1)]$ and $G[N(x_2)]$ is a union of paths. Moreover, if $t = |\{y_1, \ldots, y_{s-2}\} \cap \{z_1, \ldots, z_{s-2}\}|$, then $t \leq 1$.

Note that, if $s = 6$, then $t = 0$ and that if $s = 5$, then either $t = 0$ or $t = 1$.

If $t = 0$, then $\{x_1, y_1, y_2\}$, $\{x_1, y_{s-3}, y_{s-2}\}$, $\{x_2, z_1, z_2\}$ and $\{x_2, z_{s-3}, z_{s-2}\}$ are cliques of G, each has a non-empty intersection with Q so that Q is a vertex of degree 4 in $K(G)$.

If $t = 1$, we may take $y_1 = z_1$ so that $\{x_1, y_1, x_2\}$, $\{x_1, y_2, y_3\}$ and $\{x_2, z_2, z_3\}$ are cliques of G so that Q is a vertex of degree 3 in $K(G)$.

In either case, we have a contradiction since $K(G) \cong G$.

This completes the proof. $\square$

Proposition 5.4 *There exist no $(2, s)$-regular graphs in $\mathcal{G}(3)$ for any $3 \leq s \leq 5$.*

Proof: Suppose there is a $(2, s)$-regular graph $G \in \mathcal{G}(3)$. Let x be a vertex of degree 2 in G and let $N(x) = \{y, z\}$. Then $Q = \{x, y, z\}$ is a clique in G, by Theorem 5.2(i).

By Lemma 5.2, we may assume that $d(y) = 2$ in G. Then, clearly $d(z) = s$ for some $3 \leq s \leq 6$. Suppose $N(z) = \{x, y, y_1, \ldots, y_{s-2}\}$.

By Theorem 5.2(i) and (ii), we have $s \notin \{3, 4\}$.

Suppose $s = 5$. Then, by Theorem 5.2(ii), we may assume that $y_1y_2y_3$ is a path on 3 vertices in $G[N(z)]$, so that $Q_1 = \{z, y_1, y_2\}$ and $Q_2 = \{z, y_2, y_3\}$ are cliques of G.

Clearly, $d(y_2) = 5$. Let $N(y_2) = \{z, y_1, y_3, z_1, z_2\}$ so that $Q_3 = \{y_2, z_1, z_2\}$ is a clique of G. By taking the clique graph of G, we see that the subgraph of $K(G)$ induced by Q, Q_1, Q_2 and Q_3 is such that Q is a vertex of degree 2 in $K(G)$ not adjacent to any vertex of degree 2 in $K(G)$. This, however, contradicts Lemma 5.2 because $K(G) \cong G$. $\square$.

By using Lemma 5.2 and by following similar argument as in the proof of Proposition 5.4, one could easily see that if $G \in \mathcal{G}(3)$ is $(2, 6)$-regular, then G is an infinite graph.

Proposition 5.5 *There exist no finite* $(2, 6)$*-regular graphs in* $\mathcal{G}(3)$*.*

Proposition 5.6 *There exist no* $(3, s)$*-regular graphs in* $\mathcal{G}(3)$ *for any* $4 \leq s \leq 6$.

Proof: Let x be a vertex of degree 3 in a $(3, s)$-regular graph $G \in \mathcal{G}(3)$. Let $N(x) = \{x_1, x_2, x_3\}$. By Theorem 5.2(i), we may assume that $x_1x_2x_3$ is the path on 3 vertices in $G[N(x)]$ so that $Q_1 = \{x, x_1, x_2\}$ and $Q_2 = \{x, x_2, x_3\}$ are cliques of G.

If $d(x_2) = 3$ in G, then this implies that Q_1Q_2 is a clique of size 2 in $K(G)$. But this is impossible because $K(G) \cong G$. Hence $d(x_2) = s$.

Let $N(x_2) = \{x, x_1, x_3, y_1, \ldots, y_{s-3}\}$.

Now, $s \neq 6$, by Theorem 5.2(iii), because $G[N(x_2)]$ contains a path on 3 vertices.

Suppose $s = 4$. Since $G[N(x_2)]$ is a path on 4 vertices by Theorem 5.2(i), y_1 must be adjacent to x_1, say. Then $d(x_3)$ must be either 3 or 4. Either case leads to absurdity because $G[N(x_3)]$ is then either P_3 or P_4 which is impossible because $d(x) = 3$ and $d(x_2) = 4$.

Suppose $s = 5$. Then $Q_3 = \{x_2, y_1, y_2\}$ is a clique of G. Moreover, $d(x_1) = 3$ or 5. Let $N(x_1) = \{x, x_2, z_1, \ldots, z_t\}$ where $t \in \{1, 3\}$.

If $t = 1$, then $z_1 = y_i$ for some $i \in \{1, 2\}$ because $G[N(x_1)] \cong P_3$ by Theorem 5.2(i). But then Q_1, Q_2, Q_3 and $\{x_1, x_2, y_i\}$ form a clique K_4 in $K(G)$ which is impossible because $K(G) \cong G$.

Hence $t = 3$. By Theorem 5.2(i), we may take $z_1z_2z_3$ to be a path on 3 vertices in $G[N(x_1)]$ so that $\{x_1, z_1, z_2\}$ and $\{x_1, z_2, z_3\}$ are cliques in G. Taking the clique graph of G, we see that Q_1 is a vertex of degree 4 in $K(G)$. But this is impossible because $K(G) \cong G$.

This completes the proof. $\square$

Figure 5.5 depicts two $(4, 5)$-regular graphs all of whose cliques are of size 3. They are both drawn on the torus. It is routine to check that these two graphs are self-clique. These graphs can easily be extended to other $(4, 5)$-regular graphs in $\mathcal{G}(3)$. Moreover there exist $(4, 5)$-regular graphs in $\mathcal{G}(3)$ which do not resemble those shown in Figure 5.5.

Proposition 5.7 *There exist* $(4, 5)$*-regular graphs in* $\mathcal{G}(3)$.

Proposition 5.8 *There exist no* $(4, 6)$*-regular graphs in* $\mathcal{G}(3)$.

Proof: Let x be a vertex of degree 4 in a $(4, 6)$-regular graph $G \in \mathcal{G}(3)$. Let $N(x) = \{x_1, x_2, x_3, x_4\}$. By Theorem 5.2(i), we may assume that $x_1x_2x_3x_4$ is

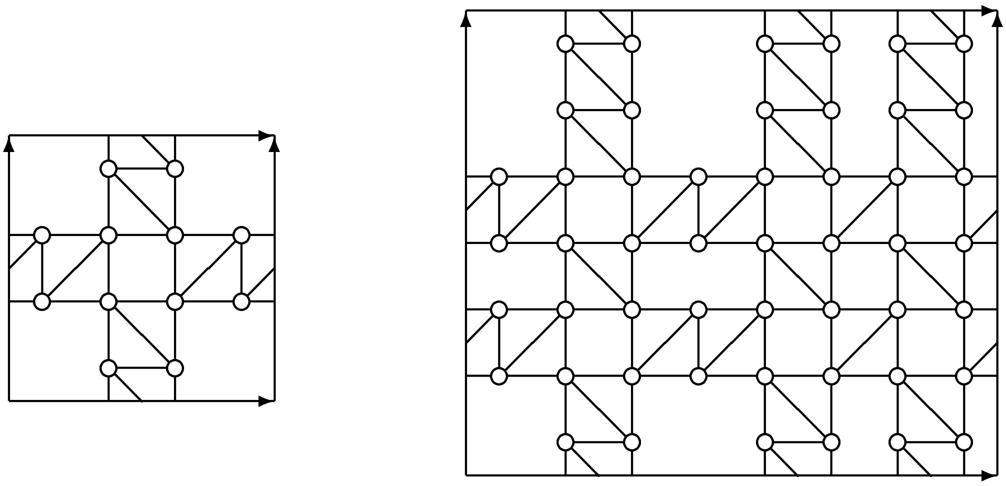


Figure 5.5: Some $(4,5)$-regular graphs in $\mathcal{G}(3)$ drawn on the torus

the path on 4 vertices in $G[N(x)]$ so that $Q_i = \{x, x_i, x_{i+1}\}$ is a clique of G for each $i = 1, 2, 3$.

Theorem 5.2(iii) implies that $d(x_2) = 4 = d(x_3)$. Let $N(x_2) = \{x, x_1, x_3, y_1\}$ and $N(x_3) = \{x, x_2, x_4, y_2\}$. Then $y_1 \neq y_2$, otherwise $\{x_2, x_3, y_1\}$ is a clique of G which, together with Q_1, Q_2 and Q_3, form a K_4 in $K(G)$ which is impossible because $K(G) \cong G$.

Hence y_1 is adjacent to x_1, and y_2 is adjacent to x_4. By Theorem 5.2(i) and (iii), $d(x_1) = 4 = d(x_4)$. This implies that there exist vertices $z_1, z_2 \in V(G) - \{x\}$ such that $xx_2y_1z_1$ and $xx_3y_2z_2$ are paths on 4 vertices in G.

Applying Theorem 5.2(iii) to the vertices y_1 and y_2, and continue with similar argument as before, we see that G is a 4-regular graph, a contradiction.

This completes the proof. $\square$

SUMMARY

The purpose of this summary is to present our main results so that they may be referred to with ease.

Chapter 4

Theorem 4.1 *Let G be a connected graph whose clique size sequence is $(2,\ldots,2,m,n)$ where $m,n \geq 3$. Then G is self-clique if and only if $G \in \mathcal{G}_i(q,r;s,t)$ for some $1 \leq i \leq 6$.*

Chapter 5

Proposition 5.1 *Suppose $G \in \mathcal{G}(k)$ where $k \geq 2$. Then for any vertex $x \in V(G)$, we have $k-1 \leq d(x) \leq k(k-1)$.*

Proposition 5.2 *For each $m,n \geq 2$, the graphs $L(m,n)$ and $L(m)$ are 5-regular and are both in $\mathcal{G}(3)$.*

Proposition 5.3 *For each $n \geq 4$, the graphs $M(n)$ and M are 6-regular and are both in $\mathcal{G}(3)$*

Proposition 5.4 *There exist no $(2,s)$-regular graphs in $\mathcal{G}(3)$ for any $3 \leq s \leq 5$.*

Proposition 5.5 *There exist no finite $(2,6)$-regular graphs in $\mathcal{G}(3)$.*

Proposition 5.6 *There exist no $(3,s)$-regular graphs in $\mathcal{G}(3)$ for any $4 \leq s \leq 6$.*

Proposition 5.7 *There exist* $(4,5)$*-regular graphs in* $\mathcal{G}(3)$.

Proposition 5.8 *There exist no* $(4,6)$*-regular graphs in* $\mathcal{G}(3)$.

Theorem 5.2 *Suppose* $G \in \mathcal{G}(3)$ *and let* x *be a vertex of degree* r *in* G. *Then* $G[N(x)]$ *is*

(i) P_r *if* $2 \leq r \leq 4$,

(ii) $P_2 \cup P_3$ *if* $r = 5$ *and*

(iii) $3P_2$ *if* $r = 6$.

Theorem 5.3 *Let* G *be a graph with no vertices of degree* 5 *or* 6. *Then* $G \in \mathcal{G}(3)$ *if and only if* G *is either the graph* C_n^2 *for some* $n \geq 7$ *or else one of the infinite graphs* J_1 *or* J_2 *of Figure 5.2.*

Corollary 5.1 *Suppose* $G \in \mathcal{G}(3)$ *and* G *is* 4*-regular. Then for any vertex* $x \in V(G)$, $G[N(x)]$ *is the path on* 4 *vertices.*

Corollary 5.2 *If there is an* r*-regular graph in* $\mathcal{G}(3)$, *then* $r \geq 4$.

Corollary 5.3 *Suppose* G *is a* 4*-regular graph. Then* $G \in \mathcal{G}(3)$ *if and only if* G *is either the graph* C_n^2 *for some* $n \geq 7$ *or else the infinite graph* J_2 *of Figure 5.2.*

References

[1] L. Alcón, M. Gutierrez, A new characterization of clique graphs, *Mat. Contemporânea* **25** (2003) 1 - 7.

[2] L. Alcón, M. Gutierrez, Cliques and extended Triangles. A necessary condition to be clique planar graph, *Discrete Appl. Math.* **141** (2004) 3 - 17.

[3] L. Alcón, L. Faria, C.M.H. de Figueiredo, M. Gutierrez, Clique graph recognition is NP-complete, *Lect. Notes Comput. Sci.*, **Vol. 4271** Springer, New York, 2006.

[4] R. Balakrishnan, P. Pauraja, Self-clique graphs and diameters of iterated clique graphs, *Util. Math.* **29** (1986) 263-268.

[5] G. Balconi, Caratterizzazione matriciale dei grafi autoduali, *Istit. Lombardo Accad. Sci. Lett. Rend. A* **113** (1979) 360-365.

[6] H. Bandelt, E. Prisner, Clique graphs and Helly graphs, *J. Combin. Theory Ser. B* **51** (1991) 34-45.

[7] J.A. Bondy, U.S.R Murty, Graph Theory: Graduate Texts in Mathematics, Springer, New York, 2008.

[8] A. Bondy, G. Durán, M.C. Lin, J.L. Szwarcfiter, A sufficient condition for self-clique graphs, *Electron. Notes Discrete Math.* **7** (2001) 174-177.

[9] A. Bondy, G. Durán, M.C. Lin, J.L. Szwarcfiter, Self-clique graphs and matrix permutations, *J. Graph Theory* **44** (2003) 178-192.

[10] A. Brandstädt, V. D. Chepoi, F. F. Dragan, V. I. Voloshin, Dually chordal graphs, *SIAM J. Discrete Math.* **11** (1998) 437-455.

[11] A. Brandstädt, V.B. Le, J. Spinrad, Graph Classes: A Survey, *Monogr. Discrete Math. Appl.* **Vol 3** SIAM, Philadelphia, 1999.

[12] B.L. Chen, K.W. Lih, Diameters of iterated clique graphs of chordal graphs, *J. Graph Theory* **14** (1990) 391-396.

[13] G.L. Chia, On self-clique graphs with given clique sizes, *Discrete Math.* **212** (2000) 185-189.

[14] G.L. Chia, Constructing graphs that are isomorphic to their clique graphs, *Proceedings of the Third Asian Mathamatical Conference 2000*, Philippines, (World Scientific, Singapore 2001) 89 - 93.

[15] G.L. Chia, Poh-Hwa Ong, On self-clique graphs with given clique sizes, II, *Discrete Math.* **309** (2009) 1538-1547.

[16] G.L. Chia, Poh-Hwa Ong, On self-clique graphs all of whose cliques have equal size, (submitted for publication).

[17] F. Escalente, Über iterierte Clique-Graphen, *Abh. Math. Sem. Univ. Hamburg* **39** (1973) 59-68.

[18] F. Escalente, B. Toft, On clique-critical graphs, *J. Combin. Theory Ser. B* **17** (1974) 170-182.

[19] J.C. George, T.D. Porter, W.D. Wallis, Characterizing balanced bipartite graphs with part-switching automorphisms, *Bull. Inst. COmbin. Appl.* **28** (2000) 85-88.

[20] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York 1980.

[21] M. Gutierrez, Tree-clique graphs, *Estud. Comun. Inst. Mat.* **63** (1996) 7-26.

[22] M. Gutierrez, J. Meidanis, On the clique operator, *Lect. Notes in Comput. Sci.* **Vol. 1380** Springer, New York, 1998.

[23] R.C. Hamelink, A partial characterization of clique graphs, *J. Combin. Theory* **5** (1968) 192-197.

[24] F. Harary, Graph Theory, Addison-Wesley, Reading, MA 1969.

[25] S.T. Hedetniemi, P.J. Slater, Line graphs of triangleless graphs and iterated clique graphs, in: Graph Theory and Applications, *Lecture Notes in Math.* **303** Springer, Berlin, 1972.

[26] B. Hedman, Clique graphs of time graphs, *J. Combin. Theory Ser. B* **37** (1984), 270 - 278.

[27] B. Hedman, Diameters of iterated clique graphs, *Hadronic J.* **9** (1986) 273-276.

[28] F. Larrión, V. Neumann-Lara, M.A. Pizaña, T.D. Porter, Self-clique graphs with prescribed clique-sizes, *Congr. Numer.* **157** (2002) 173-182.

[29] F. Larrión, V. Neumann-Lara, M.A. Pizaña, T.D. Porter, Recognizing self-clique graphs, *Mat. Contemp.* **25** (2003) 125–133.

[30] F. Larrión, V. Neumann-Lara, M.A. Pizaña, T.D. Porter, A hierarchy of self-clique graphs, *Discrete Math.* **282** (2004) 193-208.

[31] F. Larrión, M.A. Pizaña, On hereditary clique-Helly self-clique graphs, *Discrete Appl. Math.* **156** (2008) 1157–1167.

[32] C.K. Lim, A result of iterated clique graphs, *J. Aust. Math. Soc. Ser. A* **32** (1982) 289 - 294.

[33] C.K. Lim, Y.H. Peng, On graphs without multicliqual edges, *J. Graph Theory* **5** (1981) 443 - 451.

[34] E. Marczewski, Sur deux propriétés des classes d'ensembles, *Fund. Math.* **33** (1945), 303 - 307.

[35] T.A. McKee, F.R. McMorris, Topics in Intersection Graph Theory, *Monogr. Discrete Math. Appl.* **Vol. 2** SIAM, Philadelphia, 1999.

[36] E. Prisner, Graph Dynamics, *Pitman Research Notes in Mathematics* **Vol. 338** Longman, New York, 1995.

[37] E. Prisner, Hereditary clique-Helly graphs, *J. Combin. Math. Combin. Comput.* **14** (1993) 216–220.

[38] F.S. Roberts, J.H. Spencer, A characterization of clique graphs, *J. Combin. Theory Ser. B* **10** (1971), 102-108.

[39] J.L. Szwarcfiter, Recognizing clique-Helly graphs, *Ars Combin.* **45** (1997) 29-32.

[40] J.L. Szwarcfiter, *A survey on clique graphs*, in: C. Linhares-Sales, B.A. Reed (Eds.), Recent Advances in Algorithms and Combinatorics, CMS Books Math./Ouvrages Math. SMC, Vol. 11, Springer, New York, 2003.

[41] J. Szwarcfiter, C. Bornstein, Clique graphs of chordal and path graphs, *Siam J. Discrete Math.* **7** (1994) 331-336.

Appendix A

Graphs

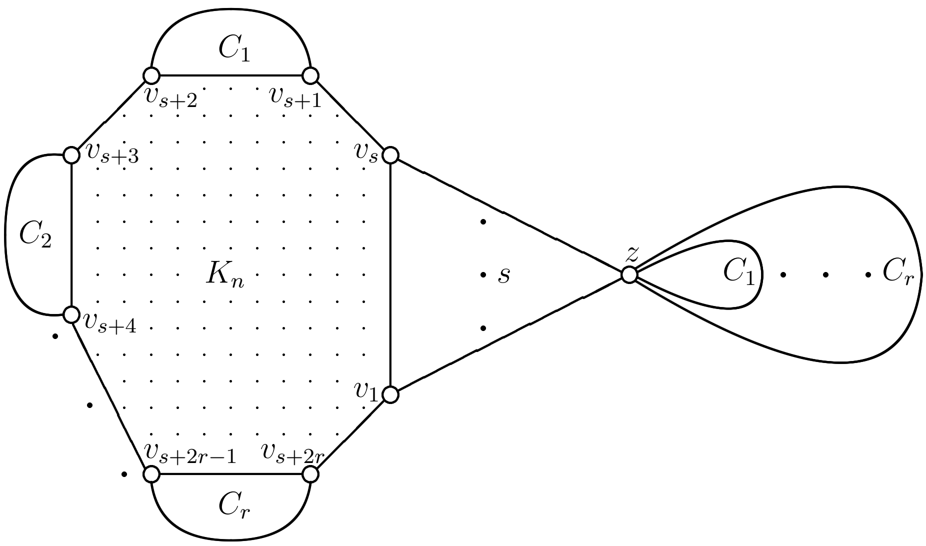


Figure A.1: A general drawing of a graph in $\mathcal{G}_{r,s}$

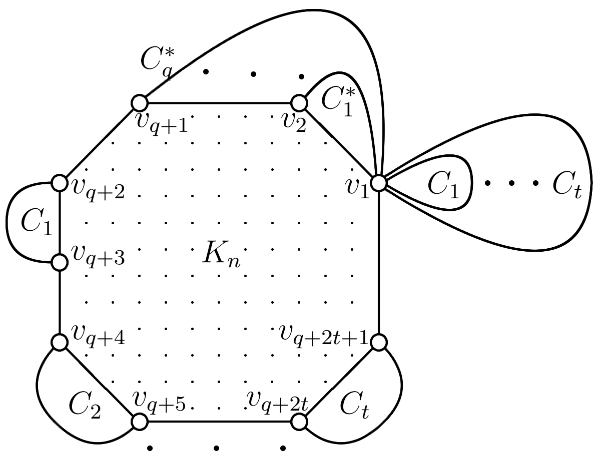


Figure A.2: A general drawing of a graph in $\mathcal{H}_{t,q}$

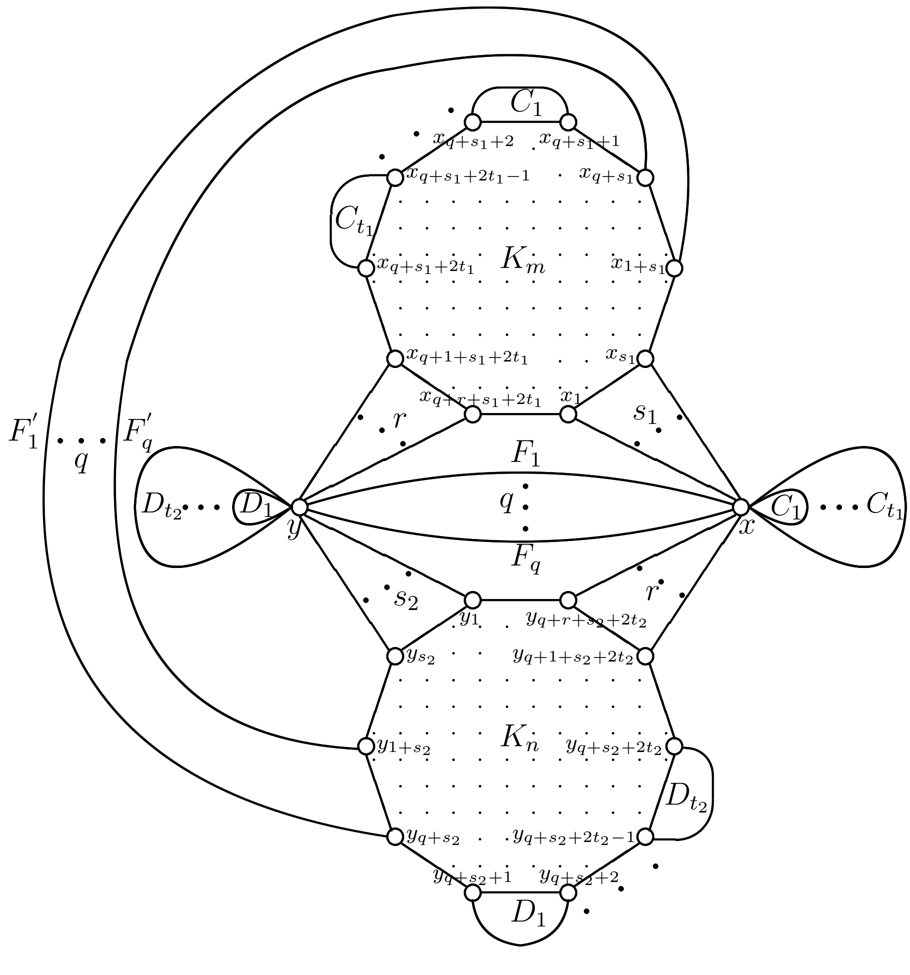


Figure A.3: A general drawing of a graph in $\mathcal{G}_1(q, r; s, t)$

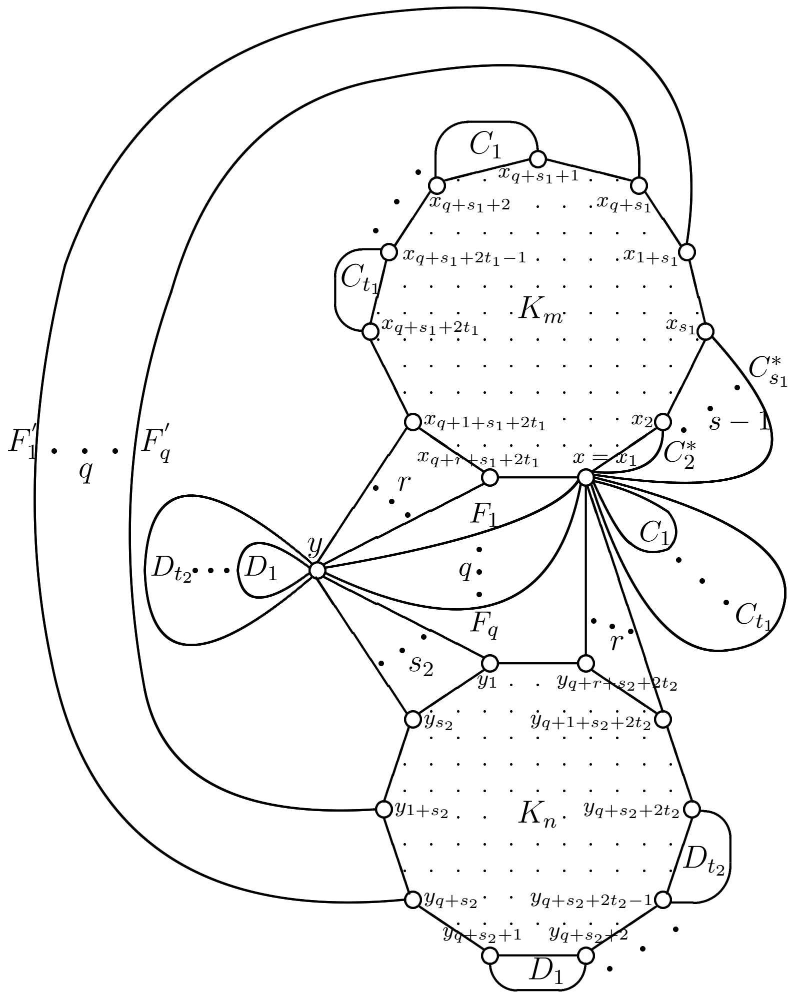


Figure A.4: A general drawing of a graph in $\mathcal{G}_2(q, r; s, t)$

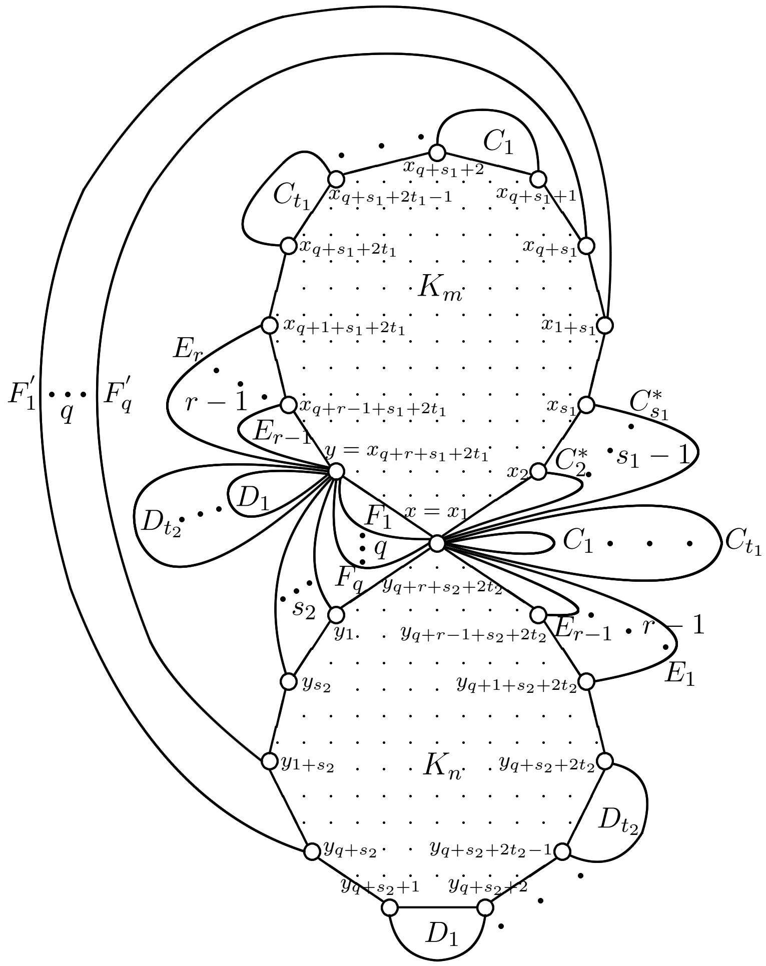


Figure A.5: A general drawing of a graph in $\mathcal{G}_3(q, r; s, t)$

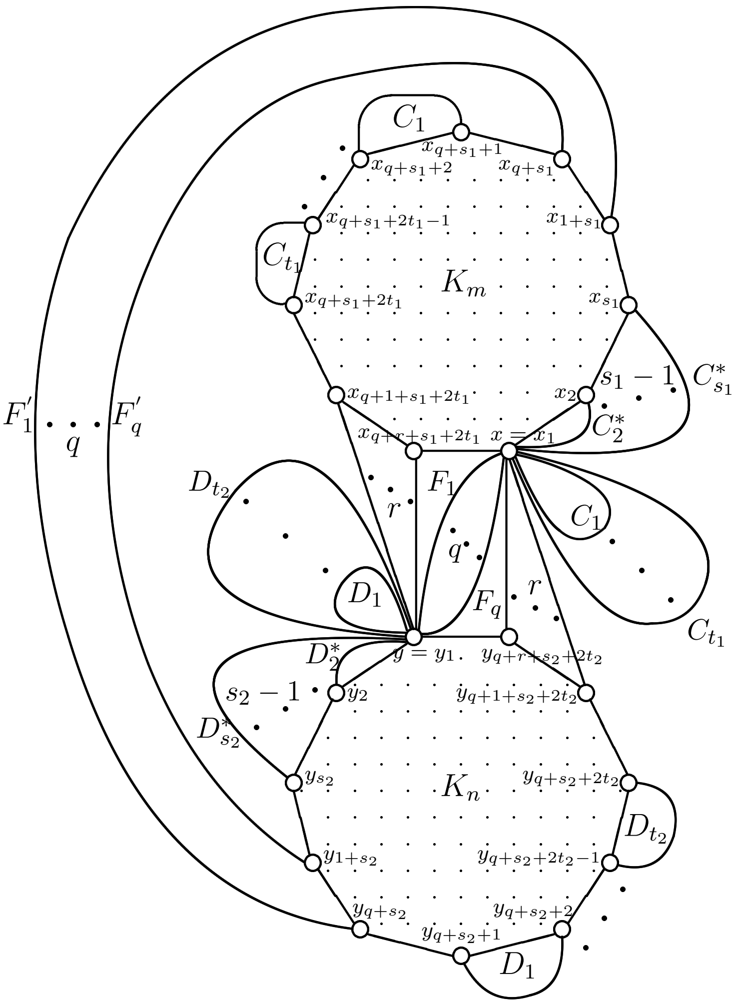


Figure A.6: A general drawing of a graph in $\mathcal{G}_4(q, r; s, t)$

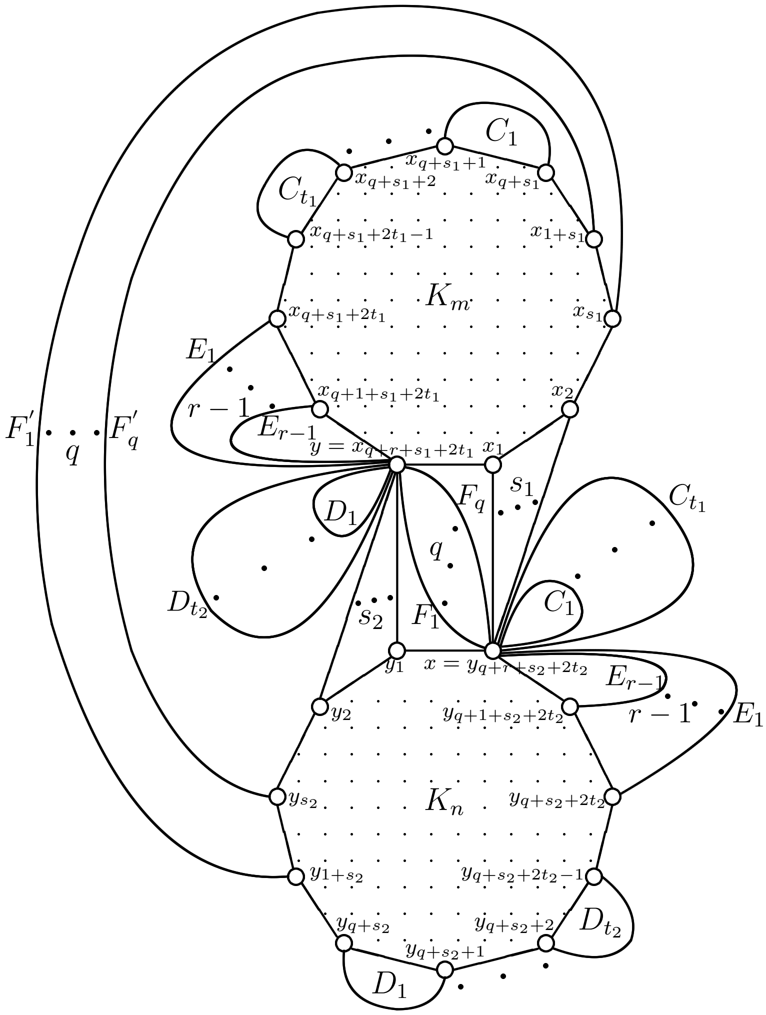


Figure A.7: A general drawing of a graph in $\mathcal{G}_5(q, r; s, t)$

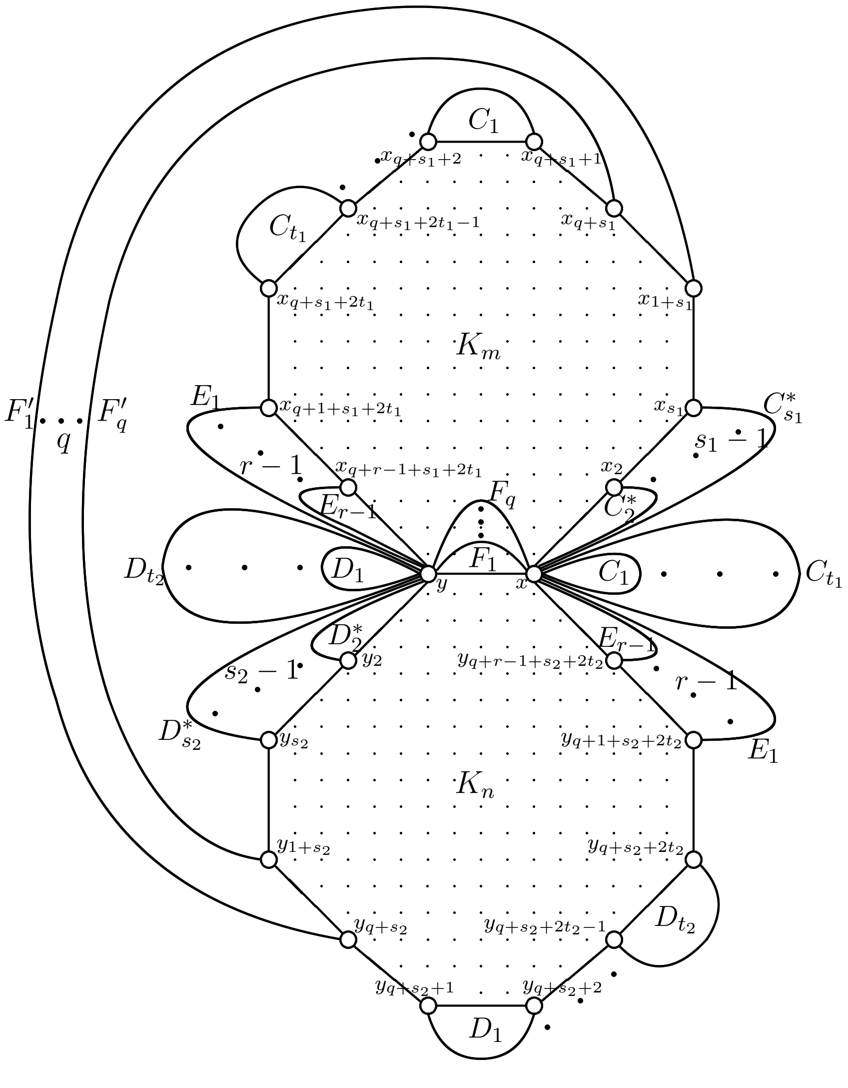


Figure A.8: A general drawing of a graph in $\mathcal{G}_6(q,r;s,t)$