NUMERICAL STUDY ON NATURAL CONVECTION IN INCLINED ENCLOSURES WITH NON-UNIFORM BOUNDARY CONDITIONS

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The understanding on the phenomena of fluid flow and heat transfer processes in enclosures is an important problem in many engineering and industrial applications. It has been a challenge to find an optimal heat transfer with numerous analyses and experiments. The geometry of the system, that is, size and inclination angle of the enclosure and boundary wall temperature can be the factors that provide the optimal heat transfer for various engineering applications. Hence, the effect of enclosure inclination and aspect ratio will be the subjects of the present study. A rectangular enclosure filled with fluid is considered. Sinusoidal variation of temperature is applied on the left wall while the opposite wall is cooled with a constant temperature. The top and bottom walls of the enclosure are adiabatic. The enclosure inclination is an inclination angle bounded between the bottom wall and the horizontal plane. The gravity is acting in vertically downward direction. The flow is considered to be two-dimensional, Newtonian, incompressible and laminar. The fluid properties are constant and the Boussinesq approximation is valid for the density variations. The governing equations and boundary conditions are derived based on the physical model proposed. The governing system of partial differential equations are non-dimensionalized using the suitable dimensionless variables. This approach reduces the complexity in physical quantities for solving the system of partial differential equations. The finite difference method is used to discretize the set of partial differential equations. An appropriate numerical algorithm is developed to find the numerical solutions of heat transfer and fluid flow in the enclosures. The numerical results are validated with previous studies. The results of the flow and temperature as well as the rate of heat transfer in enclosures are presented graphically. The flow structures and temperature distributions are affected by the enclosure inclination. The aspect ratio also affects the heat transfer in the enclosure.
ABSTRAK

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<td>et al.</td>
<td>(et alii): and others, and co-workers</td>
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<tr>
<td>FDM</td>
<td>finite difference method</td>
</tr>
<tr>
<td>i.e.</td>
<td>(id est): that is, in other words</td>
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<tr>
<td>PDE</td>
<td>partial differential equation</td>
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<tr>
<td>SOR</td>
<td>successive over-relaxation</td>
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<td>SUR</td>
<td>successive under-relaxation</td>
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NOMENCLATURE

\( \ddot{a} \) acceleration in any direction

\( A \) surface area

\( Ar \) aspect ratio

\( c_p \) specific heat capacity

\( \vec{F} \) net force in any direction

\( g \) gravitational acceleration

\( H \) enclosure height

\( I \) total enthalpy

\( k \) thermal conductivity

\( m \) mass

\( Nu \) Nusselt number

\( \overline{Nu} \) average Nusselt number

\( p \) momentum

\( P \) pressure

\( Pr \) Prandtl number

\( Q \) heat energy

\( Ra \) Rayleigh number

\( S \) source term

\( t \) time

\( T \) temperature

\( u, v \) velocity components in \( x \) - and \( y \) -direction

\( U \) velocity in any direction

\( V \) volume

\( W \) enclosure width

\( x, y \) Cartesian coordinates

\( X, Y \) dimensionless Cartesian coordinates
Greek symbols

$\alpha$  thermal diffusivity
$\beta$  volumetric coefficient of thermal expansion
$\lambda$  relaxation parameter
$\mu$  dynamic viscosity
$\nu$  kinematic viscosity
$\omega$  vorticity
$\Omega$  dimensionless vorticity
$\phi$  viscous dissipation function
$\psi$  stream function
$\Psi$  dimensionless stream function
$\rho$  density of fluid
$\sigma$  normal stress acting on a surface
$\tau$  shear stress acting on a surface
$\Theta$  dimensionless temperature
$\varphi$  inclination angle

Subscript

0  convenient reference state
c  cold
h  hot
loc  local
max  maximum
ref  reference state
CHAPTER 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

Fluid flows and heat transfer are happening during our daily life, for example, blood circulation, air ventilation, car petrol combustion and many more do involve heat transfer. These phenomena are common, but the energy waste released can be reduced through efficient heat transfer managements, such as the heat generated from radioactive reaction in the nuclear reactor is used for desalination of sea water and electricity generation. Thus, it is important to study the fluid flow and heat transfer processes to provide better understanding and improvement for most engineering, industries and medical science applications.

It has been a challenge for scientists and engineers to find an optimal heat transfer with numerous analyses and experiments. For example, the advancement in electronics industry has brought to the production of high performance electronic components with smaller sizes. However, spontaneous generation of heat loads from the working components may cause over heating and subsequently lead to the failure of an electrical device. Therefore, an efficient cooling system is needed to provide better heat transfer for the system. Another example is the designs of heat exchanger which are varied to suit for specific applications such as space heating and air-conditioning, power production, waste heat recovery and chemical processing.

Numerical methods are usually used to predict the fluid flow and heat transfer processes. This enable engineers to perform numerical investigation in virtual laboratory, thus reduces the waste of materials and the conduction of high cost experiments.
1.2 HEAT TRANSFER

Heat transfer (or heat) is thermal energy in transit due to a spatial temperature difference (Incropera et al., 2007). Heat can be defined as energy that is transferred due to the temperature difference between two systems or part of two systems. It means that heat transfer happens whenever there exists a temperature gradient or difference in a medium or between media. There exists different types of heat transfer processes due to the variation of physical mechanisms involved. Modes of heat transfer are mainly classified into three, which are conduction, convection and radiation.

![Figure 1.1: Examples of conduction, convection and radiation](http://www.ces.fau.edu/nasa/images/Energy/ConvectionConductionRadiation.jpg)

1.2.1 Conduction

Conduction is the transport of energy in a medium due to temperature gradient, and the physical mechanism is of random atomic or molecular. It can also be the transport of energy due to difference within a body or between bodies in thermal contact without the influence of mass flow and mixing. In a medium, energy is transferred from the more energetic to the less energetic particles of a substance due to interaction between particles. The energy transfer by conduction happens in the direction of decreasing temperature in presence of temperature gradient. For example, from Figure 1.1, heat is conducted along the handle where heat is transferred from the heated part to the end of the handle.
1.2.2 Convection

Convection is the mode of energy transfer between a surface and a fluid moving over the surface when they are at different temperatures. The convection heat transfer is comprised of two mechanisms, which are energy transfer due to random molecular motion (conduction or diffusion); and energy transfer by the bulk, or macroscopic, motion of the fluid (advection). This is a dominant form of heat transfer in liquids and gases. From Figure 1.1, convection heat transfer is happened between the heated pot and its contained liquid and also between the liquid at the bottom and near the surface.

1.2.3 Radiation

Thermal radiation (or radiation) is energy emitted by matter that is at a non-zero temperature. Emission can be occur from solid surfaces, liquids and gases. The energy of the radiation field is transported by electromagnetic waves (or alternatively, photons). The transfer of energy by radiation differ from the other two modes; conduction and convection require the presence of a material medium with temperature gradient, whereas radiation gives effective heat transfer in vacuum. Figure 1.1 shows that heat is radiated from the heating element.

1.3 TYPES OF CONVECTION

Convective heat transfer can be classified according to the nature of the flow. It can be classified into two categories, which are forced and natural convection.

Figure 1.2: Examples of forced convection and natural convection
1.3.1 Forced Convection

Forced convection occurs when the flow is caused by external means, such as by a fan, a pump, or atmospheric winds. For example, a fan is used to provide forced convection air cooling of hot electrical components on a stack of printed circuit board as shown in Figure 1.2(a). Central heating, air conditioner and steam turbines are derivatives of forced convection mechanisms.

1.3.2 Natural (Free) Convection

Natural (or free) convection flow is induced by buoyancy forces, which are due to density differences caused by temperature variations in the fluid. As an example, consider the heat transfer occurs in the hot components on a vertical array of circuit boards in air as shown in Figure 1.2(b). Air that makes contact with the hot components experiences an increase in temperature and hence a reduction in density. Since it is now lighter than the surrounding air, buoyancy forces induce a vertical motion for which warm air ascending from the boards is replaced by an inflow of cooler ambient air. In many systems involving multi-mode heat transfer effects, natural convection provides the largest resistance to heat transfer and therefore plays an important role in design on performance of the system. Moreover, when it is desirable to minimize heat transfer rates to minimize operating cost, natural convection is often preferred to forced convection. The process of boiling and condensation are examples of natural convection.

1.4 APPLICATIONS OF CONVECTION

There are many practical uses of natural convection in our daily life. The followings are some of the examples where natural convection is existed in nature and also its industrial applications.

Boiling and condensation are convection processes associated with the change phase of a fluid. The change from the liquid to vapour state due to boiling is sustained by heat transfer from the solid surface. However, condensation of a vapour to the liquid state results in heat transfer to the solid surface.
One of the mechanisms for cloud formation is convectional lifting. When the sun rises, the Earth’s surface is heated, and the warm air at the ground level which contains water vapour begins to rise. As the humid air rises to higher altitude, it started to cool and clouds are formed when the humid air is cooled below critical temperature. The water vapour then condenses and forms droplets in the atmosphere.

The land and sea breeze are a concept of convection current. During the daytime, the land heats up more quickly than the sea. Thus, the warm air above the land rises and is replaced by the cool air that was blown from the sea surface (sea breeze). At night, the land cools faster than the sea, the warm air above the sea surface rises and is replaced by the cool air over the land (land breeze).

Geothermal reservoir is built to generate electricity. No fuel is used to turn turbine because the heat steam is used. No smoke is emitted but only water vapour. Geothermal water also can be used for residential heating, greenhouse and agriculture.

A solar energy collector collects the solar energy and converts it into heat energy for various usages. This device absorbs the incoming solar radiation, converts it into heat energy and transfer it to a fluid flowing through the collector. The heat energy carried by the circulating fluid is then transferred to a thermal energy storage tank. Solar energy collectors are common for water heating, space heating system and solar refrigeration.

1.5 RESEARCH OBJECTIVES AND SCOPES

The objectives of the study are:

1. to construct a mathematical model for natural convection in an inclined enclosure with variable side wall temperature.

2. to develop a numerical algorithm to solve the convective flow in enclosures as stated in objective 1 using an iterative method to find the numerical solutions.

3. to find the numerical prediction of heat transfer in enclosures, the numerical algorithm achieved in objective 2 is used to develop a functional numerical simulation.

4. to analyze the results obtained from the numerical simulations in objective 3, the results are presented in the form of streamlines which demonstrate the flow patterns,
isotherms for the temperature distributions and the Nusselt number to predict the heat transfer rate in the enclosure.

5. to develop a correlation equation, that is to describe the heat transfer in enclosures based on the obtained data.

6. to investigate the effect of the enclosure inclination on the flow behaviour and heat transfer in a square enclosure with variable side wall temperature.

7. to investigate the effect of aspect ratio on the flow patterns, temperature distributions and heat transfer rate in enclosures with sinusoidal side wall temperature.

The scope of my study is to investigate natural convective flow and heat transfer in a fluid-filled inclined rectangular enclosure with sinusoidal variation of temperature on the left wall and the right wall is cooled with a constant temperature. The top and bottom walls of the enclosure are adiabatic. The enclosure inclination is an angle bounded between the inclined bottom wall and the horizontal plane. The flow is considered to be two-dimensional, Newtonian, incompressible and laminar. The finite difference method with appropriate numerical algorithm is used to solve the set of governing partial differential equations.

1.6 THESIS ORGANIZATION

There are seven chapters in this dissertation. The first chapter, Chapter 1 is an introductory chapter which gives general introduction on fluid flow and heat transfer. Examples and applications on the topic of current study will be presented in this chapter as well. Then, the scopes and objectives of study will be stated.

In Chapter 2, literature review on the earlier works will be presented. The review will be divided into several parts based on the relevant aspects to current investigation on natural convection in enclosures.

The mathematical formulation for the problems will be presented in Chapter 3. The derivation of the governing equations and boundary conditions based on the proposed model will be discussed here.
In Chapter 4, the method of solution for the problem will be discussed. The iterative method used in this study and the solution procedure will be presented as well.

Numerical results and discussion on the current problems will be given in Chapter 5 and 6. Chapter 5 will provide the study on the effect of enclosure inclination on convective flow and heat transfer in a square enclosure. The effect of aspect ratio on natural convection in an inclined rectangular enclosure will be discussed in Chapter 6.

Finally, Chapter 7 will summarize all the results from the previous chapters and conclusions will be given in the section later.
CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter, some published research works in various aspects related to the present work will be discussed. This chapter is sub-divided into four sections, which are general discussion on natural convection in enclosures, various thermal boundary conditions, the enclosure inclination and the effect of aspect ratio on convective flow in enclosures.

2.2 NATURAL CONVECTION IN ENCLOSURES

Numerous studies have been conducted on heat transfer problem and mainly are on convection (Darus, 1995). The study of natural convection in enclosures can be divided into two main groups; rectangular enclosures and non-rectangular enclosures (Basak et al., 2009b). These studies provide useful description of the confined fluids in most of the practical situations (Aydin and Yang, 2000).

For rectangular enclosures, De Vahl Davis and Jones (1983) performed a comparison study on natural convection of air in a square enclosure with the researchers available at that time. In their study, it can be noticed that various approaches were used to obtain a solution. Aydin and Yang (2000) numerically investigated the natural convection of air in a two-dimensional rectangular enclosure with localized heating from below and symmetry cooling from the sides. Rayleigh number \( R_a \) values from \( 10^3 \) to \( 10^6 \) and dimensionless heat source lengths are used to obtain the solutions. They reported that for small Rayleigh number, the heat transfer is dominated by conduction, while high Rayleigh number gives the effect of convection.

Convective fluid flow and heat transfer in enclosures also can be affected by the fluid that is contained in the enclosure. For example, Selamat et al. (2012a), Alhashash et al.
(2013) and Bhadauria et al. (2013) performed investigation on natural convection heat transfer in enclosures filled with porous medium. Selamat et al. (2012a) investigated transient natural convection in a square enclosure. They observed that the time taken to reach steady state is longer for low Rayleigh number and shorter for high Rayleigh number. Alhashash et al. (2013) studied convective flow in an enclosure with localized heating from the bottom solid wall. They found that the formation of cells in the enclosure is depending on the thickness of the bottom wall. They also reported that the heat transfer rate decreases with increasing of the wall thickness. Bhadauria et al. (2013) analyzed effects of time-periodic thermal boundary conditions and internal heating in a porous enclosure.

2.3 VARIOUS THERMAL BOUNDARY CONDITIONS

Non-isothermal thermal boundary conditions are more applicable in some engineering problems like solar energy collection and cooling of electronic components. Hence, in this section, various thermal boundary conditions such as linear wall temperature, sinusoidally varying wall temperature, non-uniform thermal conditions and others will be discussed.

2.3.1 Linearly heated wall(s)

Sathiyamoorthy et al. (2007a,b), Natarajan et al. (2007) and Basak et al. (2009a,c) studied convective flow in enclosures with linearly heated side walls, isothermally heated bottom wall and adiabatic top wall. Using square enclosure, Sathiyamoorthy et al. (2007a,b) found that for low Prandtl number fluid, the local mixing in the lower part of the enclosure is enhanced by the presence of symmetric strong secondary circulations. The presence of multiple circulations in the enclosure also causes the oscillation of the local Nusselt number. Natarajan et al. (2007) investigated the effect of the same thermal configurations in a trapezoidal enclosure. They reported that the heat transfer rate at the bottom wall is higher in the case of linearly heated left wall and cooled right wall. Then, mixed convection in a square enclosure was studied by Basak et al. (2009a). Multiple cells are observed in the enclosure and they found that the number of cells increases with the increase of the Prandtl number. Basak et al. (2009c) also imposed the same thermal
boundary conditions on trapezoidal enclosures with various inclination of the side walls. Symmetric flow pattern is observed in the enclosure for all tilting angles considered when both side walls are linearly heated. They also reported that the local Nusselt number and the average Nusselt number are higher for square enclosure.

Kaluri et al. (2010) carried a study on natural convection in a right-angled triangular enclosure with linearly or isothermally heated side walls and insulated bottom wall. They found that the local Nusselt number of the vertical wall decreases exponentially in the case of isothermal heating of the side walls. Local maxima in heat transfer at central portion of the side walls are observed for linear heating cases.

The study on natural convection in a nanofluid-filled square enclosure with linearly heated left wall, constantly cooled right wall and adiabatic top and bottom walls was investigated by Sivasankaran et al. (2010a). They concluded that the average Nusselt number is strongly depending on the types of nanoparticles.

2.3.2 Sinusoidal/non-uniform heated wall(s)

Recently, non-isothermal active side walls set more attention of many researchers. Sarris et al. (2002) work were motivated by the need to understand the heat transfer characteristics in glass melting tank. They stated that periodic heating from above of the container has strong implications for the glass industry, especially for mixing of the glass melt. They found that large Rayleigh number and large tank aspect ratio \((Ar = L/H)\) give optimum heat penetration and so give better glass melt homogenization.

Saeid (2005) performed a study on convection flow in a rectangular enclosure heated from below with cooled top wall and adiabatic side walls. A finite length of heat source is placed at the bottom surface. It is observed that the average Nusselt number increases with increasing length of the heat source. However, the ratio of heat transfer per unit area of the heat source decreases with increasing the length of the heat source.

Basak et al. (2006) studied natural convection in a square porous enclosure with uniformly and non-uniformly heated bottom wall, adiabatic top wall and cooled side walls. They found that non-uniform heating gives higher heat transfer at the center of the bottom wall for all the Rayleigh numbers considered. However, the average Nusselt number is
lower for non-uniform heating case compared to the uniform heating of the bottom wall. The local Nusselt number increases along the wall especially at the top portion of the side wall for both constant and sinusoidal heating of the bottom wall. Basak et al. (2007) also investigated the thermal configuration of uniformly and non-uniformly heating left and bottom walls, adiabatic top wall and constantly cooled right wall. They reported that the local Nusselt number decreases along the wall for conduction-dominant regime and the local Nusselt number increases for convection-dominant regimes due to highly dense temperature contour at the top portion of the cold wall.

Bilgen and Yedder (2007) studied the flow and temperature distribution in a rectangular enclosure with equally divided heated and cooled left side wall while the other walls are insulated. They found that the flow and temperature fields are symmetric with respect to the horizontal mid-plane. The local Nusselt number at the left side wall is governed by the temperature profile applied. They also found that the thermal penetration is higher if the active side wall is heated at the lower part compared to the heated upper wall. A smaller aspect ratio also gives a higher heat transfer.

Roy et al. (2008) used the finite element method to analyze the convective flows in an isosceles triangular enclosure with uniform and non-uniform bottom heating and cooled inclined walls. They calculated that the average Nusselt number for the bottom wall is $\sqrt{2}$ times of the inclined wall for the two cases of the bottom wall heating. On the other hand, Basak et al. (2008) studied the isosceles triangular enclosure with isothermally or sinusoidally heated inclined walls. Multiple flow circulations are observed in the cavity with both heating cases. They also found that geometry of the enclosure does not affect much on the flow structure in the enclosure with small values of the Prandtl number.

Other than rectangular and triangular enclosures, trapezoidal enclosures with uniform and sinusoidal heating of the bottom wall were investigated by Natarajan et al. (2008), Basak et al. (2009b) and Basak et al. (2010). The top wall is insulated while the inclined side walls are cooled with constant temperature. Natarajan et al. (2008) reported that at $Ra = 10^3$, viscous forces are dominant over the buoyancy force due to weak flow circulation in the enclosure. However, for $Ra = 10^5$, the circulation intensity increases and thermal boundary layers are observed in the enclosure. Basak et al. (2009b) found
that the average Nusselt number is invariant with respect to the side wall’s inclination at high Rayleigh number. They also concluded that the heat transfer rate is large for square enclosure compared to the trapezoidal enclosure for various wall inclination. Basak et al. (2010) observed that the thickness of the boundary layers are greater for non-uniform heating compared to the uniform heating case.

Saleh et al. (2010) studied convective flows in a porous square enclosure with non-uniform internal heating. The heated wall temperature varies sinusoidally with time about fixed mean temperature and the opposite wall is cooled with constant temperature. They observed that the location of the maximum fluid temperature varies with time according to the periodically heated wall temperature.

Later, Oztop et al. (2011) studied natural convection in an inclined square enclosure with sinusoidal heating and cooling on a side wall. They reported that sinusoidally varying wall temperature affects the fluid flow and temperature distribution in the enclosure. Heat transfer is an increasing function of the Rayleigh number. Also, addition of nanoparticles enhances the heat transfer at the heated part of the enclosure.

Convective flow in a square enclosure with sinusoidal temperature on both side walls was investigated by Sivasankaran et al. (2010b), Bhuvaneswari et al. (2011b), Sivasankaran et al. (2011), Sivasankaran and Pan (2012) and Sivasankaran and Bhuvaneswari (2013). In the presence of constant magnetic field, Bhuvaneswari et al. (2011b) found that the heat transfer is maximum at phase deviation $\phi = 3\pi/4$ and the heat transfer rate is low when the walls are of the same temperature distribution. Sivasankaran et al. (2010b) and Sivasankaran and Pan (2012) studied mixed convection in a square enclosure with sinusoidal temperature on both side walls. Also, they observed that the heat transfer is low when there is no phase deviation for the sinusoidal temperature profile at the right wall. The heat transfer is maximum when the phase deviation is $\phi = 3\pi/4$. They also concluded that non-uniform heating of both side walls produces higher heat transfer than non-uniform heating of a side wall only. Later, mixed convection in a lid-driven enclosure in the presence of the magnetic field was numerically studied by Sivasankaran et al. (2011). They found that the heat transfer increases with increasing the amplitude of the sinusoidal function. The heat transfer also increases with increasing the phase deviation.
up to $\pi/2$, and then it decreases. Sivasankaran and Bhuvaneswari (2013) studied natural convection in a porous square enclosure with sinusoidal temperature on both side walls and multiple flow structures are observed in the enclosure.

Anandalakshmi and Basak (2012) studied natural convection in rhombic enclosures with uniform or sinusoidal heating at the bottom wall. The top wall is adiabatic while the side walls are cooled with constant temperature. With constant heating of the bottom wall, it is observed that the wall inclination of $30^\circ$ gives maximum heat transfer and this orientation is suitable for liquid metal processing applications. The orientation of $75^\circ$ is efficient in solar heating applications and thermal processing of chemical solutions and oils.

### 2.3.3 Other thermal boundary conditions

Other than linear and sinusoidal wall temperature, partly active side walls with time-periodic boundary condition was investigated by Nithyadevi et al. (2006). The left active side wall is applied with time-periodic function while the right wall is kept at constant cold temperature. The rest of the walls are adiabatic. It is observed that the average Nusselt number varies non-linearly with the period.

Cheikh et al. (2007) studied natural convection in a square enclosure with partial active bottom wall. A finite length of heat source with constant heat flux is placed at the center of the bottom whereas the rest of the walls are either adiabatic or cooled with constant temperature. They found that the configuration of a cold side wall and an adiabatic side wall produces highest average Nusselt number compared to the other configurations that they have considered. Oztop and Abu-Nada (2008) conducted a study on natural convection in a partially heated rectangular enclosure and found that the heat transfer increases with the increase of the height of the heater.

Later, Varol et al. (2009) studied natural convective flow in a porous right-angled trapezoidal enclosure with partially cooled inclined wall. Constant heating is applied on the left wall while the top and bottom walls are insulated. The Nusselt number and flow strength are an increasing function of the Rayleigh number and it depends on the location of the cooler. The local and mean Nusselt numbers are the smallest in the case of middle
cooling of the inclined wall.

The effect of the location of heating and cooling zones on the flow structures and temperature distributions in an enclosure was analyzed by Sheikhzadeh et al. (2011) and Sankar et al. (2011). In the addition of nanoparticles, Sheikhzadeh et al. (2011) observed that the average Nusselt number is minimum in the case of top-bottom active walls and the maximum average Nusselt number is obtained with the case of middle-middle side walls for low Rayleigh number and bottom-middle for high Rayleigh number. Then, Sankar et al. (2011) revealed that the location of heating and cooling zones significantly affect the flow pattern and heat transfer in the porous enclosure. The heat transfer rate is found to be higher for the case of middle heating and middle cooling. They found that the partially heated and cooled walls of the enclosure also produces higher heat transfer than fully heated and cooled walls.

2.4 CONVECTION IN INCLINED ENCLOSURES

Buoyancy is due to the combined presence of a fluid density gradient and a body force that is proportional to density. Thus, buoyancy force is an important factor that drives the natural convective flow in the enclosures. Rasoul and Prinos (1997) was one of the literatures studied natural convection in an inclined square enclosure. They have considered the enclosure with isothermally heated left side wall and cooled right side wall. They also have considered three different fluids (gallium, air and silicone oil) and the Rayleigh number ranging from $10^3$ to $10^6$. This study reported that when the hot wall approaches the top position, fluid from the hot or cold wall returns back to the same wall forming two cells whose size decreases as the inclination angle decreases. The average Nusselt number increases with increasing of the Rayleigh number for all inclination angles considered. Also, the average Nusselt number increases with increasing of the Prandtl number for a fixed Rayleigh number. Then, Baytaş (2000) studied convective flow in an inclined square enclosure filled with porous medium. From this study, it is observed that as the Rayleigh number decreases, the heat transfer irreversibility begins to dominate the fluid friction irreversibility. Raos (2001) investigated that enclosure of orientation above $90^\circ$ greatly affects the convection process especially when inclination angle reached $180^\circ$. 
The orientation less than 20° shows inconsistent flow pattern in the enclosure.

Bilgen and Oztop (2005) studied convective flow in an inclined square enclosure with opening on the side wall. It is observed that the Nusselt number is maximum with inclination 30° – 90° for low Rayleigh number and 60° – 90° for high Rayleigh number. They also found that small opening at the center of side wall with enclosure inclination 90° – 120° could minimize loss for solar receiver applications. Large opening at the bottom of the side wall with inclination 60° – 90° could maximize the heat transfer for electronic cooling applications. Convective flow and heat transfer in an inclined enclosure with sinusoidal temperature profile are studied by Dalal and Das (2005). The increase of enclosure inclination gradually transforms one cell to two counter rotating cells in the enclosure. When the enclosure is heated from bottom, thermal boundary layer is formed and the thickness reduces with increasing Rayleigh number.

In the presence of magnetic field, Ece and Büyük (2006) studied natural convection in an inclined rectangular enclosure. It is observed that the anti-clockwise inclination of the enclosure causing the formation of multiple flows in the enclosure and thus it affects the temperature distribution in the enclosure as well.

Later, Ghasemi and Aminossadati (2009) studied convective flow in a square enclosure filled with copper(II) oxide (CuO) nanofluids. They observed that for inclined enclosures at high Rayleigh number, the flow structure, temperature distributions and local Nusselt number are different compared to enclosure without inclination. At low Rayleigh number, the addition of nanoparticles enhanced the heat transfer with increasing of the volume fraction. However at high Rayleigh number, there exist an optimum volume fraction which maximizes the heat transfer in the enclosure.

Munir et al. (2011) used lattice Boltzmann approach to solve natural convection in a differentially heated tilted square enclosure. They found that the vortex formation, size and flow characteristics are significantly affected by the magnitude of inclination angles.

Khezzar et al. (2012) studied natural convection in an inclined rectangular enclosure. It is observed that the Nusselt number decreases as the angle is increased in first quadrant when multiple cells flow changes to single cell flow. In addition of nanoparticles, Oztop et al. (2012) investigated convective flow and heat transfer in an inclined square enclosure
filled with copper(II) oxide nanofluid with linearly heated side wall. It is observed that small weak cell is observed at the top of the hot wall for the enclosure without inclination. Selamat et al. (2012b) studied natural convection in an inclined porous enclosure with spatially sinusoidal temperature variation. It is observed that the maximum average Nusselt number occurs at different wave number of the temperature function for different inclination angle. Later, Dou et al. (2013) analyzed convective flow in an inclined rectangular cavity. It is observed that when the inclination of the enclosure reaches a certain range, the instabilities in the top wall of the cavity moves left as the angle increases. Huelsz and Rechtman (2013) studied natural convection in an inclined square cavity using the lattice Boltzmann method. It is observed that the results are agreed well with the benchmark solution. The steady state flow can be rotated clockwise or counter-clockwise for a given inclination angle.

Recently, Basak et al. (2013) studied natural convection in isosceles triangular enclosures with linearly heated walls. It is observed that titled enclosure posses convective heat flow even for small Rayleigh number $Ra = 10^3$ due to high buoyancy force. They also concluded that the orientation of titled enclosure and straight enclosure produce higher heat transfer from fluid to wall at the central region.

### 2.5 ASPECT RATIO

The size of enclosure may affect the flow structure and temperature distribution in the enclosure. Tong (1999) studied the effect of aspect ratio on natural convection of water near its density maximum temperature. It is found that the aspect ratio has a strong impact on the flow pattern and temperature distributions in the enclosures. For small aspect ratio, the heat transfer across the wall diminishes due to the decrease of effective buoyancy force. For large aspect ratio, the heat transfer also decreases due to the enhanced shear stress effect. Later, Lartigue et al. (2000) performed experimental and numerical studies on natural convection in high aspect ratio enclosures. They observed that the cells in the core of the enclosure move downward for $Ar = 40$ and $Ra = 9222$. Rahman and Sharif (2003) studied the effect of aspect ratio on convective flow in enclosures with internal heat generation. They reported that the convection strength decreases as the aspect ratio
increases. For a particular aspect ratio ($Ar < 1$), the average heat transfer increases slightly with increase of inclination.

In the case of non-uniform heating enclosures, Saha et al. (2007) studied the effect of aspect ratio on natural convection in attics subject to periodic thermal forcing. They observed that the flow undergoes a transition between symmetry and asymmetry about the geometry symmetry plane over a diurnal cycle for aspect ratio, $Ar = 1.0$ and $0.5$. The flow remains symmetry throughout the cycle for $Ar = 2$. Later, Deng and Chang (2008) performed an investigation on natural convection in a rectangular enclosure with sinusoidal temperature profile on both side walls. It is observed that the heat transfer of the enclosure decreases with increasing of the aspect ratio when the sinusoidal temperatures are of the same phase. However, the heat transfer increases with the aspect ratio when the sinusoidal temperatures are of reversed phase ($\phi = \pi$). Next, Varol et al. (2008) studied convective flow in rectangular enclosures with sinusoidal temperature variation on the bottom wall. Multiple flows are observed inside the enclosure for all the Rayleigh numbers, aspect ratio and amplitude considered. The effect of aspect ratio is significant when the value of amplitude is high. Convective flow in a tall enclosure with linear and sinusoidal heating on a wall is studied by Yedder and Erchiqui (2009). They found that the flow is unicellular when linear profile is used and weak cell develops at the lower part of the enclosure at high Rayleigh number with a near square enclosure ($Ar \approx 1$). The flow is bi-cellular when sinusoidal temperature is used. They also concluded that tall enclosure promotes formation of multiple circulations in the enclosure. Aswatha et al. (2010) investigated the effect of aspect ratio and thermal boundary conditions on natural convection in enclosures. They observed that the average Nusselt number increases with increasing of the aspect ratio. Uniform heating at the bottom wall also produces higher Nusselt number compared to sinusoidal and linearly varying wall temperatures.

Bouabid et al. (2011) studied convective flow and entropy generation in an inclined rectangular enclosure. It is observed that the entropy generation increases with the aspect ratio. Then, Bhuvaneswari et al. (2011a) studied the effect of aspect ratio on natural convection in a porous enclosure with partially active thermal walls. They observed that the heat transfer rate decreases on increasing of the aspect ratio. The location of heating
and cooling zones affect the flow pattern and heat transfer in the enclosure. Later, Alam et al. (2012) studied natural convection in a rectangular enclosure with constant partial heating at the lower half of the left wall and partial cooling at the upper right wall. The average Nusselt number increases with increasing aspect ratio up to $Ar = 1$, then it decreases. Turan et al. (2012) investigated natural convection in rectangular enclosures with constant wall temperature or constant wall heat flux. It is observed that the mean Nusselt number increases with increasing of the aspect ratio up to a maximum value and then it decreases with further increase of the aspect ratio for the case of constant wall temperature. However in the case of constant wall heat flux, the mean Nusselt number increases monotonically with increasing of the aspect ratio. They also concluded from scaling arguments that the convective thermal transport enhances and the diffusive transport weakens with increasing of the aspect ratio.
CHAPTER 3
MATHEMATICAL FORMULATION

3.1 INTRODUCTION

In this chapter, the derivation of the convective flow and heat transfer equations will be explained. Then, the stream functions and vorticity are introduced to express the governing equations in terms of them. Finally, the boundary conditions of the convective heat transfer problem will be introduced to the system as the completion of the formulation.

3.2 THE EQUATIONS OF CONVECTIVE HEAT TRANSFER

The convective fluid flow and heat transfer can be explained by the continuity, momentum and energy equations. These governing equations are derived based on the related conservation laws which will be mentioned in the following sub-sections.

3.2.1 Continuity Equation

Consider a two-dimensional controlled volume, $\Delta V$ with element of height, $\Delta y$, length, $\Delta x$ and a unit depth in $z$-direction ($\Delta V = \Delta x \cdot \Delta y \cdot 1$). The control volume has a fixed mass, $m$. By the law of conservation of mass,

\[
\begin{array}{c|c|c}
\text{rate of change of} & \text{rate of mass convected into} & \text{rate of mass convected out of} \\
\text{mass in } \Delta V & \Delta V & \Delta V \\
\end{array}
\]

(3.1)

From Archimedes’ Principle,

\[ \rho = \frac{m}{\Delta V}, \quad \Rightarrow \quad m = \rho \Delta V, \]

(3.2)

where $\rho$ is the fluid density.
Then, consider the fluid is flowing through the control volume across any surface of an area, $A \equiv \Delta x \cdot 1$ or $\Delta y \cdot 1$. At a certain velocity, $U$ that is perpendicular to the surface, the rate of change of mass in $\Delta V$ is equal to the mass flow through the area $A$. Therefore,

$$\frac{\partial m}{\partial t} = \Delta V \frac{\partial \rho}{\partial t} = \rho AU. \quad (3.3)$$

![Figure 3.1: Fluid flow for a two-dimensional control volume](image)

Based on Figure 3.1 and equation (3.3), the equation (3.1) can be mathematically written as,

$$\Delta V \frac{\partial \rho}{\partial t} = \rho \Delta y u + \rho \Delta x v - \rho \Delta y \left( u + \frac{\partial}{\partial x} u \Delta x \right) - \rho \Delta x \left( v + \frac{\partial}{\partial y} v \Delta y \right),$$

substitute $\Delta V = \Delta x \Delta y$ into the equation, gives,

$$\Delta x \Delta y \frac{\partial \rho}{\partial t} = - \left[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) \right] \Delta x \Delta y,$$

then, divide both sides by $\Delta x \Delta y$,

$$\frac{\partial \rho}{\partial t} = - \left[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) \right].$$
and the equation can be simplify as,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0,$$  \hspace{1cm} (3.4)

where $\nabla \cdot \vec{V}$ is the rate of change of the volume of a moving fluid element, per unit volume (Wendt, 2009).

In this study, incompressible fluid flow is considered. Therefore, the density of the fluid is constant at any time. Thus, equation (3.4) becomes,

$$\frac{\partial \rho}{\partial t} = 0.$$  \hspace{1cm} (3.5)

Finally, the two-dimensional continuity equation for constant density is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$  \hspace{1cm} (3.6)

3.2.2 Momentum Equations

The momentum equations are obtained by the application of the conservation of momentum principle to the fluid flow with the assumption that mass of fluid is conserved in any closed system, that is,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} (mU) = m \frac{\partial U}{\partial t} = ma,$$  \hspace{1cm} (3.7)

where $\vec{F}$ is the net force in any direction, $\vec{p}$ is the momentum and $\vec{a}$ is the acceleration of the fluid in the direction in which the force is acting.

The conservation of momentum in the $x$- and $y$-coordinates are considered separately. The net force acting on the control volume in any of these directions can be summarize as, for example in the $x$-direction,

<table>
<thead>
<tr>
<th>net force acting on the control volume in the $x$-direction</th>
<th>rate at which $u$-momentum leaves the control volume</th>
<th>rate at which $u$-momentum enters the control volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{F}$</td>
<td>$\frac{d\vec{p}}{dt}$</td>
<td>$\frac{d}{dt} (mU)$</td>
</tr>
</tbody>
</table>

where $\vec{F}$ is the net force in any direction, $\vec{p}$ is the momentum and $\vec{a}$ is the acceleration of the fluid in the direction in which the force is acting.

The net force acting on the control volume in any of the direction, consists of two types of external forces which are surface force and body force. The surface force arises
from the pressure forces and the shearing forces acting on the faces of the control volume, whereas the body force comprises the gravity force, $g$ acting on the body. The viscous shearing forces for two-dimensional flow are expressed in terms of the velocity field by assuming the fluid to be Newtonian and are given as follows,

\[
\sigma_x = 2\mu \frac{\partial u}{\partial x},
\]
\[
\sigma_y = 2\mu \frac{\partial v}{\partial y},
\]
\[
\tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \tau_{yx},
\]

(3.9)

where $\mu$ is dynamic viscosity of the fluid.

Consider the same control volume with stresses are shown in the Figure 3.2. In the $x$-direction, the pressure force acting on the face AB is $P\Delta y$ and on the face CD is $-\left( P + \frac{\partial}{\partial y}P\Delta x \right) \Delta y$. So, the net pressure forces acting on the control volume will be,

\[
P\Delta y - \left( P + \frac{\partial}{\partial x}P\Delta x \right) \Delta y = -\frac{\partial}{\partial x}P\Delta x \Delta y.
\]

(3.10)
The viscous shearing forces comprise of two orthogonal forces, they are the normal shear stress and the tangential shear stress that act on a surface. Based on Figure 3.2, the normal viscous force acting on the face AB is $-\sigma_x \Delta y$ and on the face CD will be $\left(\sigma_x + \frac{\partial}{\partial x} \sigma_x \Delta x\right) \Delta y$. Therefore, the normal viscous forces acting on the control volume is,

$$-\sigma_x \Delta y + \left(\sigma_x + \frac{\partial}{\partial x} \sigma_x \Delta x\right) \Delta y = \frac{\partial}{\partial x} \sigma_x \Delta x \Delta y.$$  \hspace{1cm} (3.11)

The tangential forces are acting on xy-plane only. Hence from Figure 3.2, the tangential viscous force is $-\tau_{yx} \Delta x$ and $\left(\tau_{yx} + \frac{\partial}{\partial y} \tau_{yx} \Delta y\right) \Delta x$ on the face DA and BC, respectively. So, the tangential viscous forces acting on the control volume is,

$$-\tau_{yx} \Delta x + \left(\tau_{yx} + \frac{\partial}{\partial y} \tau_{yx} \Delta y\right) \Delta x = \frac{\partial}{\partial y} \tau_{yx} \Delta x \Delta y.$$ \hspace{1cm} (3.12)

![Figure 3.3: Buoyancy force components]

Finally, the body force acting on the control volume will be the gravity force. By Boussinesq approximation, the density difference is sufficiently small to be neglected except that it changes with temperature that gives rise to the buoyancy force (Oosthuizen and Naylor, 1999). Therefore, based on Figure 3.3, the body force acting on the control volume in the x-direction will be,

$$\beta g \rho (T - T_0) \sin \varphi \Delta x \Delta y,$$ \hspace{1cm} (3.13)

where $\beta$ is volumetric coefficient of thermal expansion of the fluid, $T_0$ is a convenient reference fluid temperature and $\varphi$ is the inclination angle between x-axis and the horizontal plane.
Adding up equations (3.10) – (3.13), the net force acting on the control volume in the \(x\)-direction will be,

\[- \frac{\partial}{\partial x} P \Delta x \Delta y + \frac{\partial}{\partial x} \sigma_x \Delta x \Delta y + \frac{\partial}{\partial y} \tau_{yx} \Delta x \Delta y + \beta g \rho \left( T - T_0 \right) \sin \phi \Delta x \Delta y. \]

(3.14)

From Figure 3.1 and using equation (3.3), the rate of \(u\)-momentum leaves the control volume through face CD and BC is \(\rho u \left( u + \frac{\partial}{\partial x} u \Delta x \right) \Delta y\) and \(\rho u \left( v + \frac{\partial}{\partial y} v \Delta y \right) \Delta x\) respectively. On the face AB, \(u\)-momentum enters the control volume by the rate of \(\rho u^2 \Delta y\) and \(\rho uv \Delta x\) on the face DA. Therefore, the difference between the rate at which \(u\)-momentum leaves the control volume and enters the control volume will be,

\[\rho u \left( u + \frac{\partial}{\partial x} u \Delta x \right) \Delta y + \rho u \left( v + \frac{\partial}{\partial y} v \Delta y \right) \Delta x - \rho u^2 \Delta y - \rho uv \Delta x = \rho \frac{\partial}{\partial x} u \Delta x \Delta y + \rho u \frac{\partial}{\partial y} v \Delta x \Delta y. \]

(3.15)

The rate of change of \(u\)-momentum within a control volume, \(\Delta V\) is,

\[\frac{\partial}{\partial t} (mu) = \frac{\partial}{\partial t} (\rho \Delta V u) = \frac{\partial}{\partial t} (\rho u) \Delta x \Delta y, \]

(3.16)

Using equations (3.14) – (3.16), equation (3.8) can be written as,

\[- \frac{\partial}{\partial x} P \Delta x \Delta y + \frac{\partial}{\partial x} \sigma_x \Delta x \Delta y + \frac{\partial}{\partial y} \tau_{yx} \Delta x \Delta y + \beta g \rho \left( T - T_0 \right) \sin \phi \Delta x \Delta y \]

\[= \rho \frac{\partial}{\partial x} u \Delta x \Delta y + \rho u \frac{\partial}{\partial y} v \Delta x \Delta y + \frac{\partial}{\partial t} (\rho u) \Delta x \Delta y, \]

next, divide both sides of the equation with \(\Delta x \Delta y\),

\[- \frac{\partial P}{\partial x} \frac{\partial}{\partial x} \sigma_x + \frac{\partial}{\partial y} \tau_{yx} + \beta g \rho \left( T - T_0 \right) \sin \phi = \rho \left[ \frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial y} (uv) \right] + \frac{\partial}{\partial t} (\rho u), \]
then, substitute \( \sigma_x \) and \( \tau_{yx} \) with equation (3.9),

\[
- \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left( 2\mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + \beta g \rho (T - T_0) \sin \phi \\
= \rho \left[ u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y} \right] + \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t},
\]

finally, the equation becomes,

\[
- \frac{\partial P}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 u}{\partial y^2} + \beta g \rho (T - T_0) \sin \phi \\
= \rho \left[ u \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] + \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}. \quad (3.17)
\]

From the two-dimensional continuity equation with constant density flow,

\[
\frac{\partial^2 u}{\partial x^2} = - \left( \frac{\partial^2 v}{\partial x \partial y} \right), \quad \frac{\partial^2 v}{\partial y^2} = - \left( \frac{\partial^2 u}{\partial x \partial y} \right). \quad (3.18)
\]

Using equations (3.5) and (3.18), equation (3.17) can be written as,

\[
- \frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} - \mu \left( \frac{\partial^2 v}{\partial x \partial y} \right) + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 u}{\partial y^2} + \beta g \rho (T - T_0) \sin \phi \\
= \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} \right),
\]

simplify and rearrange,

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \beta g (T - T_0) \sin \phi, \quad (3.19)
\]

where \( \nu \equiv \frac{\mu}{\rho} \) is the kinematic viscosity of the fluid.

Similarly, in the \( y \)-direction, the \( v \)-momentum equation is derived using the same procedures as deriving the \( u \)-momentum equation. Generally, the unsteady momentum equations can be written as,

\[
\frac{\partial \vec{V}}{\partial t} + \nabla \cdot (\vec{V} \vec{V}) = - \frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{V} + \vec{F}. \quad (3.20)
\]
In this study, we are considering two-dimensional flow. Therefore, the momentum equations for steady flow are,

\[
\begin{align*}
    u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \beta g (T - T_0) \sin \varphi, \\
    u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} + v \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \beta g (T - T_0) \cos \varphi.
\end{align*}
\]

\[\text{(3.21)}\]
\[\text{(3.22)}\]

### 3.2.3 Energy Equation

The energy equation is derived based on the conservation of energy principle. By the First Law of Thermodynamics (Wendt, 2009),

\[
\text{rate of change of energy inside the control volume} = \text{net flux of heat into the control volume} + \text{rate of work done on the control volume due to body and surface forces.}
\]

\[\text{(3.23)}\]

The derivation will begin from the left of the equation (3.23). The total enthalpy, \( I \) per unit mass can be defined as follow,

\[
I = \text{specific enthalpy} + \text{specific kinetic energy,}
\]

\[
= c_p T + \frac{u^2 + v^2}{2}.
\]

\[\text{(3.24)}\]

The rate of change of total enthalpy within the control volume, \( \Delta V \) of a fixed mass \( m \) is,

\[
\frac{\partial}{\partial t} (mI) = \frac{\partial m}{\partial t} I + m \frac{\partial I}{\partial t},
\]

\[
= \frac{\partial m}{\partial t} I + \rho \Delta V \frac{\partial I}{\partial t},
\]

\[
= \frac{\partial m}{\partial t} I + \rho \frac{\partial I}{\partial t} \Delta x \Delta y.
\]

\[\text{(3.25)}\]

From equation (3.3), \( \frac{\partial m}{\partial t} \) is the mass flow rate through any surface of the control volume with area \( A \). Then, the difference between the rate at which the sum of enthalpy and kinetic energy leave and enter the control volume in the \( x \)-direction is, where \( \Delta x \) is
assumed to be small,

\[ \left[ \frac{\partial m}{\partial t} I + \frac{\partial}{\partial x} \left( \frac{\partial m}{\partial t} I \right) \right] \Delta x - \frac{\partial m}{\partial t} I = \frac{\partial}{\partial x} \left( \frac{\partial m}{\partial t} I \right) \Delta x. \]

and it can be written as,

\[ \left[ \frac{\partial}{\partial x} (\rho u I) \right] \Delta x \Delta y, \quad (3.26) \]

the same arguments are used for the \( y \)-direction,

\[ \left[ \frac{\partial}{\partial y} (\rho v I) \right] \Delta x \Delta y. \quad (3.27) \]

Then, adding up equations (3.25) – (3.27), and using equation (3.24), it can be shown that the left hand side of the equation (3.23) is,

\begin{align*}
\rho \frac{\partial I}{\partial t} \Delta x \Delta y + \left[ \frac{\partial}{\partial x} (\rho u I) \right] \Delta x \Delta y + \left[ \frac{\partial}{\partial y} (\rho v I) \right] \Delta x \Delta y & = \rho \frac{\partial}{\partial t} \left[ c_p T + \left( \frac{u^2 + v^2}{2} \right) \right] \Delta x \Delta y + \frac{\partial}{\partial x} \left\{ \rho u \left[ c_p T + \left( \frac{u^2 + v^2}{2} \right) \right] \right\} \Delta x \Delta y \\
& \quad + \frac{\partial}{\partial y} \left\{ \rho v \left[ c_p T + \left( \frac{u^2 + v^2}{2} \right) \right] \right\} \Delta x \Delta y. \quad (3.28)
\end{align*}

Figure 3.4: Enthalpy and heat flows into and out of the control volume
From Figure 3.4, $Q_x$ is the rate at which heat is transferred into the control volume in the $x$-direction through the face AB. The difference between the rate at which heat is transferred into the control volume in the $x$-direction and the rate at which it is conducted out in this direction is,

$$Q_x - \left[ Q_x + \frac{\partial}{\partial x} (Q_x) \Delta x \right] = -\frac{\partial}{\partial x} (Q_x) \Delta x. \quad (3.29)$$

By Fourier’s Law, the heat transfer rate, $Q_n$, in any direction, $n$ per unit area measured normal to $n$ is given by,

$$Q_n = -k \frac{\partial T}{\partial n},$$

where $k$ is the thermal conductivity of the fluid.

The control volume has a unit depth ($\Delta z = 1$). Therefore, the net rate of heat transferred into the control volume in the $x$-direction through the face AB is,

$$Q_x = -k \Delta y \frac{\partial T}{\partial x}, \quad (3.30)$$

and substitute equation (3.30) into equation (3.29) gives,

$$-\frac{\partial}{\partial x} \left( -k \Delta y \frac{\partial T}{\partial x} \right) \Delta x = \left[ \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) \right] \Delta x \Delta y. \quad (3.31)$$

Similarly, for the $y$-direction,

$$\left[ \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) \right] \Delta x \Delta y, \quad (3.32)$$

then, combining equations (3.31) and (3.32), the net flux of heat into the control volume is,

$$\left[ \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) \right] \Delta x \Delta y + \left[ \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) \right] \Delta x \Delta y = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \Delta x \Delta y. \quad (3.33)$$
From Figure 3.2, the rate at which work is done on the control volume due to the surface forces is,

\[
\left\{ (-u \sigma_x \Delta y) + \left( (u \sigma_x \Delta y) + \frac{\partial}{\partial x} (u \sigma_x) \Delta x \Delta y \right) \right\} \\
+ \left\{ (u P \Delta y) - \left[ (u P \Delta y) + \frac{\partial}{\partial x} (u P) \Delta x \Delta y \right] \right\} \\
+ \left\{ (-u \tau_{yx} \Delta x) + \left[ (u \tau_{yx} \Delta x) + \frac{\partial}{\partial y} (u \tau_{yx}) \Delta x \Delta y \right] \right\} \\
+ \left\{ (-v \sigma_y \Delta x) + \left[ (v \sigma_y \Delta x) + \frac{\partial}{\partial y} (v \sigma_y) \Delta x \Delta y \right] \right\} \\
+ \left\{ (v P \Delta x) - \left[ (v P \Delta x) + \frac{\partial}{\partial y} (v P) \Delta x \Delta y \right] \right\} \\
+ \left\{ (-v \tau_{xy} \Delta y) + \left[ (v \tau_{xy} \Delta y) + \frac{\partial}{\partial x} (v \tau_{xy}) \Delta x \Delta y \right] \right\},
\]

and rearrange,

\[
\left[ \frac{\partial}{\partial x} (u \sigma_x) - \frac{\partial}{\partial x} (u P) + \frac{\partial}{\partial y} (u \tau_{yx}) + \frac{\partial}{\partial y} (v \sigma_y) - \frac{\partial}{\partial y} (v P) + \frac{\partial}{\partial x} (v \tau_{xy}) \right] \Delta x \Delta y
= \left( u \frac{\partial \sigma_x}{\partial x} - u \frac{\partial \tau_{yx}}{\partial x} + u \frac{\partial \tau_{yx}}{\partial y} - v \frac{\partial \sigma_y}{\partial y} + v \frac{\partial \tau_{xy}}{\partial x} \right) \Delta x \Delta y
+ \left( \sigma_x \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \frac{\partial v}{\partial x} \right) \Delta x \Delta y.
\]

Then, expand the bracketed terms on the right side of the equation (3.34) using equation (3.9),

\[
2u \mu \frac{\partial}{\partial x} \left( 2 \mu \frac{\partial u}{\partial x} \right) - u \frac{\partial P}{\partial x} + u \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\
+ v \frac{\partial}{\partial y} \left( 2 \mu \frac{\partial v}{\partial y} \right) - v \frac{\partial P}{\partial y} + v \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\
= 2u \mu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial P}{\partial x} + u \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) \\
+ 2v \mu \frac{\partial^2 v}{\partial y^2} - v \frac{\partial P}{\partial y} + v \mu \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right),
\]

(3.35)
and from the continuity equation (3.18), equation (3.35) can be rewrite as,

\[
2u\mu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial P}{\partial x} + u\mu \left( \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} \right) + 2v\mu \frac{\partial^2 v}{\partial y^2} - v \frac{\partial P}{\partial y} + v\mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = u \left[ -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right] + v \left[ -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right]. \tag{3.36}
\]

Compare equation (3.36) with equations (3.21) and (3.22), it can be rewrite as,

\[
\rho u \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \beta g (T - T_0) \sin \phi \right] + \rho v \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \beta g (T - T_0) \cos \phi \right] = \rho \left( \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t} \right) + \rho u^2 \frac{\partial u}{\partial x} + \rho uv \frac{\partial u}{\partial y} + \rho uv \frac{\partial v}{\partial x} + \rho v^2 \frac{\partial v}{\partial y} + \beta g \rho (T - T_0) (u \sin \phi + v \cos \phi). \tag{3.37}
\]

Therefore, the rate at which work is done on the control volume due to the surface forces is,

\[
\rho \left( u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t} \right) \Delta x \Delta y + \left( \rho u^2 \frac{\partial u}{\partial x} + \rho uv \frac{\partial u}{\partial y} + \rho uv \frac{\partial v}{\partial x} + \rho v^2 \frac{\partial v}{\partial y} \right) \Delta x \Delta y - \beta g \rho (T - T_0) (u \sin \phi + v \cos \phi) \Delta x \Delta y
\]

\[
+ \left( \sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} + \tau_{yx} \frac{\partial v}{\partial x} \right) \Delta x \Delta y. \tag{3.38}
\]

From the derivation of the momentum equations, the body force acting on the control volume will be the gravity force. Therefore, based on Figure 3.3, the rate at which work is done on the control volume due to the body force is,

\[
u \beta g \rho (T - T_0) \sin \phi \Delta x \Delta y + v \beta g \rho (T - T_0) \cos \phi \Delta x \Delta y
\]

\[
= \beta g \rho (T - T_0) (u \sin \phi + v \cos \phi) \Delta x \Delta y. \tag{3.39}
\]
From equations (3.38) and (3.39), the rate at which work is done on the control volume will be,

\[
\rho \left( u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t} \right) \Delta x \Delta y + \rho \left( \frac{\partial u^2}{\partial x} + \frac{\partial u v}{\partial y} + \frac{\partial u v}{\partial x} + \frac{\partial v^2}{\partial y} \right) \Delta x \Delta y
\]

\[- \beta g \rho \left( T - T_0 \right) (u \sin \phi + u \cos \phi) \Delta x \Delta y + \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial v}{\partial y} \right) \Delta x \Delta y
\]

\[= \rho \left( u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t} \right) \Delta x \Delta y + \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial v}{\partial y} \right) \Delta x \Delta y
\]

\[+ \left( \sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \frac{\partial v}{\partial x} \right) \Delta x \Delta y. \quad (3.40)
\]

From equations (3.28), (3.33) and (3.40), and after dividing by \( \Delta x \Delta y \), the equation (3.23) can be mathematically written as,

\[
\rho \frac{\partial}{\partial t} \left[ c_p T + \frac{u^2 + v^2}{2} \right] + \frac{\partial}{\partial x} \left\{ \rho u \left[ c_p T + \frac{u^2 + v^2}{2} \right] \right\} + \frac{\partial}{\partial y} \left\{ \rho v \left[ c_p T + \frac{u^2 + v^2}{2} \right] \right\}
\]

\[= k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \rho \left( u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t} \right) + \left( \rho u \frac{\partial^2 u}{\partial x^2} + \rho v \frac{\partial^2 u}{\partial y^2} + \rho v \frac{\partial^2 v}{\partial x^2} + \rho v \frac{\partial^2 v}{\partial y^2} \right)
\]

\[+ \left( \sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \frac{\partial v}{\partial x} \right). \quad (3.41)
\]

\( \rho \) and \( c_p \) are fluid properties which are being assumed constant, then equation (3.41) can be written as,

\[
\rho c_p \frac{\partial T}{\partial t} + \rho \left( u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t} \right) + \rho c_p \left( u T + \frac{1}{2} \rho \left( u \frac{\partial u}{\partial x} + \frac{1}{2} u^2 + \frac{1}{2} v^2 \right) \right)
\]

\[+ \rho c_p \left( \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{1}{2} \rho \left( u^2 + v^2 \right) + \frac{1}{2} \rho \left( u^2 + v^2 \right) + \frac{1}{2} \rho \left( u^2 + v^2 \right)
\]

\[= k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \rho \left( u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t} \right) + \left( \rho u \frac{\partial^2 u}{\partial x^2} + \rho v \frac{\partial^2 u}{\partial y^2} + \rho v \frac{\partial^2 v}{\partial x^2} + \rho v \frac{\partial^2 v}{\partial y^2} \right)
\]

\[+ \left( \sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \frac{\partial v}{\partial x} \right). \]
and simplify,
\[
\frac{\partial T}{\partial t} + \frac{\partial}{\partial x} (uT) + \frac{\partial}{\partial y} (vT) = \left( \frac{k}{\rho c_p} \right) \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \left( \frac{\mu}{\rho c_p} \right) \phi, \tag{3.42}
\]

where \(\phi\) is termed the viscous dissipation function, and it can be defined as,
\[
\phi = \frac{1}{\mu} \left( \sigma_x \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \frac{\partial v}{\partial x} \right). \tag{3.43}
\]

Using equation (3.9), the viscous dissipation function (3.43) becomes,
\[
\phi = \frac{1}{\mu} \left[ \left( 2\mu \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial x} + \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{\partial u}{\partial y} + \left( 2\mu \frac{\partial v}{\partial y} \right) \frac{\partial v}{\partial y} + \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{\partial v}{\partial x} \right],
\]
\[
= 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2.
\]

In general, the energy equation can be simplified as,
\[
\frac{\partial T}{\partial t} + \nabla \cdot (\vec{V_T}) = \alpha \nabla^2 T + \left( \frac{\mu}{\rho c_p} \right) \phi, \tag{3.44}
\]

where \(\alpha \equiv \frac{k}{\rho c_p}\) is the thermal diffusivity of the fluid.

Since the fluid properties are assumed to be constant, the velocities of the fluid flow are low and thus the dissipation term can be ignored. Therefore, the steady two-dimensional energy equation is,
\[
u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right). \tag{3.45}
\]

### 3.3 VORTICITY-STREAM FUNCTION FORMULATION

The lines of constant \(\psi\) represent streamlines, and the difference in the values of \(\psi\) between two streamlines gives the volumetric flow rate between them (Hoffman and Chiang, 2000). Thus, the plot of streamlines can illustrate the flow direction of a fluid particle. Vorticity is a measure of the angular momentum or rotational motion induced by the action of the viscous stresses acting tangentially around the surface of the fluid particles (Oosthuizen and Naylor, 1999). The vorticity-stream function formulations are advantageous in solving two-dimensional fluid flow.
Vorticity equation is acquired by eliminating the pressure term between the two momentum equations, which are equations (3.21) and (3.22). Subtract $y$-derivatives of equation (3.21) and $x$-derivatives of equation (3.22),

$$u \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} \right) + v \left( \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} \right) = v \left[ \left( \frac{\partial^3 v}{\partial x^3} - \frac{\partial^3 u}{\partial x^2 \partial y} \right) + \left( \frac{\partial^3 v}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3} \right) \right]$$

$$+ \beta g \left[ \frac{\partial}{\partial x} (T - T_0) \cos \phi - \frac{\partial}{\partial y} (T - T_0) \sin \phi \right]. \quad (3.46)$$

Stream function, $\psi$ and vorticity, $\omega$ are introduced,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad (3.47)$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (3.48)$$

The first and second derivative with respect to $x$ and $y$ for equation (3.48) are,

$$\frac{\partial \omega}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial \omega}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{\partial^3 \psi}{\partial x^3} - \frac{\partial^3 u}{\partial x^2 \partial y}, \quad \frac{\partial^2 \omega}{\partial y^2} = \frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3}. \quad (3.49)$$

Substitute equations (3.47) – (3.49) into (3.46),

$$\frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} = v \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) + \beta g \left( \frac{\partial T}{\partial x} \cos \phi - \frac{\partial T}{\partial y} \sin \phi \right), \quad (3.50)$$

which is the vorticity equation with the $u$- and $v$-velocities are expressed in terms of stream function.

In addition, the vorticity transport can be expressed in terms of stream function,

$$\omega = \frac{\partial}{\partial x} \left( -\frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right),$$

or can be written as,

$$-\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \quad (3.51)$$
Energy equation is expressed in terms of stream function by substituting equation 
(3.47) into equation (3.45),

\[
\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \left( \frac{k}{\rho c_p} \right) \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right). 
\] 

(3.52)

3.4 BOUNDARY CONDITIONS

The governing equations are important to explain the flow of a fluid. However, the flow fields are quite different for various cases of heat transfer even though the governing equations are the same. So, the boundary conditions are important to govern the flow fields for different cases. For natural convection in enclosures, the boundary conditions are derived based on three criteria, the velocity, pressure, and temperature of the flow. However, with the vorticity-stream function formulations, the boundary conditions for pressure are not necessary required in solving the governing equations.

3.4.1 Velocity Conditions

Based on Figure 3.5, \( u \) and \( v \) are the velocities of the fluid in the \( x \)- and \( y \)-directions, respectively. The boundaries AB, BC, CD, and DA are the solid surfaces of the enclosure. There are two conditions to be considered in setting the boundary conditions for velocity, which are the Dirichlet and Neumann conditions.

**Dirichlet condition:**

Fluid that flows on a solid surface experiences no-slip condition. At the point of contact between a viscous fluid and a solid surface, the fluid velocity is the same as that of the
solid surface. So, the fluid particle at the boundary is of zero velocity in relative to the boundary. Therefore, the Dirichlet boundary condition for all solid surfaces are,

\[ u = v = 0. \quad (3.53) \]

**Neumann condition:**

Mass is conserved in a closed system. Therefore, no fluid flows in or out from the system. This implies that there is no velocity gradient on every solid wall. Hence, the Neumann boundary condition implies that,

\[
\begin{align*}
on \text{AB and CD} & : \quad \frac{\partial u}{\partial y} = 0, \\
on \text{BC and DA} & : \quad \frac{\partial v}{\partial x} = 0. \quad (3.54)
\end{align*}
\]

### 3.4.2 Temperature Conditions

Based on Figure 3.6, the enclosure is heated at temperature \( T_h \) on the left wall. The temperature \( T_h \) can be a function based on the enclosure’s height where the enclosure can be uniformly or non-uniformly heated along the wall AB. At wall CD, it is cooled with constant temperature \( T_c \). However, the thermal boundary conditions on the walls BC and DA depend on whether the walls are adiabatic or perfectly conducting. In this study, the walls are considered to be adiabatic, therefore the temperature conditions of the side walls...
on AB : \( T = T_h \) or \( T(y) \),
on CD : \( T = T_c \),
on BC and DA : \( \frac{\partial T}{\partial y} = 0 \). \hfill (3.55)

### 3.4.3 Boundary Conditions on Stream Function

Based on equation (3.47), the boundary conditions of the stream function is depending on the boundary conditions of \( u \) and \( v \)-velocities at the boundaries. From the derivation of equations (3.53) and (3.54),

on all solid boundaries AB, BC, CD and DA : \( \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x} = 0 \), \hfill (3.56)
on AB and CD : \( \frac{\partial^2 \psi}{\partial y^2} = 0 \), \hfill (3.57)
on BC and DA : \( \frac{\partial^2 \psi}{\partial x^2} = 0 \). \hfill (3.58)

The actual value of \( \psi \) is unknown because only the derivatives of \( \psi \) are available in hand. However, from Figure 3.5, the value of \( \psi \) at point A can be arbitrary taken as,

\[ \psi_A = 0. \]

Since \( \frac{\partial \psi}{\partial y} = 0 \) along AB, this suggests that \( \psi = 0 \) along AB and,

\[ \psi_B = 0. \]

Similarly, \( \frac{\partial \psi}{\partial x} = 0 \) along BC, this also implies that \( \psi = 0 \) along BC and,

\[ \psi_C = 0. \]

Using the same arguments for walls CD and DA, it can be conclude that on walls AB, BC, CD and DA,

\[ \psi = 0, \quad \frac{\partial \psi}{\partial n} = 0, \] \hfill (3.59)
where \( n \) is the coordinate measured normal to the surface being considered.

### 3.5 DIMENSIONLESS EQUATIONS

The governing equations for convective heat transfer comprises of a system of partial differential equations. The equations are dimensional and the physical quantities involved are depending on the properties of the fluid considered. Thus, dimensionless method is introduced to reduce the complexity in solving the equations. For natural convection in a square enclosure with height and width \( W \), the dimensionless variables are introduced as follow,

\[
X = \frac{x}{W}, \quad Y = \frac{y}{W}, \quad \Psi = \frac{\psi Pr}{\nu}, \quad \Omega = \frac{\omega W^2 Pr}{\nu}, \quad \Theta = \frac{T - T_c}{T_h - T_c},
\]

where \( Pr = \frac{\nu}{\alpha} \) is the Prandtl number. Prandtl number is expressed as the ratio of kinematic viscosity of the fluid, \( \nu \) to the thermal diffusivity of the fluid, \( \alpha \). Prandtl number influences the thickness of thermal and velocity boundary layer. The thickness is large if the viscosity is particularly large, i.e. \( Pr \) is large.

From equation (3.60), the terms involve in the governing equations can be expressed in the dimensionless form as the following,

\[
\frac{\partial \psi}{\partial x} = \left( \frac{\nu}{W Pr} \right) \frac{\partial \Psi}{\partial X}, \quad \frac{\partial \psi}{\partial y} = \left( \frac{\nu}{W Pr} \right) \frac{\partial \Psi}{\partial Y}, \quad \frac{\partial \omega}{\partial x} = \left( \frac{\nu}{W^3 Pr} \right) \frac{\partial \Omega}{\partial X}, \quad \frac{\partial \omega}{\partial y} = \left( \frac{\nu}{W^3 Pr} \right) \frac{\partial \Omega}{\partial Y},
\]

\[
\frac{\partial^2 \omega}{\partial x^2} = \left( \frac{\nu}{W^4 Pr} \right) \frac{\partial^2 \Omega}{\partial X^2}, \quad \frac{\partial^2 \omega}{\partial y^2} = \left( \frac{\nu}{W^4 Pr} \right) \frac{\partial^2 \Omega}{\partial Y^2}, \quad \frac{\partial T}{\partial x} = \left( \frac{T_h - T_c}{W} \right) \frac{\partial \Theta}{\partial X}, \quad \frac{\partial T}{\partial y} = \left( \frac{T_h - T_c}{W} \right) \frac{\partial \Theta}{\partial Y},
\]

\[
\frac{\partial^2 T}{\partial x^2} = \left( \frac{T_h - T_c}{W^2} \right) \frac{\partial^2 \Theta}{\partial X^2}, \quad \frac{\partial^2 T}{\partial y^2} = \left( \frac{T_h - T_c}{W^2} \right) \frac{\partial^2 \Theta}{\partial Y^2}.
\]

Substitute the dimensionless equations (3.61) into the stream equation (3.51),

\[
\left( \frac{\nu}{W^2 Pr} \right) \frac{\partial^2 \Psi}{\partial X^2} + \left( \frac{\nu}{W^2 Pr} \right) \frac{\partial^2 \Psi}{\partial Y^2} = - \left( \frac{\nu}{W^2 Pr} \right) \Omega.
\]
and simplify to obtain dimensionless stream equation,

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -\Omega. \quad (3.62)$$

Similarly, the non-dimensionalized vorticity equation (3.50) is,

$$\left( \frac{v}{W Pr} \right) \frac{\partial \Psi}{\partial Y} \left( \frac{v}{W^3 Pr} \right) \frac{\partial \Omega}{\partial X} - \left( \frac{v}{W Pr} \right) \frac{\partial \Psi}{\partial X} \left( \frac{v}{W^3 Pr} \right) \frac{\partial \Omega}{\partial Y}$$

$$= v \left[ \left( \frac{v}{W^4 Pr} \right) \frac{\partial^2 \Omega}{\partial X^2} + \left( \frac{v}{W^4 Pr} \right) \frac{\partial^2 \Omega}{\partial Y^2} \right]$$

$$+ \beta g \left[ \left( \frac{T_h - T_c}{W} \right) \frac{\partial \Theta}{\partial X} \cos \phi - \left( \frac{T_h - T_c}{W} \right) \frac{\partial \Theta}{\partial Y} \sin \phi \right],$$

and the simplified form is,

$$\frac{\partial \Psi}{\partial Y} \frac{\partial \Omega}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Omega}{\partial Y} = \frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2} + Ra Pr \left( \frac{\partial \Theta}{\partial X} \cos \phi - \frac{\partial \Theta}{\partial Y} \sin \phi \right), \quad (3.63)$$

where $Ra = \frac{\beta g (T_h - T_c) W^3}{\alpha \nu}$ is the Rayleigh number. It shows the relative magnitude of the buoyancy and viscous forces acting on the fluid. Rayleigh number is a dimensionless number associated with natural convection heat transfer to determine whether the fluid undergoes conductive or convective transport.

The dimensionless energy equation (3.52) is,

$$\left( \frac{v}{W Pr} \right) \frac{\partial \Psi}{\partial Y} \left( T_h - T_c \right) \frac{\partial \Theta}{\partial X} - \left( \frac{v}{W Pr} \right) \frac{\partial \Psi}{\partial X} \left( \frac{T_h - T_c}{W} \right) \frac{\partial \Theta}{\partial Y}$$

$$= \alpha \left[ \left( \frac{T_h - T_c}{W^2} \right) \frac{\partial^2 \Theta}{\partial X^2} + \left( \frac{T_h - T_c}{W^2} \right) \frac{\partial^2 \Theta}{\partial Y^2} \right],$$

and divide both sides with $\frac{v(T_h - T_c)}{W^2 Pr}$,

$$\frac{\partial \Psi}{\partial Y} \frac{\partial \Theta}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Theta}{\partial Y} = \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2}. \quad (3.64)$$

Besides the governing equations, the boundary conditions also have to express in the dimensionless form. Based on the dimensionless variables (3.60), the boundary condi-
tions of the stream equation (3.59) is,

\[
\text{On all solid boundaries: } \Psi = 0, \quad \frac{\partial \Psi}{\partial N} = 0, \quad (3.65)
\]

where \( N \equiv \frac{u}{W} \) is the dimensionless coordinate measured normal to the surface being considered.

The boundary conditions for the vorticity are derived based on the boundary conditions of the stream function. Consider the walls AB and CD, from equation (3.57), \( \frac{\partial^2 \Psi}{\partial Y^2} = 0 \). Substitute into dimensionless stream equation (3.62), this implies that,

\[
\text{on } X = 0 \text{ and } 1, \quad 0 \leq Y \leq 1 : \quad \Omega = -\frac{\partial^2 \Psi}{\partial X^2}. \quad (3.66)
\]

Similarly, from equation (3.58), on walls BC and DA, \( \frac{\partial^2 \Psi}{\partial X^2} = 0 \), thus,

\[
\text{on } Y = 0 \text{ and } 1, \quad 0 \leq X \leq 1 : \quad \Omega = -\frac{\partial^2 \Psi}{\partial Y^2}. \quad (3.67)
\]

The dimensionless form of the temperature conditions (3.55) is,

\[
\text{on } X = 0, \quad 0 \leq Y \leq 1 : \quad \Theta = 1,
\]

\[
\text{on } X = 1, \quad 0 \leq Y \leq 1 : \quad \Theta = 0,
\]

\[
\text{on } Y = 0 \text{ and } 1, \quad 0 \leq X \leq 1 : \quad \frac{\partial \Theta}{\partial Y} = 0, \quad (3.68)
\]

for the case with constant left wall temperature \( T = T_h \).

The overall heat transfer across the enclosure is a very important parameter in engineering applications. The local, \( Nu_{loc} \) and average Nusselt number, \( \overline{Nu} \) along the side walls are calculated using,

\[
Nu_{loc} = -\frac{\partial \Theta}{\partial X}, \quad (3.69)
\]

\[
\overline{Nu} = \int_0^1 Nu_{loc} dY. \quad (3.70)
\]
CHAPTER 4

METHOD OF SOLUTION

4.1 INTRODUCTION

In this chapter, the numerical methods used to solve the problems considered will be discussed. The finite difference method is used to discretise the dimensionless governing equations (3.62) – (3.64) and boundary conditions (3.65) – (3.68). Later, the iterative algorithm will be explained step by step and finally validation with previous researches will be given.

4.2 FINITE DIFFERENCE APPROXIMATIONS

In numerical analysis, the derivatives of the dependent variables in partial differential equations are required to express in approximate expressions to obtain a solution. Finite difference methods are one of the numerical methods for approximating the solutions of differential equations using the finite difference equations to approximate derivatives.

In general, a finite difference approximation approximates the value of some derivatives of a function, \( f(x) \) at point \( x \) in its domain relies on a suitable combination of sampled function values of nearby points. Given a function \( f(x) \), which is analytical, \( f(x + \Delta x) \) can be expanded in a Taylor series about \( x \) as,

\[
f(x + \Delta x) = f(x) + (\Delta x) \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots \tag{4.1}
\]

Solving for \( \frac{\partial f}{\partial x} \) and obtain,

\[
\frac{\partial f}{\partial x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} + O(\Delta x), \tag{4.2}
\]

where \( O(\Delta x) \) is the terms of order \( \Delta x \). This is the forward difference approximation of \( \frac{\partial f}{\partial x} \).
of order $\Delta x$. Now, expanding $f(x + 2\Delta x)$ about $x$,
\[
f(x + 2\Delta x) = f(x) + (2\Delta x) \frac{\partial f}{\partial x} + \frac{(2\Delta x)^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{(2\Delta x)^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots \quad (4.3)
\]
Multiplying equation (4.1) by 4, and subtract equation (4.3) gives,
\[
4f(x + \Delta x) - f(x + 2\Delta x) = 3f(x) + (2\Delta x) \frac{\partial f}{\partial x} + (-4)(\Delta x)^3 \frac{\partial^3 f}{\partial x^3} + \ldots
\]
Again, solve for $\frac{\partial f}{\partial x}$ and obtain,
\[
\frac{\partial f}{\partial x} = \frac{-f(x + 2\Delta x) + 4f(x + \Delta x) - 3f(x)}{2\Delta x} + O(\Delta x)^2. \quad (4.4)
\]
This is called the forward difference approximation for $\frac{\partial f}{\partial x}$ of order $(\Delta x)^2$. Second-order accurate finite difference approximations produce higher accuracy of derivatives, by adding more terms from the Taylor series expansions.

Now, consider the Taylor series expansion of $f(x - \Delta x)$ about $x$,
\[
f(x - \Delta x) = f(x) + (-\Delta x) \frac{\partial f}{\partial x} + \frac{(-\Delta x)^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{(-\Delta x)^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots \quad (4.5)
\]
Solving for $\frac{\partial f}{\partial x}$,
\[
\frac{\partial f}{\partial x} = \frac{f(x) - f(x - \Delta x)}{\Delta x} + O(\Delta x), \quad (4.6)
\]
and the backward difference approximation of $\frac{\partial f}{\partial x}$ is found. Also, consider the Taylor series expansion of $f(x - 2\Delta)$ about $x$,
\[
f(x - 2\Delta) = f(x) + (-2\Delta x) \frac{\partial f}{\partial x} + \frac{(-2\Delta x)^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{(-2\Delta x)^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots \quad (4.7)
\]
Using the similar technique, i.e. multiply equation (4.5) by 4, and then subtract equation (4.7),
\[
4f(x - \Delta x) - f(x - 2\Delta x) = 3f(x) + (-2\Delta x) \frac{\partial f}{\partial x} + (-4)(\Delta x)^3 \frac{\partial^3 f}{\partial x^3} + \ldots
\]
and solve for $\frac{\partial f}{\partial x}$ give the second-order backward approximation of the first derivative of
that is,
\[ \frac{\partial f}{\partial x} = -\frac{f(x - 2\Delta x) + 4f(x - \Delta x) - 3f(x)}{-2\Delta x} + O(\Delta x)^2. \]  
(4.8)

Now consider the Taylor series expansions (4.1) and (4.5). Subtracting equation (4.5) from equation (4.1) to obtain,

\[ f(x + \Delta x) - f(x - \Delta x) = 2\Delta x \frac{\partial f}{\partial x} + 2 \frac{(\Delta x)^3}{3!} \frac{\partial^3 f}{\partial x^3} + \ldots \]

Solving for \( \frac{\partial f}{\partial x} \),
\[ \frac{\partial f}{\partial x} = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + O(\Delta x)^2, \]  
(4.9)
give the central difference approximation of \( \frac{\partial f}{\partial x} \) of order \( (\Delta x)^2 \).

To approximate expressions for the higher order derivatives of the function \( f \), say \( \frac{\partial^2 f}{\partial x^2} \), consider the equations (4.1) and (4.5) again. Adding up equations (4.1) and (4.5),

\[ f(x + \Delta x) + f(x - \Delta x) = 2f(x) + 2 \frac{(\Delta x)^2}{2!} \frac{\partial^2 f}{\partial x^2} + 2 \frac{(\Delta x)^4}{4!} \frac{\partial^4 f}{\partial x^4} + \ldots \]

Solving for \( \frac{\partial^2 f}{\partial x^2} \), the central difference approximation for the second derivative of \( f \) can be written as,
\[ \frac{\partial^2 f}{\partial x^2} = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2} + O(\Delta x)^2. \]  
(4.10)

If the subscript index \( i \) is used to represent the discrete points in the \( x \)-direction, the finite difference approximation equations of \( f \) can be written as the following.

First order derivative with first and second order forward difference approximations are,
\[ \frac{\partial f}{\partial x} \approx \frac{f_{i+1} - f_i}{\Delta x}, \]  
\[ \frac{\partial f}{\partial x} \approx \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2\Delta x}, \]  
respectively.
While, the first order derivative with first and second order backward difference approximations are,

\[
\frac{\partial f}{\partial x} \approx \frac{f_i - f_{i-1}}{\Delta x}, \quad (4.13)
\]
\[
\frac{\partial f}{\partial x} \approx -\frac{f_{i-2} + 4f_{i-1} - 3f_i}{-2\Delta x}. \quad (4.14)
\]

Finally, first and second order derivatives with second order central difference approximation are,

\[
\frac{\partial f}{\partial x} \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}, \quad (4.15)
\]
\[
\frac{\partial^2 f}{\partial x^2} \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2}. \quad (4.16)
\]

The equations (4.11) – (4.16) are used to approximate the partial differential equations of the governing equations and boundary conditions considered in the study.
4.3 ITERATIVE METHODS

The finite difference method is used to discretize partial differential equations. Consider a two-dimensional elliptic equation,

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{b} S(x, y) = 0, \]  

(4.17)

where \( b \neq 0 \). Using central difference approach from equation (4.16),

\[ \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2} + \frac{1}{b} S_{i,j} = 0, \]  

(4.18)

and rearrange,

\[ f_{i,j} = \frac{1}{2(1 + R^2)} \left[ f_{i+1,j} + f_{i-1,j} + R^2(f_{i,j+1} + f_{i,j-1}) + (\Delta x)^2 \frac{1}{b} S_{i,j} \right], \]  

(4.19)

where \( R = \frac{\Delta x}{\Delta y} \).

![Grid points employed in the Gauss-Seidel iteration method](image)

**Figure 4.2:** Grid points employed in the Gauss-Seidel iteration method

To solve equation (4.19) within a chosen domain with large amount of nodal points, it is more appropriate to use iterative methods than that of direct methods such as Cramer’s
rule and Gaussian elimination. In order to solve for the value of \( f \) at grid point \( i, j \), the values of \( f \) on the right-hand side of equation (4.19) must be provided. If the current values of \( f \) are used to compute the neighbouring points as soon as they are available, \( f^{(k+1)}_{i,j} = \frac{1}{2(1+R^2)} \left[ f^{(k)}_{i+1,j} + f^{(k+1)}_{i-1,j} + R^2 \left( f^{(k)}_{i,j+1} + f^{(k+1)}_{i,j-1} \right) + (\Delta x)^2 \frac{b}{1} S^{(k)}_{i,j} \right], \) (4.20)

this gives the Gauss-Seidel iteration method as shown in Figure (4.2). Here, the computation is assumed to start from left to right and bottom to top, and superscript \((k)\) is the iteration number. Gauss-Seidel iteration method increases the convergence rate over the Jacobi method which uses initial guessed values of neighbouring points or previous computed values.

Perform \( f^{(k)}_{i,j} - f^{(k)}_{i,j} \) to the right-hand side of the equation (4.20), and rearrange,

\[
\begin{align*}
f^{(k+1)}_{i,j} &= f^{(k)}_{i,j} + \frac{1}{2(1+R^2)} \left[ f^{(k)}_{i+1,j} + f^{(k+1)}_{i-1,j} + R^2 \left( f^{(k)}_{i,j+1} + f^{(k+1)}_{i,j-1} \right) - 2 \left( 1 + R^2 \right) f^{(k)}_{i,j} \right. \\
&\hspace{1cm} \left. + (\Delta x)^2 \frac{1}{b} S^{(k)}_{i,j} \right].
\end{align*}
\]

As the computation proceeds, \( f^{(k)}_{i,j} \) will approach to \( f^{(k+1)}_{i,j} \). To accelerate the convergence, the bracket terms on the right-hand side is multiplied by the relaxation parameter, \( \lambda \) and obtain,

\[
\begin{align*}
f^{(k+1)}_{i,j} &= (1 - \lambda) f^{(k)}_{i,j} + \frac{\lambda}{2(1+R^2)} \left[ f^{(k)}_{i+1,j} + f^{(k+1)}_{i-1,j} + R^2 \left( f^{(k)}_{i,j+1} + f^{(k+1)}_{i,j-1} \right) \right. \\
&\hspace{1cm} \left. + (\Delta x)^2 \frac{1}{b} S^{(k)}_{i,j} \right]. \quad (4.21)
\end{align*}
\]

The values of the relaxation parameter \( \lambda \) must be in the range of \( 0 < \lambda < 2 \) for convergence. If \( 0 < \lambda < 1 \), it is called the successive under-relaxation (SUR) method and the range \( 1 < \lambda < 2 \) gives the successive over-relaxation (SOR) method. For \( \lambda = 1 \), the equation (4.21) reduces to the Gauss-Seidel iteration method.
4.4 DISCRETIZATION OF GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The dimensionless governing equations (3.62) – (3.64) and boundary conditions (3.65) – (3.68), are discretized using the finite difference formulations and then solve using the iterative methods discussed in the previous section.

The dimensionless stream equation (3.62) is discretized as,

$$
\Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j} + \frac{\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}}{(\Delta Y)^2} + \Omega_{i,j} = 0,
$$

and iteratively can be express as,

$$
\Psi_{i,j}^{(k+1)} = (1 - \lambda_\Psi) \Psi_{i,j}^{(k)}
+ \frac{\lambda_\Psi}{2(1 + R^2)} \left[ \Psi_{i+1,j}^{(k)} + \Psi_{i-1,j}^{(k)} + R^2 \left( \Psi_{i,j+1}^{(k)} + \Psi_{i,j-1}^{(k)} \right) + (\Delta X)^2 \Omega_{i,j}^{(k)} \right].
$$

Similarly, the finite difference approximations for the dimensionless vorticity equation (3.63) is,

$$
\frac{\Omega_{i+1,j} - 2\Omega_{i,j} + \Omega_{i-1,j}}{(\Delta X)^2} + \frac{\Omega_{i,j+1} - 2\Omega_{i,j} + \Omega_{i,j-1}}{(\Delta Y)^2} - (S_\Omega)_{i,j} = 0,
$$

where,

$$
(S_\Omega)_{i,j} = \frac{1}{Pr} \left[ \left( \frac{\Psi_{i,j+1} - \Psi_{i,j-1}}{2\Delta Y} \right) \left( \frac{\Omega_{i+1,j} - \Omega_{i-1,j}}{2\Delta X} \right) - \left( \frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{2\Delta X} \right) \left( \frac{\Omega_{i,j+1} - \Omega_{i,j-1}}{2\Delta Y} \right) \right]
- Ra \left[ \left( \frac{\Theta_{i,j+1} - \Theta_{i,j-1}}{2\Delta X} \cos \phi \right) - \left( \frac{\Theta_{i,j+1} - \Theta_{i,j-1}}{2\Delta Y} \right) \sin \phi \right].
$$

The iterative expression for the vorticity equation is,

$$
\Omega_{i,j}^{(k+1)} = (1 - \lambda_\Omega) \Omega_{i,j}^{(k)}
+ \frac{\lambda_\Omega}{2(1 + R^2)} \left[ \Omega_{i+1,j}^{(k)} + \Omega_{i-1,j}^{(k)} + R^2 \left( \Omega_{i,j+1}^{(k)} + \Omega_{i,j-1}^{(k)} \right) - (\Delta X)^2 (S_\Omega)_{i,j}^{(k)} \right].
$$
Also, for dimensionless energy equation (3.64),

\[
\frac{\Theta_{i+1,j} - 2\Theta_{i,j} + \Theta_{i-1,j}}{(\Delta X)^2} + \frac{\Theta_{i,j+1} - 2\Theta_{i,j} + \Theta_{i,j-1}}{(\Delta Y)^2} - (S_{\Theta})_{i,j} = 0,
\]

where,

\[
(S_{\Theta})_{i,j} = \left(\frac{\Psi_{i,j+1} - \Psi_{i,j-1}}{2\Delta Y}\right) \left(\frac{\Theta_{i+1,j} - \Theta_{i-1,j}}{2\Delta X}\right) - \left(\frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{2\Delta X}\right) \left(\frac{\Theta_{i,j+1} - \Theta_{i,j-1}}{2\Delta Y}\right).
\]

In the iterative expression is,

\[
\Theta_{i,j}^{(k+1)} = (1 - \lambda_{\Theta}) \Theta_{i,j}^{(k)} + \frac{\lambda_{\Theta}}{2(1 + R^2)} \left[\Theta_{i+1,j}^{(k)} + \Theta_{i-1,j}^{(k+1)} + R^2 \left(\Theta_{i,j+1}^{(k)} + \Theta_{i,j-1}^{(k+1)}\right) - (S_{\Theta})_{i,j}^{(k)}\right].
\] (4.24)

The equations (4.22) – (4.24) are used to solve internal nodal points, that is for points \(i = 2, \ldots, nX\) and \(j = 2, \ldots, nY\), where \(nX\) and \(nY\) are the number of grids in the \(X\)- and \(Y\)-directions respectively. The values of the boundaries are given by the boundary conditions of each variables considered. The boundary conditions of the stream function (3.65) give,

\[
\Psi_{1,j} = 0, \quad \Psi_{nX+1,j} = 0, \quad \text{for } j = 1, 2, 3, \ldots, nY + 1,
\]

\[
\Psi_{i,1} = 0, \quad \Psi_{i,nY+1} = 0, \quad \text{for } i = 1, 2, 3, \ldots, nX + 1.
\] (4.25)

The boundary conditions of vorticity (3.66) give that, say along the left wall,

\[
\Omega_{1,j} = -\frac{\partial^2 \Psi}{\partial X^2} \bigg|_{1,j}.
\] (4.26)

Consider the Taylor series expansions,

\[
\Psi_{2,j} = \Psi_{1,j} + (\Delta X) \frac{\partial \Psi}{\partial X} \bigg|_{1,j} + \frac{(\Delta X)^2}{2!} \frac{\partial^2 \Psi}{\partial X^2} \bigg|_{1,j} + \frac{(\Delta X)^3}{3!} \frac{\partial^3 \Psi}{\partial X^3} \bigg|_{1,j} + \ldots
\] (4.27)
and,

\[ \Psi_{3,j} = \Psi_{1,j} + (2\Delta X) \frac{\partial \Psi}{\partial X} \bigg|_{1,j} + \frac{(2\Delta X)^2 \partial^2 \Psi}{2!} \bigg|_{1,j} + \frac{(2\Delta X)^3 \partial^3 \Psi}{3!} \bigg|_{1,j} + \ldots \quad (4.28) \]

Eliminating \( \frac{\partial^3 \Psi}{\partial X^3} \) term by subtracting (4.28) from 8 times of equation (4.27) and get,

\[ 8\Psi_{2,j} - \Psi_{3,j} = 7\Psi_{1,j} + 6(\Delta X) \frac{\partial \Psi}{\partial X} \bigg|_{1,j} + 4 \frac{(\Delta X)^2 \partial^2 \Psi}{2!} \bigg|_{1,j} + \ldots \]

Since \( \Psi_{1,j} = 0 \) and \( \frac{\partial \Psi}{\partial X} \big|_{1,j} = 0 \), rearrange and substitute into equation (4.26) to obtain,

\[ \Omega_{1,j} = -\frac{8\Psi_{2,j} - \Psi_{3,j}}{2(\Delta X)^2}, \quad \text{for } j = 1, 2, 3, \ldots, nY + 1. \quad (4.29) \]

Now, the boundary conditions on the right wall is considered,

\[ \Omega_{nX+1,j} = - \frac{\partial^2 \Psi}{\partial X^2} \bigg|_{nX+1,j}. \quad (4.30) \]

With the Taylor expansion,

\[ \Psi_{nX,j} = \Psi_{nX+1,j} + (-\Delta X) \frac{\partial \Psi}{\partial X} \bigg|_{nX+1,j} + \frac{(-\Delta X)^2 \partial^2 \Psi}{2!} \bigg|_{nX+1,j} + \frac{(-\Delta X)^3 \partial^3 \Psi}{3!} \bigg|_{nX+1,j} + \ldots \quad (4.31) \]

and,

\[ \Psi_{nX-1,j} = \Psi_{nX+1,j} + (-2\Delta X) \frac{\partial \Psi}{\partial X} \bigg|_{nX+1,j} + \frac{(-2\Delta X)^2 \partial^2 \Psi}{2!} \bigg|_{nX+1,j} + \frac{(-2\Delta X)^3 \partial^3 \Psi}{3!} \bigg|_{nX+1,j} + \ldots \quad (4.32) \]

Multiplying 8 to the equation (4.31) and then subtract equation (4.31),

\[ 8\Psi_{nX,j} - \Psi_{nX-1,j} = 7\Psi_{nX+1,j} + 6(-\Delta X) \frac{\partial \Psi}{\partial X} \bigg|_{nX+1,j} + 4 \frac{(-\Delta X)^2 \partial^2 \Psi}{2!} \bigg|_{nX+1,j} + \ldots \]
After eliminating terms with $\Psi_{nX+1,j} = 0$ and $\frac{\partial \Psi}{\partial X} \big|_{nX+1,j} = 0$, and substitute into equation (4.30) to obtain,

$$
\Omega_{nX+1,j} = -\frac{8\Psi_{nX,j} - \Psi_{nX-1,j}}{2(\Delta X)^2}, \quad \text{for } j = 1, 2, 3, \ldots, nY + 1. \quad (4.33)
$$

Using the same procedure, the boundary conditions for dimensionless vorticity (3.67) along the bottom and top walls are,

$$
\Omega_{i,1} = -\frac{8\Psi_{i,2} - \Psi_{i,3}}{2(\Delta Y)^2}, \quad \text{for } i = 1, 2, 3, \ldots, nX + 1, \quad (4.34)
$$

$$
\Omega_{i,nY+1} = -\frac{8\Psi_{i,nY} - \Psi_{i,nY-1}}{2(\Delta Y)^2}, \quad \text{for } i = 1, 2, 3, \ldots, nX + 1. \quad (4.35)
$$

The boundary conditions for the dimensionless temperature (3.68) are,

$$
\Theta_{1,j} = 1, \quad \text{for } j = 1, 2, 3, \ldots, nY + 1,
$$

$$
\Theta_{nX+1,j} = 0, \quad \text{for } j = 1, 2, 3, \ldots, nY + 1. \quad (4.36)
$$

Using second order forward and backward approximations, i.e. equations (4.12) and (4.14), the temperature boundary conditions along the top and bottom walls are,

$$
\Theta_{i,1} = \frac{1}{3} (4\Theta_{i,2} - \Theta_{i,3}), \quad \text{for } i = 1, 2, 3, \ldots, nX + 1,
$$

$$
\Theta_{i,nY+1} = \frac{1}{3} (4\Theta_{i,nY} - \Theta_{i,nY-1}), \quad \text{for } i = 1, 2, 3, \ldots, nX + 1. \quad (4.37)
$$

### 4.5 NUMERICAL PROCEDURE

The set of the finite-difference equations and the associated boundary conditions are solved iteratively with the initial guessed values of the three variables, which are stream function, vorticity and temperature at all points of fluid in the domain considered. The algorithm of the numerical solution procedure is given as follow:

1. **State the initial values.**

   The initial values of the stream function $\Psi$, vorticity $\Omega$ and temperature $\Theta$ are described by the conditions of fluid at the initial state. Initially, the fluid is at rest...
and only pure conduction across the fluid layer where the temperature is varying linearly with $X$ in the enclosure. Therefore, on every points in the enclosure, for $i = 1, ..., nX + 1$ and $j = 1, ..., nY + 1$,

$$\Psi_{i,j}^{(0)} = 0, \quad \Omega_{i,j}^{(0)} = 0, \quad \Theta_{i,j}^{(0)} = 1 - X_{i,j}.$$ 

2. **Set temperature boundary conditions.**

The temperature difference at the side walls initiates the process of heat transfer across the enclosure. Therefore, temperature boundary conditions given by equation (4.36) are, for $j = 1$ to $nY + 1$,

$$\Theta_{1,j}^{(0)} = 1, \quad \Theta_{nX+1,j}^{(0)} = 0.$$ 

3. **Calculate internal values of stream function, $\Psi_{i,j}$.**

The source term, $S(\Psi)_{i,j}$ for all internal points are updated from the terms of previous iteration. Then, equation (4.22) is used to evaluate the values of $\Psi_{i,j}$ at the internal points of the enclosure, that is for $i = 2, ..., nX$ and $j = 2, ..., nY$. Successive over-relaxation (SOR) method is used, where $1 < \lambda_\Psi < 2$. The SOR method can accelerate the convergence of a linear partial differential equation, that is the stream function.

4. **Evaluate internal values of temperature, $\Theta_{i,j}$.**

The source term, $S(\Theta)_{i,j}$ for all internal points are updated using the terms of previous iteration and new computed values of $\Psi_{i,j}$. The new values of $\Theta_{i,j}$ for all internal points are evaluate using the equation (4.24). The relaxation parameter, $\lambda_\Theta$ is used in the range $0 < \lambda_\Theta < 1$, that is successive under-relaxation (SUR) method is used to solve the equation (4.24). The SUR method is used for non-linear partial differential equation in which the source term is product of the dependent variable considered.

5. **Update boundary values of temperature, $\Theta_{i,j}$.**

The fluid was assumed to be pure conduction at the beginning, but now fluid motion
has been triggered and heat should be transferred due to the flow. Therefore, the temperature conditions on the adiabatic top and bottom walls should be updated from the newly computed temperature values using equation (4.37).

6. **Update boundary values of vorticity transport, \( \Omega_{i,j} \).**

The values of the \( \Omega_{i,j} \) along all the boundaries are calculated using the equations (4.30) and (4.33) – (4.36) with the newly computed values of \( \Psi_{i,j} \).

7. **Approximate internal values of vorticity transport, \( \Omega_{i,j} \).**

The source term, \( S(\Omega)_{i,j} \) is updated from the values of previous iteration as well as newly computed values of \( \Psi_{i,j} \) and \( \Theta_{i,j} \). Then under-relaxation approach is used in the equation (4.23) to solve for \( \Omega_{i,j} \) at the internal points.

8. **Stopping criteria.**

Step 3 to 7 are repeated until convergence is obtain with the condition that,

\[
\sum_{i,j} \left( \xi_{i,j}^{(k+1)} - \xi_{i,j}^{(k)} \right) < \epsilon,
\]

where \( \xi \) is either \( \Psi \), \( \Omega \) or \( \Theta \). Here, \((k)\) represents the iteration number and \( \epsilon \) is the convergence criterium.

9. **Contour plot and heat transfer analysis.**

Once the converge solution is obtained, the data are saved and plotted to obtain contour plots of streamlines and isotherms. The local Nusselt number, \( Nu_{loc} \) which shows the heat transfer along the walls are calculated using finite difference approximations,

\[
Nu_{1,j} = -\frac{\Theta_{2,j} - \Theta_{1,j}}{\Delta X}.
\]

Numerical integration (i.e. trapezoidal rule) is used to approximate the average Nusselt number, \( Nu \) at the side walls, that is,

\[
Nu = \frac{\Delta Y}{2} \left( Nu_{1,1} + 2Nu_{1,2} + \ldots + 2Nu_{1,nY} + Nu_{1,nY+1} \right).
\]

This algorithm can be illustrated in the Figure 4.3.
Start

- Input parameter: $Pr$, $Ra$.
- Input geometrical information: $Ar$, $\varphi$.
- Input number of grids: $nX$, $nY$.

- Initial guess: $\Psi$, $\Theta$, $\Omega$.
- Set $\Theta$ boundary conditions.

- Calculate $S(\Psi)_{i,j}$.
- Solve $\Psi_{i,j}$.

- Calculate $S(\Theta)_{i,j}$.
- Solve $\Theta_{i,j}$.
- Correct $\Theta$ boundaries.

- Update $\Omega$ boundaries.
- Calculate $S(\Omega)_{i,j}$.
- Solve $\Omega_{i,j}$.

Convergence?

- Yes
  - Calculate and display $\overline{Nu}$.
  - Display contours of $\Psi$ and $\Theta$.

- No

Stop

Figure 4.3: Flow chart for numerical procedure
4.6 GRID TEST AND CODE VALIDATION

Several grid sensitivity tests were performed to ensure that the results are grid independence. Uniform grid in $X$- and $Y$-directions were used in all computations. A grid test was performed for $Ra = 10^5$ and $Pr = 0.71$ in the range of $21 \times 21$ to $151 \times 151$ grids. The results are shown in Figure 4.4. We found that the grid size of $111 \times 111$ is sufficient to perform as good as the finer mesh sizes and so we adapt to grid size $111 \times 111$ for further calculations for the case of natural convection in square enclosures heated with constant temperature.

The verification of the developed computer code is very important in the numerical study. Therefore, the results of present code are tested to verify the present numerical code and compared with the solutions available in the literature for square enclosure heated with constant temperature at the side walls. The comparison of some results for natural convection in a square enclosure is tabulated in Table 4.1. From Table 4.1, the results predicted by current computer code are agreed well with the previous studies.

<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ra$</td>
<td>$\overline{Nu}$</td>
<td>$</td>
<td>\Psi</td>
<td>_{max}$</td>
</tr>
<tr>
<td>$10^3$</td>
<td>1.118</td>
<td>-</td>
<td>-</td>
<td>1.177</td>
</tr>
<tr>
<td>$10^4$</td>
<td>2.243</td>
<td>-</td>
<td>-</td>
<td>5.076</td>
</tr>
</tbody>
</table>

The streamlines (contour of $\Psi$) and isotherms (contour of $\Theta$) for the validation are presented in the Figure 4.5. The negative sign of the stream function $\Psi$ indicates clockwise flow and positive sign shows counter-clockwise fluid flow. The contour of $\Theta$ shows the isoline of dimensionless temperature where red is for high temperature and blue indicates lower temperature.
Figure 4.4: Grid independency test at Ra = 10^5 with (a) $|\Psi|_{\text{max}}$ and (b) $\overline{Nu}$
Figure 4.5: Streamlines (left) and isotherms (right) for natural convection in square enclosures heated with constant temperature on the left wall at (a) $Ra = 10^3$ (b) $Ra = 10^4$ (c) $Ra = 10^5$ and (d) $Ra = 10^6$
CHAPTER 5

NATURAL CONVECTION IN SQUARE ENCLOSURE

5.1 INTRODUCTION

In this chapter, the effect of enclosure inclination on natural convection in a square enclosure with sinusoidal temperature conditions is considered. The problem formulation will be given followed by the discussion on the numerical results obtained. The fluid flow and temperature distribution as well as heat transfer will be presented graphically in terms of streamlines, isotherms and Nusselt numbers.

5.2 PROBLEM FORMULATION

Consider an air-filled two dimensional square enclosure of width and height, $W$ as shown in Figure 5.1. The top and bottom walls of the enclosure are insulated. The sinusoidal temperature profile is applied on the vertical left wall while the right wall is cooled.
with constant temperature $T_c$. The inclination angle of the enclosure, $\varphi$ is the angle between the bottom wall and the horizontal plane. The gravity acts in the vertically downward direction. The velocity components, $u$ and $v$ are taken in $x$- and $y$-directions, respectively. The fluid in the enclosure is incompressible and Newtonian. The fluid properties are constant, the density variation are neglected except in buoyancy term (by Boussinesq approximation). Furthermore, the flow is assumed to be steady and the viscous dissipation is negligible. By the law of conservation for mass, momentum and energy, the governing equations for natural convection are;

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (5.1)
$$

$$
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \beta g (T - T_c) \sin \varphi, \quad (5.2)
$$

$$
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \beta g (T - T_c) \cos \varphi, \quad (5.3)
$$

$$
u \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right). \quad (5.4)
$$

The boundary conditions are;

on all solid boundaries : $u = v = 0$,

on $x = 0$, $0 \leq y \leq W$ : $T = T_c + (T_{ref} - T_c) \cos \left( \frac{2\pi y}{W} \right)$,

on $x = W$, $0 \leq y \leq W$ : $T = T_c$,

on $y = 0$ and $W$, $0 \leq x \leq W$ : \( \frac{\partial T}{\partial y} = 0 \). \hspace{1cm} (5.5)

The stream function and vorticity are defined as;

$$
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}; \quad (5.6)
$$

$$
u = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad (5.7)
$$

and the following dimensionless variables are introduced;

$$
X = \frac{x}{W}, \quad Y = \frac{y}{W}, \quad \Psi = \frac{\psi Pr}{\nu}, \quad \Omega = \frac{\omega W^2 Pr}{\nu}, \quad \Theta = \frac{T - T_c}{T_{ref} - T_c}. \quad (5.8)
$$
Using equations (5.6) – (5.8), the governing equations (5.1) – (5.4) and boundary conditions (5.5) in terms of dimensionless form are:

\[
\begin{align*}
\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} &= -\Omega, \quad (5.9) \\
\frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2} &= \frac{1}{Pr} \left( \frac{\partial \Psi \partial \Omega}{\partial Y \partial X} - \frac{\partial \Psi \partial \Omega}{\partial X \partial Y} \right) - Ra \left( \frac{\partial \Theta}{\partial X} \cos \varphi - \frac{\partial \Theta}{\partial Y} \sin \varphi \right), \quad (5.10) \\
\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} &= \frac{\partial \Psi}{\partial Y} \frac{\partial \Theta}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Theta}{\partial Y}. \quad (5.11)
\end{align*}
\]

with boundary conditions:

\[
\begin{align*}
\text{on all solid walls} : & \quad \Psi = 0, \\
\text{on } X = 0, \quad 0 \leq Y \leq 1 : & \quad \Theta = \cos(2\pi Y), \quad \Omega = -\frac{\partial^2 \Psi}{\partial X^2}, \\
\text{on } X = 1, \quad 0 \leq Y \leq 1 : & \quad \Theta = 0, \quad \Omega = -\frac{\partial^2 \Psi}{\partial X^2}, \\
\text{on } Y = 0 \text{ and } 1, \quad 0 \leq X \leq 1 : & \quad \frac{\partial \Theta}{\partial Y} = 0, \quad \Omega = -\frac{\partial^2 \Psi}{\partial Y^2}. \quad (5.12)
\end{align*}
\]

5.3 RESULTS AND DISCUSSION

A numerical study is conducted to investigate the effects of non-isothermal boundary conditions on square enclosures. In this study, air (i.e. \( Pr = 0.71 \)) is taken as working medium in all computations. All the results will be demonstrate in terms of streamlines, isotherms and average Nusselt number. Various Rayleigh number \( Ra \) will be considered in the range \( 10^3 \) to \( 10^6 \) and inclination angles in the range \( 0^\circ \) to \( 90^\circ \).

Figures 5.2 and 5.3 show the streamlines and isotherms for square enclosures with inclination angles \( 0^\circ, 45^\circ \) and \( 90^\circ \). Generally, the fluid circulation is strongly depending on the Rayleigh number and inclination angles as shown in Figure 5.2. For \( \varphi = 0^\circ \), as the Rayleigh number is varying from \( 10^3 \) to \( 10^6 \), the main inner cell is shifting towards the right wall. It means that the flow penetrates into the enclosure more and more as the Rayleigh number increases. For \( Ra = 10^3 \), the flow is seen to be very weak as observed from stream function and temperature contours. The temperature distribution is similar to that with stationary fluid and the heat transfer is purely conduction. One main cell and four small cells are formed for the flow field in the enclosure. At the top and bottom portions
Figure 5.2: Streamlines for different values of enclosure inclination at (a) $Ra = 10^3$ (b) $Ra = 10^4$ (c) $Ra = 10^5$ and (d) $Ra = 10^6$
Figure 5.3: Isotherms for different values of enclosure inclination at (a) $Ra = 10^3$ (b) $Ra = 10^4$ (c) $Ra = 10^5$ and (d) $Ra = 10^6$
of the side wall of the enclosure, the temperature of the fluid will be high, produces a reduction of fluid density, the buoyancy forces tend to cause fluid motion, forming clockwise circulation at the corners. Lower temperature is gradually applied to the center of the left side wall causing cold fluid to move down and meets the hot fluid at the lower surface, then the fluid starts to move up when reaches the right side wall, forming counter-clockwise flow.

The isotherm shows that conduction is the dominant heat transfer mechanism as depicted in Figure 5.3 for $Ra = 10^3$. The same phenomenon for $Ra = 10^4$ but the top left cell of the enclosure is larger compared to $Ra = 10^3$. While for $Ra = 10^5$, three cells are formed for the streamlines. The upper cell is larger than the small cell at the bottom. It is because cold fluid is moving down from the center suppresses the penetration of the hot fluid at the bottom of the enclosure. The isotherms are concentrated near the left wall, indicating that heat transfer mode is starting to be convection dominant and the temperature gradients are confined to the active side wall in the form of thermal boundary layer. It becomes thinner with increasing of the Rayleigh number and indicating higher average heat transfer. The same phenomenon apply for $Ra = 10^6$. The center of the upper cell is moving to the right of the enclosure implies that the flow penetration of the hot fluid at the top of the enclosure is higher compared to the fluid at the bottom. For $\phi = 45^\circ$, two cells are formed in the enclosure for all Rayleigh numbers considered. When the Rayleigh number is varying from $10^3$ to $10^6$, the upper cell is getting larger, this shows that the penetration of the hot fluid is getting deeper into the enclosure as it can be seen from the isotherms. There are two main cells of opposing flow directions formed vertically symmetry for $Ra = 10^3$ to $10^5$ with $\phi = 90^\circ$. The enclosure actually is heated from bottom with this orientation but $Ra = 10^6$ show a different flow pattern among the other. This is due to the cooling at the top wall and high concentration of cold fluid at the bottom of the enclosure, causing non-symmetrical circulation of fluid at the steady state.

The variation of local Nusselt number along the heated wall at different inclination angles for various Rayleigh number are shown in Figure 5.4. For $Ra = 10^3$, the local heat transfer, $Nu_{loc}$ is almost the same for the inclination angles plotted, indicates that effect of inclination angles is indifferent to local heat transfer. For $Ra = 10^4$ to $10^6$, the inclination
Figure 5.4: Variation of the local Nusselt number for different values of enclosure inclination at (a) $Ra = 10^3$ (b) $Ra = 10^4$ (c) $Ra = 10^5$ and (d) $Ra = 10^6$

Figure 5.5: Variation of the average Nusselt number for different values of Rayleigh number and inclination angle
angles do affect the local heat transfer as seen from Figure 5.4. The shape of local Nusselt number curves clearly shows the effect of sinusoidal temperature distribution on local heat transfer directly.

Figure 5.5(a) shows the variation of mean Nusselt numbers with the Rayleigh numbers for different inclination angles. It is found that the average heat transfer along the hot wall is increasing with increasing of the Rayleigh number. There is no significant change in the average Nusselt number for the inclination angles 45° and 90°. However, much difference in average Nusselt number is found between the vertical cavity (0°) and inclined cavity (45° and 90°). Figure 5.3 shows that the isotherms for enclosure inclinations of 45° and 90° are clustered near the thermal active side wall, and the thermal boundary layer thicknesses are almost the same, so the heat transfer rates are similar for 45° and 90° of enclosure inclinations. It is also observed from Figure 5.5(a) that this difference is increased on increasing of the Rayleigh number. The variation of mean Nusselt numbers with inclination angle is shown in Figure 5.5(b). The average heat transfer is increasing for $Ra = 10^3$ and $10^4$, but $Ra = 10^5$ and $10^6$ show a different trend. It increases first then decreases with the increasing inclination of the enclosure. These show that the heat transfer varies non-linearly with the increasing of the enclosure inclination for $Ra = 10^5$ and $10^6$. The enclosure of 0° inclination gives lowest heat transfer rate whereas 90° inclination provides highest heat transfer rate for $Ra = 10^3$ and $10^4$. For $Ra = 10^5$ and $10^6$, the lowest and highest heat transfer can be found at 0° and 60° respectively.

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CHAPTER 6

NATURAL CONVECTION IN RECTANGULAR ENCLOSURES

6.1 INTRODUCTION

In this chapter, the effect of aspect ratio on natural convection in an inclined rectangular with sinusoidal temperature profile will be considered. The sections will begin with the mathematical formulation of the problem followed by the results and discussion.

6.2 MATHEMATICAL FORMULATION

Consider an air-filled two dimensional rectangular enclosure of height, \( H \) and width, \( W \) as shown in Figure 6.1. The top and bottom walls of the enclosure are insulated. Sinusoidal temperature profile is applied on the vertical left wall while the right wall is cooled with constant temperature \( T_c \). The inclination angle of the enclosure \( \phi \) is the angle between the bottom wall and the horizontal plane. The gravity acts in the vertically downward direction. The velocity components, \( u \) and \( v \) are taken in \( x \) and \( y \) directions, respec-

\[
T(y) = T_c + (T_{ref} - T_c) \cos \left( 2\pi \frac{y}{H} \right)
\]

Figure 6.1: Schematic diagram
tively. The fluid in the enclosure is incompressible and Newtonian. The fluid properties are constant, the density variation are neglected except in buoyancy term (by Boussinesq approximation). Furthermore, we assume that the flow is steady and the viscous dissipation is negligible. By the law of conservation for mass, momentum and energy, the governing equations are;

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (6.1)
\]

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \beta g (T - T_c) \sin \phi, \quad (6.2)
\]

\[
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \beta g (T - T_c) \cos \phi, \quad (6.3)
\]

\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right). \quad (6.4)
\]

The boundary conditions are;

on all solid boundaries : \( u = v = 0, \)

on \( x = 0, \ 0 \leq y \leq H : \ T = T_c + (T_{\text{ref}} - T_c) \cos \left( 2\pi \frac{y}{H} \right), \)

on \( x = W, \ 0 \leq y \leq H : \ T = T_c, \)

on \( y = 0 \) and \( H, \ 0 \leq x \leq W : \ \frac{\partial T}{\partial y} = 0. \quad (6.5)\)

The governing equations are expressed in terms of stream function and vorticity which are defined as;

\[
u = -\frac{\partial \psi}{\partial x}, \quad \psi = \frac{\partial \psi}{\partial y}, \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and} \quad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}, \quad (6.7)
\]

and then the following dimensionless variables are introduced;

\[
X = \frac{x}{W}, \quad Y = \frac{y}{H}, \quad \Psi = \frac{\psi Pr}{\nu}, \quad \Omega = \frac{\omega HW Pr}{\nu}, \quad \Theta = \frac{T - T_c}{T_{\text{ref}} - T_c}, \quad Ar = \frac{H}{W}. \quad (6.8)
\]
Using equations (6.6) – (6.8), the governing equations (6.1) – (6.4) and boundary conditions (6.5) are written in dimensionless form as follows;

\[ \frac{\partial^2 \Psi}{\partial X^2} + \frac{1}{Ar^2} \frac{\partial^2 \Psi}{\partial Y^2} = -\frac{1}{Ar} \Omega, \]  
\[ \frac{\partial^2 \Omega}{\partial X^2} + \frac{1}{Ar^2} \frac{\partial^2 \Omega}{\partial Y^2} = \frac{1}{Ar} \left[ \frac{1}{Pr} \left( \frac{\partial \Psi}{\partial Y} \frac{\partial \Omega}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Omega}{\partial Y} \right) - Ra \left( \frac{1}{Ar} \frac{\partial \Theta}{\partial X} \cos \phi - \frac{1}{Ar^2} \frac{\partial \Theta}{\partial Y} \sin \phi \right) \right], \]
\[ \frac{\partial^2 \Theta}{\partial X^2} + \frac{1}{Ar^2} \frac{\partial^2 \Theta}{\partial Y^2} = \frac{1}{Ar} \left( \frac{\partial \Psi}{\partial Y} \frac{\partial \Theta}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Theta}{\partial Y} \right), \]

with boundary conditions:

on all solid walls : \( \Psi = 0, \)

on \( X = 0, \ 0 \leq Y \leq 1 : \) \( \Theta = \cos(2\pi Y), \ \Omega = -Ar \frac{\partial^2 \Psi}{\partial X^2}, \)

on \( X = 1, \ 0 \leq Y \leq 1 : \) \( \Theta = 0, \ \Omega = -Ar \frac{\partial^2 \Psi}{\partial X^2}, \)

on \( Y = 0 \text{ and } 1, \ 0 \leq X \leq 1 : \) \( \frac{\partial \Theta}{\partial Y} = 0, \ \Omega = -\frac{1}{Ar} \frac{\partial^2 \Psi}{\partial Y^2}. \)

Note that when \( Ar = 1, \) this problem reduces to the problem discussed in Chapter 5.

6.3 RESULTS AND DISCUSSION

A numerical study is conducted to investigate the effect of aspect ratio and inclination angles on non-isothermal rectangular enclosure. In this study, we use \( Pr = 0.71 \) for all computations. We demonstrate the results in terms of streamlines, isotherms, local and average Nusselt numbers. The aspect ratio, \( Ar \) considered are 0.25, 0.5, 1, 2, 5 and 10. The Rayleigh number, \( Ra \) varies in the range \( 10^3 \) to \( 10^6 \) and inclination angle \( \phi \) from 0° to 90°.

Figures 6.2 and 6.3 present the streamlines and isotherms, respectively for \( Ra = 10^3 \) with \( \phi = 0^\circ, \ 45^\circ \) and \( 90^\circ \). For \( \phi = 0^\circ \), the strength of the flow circulation, \( |\Psi|_{max} \) is increased slightly first up to \( Ar = 0.5 \) and then decreases on increasing of the aspect ratio. But for a given aspect ratio, for example \( Ar = 1, \ |\Psi|_{max} \) decreases on increasing of the enclosure inclination from 0° to 90°. Thus, \( |\Psi|_{max} \) is decreasing with increasing
Figure 6.2: Streamlines for different values of enclosure inclination at $Ra = 10^3$ with (a) $Ar = 0.25$ (b) $Ar = 0.5$ (c) $Ar = 1$ (d) $Ar = 2$ (e) $Ar = 5$ and (f) $Ar = 10$. 
Figure 6.3: Isotherms for different values of enclosure inclination at $Ra = 10^3$ with (a) $Ar = 0.25$ (b) $Ar = 0.5$ (c) $Ar = 1$ (d) $Ar = 2$ (e) $Ar = 5$ and (f) $Ar = 10$
of the aspect ratio and enclosure inclination. It can be seen from the streamlines that one main cell occupies the majority of the enclosure and two weak cells are forming near the thermal active side wall for \( \varphi = 0^\circ \) with \( Ar < 2 \). When \( Ar \geq 2 \), three cells are forming in the enclosure for \( \varphi = 0^\circ \) and \( 45^\circ \). However, the streamlines show totally different flow structure for \( \varphi = 90^\circ \). That is, the streamlines show symmetrical flow pattern for \( Ar = 0.5 \) to \( 10 \) and nearly symmetric for \( Ar = 0.25 \). The isotherms reveal that the heat transfer mode for all aspect ratios considered is conduction. For small aspect ratio, the temperature fields are concentrated near the side wall, indicating that the heat transfer at the wall is higher for smaller aspect ratio.

The streamlines and isotherms for \( Ra = 10^6 \) with \( \varphi = 0^\circ, 45^\circ \) and \( 90^\circ \) are shown in Figures 6.4 and 6.5, respectively. The stream circulation \( |\Psi|_{\text{max}} \) decreases with increasing of the aspect ratio. There is no common trend in variation of \( |\Psi|_{\text{max}} \) on increasing the enclosure inclination for \( Ra = 10^6 \). It can be seen from Figure 6.4 that mainly three cells are forming in the enclosure for all aspect ratios with \( \varphi = 0^\circ \) and \( 45^\circ \). However, the flow pattern at \( 90^\circ \) shows two trends for low and high values of the aspect ratio. Two symmetrical cells about the vertical axis are formed for \( Ar \geq 2 \), but multiple cells are observed for \( Ar < 2 \). The streamlines also reveal that the strength of circulation in small aspect ratio is higher compared to larger aspect ratio as the main (inner) cell is more elongated for small aspect ratio. In Figure 6.5, the isotherms show that convection is dominant for all values of aspect ratio considered. The heat transfer at the thermal active side wall is higher for smaller aspect ratio as depicted in the isotherms because the temperature field is more concentrated near the wall. That is, thermal boundary layers are formed along the left wall for \( Ar < 2 \).

Figure 6.6 shows the local Nusselt number for various aspect ratio with \( Ra = 10^5 \), \( \varphi = 0^\circ, 45^\circ \) and \( 90^\circ \). For all aspect ratio considered, it can be seen that the shape of the local Nusselt number curves clearly shows the effect of the sinusoidal temperature distribution on the local heat transfer directly. The shape of the curves is quite similar for a given enclosure inclination with \( 0.25 \leq Ar \leq 5 \). However for \( Ar = 10 \), the shape of the curves for the inclination angle considered are quite similar, indicating that the inclination of the enclosure is insignificant on the local heat transfer along the wall of the enclosure.
Figure 6.4: Streamlines for different values of enclosure inclination at $Ra = 10^6$ with (a) $Ar = 0.25$ (b) $Ar = 0.5$ (c) $Ar = 1$ (d) $Ar = 2$ (e) $Ar = 5$ and (f) $Ar = 10$
Figure 6.5: Isotherms for different values of enclosure inclination at $Ra = 10^6$ with (a) $Ar = 0.25$ (b) $Ar = 0.5$ (c) $Ar = 1$ (d) $Ar = 2$ (e) $Ar = 5$ and (f) $Ar = 10$
Figure 6.6: Variation of the local Nusselt number for different values of enclosure inclination at $Ra = 10^5$ with (a) $Ar = 0.25$ (b) $Ar = 0.5$ (c) $Ar = 1$ (d) $Ar = 2$ (e) $Ar = 5$ and (f) $Ar = 10$
The range of the local heat transfer is decreasing with increasing of the aspect ratio. The heat transfer becomes poor in the high aspect ratio cases. It is because at high aspect ratio, the heat transfer mode is mainly of conduction due to the short distance between the heated and cooled side walls.

The variation of local Nusselt number for different aspect ratio for the case of $\varphi = 0^\circ, 45^\circ$ and $90^\circ$ with $Ra = 10^5$ is shown in Figure 6.7. For all enclosure inclination considered, the shape of the curves is similar of decreasing amplitude with increasing of the aspect ratio. For $\varphi = 0^\circ$ and $45^\circ$, the local heat transfer is higher at the lower part of the heated wall whereas for $\varphi = 90^\circ$, the heat transfer is symmetrical along the vertical mid-plane. It also can be noticed that for all inclination angles, the local heat transfer for $Ar = 5$ and $10$ are nearly constant by comparing with the other aspect ratio.

Figure 6.8 shows the average Nusselt number at the heated part of the thermal active side wall for various aspect ratios and enclosure inclinations. The points are representing the numerical data obtained from simulation while the lines are for the correlation equations in the figures. It is found that the average heat transfer increases on increasing of the Rayleigh number. The heat transfer increases first then decreases with increasing the inclination of the enclosure. By comparing the range of the average Nusselt number, it is getting larger with increasing of the Rayleigh number. So, the average Nusselt number is actually increasing with increasing of the Rayleigh number for all aspect ratios and enclosure inclinations considered. It is also observed that the average Nusselt number decreases with increasing of the aspect ratio for a given Rayleigh number and enclosure inclination. The average Nusselt number is almost constant on increasing the inclination angle for low values of the Rayleigh number. The average Nusselt number behaves non-linearly with enclosure inclination for high Rayleigh number and $Ar < 5$.

The correlation equations for the average Nusselt number are derived based on the relation of the average heat transfer at the heated part of the wall with enclosure inclination, $\varphi$, aspect ratio, $Ar$ and Rayleigh number, $Ra$. The equations are as follows,

1. For $Ra = 10^3$ and $Ra = 10^4$,

$$
Nu = \frac{1}{Ar} \left( 1.63 + 0.07Ra \right), \quad (6.13)
$$
Figure 6.7: Variation of the local Nusselt number for different values of aspect ratio at $Ra = 10^5$ with enclosure inclination (a) $\phi = 0^\circ$ (b) $\phi = 45^\circ$ and (c) $\phi = 90^\circ$
For $Ra = 10^5$,

$$ Nu = \frac{Ra}{Ar} \left( 0.15\varphi^6 - 0.76\varphi^5 + 1.60\varphi^4 - 1.64\varphi^3 ight. $$

$$ \left. + 0.76\varphi^2 - 0.03\varphi + 0.25 \right) \quad (6.14) $$

3. For $Ra = 10^6$,

$$ Nu = \frac{Ra}{Ar} \left( 0.13\varphi^6 - 0.47\varphi^5 + 0.24\varphi^4 + 0.76\varphi^3 ight. $$

$$ \left. - 1.05\varphi^2 + 0.43\varphi + 0.29 \right) \quad (6.15) $$

where $\tilde{Ra} = Ra^{-\frac{1}{2.10}} + 0.3$ and $\varphi$ is calculated in radians. The correlation equations are agreed well with the numerical data obtained as depicted in Figure 6.8.

**Figure 6.8:** Variation of the average Nusselt number for different values of aspect ratio and enclosure inclination at (a) $Ra = 10^3$ (b) $Ra = 10^4$ (c) $Ra = 10^5$ and (d) $Ra = 10^6$.
CHAPTER 7

SUMMARY AND CONCLUSIONS

7.1 SUMMARY

Buoyancy-driven flows are of great importance for a wide range of natural phenomena, geophysics, medical sciences and engineering applications. The behaviour of natural convection heat transfer in rectangular enclosures for the case of non-uniform thermal boundary conditions has been investigated. The major components of this study are to analyze the buoyancy effect and the size of the enclosures on buoyancy-driven flows in the case of non-uniform heated enclosures.

A two-dimensional model is considered in this study. The fluid flow is Newtonian, incompressible, laminar and Boussinesq approximation is valid for the density variation. Sinusoidal temperature variation is applied on the left wall of the enclosure while the right wall is kept at constant cold temperature, and the bottom and top walls are insulated. The enclosure inclination is taken as the angle between the bottom wall and the horizontal plane. Based on these assumptions and physical configurations, the governing equations, i.e. the continuity equation, momentum equations and energy equation, together with appropriate boundary conditions are derived to describe the model proposed in this study. Later, vorticity-stream function formulation is used to eliminate the pressure terms in the momentum equations. Suitable dimensionless variables are introduced to non-dimensionalize the governing equations and boundary conditions.

The governing equations are comprised of a system of partial differential equations. Finite difference approximations are used to discretize partial differential equations for approximating the solutions of dependent variables considered. Then, an iterative method is chosen and numerical algorithm is introduced to solve the equations iteratively. Numerical code is tested and verified with previous researches. A good agreement is found
when the results are compared with the literatures available.

The effect of enclosure inclination on natural convective flow and heat transfer in a square enclosure is studied in Chapter 5. The governing equations and boundary conditions are expressed in dimensionless form by introducing suitable dimensionless variables. Then, numerical data are obtained in the form of streamlines, isotherms as well as Nusselt numbers. The data are analyzed and plotted using suitable graphical representations.

Later, the effect of aspect ratio on natural convective fluid flow and heat transfer in a rectangular enclosure is investigated. Appropriate dimensionless variables are introduced to get the dimensionless governing equations and boundary conditions. The results obtained from the simulations are analyzed and presented in Chapter 6.

The manuscripts produced from the two problems are published in the ISI international journal and proceedings, see Appendix A.

7.2 CONCLUSIONS

The natural convective fluid flow and heat transfer in an inclined enclosure with sinusoidal temperature profile on the side wall has been studied numerically. The governing equations and boundary conditions are solved by using the finite difference method. From this study, the following conclusions are drawn:

1. Multiple flow pattern is observed for tall vertical enclosures.

2. For small Rayleigh number, the heat transfer is dominated by conduction across the fluid layers. Then, convection is dominated with increasing of the Rayleigh number for aspect ratio in the range of 0.25 to 5.

3. The heat transfer in tall enclosure (Ar = 10) is mostly by conduction for all the Rayleigh numbers and enclosure inclination considered.

4. Heat transfer is enhanced in the shallow enclosures than that of in the tall vertical enclosures for all the Rayleigh numbers considered.
5. The local Nusselt number affects much by the enclosure inclination for low aspect ratio \((Ar \leq 5)\). But, the effect of enclosure inclination on the variation of the local Nusselt number is not significant for higher aspect ratio \((Ar = 10)\).

6. The average Nusselt number increases with increasing the Rayleigh number, but it decreases with increasing of the aspect ratio.

7. The average Nusselt number increases first then decreases with increasing inclination of the enclosure for high Rayleigh number \((10^5 \leq Ra \leq 10^6)\).
REFERENCES


APPENDIX A

LIST OF PUBLICATIONS

Journal:


Proceeding: