ABSTRACT

This thesis is a study of certain Engel conditions. First, we will define the set of all the X-relative left Engel elements $L(G, X)$ and the set of all the bounded X-relative left Engel elements $\overline{L}(G, X)$, where X is a subset of G. When $X = G$, $L(G, X) = L(G)$ and $\overline{L}(G, X) = \overline{L}(G)$, where $L(G)$ is the set of all the usual left Engel elements and $\overline{L}(G)$ is the set of all the usual bounded left Engel elements. Next, we define the X-relative Hirsch-Plotkin radical $HP(G, X)$ and the X-relative Baer radical $B(G, X)$. When $X = G$, $HP(G, X) = HP(G)$ and $B(G, X) = B(G)$ where $HP(G)$ is the usual Hirsch-Plotkin radical and $B(G)$ is the usual Baer radical. We will show that if X is a normal solvable subgroup of G, then $B(G, X) = \overline{L}(G, X)$ and $HP(G, X) = L(G, X)$. This is an extension of the classical results $B(G) = \overline{L}(G)$ and $HP(G) = L(G)$ provided that G is solvable. Next, we show that if X is a normal subgroup of G and G satisfies the maximal condition, then $L(G, X) = HP(G, X) = B(G, X) = \overline{L}(G, X)$, which is also an extension of the classical result $L(G) = HP(G) = B(G) = \overline{L}(G)$. We also proved similar results when X is a subgroup of certain linear groups.

Let G be a group and $h, g \in G$. The 2-tuple (h, g) is said to be an n-Engel pair, $n \geq 2$, if $h = [h,_n g], g = [g,_n h]$ and $h \neq 1$. We will show that the subgroup generated by the 5-Engel pair (x, y) satisfying $yxy = xyx$ and $x^5 = 1$ is the alternating group A_5. Next, we show that if (x, y) is an n-Engel pair, $xyx^{-2}yx = yxy$ and $yxy^{-2}xy = xyx$, then $n = 2k$ where $k = 4$ or $k \geq 6$. Furthermore, the subgroup generated by $\{x, y\}$ is determined for $k = 4, 6, 7$ and 8.

Finally, we prove the existence of Engel pairs in certain special linear groups of order 2. In particular, we show that if $SL(2, F)$ is the special linear group of order 2 over the field F, then given any field L, there is a field extension F' of L with $[F' : L] \leq 6$ such that $SL(2, F')$ has an n-Engel pair for some integer $n \geq 4$. We will also show that $SL(2, F)$ has a 5-Engel pair if F is a field of characteristic $p \equiv \pm 1 \mod 5$.

ABSTRAK

Tesis ini merupakan suatu kajian bagi kondisi Engel. Yang pertama, kami menakrifkan set unsur Engel kiri relatif-X, $L(G,X)$, dan set unsur Engel kiri relatif-X terkurung $\overline{L}(G,X)$, di mana X ialah subset bagi G. Apabila $X = G$, $L(G,X) = L(G)$ dan $\overline{L}(G,X) = \overline{L}(G)$, di mana $L(G)$ ialah set bagi semua unsur Engel kiri dan $\overline{L}(G)$ ialah set bagi semua unsur Engel kiri terkurung. Kemudian, kami menakrifkan Radikal Hirsch-Plotkin relatif-X $HP(G,X)$ dan radikal Baer relatif-X, $B(G,X)$. Apabila $X = G$, $HP(G,X) = HP(G)$ dan $B(G,X) = B(G)$ di mana $HP(G)$ merupakan radikal Hirsch-Plotkin dan $B(G)$ ialah radikal Baer. Kami akan buktikan bahawa jikalau X subkumpulan normal boleh selesai bagi G, maka $B(G,X) = \overline{L}(G,X)$ dan $HP(G,X) = L(G,X)$. Ini ialah suatu generalisasi bagi $B(G) = \overline{L}(G)$ dan $HP(G) = L(G)$ di mana G boleh selesai. Kemudian, kami buktikan jikalau X ialah subkumpulan normal bagi G dan G memuaskan kondisi maksimal, maka $L(G,X) = HP(G,X) = B(G,X) = \overline{L}(G,X)$, juga generalisasi bagi $L(G) = HP(G) = B(G) = \overline{L}(G)$. Kami juga buktikan hasil serupa bila X ialah subkumpulan bagi kumpulam linear tertentu.

Biar G suatu kumpulan dan $h, g \in G$. Suatu rangkap-2 (h,g) ialah pasangan n-Engel , $n \geq 2$, jika $h = [h,_n g], g = [g,_n h]$ dan $h \neq 1$. Kami akan tunjukkan bahawa subkumpulan yang dijana oleh pasangan 5-Engel (x,y) yang memuaskan $yxy = xyx$ dan $x^5 = 1$ ialah kumpulan A_5. Kemudian, kami tunjukkan bakawa jikalau (x,y) ialah pasangan n-Engel, $xyx^{-2}yx = yxy$ dan $yxy^{-2}xy = xyx$, maka $n = 2k$ di mana $k = 4$ atau $k \geq 6$. Di samping itu, subkumpulan yang dijana oleh $\{x,y\}$ dikenalpasti bagi $k = 4, 6, 7$ dan 8.

Akhirnya, kami buktikan wujudnya pasangan Engel dalam kumpulan linear istimewa berperingkat 2 yang tertentu. Terutamanya, kami buktikan jikalau $SL(2, F)$ merupakan kumpulan linear istimewa berperingkat 2 di atas medan F, maka bagi semua medan L, wujud suatu peluasan medan F' oleh L dengan $[F' : L] \leq 6$ di mana $SL(2, F')$ ada suatu pasangan n-Engel bagi sesuatu integer $n \geq 4$. Kami juga akan buktikan $SL(2, F)$ mempunyai pasangan 5-Engel jikalau F ialah suatu medan sengan cirian $p \equiv \pm 1 \mod 5$.

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Chapter 1

Introduction

1.1 General Introduction

The aim of this thesis is to give some generalizations on Engel elements, and to characterize finite groups having Engel pairs.

In recent decades, increasingly more mathematicians are involved in the study of Engel elements in groups. The study of such elements are facilitated by the increasing processing power of computer and its software. With such computational power, we are able to visualize abstract objects and find new examples, which lead to the finding of new theorems.

1.2 Relative Engel Elements

Let G be a group. Let $x_1, x_2, \ldots, x_m \in G$. The *commutator* of x_1 and x_2 is $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, and a *simple commutator of weight* $m \geq 2$ is defined recursively as

$$[x_1, x_2, \ldots, x_m] = [[x_1, x_2, \ldots, x_{m-1}], x_m],$$

where by convention $[x_1] = x_1$. Let $x, y \in G$. A useful shorthand notation is

$$[x,_m y] = [x, \overbrace{y, y, \ldots, y}^{m}],$$

where by convention $[x,_0 y] = [x]$.

An element $g \in G$ is called a *left Engel element* of G, if for each $x \in G$, there is a positive integer $n = n(g, x)$ such that $[x,_n g] = 1$. The set of all left Engel elements is denoted by $L(G)$. An element $g \in G$ is called a *left m-Engel element* of G, if $[x,_m g] = 1$ for all $x \in G$. The set of all left m-Engel elements is denoted by $L_m(G)$. Let $\mathbb{N}$ be the set of all positive integers. The elements in $\overline{L}(G) = \bigcup_{m \in \mathbb{N}} L_m(G)$ are called *bounded left Engel elements* of G.

An element $g \in G$ is called a *right Engel element* of G, if for each $x \in G$, there is a positive integer $n = n(g, x)$ such that $[g,_n x] = 1$. The set of all right Engel elements is denoted by $R(G)$. An element $g \in G$ is called a *right m-Engel element* of G, if $[g,_m x] = 1$ for all $x \in G$. The set of all right m-Engel elements is denoted by $R_m(G)$. The elements in $\overline{R}(G) = \bigcup_{m \in \mathbb{N}} R_m(G)$ are called *bounded right Engel elements* of G.

Let X be a subset of a group G. An element $g \in G$ is called an *X-relative left Engel element* of G, if for each $x \in X$, there is a positive integer $n = n(g, x)$ such that $[x,_n g] = 1$. The set of all X-relative left Engel elements is denoted by $L(G, X)$. An element $g \in G$ is called an *X-relative left m-Engel element* of G, if $[x,_m g] = 1$ for all $x \in X$. The set of all X-relative left m-Engel elements is denoted by $L_m(G, X)$. Let $\mathbb{N}$ be the set of all positive integers. The elements in $\overline{L}(G, X) = \bigcup_{m \in \mathbb{N}} L_m(G, X)$ are called *bounded X-relative left Engel elements* of G.

The set of *X-relative right Engel elements* of G, *X-relative right m-Engel elements* of G, and *bounded X-relative right Engel elements* of G, denoted by $R(G, X)$, $R_m(G, X)$ and $\overline{R}(G, X)$ respectively, are defined similarly.

Note that when $X = G$, we have $L(G,G) = L(G)$, $L_m(G,G) = L_m(G)$ and $\overline{L}(G,G) = \overline{L}(G)$. The same are true for $R(G,G)$, $R_m(G,G)$ and $\overline{R}(G,G)$.

1.3 Relative Hirsch-Plotkin and Baer Radicals

Let $a, b \in G$, H be a subgroup of G, and X be a subset of G. We shall use the following notations:

(a) $a^b = b^{-1}ab$,

(b) $H^b = b^{-1}Hb$,

(c) $\langle X \rangle$ is the subgroup generated by X.

(d) $H^X = \langle \{h^x \ : \ h \in H, x \in X\} \rangle$.

The *Hirsch-Plotkin radical* of a group G, denoted by $HP(G)$, is the unique maximal normal locally nilpotent subgroup of G (see [28, 12.1.3 on p. 343]). In fact

$$HP(G) = \{a \in G \ : \ \langle a \rangle^G \text{ is locally nilpotent}\}.$$

This motivates us to define the *X-relative Hirsch-Plotkin radical* by

$$HP(G,X) = \{a \in G \ : \ \langle a \rangle^X \text{ is locally nilpotent}\}.$$

Note that $HP(G,X)$ may not be a group.

The *Baer radical* of a group G, denoted by $B(G)$ is the set of all $a \in G$ such that $\langle a \rangle$ is subnormal in G. In fact

$$B(G) = \{a \in G \ : \ \langle a \rangle \text{ is subnormal in } \langle a \rangle^G\}.$$

This motivates us to define the *X-relative Baer radical* by

$$B(G,X) = \{a \in G \ : \ \langle a \rangle \text{ is subnormal in } \langle a \rangle^X\}.$$

Note that $HP(G,G) = HP(G)$ and $B(G,G) = B(G)$.

1.4 Classical and New Results

A common problem in the theory of Engel elements is to find conditions on G, so that $HP(G) = L(G)$ and $B(G) = \overline{L}(G)$. In the sense of relativity, the problem is to find conditions on X, so that $HP(G,X) = L(G,X)$ and $B(G,X) = \overline{L}(G,X)$.

Gruenberg [14] proved the following classical theorem for solvable groups (see also [28, 12.3.3 on p. 357]).

Theorem 1.4.1. *If G is a solvable group, then $B(G) = \overline{L}(G)$ and $HP(G) = L(G)$.*

A group G is said to satisfy the *maximal condition* if there is no infinite ascending chain of subgroups

$$H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq \cdots .$$

Baer ([28, 12.3.7 on p. 360]) proved that similar identities hold if G satisfies the maximal condition.

Theorem 1.4.2. *If G satisfies the maximal condition, then $L(G) = HP(G) = B(G) = \overline{L}(G)$.*

The following theorems which are generalizations of Theorems 1.4.1 and 1.4.2 will be proved in Chapter 2.

Main Theorem 1. *Let X be a normal solvable subgroup of a group G. Then*

(a) $B(G,X) = \overline{L}(G,X)$,

(b) $HP(G,X) = L(G,X)$.

Main Theorem 2. *Let X be a normal subgroup of a group G. If G satisfies the maximal condition, then $L(G,X) = HP(G,X) = B(G,X) = \overline{L}(G,X)$.*

Let R be a commutative ring with identity and A be an R-module. We shall denote the group of all R-automorphisms of A by $\mathrm{Aut}_R A$. Let

$$F\mathrm{Aut}_R A = \{\alpha \in \mathrm{Aut}_R A \ : \ (\alpha - 1)A \text{ is a Noetherian } R\text{-module}\}.$$

Note that $F\mathrm{Aut}_R A$ is a subgroup of $\mathrm{Aut}_R A$ and it is called the *finitary automorphisms* group of A over R (see [33, Section 1]).

Theorem 1.4.3. *Let R be a commutative Noetherian ring with identity and A be a finitely generated R-module. If G is a subgroup of $\mathrm{Aut}_R A$, then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$.*

Theorem 1.4.4. *Let G be a subgroup of a finitary automorphisms group of a module over a commutative ring with identity. Then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$.*

Theorem 1.4.3 was proved by Gruenberg [15, Theorem 0] and Theorem 1.4.4 was proved by Wehrfritz [33, 4.4]. We will give a generalization of these two results (see Main Theorem 3 and 4).

Definition 1.4.5. Let H, K be subgroups of a group G and $H \triangleleft K$. Let $\mathbb{N}_0$ be the set of non-negative integers. An element $b \in G$ is said to be (H,K)-*centralizable* if there is a sequence of normal subgroups of K, say $\{H_i\}_{i\in\mathbb{N}_0}$ such that

(a) $H_0 = H$,

(b) $H_{i+1} = \{d \in K \ : \ [d,b] \in H_i\}$ for all $i \in \mathbb{N}_0$.

It is not hard to see that $H = H_0 \lhd H_1 \lhd H_2 \lhd \cdots$. The sequence $\{H_i\}_{i\in\mathbb{N}_0}$ shall be called the (H, K)-*centralized normal sequence* of b.

A set $W \subseteq G$ is said to be (H, K)-*centralizable* if every element in W is (H, K)-centralizable.

Main Theorem 3. *Let G be a group, R be a commutative Noetherian ring with identity and A be a finitely generated R-module. Let S be a normal subgroup of G such that $\langle L(S)\rangle$ is a subgroup of $\mathrm{Aut}_R A$. If $L(G, S)$ is $(HP(S), S)$-centralizable, then*

(a) $B(G, S) = \overline{L}(G, S)$,

(b) $HP(G, S) = L(G, S)$.

Main Theorem 4. *Let G be a group, R be a commutative ring with identity and A be an R-module. Let S be a normal subgroup of G such that $\langle L(S)\rangle$ is a subgroup of $F\mathrm{Aut}_R A$. If $L(G, S)$ is $(HP(S), S)$-centralizable, then $HP(G, S) = L(G, S)$.*

These two theorems will be proved in Chapter 3.

1.5 Engel Pairs

Every finite group G satisfies a law $[x,_r y] = [x,_s y]$ for some positive integers $r < s$. The minimal value of r is called the Engel depth of G (see [5, 6]). So, given any non-left Engel element $g \in G$, there exists a $h \in G$ and a positive integer n such that $h = [h,_n g]$. However we do not know whether $g = [g,_n h]$ or not.

Let $G(a, b) = \{x, y | x = [x,_a y], y = [y,_b x]\}$. It can be shown that $G(1, b) = 1$ and $G(2, 2) = 1$. However, we wonder if $G(a, b)$ is also finite for other values of a and b (see Problem 11.18 of [24]: Note that Problem 17.80 is a special case for Problem 11.18 in which $a = b$).

Definition 1.5.1. Let G be a group and $h, g \in G$. The 2-tuple (h, g) is said to be an *n-Engel pair*, $n \geq 2$, if $h = [h,_n g]$, $g = [g,_n h]$ and $h \neq 1$.

In Chapter 4, we will show that if (h, g) is an n-Engel pair and $hgh = ghg$, then n must be a multiple of 5. Furthermore, the subgroup generated by $\{h, g\}$ is isomorphic to A_5 if the order of h is 5, and is isomorphic to H_2 if the order of h is 10. Here, A_5 is the alternating subgroup on 5 elements and H_2 is the central extension of the cyclic group of order 2 by A_5 (see Main Theorem 5).

In Chapter 5, we will show that if (h, g) is an n-Engel pair in a group H satisfying the conditions $hgh^{-2}gh = ghg$ and $ghg^{-2}hg = hgh$, then $n = 2k$ where $k = 4$ or $k \geq 6$ (see Main Theorem 6). For $k = 4, 6, 7, 8$, we will give characterizations of the subgroup generated by this n-Engel pair (see Main Theorem 7).

Let $SL(2, F)$ be the special linear group of order 2 over the field F. If $F = \mathbb{Z}_p$ for a prime p, then we shall write $SL(2, p)$ instead of $SL(2, F)$. In Chapter 6, we will prove the following theorems.

Main Theorem 8. *Given any field L, there is a field extension F of L with $[F : L] \leq 6$ such that $SL(2, F)$ has an n-Engel pair for some integer $n \geq 4$.*

Main Theorem 9. *Given any field F of characteristic $p \equiv \pm 1 \mod 5$, $SL(2, F)$ has a 5-Engel pair.*

Chapter 2

Relative Engel Elements I

2.1 A Brief Introduction

Let X be a subset of a group G. Recall that an element $g \in G$ is called an X-relative left Engel element of G, if for each $x \in X$, there is a positive integer $n = n(g, x)$ such that $[x,_n g] = 1$. The set of all X-relative left Engel elements is denoted by $L(G, X)$. An element $g \in G$ is called an X-relative left m-Engel element of G, if $[x,_m g] = 1$ for all $x \in X$. The set of all X-relative left m-Engel elements is denoted by $L_m(G, X)$. Let $\mathbb{N}$ be the set of all positive integers. The elements in $\overline{L}(G, X) = \bigcup_{m \in \mathbb{N}} L_m(G, X)$ are called bounded X-relative left Engel elements of G.

The set of X-relative right Engel elements of G, X-relative right m-Engel elements of G, and bounded X-relative right Engel elements of G, denoted by $R(G, X)$, $R_m(G, X)$ and $\overline{R}(G, X)$ respectively, are defined similarly.

Let $Z(G)$ be the center of G. Note that $L_1(G) = R_1(G) = Z(G)$. Let $C_G(X) = \{g \in G \ : \ gx = xg \ \forall x \in X\}$. Clearly $C_G(X)$ is a subgroup of G, and $L_1(G, X) = R_1(G, X) = C_G(X)$. We shall characterize $L_2(G, X)$ and $R_2(G, X)$ in Section 2.2.

We will prove Main Theorem 1 and Main Theorem 2 in Sections 2.3 and 2.4, respectively.

The main results in this chapter have been published (see S. G. Quek, K. B. Wong, P. C. Wong, *On Engel elements of a group relative to certain subgroup*, Comm. Algebra **40** (2012), 4693–4701).

2.2 2-Engel elements

Let G be a group. Let $a, b \in G$, H be a subgroup of G, and X be a subset of G. We shall use the following notations:

(a) $a^b = b^{-1}ab$,

(a) $H^b = b^{-1}Hb$,

(c) $\langle X \rangle$ is the subgroup generated by X.

(d) $H^X = \langle \{h^x \ : \ h \in H, x \in X\} \rangle$.

(e) $N_G(H) = \{g \in G \ : \ H^g = H\}$ (note that $N_G(H)$ is called the *normalizer* of H in G).

Now Theorem 2.2.1 and Theorem 2.2.2 follow from the fact that $[x, g, g] = 1$ if and only if g commutes with $[x, g] = g^{-x}g$ if and only if g commutes with g^x.

Theorem 2.2.1. *Let X be a subset of a group G. Then*

$$L_2(G, X) = \{g \in G \ : \ [g, g^x] = 1 \text{ for all } x \in X\}.$$

Furthermore if X is a subgroup, then $L_2(G, X) = \{g \in G \ : \ \langle g \rangle^X$ is abelian $\}$.

Theorem 2.2.2. *Let X be a subset of a group G. Then*

$$R_2(G,X) = \{g \in G \ : \ [x, x^g] = 1 \text{ for all } x \in X\}.$$

Note that Theorem 2.2.1 is a generalization of the well-known fact $L_2(G) = \{g \in G \ : \ \langle g \rangle^G \text{ is abelian}\}$ (see [23] and [28, 12.3.6 on p. 358]). It was shown by Kappe [22] that when $X = G$, $R_2(G,G) \subseteq L_2(G,G)$, and $R_2(G,G)$ is a group. However in general we do not know whether $R_2(G,X)$ is a group or not. We give a characterization of $R_2(G,X)$ in Corollary 2.2.3.

Corollary 2.2.3. *Let X be a subset of a group G and $g \in R_2(G,X)$. Then*

(a) $g^{-1} \in R_2(G,X)$,

(b) *if $a \in G$ and $aXa^{-1} \subseteq X$, then $a^{-1}ga \in R_2(G,X)$,*

(c) *$R_2 = R_2(G,X)$ is a subgroup of G if and only if $\langle x \rangle^{R_2}$ is abelian for all $x \in X$.*

Proof. (a) By Theorem 2.2.2, for all $x \in X$, $[x, x^g] = 1$, i.e, $[x^{g^{-1}}, x] = 1$, and this implies $[x, x^{g^{-1}}] = 1$. Hence $g^{-1} \in R_2(G,X)$.

(b) Again by Theorem 2.2.2, for all $x \in X$, $[x, x^g] = 1$, i.e., $[x^a, x^{ga}] = 1$. Now let $y \in X$ and set $x = y^{a^{-1}}$, then $[y, y^{a^{-1}ga}] = 1$. Hence $a^{-1}ga \in R_2(G,X)$.

(c) Suppose R_2 is a subgroup of G. Let $x \in X$ and $a, b \in R_2$. Then $ab^{-1} \in R_2$, and by Theorem 2.2.2, $[x, x^{ab^{-1}}] = 1$, i.e., $[x^b, x^a] = 1$. This implies that $\langle x \rangle^{R_2}$ is abelian.

The converse is proved similarly. $\square$

Note that Abdollahi [1] and Newell [27] have given characterizations of $L_3(G) = L_3(G,G)$ and $R_3(G) = R_3(G,G)$, respectively. It would be interesting to know how elements in $L_3(G,X)$ and $R_3(G,X)$ behave. See also a more recent paper by

Abdollahi and Khosravi [2] on 4-Engel elements. The reader may also refer to a chapter by Abdollahi in a recent textbook [3] on Engel elements.

2.3 Main Theorem 1

Recall that the X-relative Hirsch-Plotkin radical is defined by

$$HP(G,X) = \{a \in G \ : \ \langle a \rangle^X \text{ is locally nilpotent}\},$$

and the X-relative Baer radical is defined by

$$B(G,X) = \{a \in G \ : \ \langle a \rangle \text{ is subnormal in } \langle a \rangle^X\}.$$

Lemma 2.3.1. *Let G be a group and A, K_1, K_2 be subgroups of G such that $K_1 \lhd K_2$ and $A \subseteq K_1$. If A^{K_1} is locally nilpotent, then A^{K_2} is locally nilpotent.*

Proof. First note that A^{K_2} is a subgroup of K_1. So $A^{K_1} \lhd A^{K_2}$. Let $x \in K_2$. If $u \in K_1$, then $u^{x^{-1}} \in K_1$, and $u^{-1}(A^{K_1})^x u = u^{-1}x^{-1}(A^{K_1})xu = x^{-1}u^{-x^{-1}}(A^{K_1})u^{x^{-1}}x = (A^{K_1})^x$. So $(A^{K_1})^x \lhd K_1$, and thus $(A^{K_1})^x \lhd A^{K_2}$. Now $(A^{K_1})^x$ is locally nilpotent implies that $(A^{K_1})^x \subseteq HP(A^{K_2})$. Since $A^{K_2} = \left\langle \bigcup_{x \in K_2} (A^{K_1})^x \right\rangle$, $A^{K_2} = HP(A^{K_2})$ is locally nilpotent. $\square$

Theorem 2.3.2. *Let X be a subgroup of a group G. Then*

(a) $B(G,X) \subseteq HP(G,X)$,

(b) $B(G,X) \subseteq \overline{L}(G,X)$,

(c) $HP(G,X) \subseteq L(G,X)$.

Proof. (a) Let $a \in B(G,X)$. Then $\langle a \rangle$ is subnormal in $\langle a \rangle^X$. Since $\langle a \rangle^X \lhd \langle a, X \rangle$, there is a subnormal chain

$$\langle a \rangle = A_0 \lhd A_1 \lhd A_2 \lhd \cdots \lhd A_n = \langle a, X \rangle.$$

Now $\langle a\rangle = \langle a\rangle^{A_1}$ is locally nilpotent implies that $\langle a\rangle^{A_2}$ is locally nilpotent (Lemma 2.3.1). So by applying Lemma 2.3.1 repeatedly, we conclude that $\langle a\rangle^{\langle a,X\rangle}$ is locally nilpotent. Therefore $\langle a\rangle^X$ is locally nilpotent as it is a subgroup of $\langle a\rangle^{\langle a,X\rangle}$. Hence $a \in HP(G, X)$.

(b) Let $a \in B(G, X)$. Again we have $\langle a\rangle = A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \cdots \triangleleft A_n = \langle a, X\rangle$. Let $x \in X$. As $a \in A_{n-1}$ and $x \in A_n$, we have $[x, a] \in A_{n-1}$. As $a \in A_{n-2}$ and $[x, a] \in A_{n-1}$, we have $[x, a, a] \in A_{n-2}$. We can continue this process to obtain $[x,_n a] \in A_0$. So $[x,_{n+1} a] = 1$ for all $x \in X$. Thus $a \in L_{n+1}(G, X) \subseteq \overline{L}(G, X)$.

(c) Let $a \in HP(G, X)$. Then $\langle a\rangle^X$ is locally nilpotent. Let $x \in X$. Note that $[x, a], a \in \langle a\rangle^X$, and so $\langle a, [x, a]\rangle$ is nilpotent. Therefore there is a positive integer $n = n(x, a)$ such that $[x,_n a] = 1$. Hence $a \in L(G, X)$. □

Lemma 2.3.3. *Let C, D be normal subgroups of a group G such that $C \subseteq D$ and D/C is abelian. Let $a \in L(G, D)$ be fixed. Inductively set $C_0 = C$, and for $i \geq 1$, $C_i = \{d \in D \ : \ [d, a] \in C_{i-1}\}$. Then*

(a) *C_i is a normal subgroup of D and D/C_i is abelian for all $i \geq 0$,*

(b) *$C_i \subseteq C_{i+1}$ for all $i \geq 0$,*

(c) *$D = \bigcup_{i\geq 0} C_i$,*

(d) *if $a \in \overline{L}(G, D)$, then $D = C_m$ for some positive integer m.*

Proof. (a) We shall prove by induction on i. Clearly it is true for $i = 0$. Suppose $i \geq 1$. Assume that C_{i-1} is a normal subgroup of D and D/C_{i-1} is abelian.

Note that C_i/C_{i-1} is the centralizer of aC_{i-1} in D/C_{i-1} and $C_i/C_{i-1} \lhd D/C_{i-1}$ for D/C_{i-1} is abelian. Hence C_i is a normal subgroup of D and D/C_i is abelian, being a quotient of D/C_{i-1}.

(b) This follows from part (a) of this theorem.

(c) Clearly $\bigcup_{i\geq 0} C_i \subseteq D$. Let $d \in D$. If $d = 1$, then $d \in \bigcup_{i\geq 0} C_i$. We may assume $d \neq 1$. Note that $[d,_n a] = 1$ for some positive integer n, and $[d,_l a] \in D$ for $l = 1, 2, \ldots, n-1$ (because $D \lhd G$). Now $[[d,_{n-1} a], a] = [d,_n a] = 1$ implies that $[d,_{n-1} a] \in C_1$. As $[[d,_{n-2} a], a] = [d,_{n-1} a] \in C_1$, we have $[d,_{n-2} a] \in C_2$. By continuing this way, we see that $[d, a] \in C_{n-1}$ and $d \in C_n$. Hence $D = \bigcup_{i\geq 0} C_i$.

(d) If $a \in \overline{L}(G, D)$, then there is a positive integer m such that $[d,_m a] = 1$ for all $d \in D$. By using a similar argument as in the proof of part (c) of this theorem, we see that $D = C_m$. $\square$

Main Theorem 1. *Let X be a normal solvable subgroup of a group G. Then*

(a) $B(G, X) = \overline{L}(G, X)$,

(b) $HP(G, X) = L(G, X)$.

Proof. Let the derived length of X be d. Note that $X^{(i+1)}$ and $X^{(i)}$ are normal in G and $X^{(i)}/X^{(i+1)}$ is abelian for $i = 0, 1, 2, \ldots, d-1$, and furthermore $X^{(0)} = X$ and $X^{(d)} = 1$.

(a) By part (b) of Theorem 2.3.2, it is sufficient to show that $\overline{L}(G, X) \subseteq B(G, X)$. Let $a \in \overline{L}(G, X)$. Then $a \in \overline{L}(G, X^{(i)})$ (for $X^{(i)} \subseteq X$). By part (d) of Lemma 2.3.3, there is a positive integer m_i such that $X^{(i)} = C_{im_i}$ where $C_{i0} = X^{(i+1)}$ and for $j = 1, 2, \ldots, m_i$, $C_{ij} = \{d \in X^{(i)} \,:\, [d, a] \in C_{i(j-1)}\}$. Note that $\langle a, C_{ij}\rangle \lhd \langle a, C_{i(j+1)}\rangle$.

Therefore $\langle a, X^{(i+1)}\rangle$ is subnormal in $\langle a, X^{(i)}\rangle$. This implies that $\langle a\rangle = \langle a, X^{(d)}\rangle$ is subnormal in $\langle a, X\rangle$. As $\langle a\rangle^X$ is a subgroup of $\langle a, X\rangle$, $\langle a\rangle$ is subnormal in $\langle a\rangle^X$. Hence $a \in B(G, X)$.

(b) By part (c) of Theorem 2.3.2, it is sufficient to show that $L(G, X) \subseteq HP(G, X)$. Let $a \in L(G, X)$. Then $a \in L(G, X^{(i)})$. By Lemma 2.3.3, $X^{(i)} = \bigcup_{j\geq 0} C_{ij}$ where $C_{i0} = X^{(i+1)}$ and for $j \geq 1$, $C_{ij} = \{d \in X^{(i)} : [d, a] \in C_{i(j-1)}\}$. Note that $\langle a, C_{ij}\rangle \triangleleft \langle a, C_{i(j+1)}\rangle$.

When $i = d-1$, we have $C_{(d-1)0} = X^{(d)} = 1$ and

$$\langle a\rangle = \langle a, C_{(d-1)0}\rangle \triangleleft \langle a, C_{(d-1)1}\rangle \triangleleft \langle a, C_{(d-1)2}\rangle \triangleleft \cdots .$$

Note that $\langle a\rangle = \langle a\rangle^{\langle a, C_{(d-1)1}\rangle}$ is locally nilpotent. By Lemma 2.3.1, $\langle a\rangle^{\langle a, C_{(d-1)2}\rangle}$ is locally nilpotent. In fact inductively, we see that $\langle a\rangle^{\langle a, C_{(d-1)j}\rangle}$ is locally nilpotent for all $j \geq 1$. Furthermore $\langle a\rangle^{\langle a, C_{(d-1)1}\rangle} \subseteq \langle a\rangle^{\langle a, C_{(d-1)2}\rangle} \subseteq \langle a\rangle^{\langle a, C_{(d-1)3}\rangle} \subseteq \cdots$ is an ascending chain of locally nilpotent groups. Therefore $\langle a\rangle^{\langle a, X^{(d-1)}\rangle} = \bigcup_{j\geq 1} \langle a\rangle^{\langle a, C_{(d-1)j}\rangle}$ is locally nilpotent.

When $i = d-2$, we have $C_{(d-2)0} = X^{(d-1)}$ and

$$\langle a, C_{(d-2)0}\rangle \triangleleft \langle a, C_{(d-2)1}\rangle \triangleleft \langle a, C_{(d-2)2}\rangle \triangleleft \cdots .$$

Note that $\langle a\rangle^{\langle a, C_{(d-2)0}\rangle} = \langle a\rangle^{\langle a, X^{(d-1)}\rangle}$ is locally nilpotent. By using similar argument as in the previous paragraph, we deduce that $\langle a\rangle^{\langle a, X^{(d-2)}\rangle}$ is locally nilpotent.

By continuing this process, we see that $\langle a\rangle^{\langle a, X\rangle}$ is locally nilpotent. Hence $\langle a\rangle^X$ is locally nilpotent, and $a \in HP(G, X)$. $\square$

Note that Main Theorem 1 is a generalization of a theorem of Gruenberg which states that $B(G) = \overline{L}(G)$ and $HP(G) = L(G)$ for any solvable group G (see [14] and [28, 12.3.3 on p. 357]).

2.4 Main Theorem 2

Lemma 2.4.1. *Let X be a subgroup of a group G. If G satisfies the maximal condition, then $HP(G, X) = B(G, X)$.*

Proof. By part (a) of Theorem 2.3.2, it is sufficient to show that $HP(G, X) \subseteq B(G, X)$. Let $a \in HP(G, X)$. Then $\langle a \rangle^X$ is locally nilpotent. Since G satisfies maximal condition, $\langle a \rangle^X$ is finitely generated. Thus $\langle a \rangle^X$ is nilpotent, and $\langle a \rangle$ is subnormal in $\langle a \rangle^X$. So $a \in B(G, X)$. □

Lemma 2.4.2. *Let X be a subgroup of a group G. If $a \in L(G, X)$, then $a^u \in L(G, X)$ for all $u \in X$.*

Proof. Let $u \in X$ be fixed. Let $x \in X$. Then $x^{u^{-1}} \in X$ (as X is a group), and there is a positive integer n such that $[x^{u^{-1}},_n a] = 1$. So $[x^{u^{-1}},_n a]^u = 1$ and $[x,_n a^u] = 1$. Hence $a^u \in L(G, X)$. □

Lemma 2.4.3. *Let X be a normal subgroup of a group G, and $a \in G$. Let $\{a\}^X = \{a^x \ : \ x \in X\}$. Then $v^u, v^{u^{-1}} \in \{a\}^X$ for all $u, v \in \{a\}^X$.*

Proof. Let $u = a^{x_1}$ and $v = a^{x_2}$ where $x_1, x_2 \in X$. Then

$$\begin{aligned} v^u &= x_1^{-1}a^{-1}x_1x_2^{-1}ax_2x_1^{-1}ax_1 \\ &= (x_1^{-1}a^{-1}x_1x_2^{-1}a)a(a^{-1}x_2x_1^{-1}ax_1) \in \{a\}^X, \end{aligned}$$

for $x_1^{-1}a^{-1}x_1x_2^{-1}a \in X$. Similarly $v^{u^{-1}} \in \{a\}^X$. □

Let $u \in G$. A subgroup A of G is said to be (X, u)-*generated* if A is generated by elements in $\{u\}^X$, i.e., $A = \langle A \cap \{u\}^X \rangle$.

Lemma 2.4.4. *Let X be a normal subgroup of a group G, and $u \in G$. Suppose A, B are (X, u)-generated subgroups of G. If $A \subsetneq B$ and B is nilpotent, then there is an element $x \in X$ such that $u^x \in B \setminus A$ and $u^x \in N_G(A)$.*

Proof. Since B is nilpotent, A is subnormal in B, i.e., $A = A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \cdots \triangleleft A_n = B$. As both A and B are (X, u)-generated, and $A \subsetneq B$, we deduce that $A \cap \{u\}^X \subsetneq B \cap \{u\}^X$. So there is a i_0 such that $A \cap \{u\}^X = A_1 \cap \{u\}^X = \cdots = A_{i_0} \cap \{u\}^X \subsetneq A_{i_0+1} \cap \{u\}^X$. Let $u^x \in A_{i_0+1} \cap \{u\}^X \setminus (A_{i_0} \cap \{u\}^X)$. Since $A_{i_0}^{u^x} = A_{i_0}$, by Lemma 2.4.3, $(A \cap \{u\}^X)^{u^x} = (A_{i_0} \cap \{u\}^X)^{u^x} \subseteq A_{i_0} \cap \{u\}^X = A \cap \{u\}^X$. Similarly $(A \cap \{u\}^X)^{u^{-x}} \subseteq A \cap \{u\}^X$. Hence $u^x \in N_G(A)$. $\square$

Lemma 2.4.5. *Let M be a subgroup of a group G. Let $a \in N_G(M)$. Suppose that $M = \langle w_1, w_2, \ldots, w_n \rangle$ and there is a positive integer m such that $[w_{i,m}\, a] = 1$ for $i = 1, 2, \ldots, n$. If M is nilpotent, then $\langle a, M \rangle$ is nilpotent.*

Proof. Note that $M \triangleleft \langle a, M \rangle$. This implies that $[M, M] \triangleleft \langle a, M \rangle$. It is clear that $\langle a, M \rangle / [M, M]$ is nilpotent of class at most m. Since M is nilpotent, $\langle a, M \rangle$ is nilpotent (see [28, 5.2.10 on p. 129]). $\square$

Lemma 2.4.6. *Let x, y, a be elements in a group G. If l is a positive integer, then*

$$[a^{-x},_l\, a^y] = [y^{-1}x,_{l+1}\, a^y]^{a^{-y}}.$$

Proof. Note that

$$\begin{aligned}
[a^{-x},_{l} a^{y}] &= [a^{-yy^{-1}x},_{l} a^{y}] \\
&= [(a^{y})^{-y^{-1}x},_{l} a^{y}] \\
&= [[y^{-1}x, a^{y}]a^{-y},_{l} a^{y}] \\
&= [[[y^{-1}x, a^{y}]a^{-y}, a^{y}],_{l-1} a^{y}] \\
&= [[[y^{-1}x, a^{y}], a^{y}]^{a^{-y}}[a^{-y}, a^{y}],_{l-1} a^{y}] \\
&= [[y^{-1}x, a^{y}, a^{y}]^{a^{-y}},_{l-1} a^{y}] \\
&= [[y^{-1}x, a^{y}, a^{y}],_{l-1} a^{y}]^{a^{-y}} \\
&= [y^{-1}x,_{l+1} a^{y}]^{a^{-y}}.
\end{aligned}$$

□

Main Theorem 2. *Let X be a normal subgroup of a group G. If G satisfies the maximal condition, then $L(G,X) = HP(G,X) = B(G,X) = \overline{L}(G,X)$.*

Proof. By Theorem 2.3.2 and Lemma 2.4.1, it is sufficient to show that $L(G,X) \subseteq HP(G,X)$. Let $a \in L(G,X)$. We need to show that $\langle a \rangle^X$ is nilpotent.

Let $\mathcal{T}$ be the set of all (X,a)-generated nilpotent subgroups of G. Note that $\mathcal{T} \neq \varnothing$ for $\langle a \rangle \in \mathcal{T}$. Since G satisfies the maximal condition, it has a maximal (X,a)-generated nilpotent subgroup in $\mathcal{T}$.

Claim 1. Let $U \in \mathcal{T}$. If $U^{a^y} = U$ for some $y \in X$, then $\langle U, a^y \rangle \in \mathcal{T}$.

Proof of Claim 1. Note that $U \lhd \langle U, a^y \rangle$. As U is (X,a)-generated and G satisfies the maximal condition, we may assume that $U = \langle a^{x_1}, a^{x_2}, \ldots, a^{x_m} \rangle$ where $x_1, x_2, \ldots, x_m \in X$. By Lemma 2.4.2, $a^y \in L(G,X)$. So there is a positive integer l such that $[y^{-1}x_{i,l+1}\, a^y] = 1$ for $i = 1, 2, \dots, m$. It is not hard to see that $U = \langle a^{-x_1}, a^{-x_2}, \dots, a^{-x_m}\rangle$.

Now for $i = 1, 2, \dots, m$, by Lemma 2.4.6, we have $[a^{-x_i},_l\, a^y] = [y^{-1}x_{i,l+1}\, a^y]^{a^{-y}} = 1$. By Lemma 2.4.5, $\langle U, a^y\rangle$ is nilpotent, and therefore $\langle U, a^y\rangle \in \mathcal{T}$. $\square$

Case 1. G has only one maximal (X, a)-generated nilpotent subgroup in $\mathcal{T}$, say M. If $M = \langle a\rangle^X$, then $\langle a\rangle^X$ is nilpotent, and $a \in HP(G, X)$. Suppose $M \subsetneq \langle a\rangle^X$. Then there is a $y \in X$ with $a^y \notin M$. Note that by Lemma 2.4.3, M^{a^y} is (X, a)-generated, and it is nilpotent. Since G satisfies the maximal condition, M^{a^y} is contained in a maximal (X, a)-generated nilpotent subgroup. This means $M^{a^y} \subseteq M$, for G has only one maximal (X, a)-generated nilpotent subgroup. Similarly $M^{a^{-y}} \subseteq M$. Therefore $M^{a^y} = M$, and by Claim 1, $\langle M, a^y\rangle \in \mathcal{T}$, but this contradicts the maximality of M. Hence $M = \langle a\rangle^X$.

Case 2. G has at least two maximal (X, a)-generated nilpotent subgroups in $\mathcal{T}$. We will show that this case cannot happen. Consider the following set

$$\mathcal{I} = \left\{\langle U \cap V \cap \{a\}^X\rangle\right\},$$

where U, V are distinct maximal (X, a)-generated nilpotent subgroups in $\mathcal{T}$. Since G satisfies the maximal condition, there is a maximal element $I = \langle U_0 \cap V_0 \cap \{a\}^X\rangle \in \mathcal{I}$, where U_0, V_0 are distinct maximal (X, a)-generated nilpotent subgroups in $\mathcal{T}$. Let $W = \langle N_{U_0}(I) \cap \{a\}^X\rangle$. So W is also (X, a)-generated, and nilpotent (for W is a subgroup of the nilpotent group U_0).

Note that $I \neq U_0$, for otherwise $U_0 = V_0$. So $I \subsetneq U_0$, and by Lemma 2.4.4, there is a $u \in X$ with $a^u \in U_0 \setminus I$ and $a^u \in N_G(I)$. This means $I \subsetneq W$, as $a^u \in W$. Similarly $I \neq V_0$, and there is a $v \in X$ with $a^v \in V_0 \setminus I$ and $a^v \in N_G(I)$. Note that

$a^v \notin U_0$ and $a^u \notin V_0$.

Claim 2. There is no $y \in X$ with $a^y \in N_G(W)$ and $a^y \notin U_0$.

Proof of Claim 2. Suppose the contrary. Then there is a $y \in X$ with $a^y \in N_G(W)$ and $a^y \notin U_0$. So $W \triangleleft \langle W, a^y \rangle$. By Claim 1, $\langle W, a^y \rangle \in \mathcal{T}$. So it is contained in a maximal element $T \in \mathcal{T}$. Note that $T \neq U_0$, as $a^y \notin U_0$. It is not hard to see that $W \subseteq \langle U_0 \cap T \cap \{a\}^X \rangle \in \mathcal{I}$. This means $I \subsetneq W \subseteq \langle U_0 \cap T \cap \{a\}^X \rangle$, a contradiction to the maximality of I. $\square$

Now by Claim 2, $a^v \notin N_G(W)$. By Lemma 2.4.2, $a^u \in L(G, X)$. So there is a positive integer l such that $[u^{-1}v,_{l+1} a^u] = 1$. By Lemma 2.4.6, $[a^{-v},_l a^u] = 1$. From this we deduce that there is a positive integer k such that $[a^{-v},_k a^u] \in N_G(W)$ and $[a^{-v},_{k-1} a^u] \notin N_G(W)$ (note that $k \geq 1$, as $a^v \notin N_G(W)$). Let $z = [a^{-v},_{k-1} a^u]$. Then $z \in N_G(I)$, and $a^{-uz}a^u = [z, a^u] \in N_G(W)$. Since $a^u \in W$, $a^{uz} \in N_G(W)$. By Lemma 2.4.3, $a^{uz} \in \{a\}^X$. So by Claim 2, we conclude that $a^{uz} \in U_0$. Since $z \in N_G(I)$ and $a^u \notin I$, $a^{uz} \notin I$.

Now $a^{uz} \in W^z$ implies that $I \subsetneq \langle U_0 \cap W^z \cap \{a\}^X \rangle$. By Lemma 2.4.3, $W^z \in \mathcal{T}$. So W^z is contained in a maximal element P in $\mathcal{T}$. If $P \neq U_0$, then $\langle U_0 \cap P \cap \{a\}^X \rangle \in \mathcal{I}$, and this contradicts the maximality of I. Hence $P = U_0$, and $W^z \subseteq U_0$. This means $(N_{U_0}(I) \cap \{a\}^X)^z \subseteq U_0$.

By Lemma 2.4.3 and the fact that $z \in N_G(I)$, we have $(N_{U_0}(I) \cap \{a\}^X)^z \subseteq U_0 \cap N_G(I) \cap \{a\}^X = N_{U_0}(I) \cap \{a\}^X$. Therefore $W^z \subseteq W$. By the choice of z, $W^z \subsetneq W$, but then we have an ascending chain of subgroups $W \subsetneq W^{z^{-1}} \subsetneq W^{z^{-2}} \subsetneq \cdots$, a contradiction. Hence Case 2 cannot happen. $\square$

Note that Main Theorem 2 is a generalization of a theorem of Baer which states that $L(G) = HP(G) = B(G) = \overline{L}(G)$ for any group G that satisfies the maximal condition (see [28, 12.3.7 on p. 360]).

Chapter 3

Relative Engel Elements II

3.1 A Brief Introduction

This chapter is motivated by the following two results, one by Gruenberg [15, Theorem 0] and the other by Wehrfritz [33, 4.4].

Theorem 1.4.3. [Gruenberg's Theorem] *Let R be a commutative Noetherian ring with identity and A be a finitely generated R-module. If G is a subgroup of* $\mathrm{Aut}_R A$, *then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$.*

Theorem 1.4.4. [Wehrfritz's Theorem] *Let G be a subgroup of a finitary automorphisms group of a module over a commutative ring with identity. Then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$.*

Considering the work in Chapter 2, it is quite natural to ask whether similar results hold for relative left Engel elements. The answers are affirmative (see Main Theorem 3 and Main Theorem 4). We will also show that if X is a normal locally solvable subgroup of G, then $HP(G, X) = L(G, X)$ (see Theorem 3.4.3).

The materials in this chapter have been published (see S. G. Quek, K. B. Wong, P. C. Wong, *On left Engel elements of a group relative to subgroup of certain linear groups*, J. Pure Appl. Algebra **217** (2013) 427–431).

3.2 A generalization of Gruenberg's Theorem

We shall need the following theorem.

Theorem 3.2.1. [15, Theorem 2] *Let $\mathfrak{X}$ be a class of groups. Suppose that*

(i) *$\mathfrak{X}$ is closed with respect to formation of images, i.e., if $G \in \mathfrak{X}$, then $G/N \in \mathfrak{X}$ for all $N \lhd G$,*

(ii) *if $G \in \mathfrak{X}$, then every finitely generated subgroup of G lies in a finitely generated $\mathfrak{X}$-subgroup,*

(iii) *if $G \in \mathfrak{X}$ and G is finite, then G is solvable.*

Let R be a commutative Noetherian ring with identity and A be a finitely generated R-module. If G is a subgroup of $\mathrm{Aut}_R A$ and $G \in \mathfrak{X}$, then G is solvable.

Let $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{S}$ be the class of all finite, finitely generated and solvable groups, respectively. If we use the Hall's calculus of closure operations [16, Section 1.3], say Q (quotient group closure) and L (local closure), then conditions (i), (ii) and (iii) of Theorem 3.2.1 can be written as (i) $\mathsf{Q}\mathfrak{X} = \mathfrak{X}$; (ii) $\mathfrak{X} \leqslant \mathsf{L}(\mathfrak{G} \cap \mathfrak{X})$; (iii) $\mathfrak{X} \cap \mathfrak{F} \leqslant \mathfrak{S}$, respectively.

Lemma 3.2.2. *If S and T are normal subgroups of a group G, then $\langle L(S, S \cap T)\rangle$ is a normal subgroup of G.*

Proof. It is sufficient to show that $a^g = g^{-1}ag \in L(S, S \cap T)$ for all $a \in L(S, S \cap T)$ and $g \in G$. Let $x \in S \cap T$. Then $x^{g^{-1}} = gxg^{-1} \in S \cap T$ and there is a positive integer $n = n(a, x^{g^{-1}})$ with $[x^{g^{-1}},_n a] = 1$. Note that $[x,_n a^g] = g^{-1}[x^{g^{-1}},_n a]g = 1$. Since x was arbitrary, we conclude that $a^g \in L(S, S \cap T)$. ∎

The following lemma is obvious.

Lemma 3.2.3. *Let S and T be subgroups of a group G. If $S \subseteq T$, then $L(T) \cap S \subseteq L(S)$ and $\overline{L}(T) \cap S \subseteq \overline{L}(S)$.*

Theorem 3.2.4. *The following hold for any group G.*

(a) *If $L(\langle L(G) \rangle) = HP(\langle L(G) \rangle)$, then $L(G) = HP(G)$.*

(b) *If $L(\langle L(G) \rangle) = HP(\langle L(G) \rangle)$ and $\overline{L}(\langle L(G) \rangle) = B(\langle L(G) \rangle)$, then $\overline{L}(G) = B(G)$.*

Proof. (a) By Theorem 2.3.2, it is sufficient to show that $L(G) \subseteq HP(G)$. By Lemma 3.2.3, $L(G) = L(G) \cap \langle L(G) \rangle \subseteq L(\langle L(G) \rangle)$. Since $HP(\langle L(G) \rangle)$ is a characteristic subgroup of $\langle L(G) \rangle$ and $\langle L(G) \rangle$ is normal in G (by taking $T = S = G$ in Lemma 3.2.2), we have $HP(\langle L(G) \rangle)$ is normal in G. Hence $L(\langle L(G) \rangle) = HP(\langle L(G) \rangle) \subseteq HP(G)$ and $L(G) \subseteq HP(G)$.

(b) By Theorem 2.3.2, it is sufficient to show that $\overline{L}(G) \subseteq B(G)$. By part (a), $L(G) = HP(G) = \langle L(G) \rangle$. Therefore $\overline{L}(HP(G)) = B(HP(G))$, and

$$\overline{L}(G) = \overline{L}(G) \cap L(G) = \overline{L}(G) \cap HP(G).$$

It then follows from Lemma 3.2.3 that $\overline{L}(G) \subseteq \overline{L}(HP(G)) = B(HP(G))$. So it is sufficient to show that $B(HP(G)) \subseteq B(G)$.

Let $g \in B(HP(G))$. Then $\langle g \rangle$ is subnormal in $\langle g \rangle^{HP(G)}$. Since $\langle g \rangle^{HP(G)} \lhd HP(G) \lhd G$, we conclude that $\langle g \rangle$ is subnormal in G and thus in $\langle g \rangle^{G}$. So $g \in B(G)$ and $B(HP(G)) \subseteq B(G)$. □

Corollary 3.2.5. *Let G be a group. If $\langle L(G)\rangle$ is solvable, then*

(a) $L(G) = \langle L(G)\rangle = HP(G)$,

(b) $\overline{L}(G) = B(G)$.

Proof. Since $\langle L(G)\rangle$ is solvable,

$$L(\langle L(G)\rangle) = HP(\langle L(G)\rangle), \text{ and } \overline{L}(\langle L(G)\rangle) = B(\langle L(G)\rangle),$$

(see [14] and [28, 12.3.3 on p. 357]). The corollary then follows from Theorem 3.2.4. □

Lemma 3.2.6. *For any group G, $\langle L(G)\rangle = \langle L(\langle L(G)\rangle)\rangle$.*

Proof. By Lemma 3.2.3, $L(G) = L(G) \cap \langle L(G)\rangle \subseteq \langle L(\langle L(G)\rangle)\rangle$. So $\langle L(G)\rangle \subseteq \langle L(\langle L(G)\rangle)\rangle$. The lemma follows by noticing that $L(\langle L(G)\rangle) \subseteq \langle L(G)\rangle$. □

Lemma 3.2.7. *Let $\mathfrak{X}$ be the class of groups that satisfies $G = \langle L(G)\rangle$. Then $\mathfrak{X}$ satisfies conditions* (i), (ii) *and* (iii) *of Theorem 3.2.1.*

Proof. Let $G \in \mathfrak{X}$. Then $G = \langle L(G)\rangle$.

(i) Let N be a normal subgroup of G. We need to show that $G/N = \langle L(G/N)\rangle$. This follows by noting that $aN \in L(G/N)$ for all $a \in L(G)$.

(ii) Let S be a finitely generated subgroup of G, say $S = \langle s_1, s_2, \dots, s_n\rangle$. For each i, let

$$s_i = \prod_{1\le j\le m_i} t_{ij}^{\epsilon_{ij}},$$

where $\epsilon_{ij} = \pm 1$ and $t_{ij} \in L(G)$. Let T be the subgroup of G generated by all t_{ij}'s. We need to show that $T \in \mathfrak{X}$. Clearly $\langle L(T)\rangle \subseteq T$. By Lemma 3.2.3, $T = L(G) \cap T \subseteq L(T) \subseteq \langle L(T)\rangle$. Thus $T = \langle L(T)\rangle$ and $T \in \mathfrak{X}$.

(iii) We need to show that if G is finite, then G is solvable. By [28, 12.3.7 on p. 360], $L(G) = HP(G)$. So $G = HP(G)$ is nilpotent, and thus solvable. □

Theorem 3.2.8. *Let G be a group, R be a commutative Noetherian ring with identity and A be a finitely generated R-module. If $\langle L(G)\rangle$ is a subgroup of $\mathrm{Aut}_R A$, then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$. Furthermore, $\langle L(G)\rangle$ is solvable.*

Proof. Let $\mathfrak{X}$ be defined as in Lemma 3.2.7. By Lemma 3.2.6, $\langle L(G)\rangle \in \mathfrak{X}$. It then follows from Lemma 3.2.7 and Theorem 3.2.1 that $\langle L(G)\rangle$ is solvable. Therefore $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$ by Corollary 3.2.5. □

Note that in Theorem 3.2.8, we have replaced the condition 'G is a subgroup of $\mathrm{Aut}_R A$' in Theorem 1.4.3 with '$\langle L(G)\rangle$ is a subgroup of $\mathrm{Aut}_R A$'.

3.3 Main Theorem 3

Let us recall the following definition.

Definition 1.4.5. Let H, K be subgroups of a group G and $H \triangleleft K$. Let $\mathbb{N}_0$ be the set of non-negative integers. An element $b \in G$ is said to be (H, K)-*centralizable* if there is a sequence of normal subgroups of K, say $\{H_i\}_{i\in\mathbb{N}_0}$ such that

(a) $H_0 = H$,

(b) $H_{i+1} = \{d \in K \ : \ [d, b] \in H_i\}$ for all $i \in \mathbb{N}_0$.

It is not hard to see that $H = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots$. The sequence $\{H_i\}_{i\in\mathbb{N}_0}$ shall be called the (H, K)-*centralized normal sequence* of b.

A set $W \subseteq G$ is said to be (H, K)-*centralizable* if every element in W is (H, K)-centralizable.

Lemma 3.3.1. *Let S be a subgroup of a group G, and S' be the commutator subgroup of S. Let $b \in N_G(S) = \{g \in G \ : \ S^g = S\}$. Then b is (S', S)-centralizable.*

Proof. Let $H_0 = S'$, inductively let

$$H_{i+1} = \{d \in S \ : \ [d, b] \in H_i\},$$

for all $i \in \mathbb{N}_0$. Suppose $H_i \lhd S$. We shall show that $H_{i+1} \lhd S$. Let $d_1, d_2 \in H_{i+1}$. Note that

$$[d_1 d_2^{-1}, b] = [d_1, b]^{d_2^{-1}} [d_2^{-1}, b] = [d_1, b]^{d_2^{-1}} ([d_2, b]^{d_2})^{-1} \in H_i.$$

So $d_1 d_2^{-1} \in H_{i+1}$ and H_{i+1} is a subgroup of S. Since S/S' is abelian and H_{i+1}/S' is a subgroup of S/S', we conclude that $H_{i+1} \lhd S$. Hence $\{H_i\}_{i \in \mathbb{N}_0}$ is a (S', S)-centralized normal sequence of b. $\square$

Lemma 3.3.2. *Let H, K be subgroups of a group G and $H \lhd K$. Let $b \in N_G(H)$ be (H, K)-centralizable. Then the following hold.*

(a) *If $b \in L(G, K)$ and $\langle b \rangle^H$ is locally nilpotent, then $\langle b \rangle^K$ is locally nilpotent.*

(b) *If $b \in \overline{L}(G, K)$, then $\langle b \rangle^H$ is subnormal in $\langle b \rangle^K$.*

Proof. Let $\{H_i\}_{i \in \mathbb{N}_0}$ be the (H, K)-centralized normal sequence of b.

(a) First we show that $K = \bigcup_{i \in \mathbb{N}_0} H_i$. Clearly $\bigcup_{i \in \mathbb{N}_0} H_i \subseteq K$. Let $k \in K$. Then $[k,_n b] = 1$ for a positive integer n. Then $[k,_{n-1} b] \in H_1$, and then $[k,_{n-2} b] \in H_2$. So by continuing this way, we see that $k \in H_n$. Hence $K = \bigcup_{i \in \mathbb{N}_0} H_i$.

Note that

$$\langle b, H_0 \rangle \lhd \langle b, H_1 \rangle \lhd \langle b, H_2 \rangle \lhd \cdots .$$

Since $b \in N_G(H_0)$, every element in $\langle b, H_0\rangle$ can be written in the form of $b^l h$ where $h \in H_0$ and l an integer. So $\langle b\rangle^{\langle b,H_0\rangle} = \langle b\rangle^{H_0}$ is locally nilpotent. By Lemma 2.3.1, $\langle b\rangle^{\langle b,H_1\rangle}$ is locally nilpotent. Inductively, $\langle b\rangle^{\langle b,H_i\rangle}$ is locally nilpotent for all $i \in \mathbb{N}_0$. Now

$$\langle b\rangle^{\langle b,H_0\rangle} \subseteq \langle b\rangle^{\langle b,H_1\rangle} \subseteq \langle b\rangle^{\langle b,H_2\rangle} \subseteq \cdots$$

is an ascending chain of locally nilpotent groups. Therefore $\langle b\rangle^{\langle b,K\rangle} = \bigcup_{i\in\mathbb{N}_0} \langle b\rangle^{\langle b,H_i\rangle}$ is locally nilpotent. This implies that $\langle b\rangle^K$ is locally nilpotent, for it is a subgroup of $\langle b\rangle^{\langle b,K\rangle}$.

(b) Since $b \in \overline{L}(G, K)$, there is a fixed positive integer n such that $[k,_n b] = 1$ for all $k \in K$. This implies that $K = H_n = H_{n+1} = \cdots$, and

$$\langle b, H_0\rangle \lhd \langle b, H_1\rangle \lhd \langle b, H_2\rangle \lhd \cdots \lhd \langle b, H_n\rangle = \langle b, K\rangle.$$

Therefore $\langle b\rangle^{H_0} = \langle b\rangle^{\langle b,H_0\rangle}$ is subnormal in $\langle b\rangle^{\langle b,K\rangle}$, and thus subnormal in $\langle b\rangle^K$. $\square$

The following theorem can be proved easily by using Lemmas 3.3.1 and 2.3.1, and by noting that every element in $L(G, S)$ is $(S^{(i+1)}, S^{(i)})$-centralizable where $S^{(i)}$ is the ith derived subgroup of S.

Theorem 3.3.3. *Let S be a normal solvable subgroup of a group G. Then*

(a) *$B(G, S) = \overline{L}(G, S)$,*

(b) *$HP(G, S) = L(G, S)$.*

The following lemma is obvious and it is an analogue of Lemma 3.2.3 for relative Engel elements.

Lemma 3.3.4. *Let S and T be subgroups of a group G. If $S \subseteq T$, then $L(G,T) \subseteq L(G,S)$ and $\overline{L}(G,T) \subseteq \overline{L}(G,S)$.*

Main Theorem 3. *Let G be a group, R be a commutative Noetherian ring with identity and A be a finitely generated R-module. Let S be a normal subgroup of G such that $\langle L(S)\rangle$ is a subgroup of $\mathrm{Aut}_R A$. If $L(G,S)$ is $(HP(S),S)$-centralizable, then*

(a) $B(G,S) = \overline{L}(G,S)$,

(b) $HP(G,S) = L(G,S)$.

Proof. By Theorem 3.2.8, $L(S) = HP(S)$ and $\overline{L}(S) = B(S)$. Furthermore, $HP(S) = \langle L(S)\rangle$ is solvable and $HP(S) \lhd G$.

(a) By Theorem 2.3.2, it is sufficient to show that $\overline{L}(G,S) \subseteq B(G,S)$. By part (a) of Main Theorem 1, $B(G,HP(S)) = \overline{L}(G,HP(S))$. Let $b \in \overline{L}(G,S)$. Then $b \in \overline{L}(G,HP(S))$ by Lemma 3.3.4. So $b \in B(G,HP(S))$, i.e., $\langle b\rangle$ is subnormal in $\langle b\rangle^{HP(S)}$. By part (b) of Lemma 2.3.1, $\langle b\rangle^{HP(S)}$ is subnormal in $\langle b\rangle^S$. Hence $\langle b\rangle$ is subnormal in $\langle b\rangle^S$, and $\overline{L}(G,S) \subseteq B(G,S)$.

(b) By Theorem 2.3.2, it is sufficient to show that $L(G,S) \subseteq HP(G,S)$. By part (b) of Main Theorem 1, $HP(G,HP(S)) = L(G,HP(S))$. Let $b \in L(G,S)$. Then $b \in L(G,HP(S))$ by Lemma 3.3.4. So $b \in HP(G,HP(S))$, i.e., $\langle b\rangle^{HP(S)}$ is locally nilpotent. By part (a) of Lemma 2.3.1, $\langle b\rangle^S$ is locally nilpotent, and thus $b \in HP(G,S)$. Hence $L(G,S) \subseteq HP(G,S)$. □

We note here that when $S = G$ in Main Theorem 3, we have $L(G) = L(G, G) = HP(G)$. Let $H_0 = HP(G)$ and $H_i = G$ for all $i \geq 1$. If $b \in L(G)$, then $\{H_i\}_{i \in \mathbb{N}_0}$ is the $(HP(G), G)$-centralized normal sequence of b. So the condition $L(G)$ is $(HP(G), G)$-centralizable is redundant. Therefore Main Theorem 3 is a generalization of Theorem 3.2.8, and thus a generalization of Theorem 1.4.3.

3.4 Main Theorem 4

Theorem 3.4.1. *Let G be a group, R be a commutative ring with identity and A be an R-module. If $\langle L(G) \rangle$ is a subgroup of $F\mathrm{Aut}_R A$, then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$.*

Proof. By Theorem 1.4.4, $L(\langle L(G) \rangle) = HP(\langle L(G) \rangle)$ and $\overline{L}(\langle L(G) \rangle) = B(\langle L(G) \rangle)$. Therefore $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$ by Theorem 3.2.4. □

Note that in Theorem 3.4.1, we have replaced the condition 'G is a subgroup of $F\mathrm{Aut}_R A$' in Theorem 1.4.4 with '$\langle L(G) \rangle$ is a subgroup of $F\mathrm{Aut}_R A$'.

Lemma 3.4.2. [28, Exercise 12.3.6 on p. 362] *Let x, a be two elements of a group such that $[x,_n a] = 1$ for some positive integer n. Then $\langle x \rangle^{\langle a \rangle}$ is finitely generated. In fact,*

$$\langle x \rangle^{\langle a \rangle} = \langle x, [x, a], [x,_2 a], \ldots, [x,_{n-1} a] \rangle .$$

Theorem 3.4.3. *Let S be a normal locally solvable subgroup of a group G. Then $HP(G, S) = L(G, S)$.*

Proof. By Theorem 2.3.2, it is sufficient to show that $L(G, S) \subseteq HP(G, S)$. Let $a \in L(G, S)$. We need to show that $\langle a \rangle^S$ is locally nilpotent. Let K be a finitely generated subgroup of $\langle a\rangle^S$. Then $K = \langle k_1, \ldots, k_m\rangle$, where

$$k_i = \prod_{j=1}^{l_i} s_{ij}^{-1} a^{z_{ij}} s_{ij},$$

$s_{ij} \in S$ and z_{ij} is an integer.

Let T be the subgroup generated by all $\langle s_{ij}\rangle^{\langle a\rangle}$, i.e.,

$$T = \langle \{\langle s_{ij}\rangle^{\langle a\rangle} \ : \ \text{for all } i, j\}\rangle.$$

Since S is normal in G, T is a subgroup of S. Furthermore, by Lemma 3.4.2, T is finitely generated. So T is solvable.

Let the derived length of T be d. Note that $a \in N_G(T)$. Since the ith derived subgroup $T^{(i)}$ is a characteristic subgroup of T, we have $a \in N_G(T^{(i)})$. Therefore $T^{(i)}$ is a normal subgroup of $\langle T, a\rangle$. By Lemma 3.3.4, $a \in L(G, T^{(i)})$, and by Lemma 3.3.1, a is $(T^{(i+1)}, T^{(i)})$-centralizable. Now $\langle a\rangle = \langle a\rangle^{T^{(d)}}$ is abelian, and thus locally nilpotent. So, by part (a) of Lemma 2.3.1, $\langle a\rangle^{T^{(d-1)}}$ is locally nilpotent. By applying Lemma 2.3.1 repeatedly, we see that $\langle a\rangle^T$ is locally nilpotent. Since K is a subgroup of $\langle a\rangle^T$, K is nilpotent. Hence $\langle a\rangle^S$ is locally nilpotent, and $HP(G, S) = L(G, S)$.

□

Main Theorem 4. *Let G be a group, R be a commutative ring with identity and A be an R-module. Let S be a normal subgroup of G such that $\langle L(S)\rangle$ is a subgroup of $F\mathrm{Aut}_R A$. If $L(G, S)$ is $(HP(S), S)$-centralizable, then $HP(G, S) = L(G, S)$.*

Proof. By Theorem 2.3.2, it is sufficient to show that $L(G, S) \subseteq HP(G, S)$. Note that $HP(S)$ is a normal locally solvable subgroup of G. By Theorem 3.4.3, $HP(G, HP(S)) = L(G, HP(S))$. Let $b \in L(G, S)$. Then $b \in L(G, HP(S))$ by

Lemma 3.3.4. So $b \in HP(G, HP(S))$, i.e., $\langle b \rangle^{HP(S)}$ is locally nilpotent. By part (a) of Lemma 2.3.1, $\langle b \rangle^{S}$ is locally nilpotent, and thus $b \in HP(G, S)$. Hence $L(G, S) \subseteq HP(G, S)$. $\square$

We note here that when $S = G$ in Main Theorem 4, we have $L(G) = L(G, G) = HP(G)$. Furthermore, $L(G)$ is $(HP(G), G)$-centralizable. Therefore Main Theorem 4 is a generalization of Theorem 3.4.1, and thus a generalization of Theorem 1.4.4. However we do not know whether $\overline{L}(G, S) = B(G, S)$ or not, under the hypothesis of Theorem 3.4.3 and Main Theorem 4.

Finally we would like to refer the reader to Abdollahi [1], Abdollahi and Khosravi [2], Crosby and Traustason [9, 10], and Newell [27], for some recent results on left and right Engel elements. It is natural to ask whether similar results hold for relative left and right Engel elements.

Chapter 4

Non-Engel Elements

4.1 A Brief Introduction

Note that every finite group G satisfies a law $[x,_r y] = [x,_s y]$ for some positive integers $r < s$. The minimal value of r is called the Engel depth of G (see [5, 6]). So, given any non-left Engel element $g \in G$, there exists a $h \in G$ and a positive integer n such that $h = [h,_n g]$. However we do not know whether $g = [g,_n h]$ or not. This motivates us to propose the following problem.

Problem 4.1.1. Let G be a finite group. Given any positive integer n, does there exist $h, g \in G$ such that $h = [h,_n g]$ and $g = [g,_n h]$?

Note that when $n = 1$, we have $h = 1 = g$. So we shall only consider $n \geq 2$. Since every finite group can be embedded into a Symmetric group, we may first consider the following problem.

Problem 4.1.2. Let $[l] = \{1, 2, \ldots, l\}$ and S_l be the symmetric group on $[l]$, $l \geq 2$. Given any positive integer $n \geq 2$, does there exist $\alpha, \beta \in S_l$ such that $\beta = [\beta,_n \alpha]$ and $\alpha = [\alpha,_n \beta]$?

Let us recall the following definition.

Definition 1.5.1. Let G be a group and $h, g \in G$. The 2-tuple (h, g) is said to be an *n-Engel pair*, $n \geq 2$, if $h = [h,_n g]$, $g = [g,_n h]$ and $h \neq 1$.

Now consider the following elements in S_{40}:

$$\alpha_1 = (1\ \ 2\ \ 3\ \ 4\ \ 5),$$

$$\beta_1 = (1\ \ 5\ \ 2\ \ 4\ \ 3),$$

$$u_1 = (1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10),$$

$$u_2 = (11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18\ \ 19\ \ 20),$$

$$u_3 = (21\ \ 22\ \ 23\ \ 24\ \ 25\ \ 26\ \ 27\ \ 28\ \ 29\ \ 30),$$

$$u_4 = (31\ \ 32\ \ 33\ \ 34\ \ 35\ \ 36\ \ 37\ \ 38\ \ 39\ \ 40),$$

$$v_1 = (1\ \ 8\ \ 11\ \ 21\ \ 18\ \ 6\ \ 3\ \ 16\ \ 26\ \ 13),$$

$$v_2 = (2\ \ 31\ \ 30\ \ 27\ \ 39\ \ 7\ \ 36\ \ 25\ \ 22\ \ 34),$$

$$v_3 = (4\ \ 37\ \ 12\ \ 33\ \ 5\ \ 9\ \ 32\ \ 17\ \ 38\ \ 10),$$

$$v_4 = (14\ \ 40\ \ 20\ \ 24\ \ 28\ \ 19\ \ 35\ \ 15\ \ 29\ \ 23),$$

$$\alpha_2 = u_1 u_2 u_3 u_4,$$

$$\beta_2 = v_1 v_2 v_3 v_4.$$

Note that (α_1, β_1) and (α_2, β_2) are 5-Engel pairs. Let H_1 and H_2 be the subgroups generated by $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$, respectively. Then H_1 has 60 elements and H_2 has 120 elements. Since H_1 is also a subgroup of S_5 and has 60 elements (thus of index 2), it must be the alternating subgroup A_5. Note that α_2^5 is in the centre of H_2 and $H_2/\langle\alpha_2^5\rangle \cong H_1 = A_5$. Therefore H_2 is the central extension of the cyclic group of order 2 by A_5.

It can be verified that $\alpha_i\beta_i\alpha_i = \beta_i\alpha_i\beta_i$ for $i = 1, 2$. This motivates us to consider n-Engel pairs with such property. In other words, given any 5-Engel pair (α, β) with $\alpha\beta\alpha = \beta\alpha\beta$, we would like to know about the structure of the the subgroups generated by $\{\alpha, \beta\}$.

In this chapter, we will show that if (h, g) is an n-Engel pair and $hgh = ghg$, then n must be a multiple of 5. Furthermore, the subgroup generated by $\{h, g\}$ is either isomorphic to A_5 or H_2 (see Main Theorem 5). We will also show that if (h, g) is an n-Engel pair, $hg^th = ghg$, and $gh^tg = hgh$, then n and $t - 1$ must be a multiple of 5, and $hgh = ghg$ (see Theorem 4.4.2).

The main results in this chapter have been published (see S. G. Quek, K. B. Wong, P. C. Wong, *On certain pairs of non-Engel elements in finite groups*, J. Algebra Appl. **12** (2013), #1250213).

4.2 Equivalent forms

In this section, we shall assume x, y are elements in a group G.

Lemma 4.2.1. *Let k be the smallest positive integer such that (x, y) is a k-Engel pair. If (x, y) is an n-Engel pair, then n is a multiple of k.*

Proof. By the Division Algorithm, $n = qk + r$ for some positive integers q, r and $0 \leq r < k$. If $r \neq 0$, then $x = [x,_n y] = [x,_{qk+r} y] = [x,_r y]$ and $y = [y,_r x]$. If $r = 1$, then $y = 1 = x$, a contradiction. So $r \geq 2$, and (x, y) is an r-Engel pair, again a contradiction, for $r < k$. Hence $r = 0$ and n is a multiple of k. $\square$

The following corollary follows from Lemma 4.2.1.

Corollary 4.2.2. *Let k be the smallest positive integer such that (x, y) is a k-Engel pair. If (x, y) is a p-Engel pair and p is a prime, then $p = k$.*

Note that the following are two equivalent variants of $yxy = xyx$.

$$x^y = y^{x^{-1}}, \tag{4.2.1}$$

$$[x, y] = yx^{-1}. \tag{4.2.2}$$

From these, we get the following useful consequences.

$$x^y = xyx^{-1} = y(x^y)x^{-1}, \tag{4.2.3}$$

$$[x,_2 y] = [x^{-1}, y] = y^{-x^{-1}}y = x^{-y}y = (x^{-1}y)^y = (yx^{-1})^{y^2} = [x, y]^{y^2}. \tag{4.2.4}$$

By induction on n and (4.2.4), we derive that

$$[x,_{n+1} y] = [x, y]^{y^{2n}}. \tag{4.2.5}$$

All consequences have a variant where we swap x and y.

Lemma 4.2.3. *If $x = [x,_n y]$, $y = [y,_n x]$, and $yxy = xyx$, then $x^{y^2} = yx^{-1}$ and $y^{x^2} = xy^{-1}$.*

Proof. By (4.2.5), $[x, y] = [x,_{n+1} y] = [x, y]^{y^{2n}}$. It then follows from (4.2.2) that $x^{-1} = (x^{-1})^{y^{2n}}$. So, y^{2n} commutes with x. Then by using (4.2.4),

$$x^{y^2} = [x,_n y]^{y^2} = [[x, y]^{y^2},_{n-1} y] = [x,_{n+1} y] = [x, y]^{y^{2n}} = [x, y] = yx^{-1}.$$

By symmetry, $y^{x^2} = xy^{-1}$. $\square$

Lemma 4.2.4.

(a) *If* $x^{y^2} = yx^{-1}$, *then* $y^5x = xy^5$.

(b) *If* $x^{y^2} = yx^{-1}$, $y^{x^2} = xy^{-1}$ *and* $yxy = xyx$, *then* $y^5 = x^5$, $y^{10} = x^{10} = 1$, $x = [x,_5 y]$, *and* $y = [y,_5 x]$.

Proof. (a) It follows from $x^{y^4} = y(x^{y^2})^{-1} = yxy^{-1} = x^{y^{-1}}$.

(b) By using (4.2.3) repeatedly, we obtain $x^y = y^5(x^y)x^{-5}$. It then follows from part (a) of this lemma that $y^5 = x^5$. Therefore x^5 commutes with y and $x^5 = (x^{y^2})^5 = (yx^{-1})^5$. By symmetry, $y^5 = (xy^{-1})^5$. So, $x^5 = y^{-5} = x^{-5}$, and thus $y^{10} = x^{10} = 1$. By (4.2.2), $x^{y^2} = yx^{-1} = [x, y]$. Then by (4.2.5), $[x,_5 y] = [x, y]^{y^8} = x^{y^{10}} = x$. By symmetry, $y = [y,_5 x]$. $\square$

Theorem 4.2.5. *Let* n *be a positive integer and*

$$G_n = \langle x, y\ ;\ x = [x,_n y], y = [y,_n x], yxy = xyx\rangle.$$

Then G_n *is the trivial group if* n *is not a multiple of 5. Furthermore, for all positive integers* l,

$$G_{5l} \cong G_5 \cong \langle x, y\ ;\ xy^2x = y^3, yx^2y = x^3, yxy = xyx\rangle.$$

Proof. Suppose G_n is not the trivial group. Then (x, y) is an n-Engel pair. By Lemma 4.2.3 and part (b) of Lemma 4.2.4, (x, y) is a 5-Engel pair. Since 5 is a prime, it follows from Lemma 4.2.1 and Corollary 4.2.2 that n is a multiple of 5.

Note that $x^{y^2} = yx^{-1}$ and $y^{x^2} = xy^{-1}$ are equivalent to $xy^2x = y^3$ and $yx^2y = x^3$, respectively. By Lemma 4.2.3 and part (b) of Lemma 4.2.4, we deduce that

$$
\begin{aligned}
G_{5l} &= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx \rangle \\
&= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx, xy^2x = y^3, yx^2y = x^3 \rangle \\
&= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx, xy^2x = y^3, \\
&\qquad\qquad\qquad\qquad yx^2y = x^3, x = [x,_5 y], y = [y,_5 x] \rangle \\
&= \langle x, y \ ; \ x = [x,_5 y], y = [y,_5 x], yxy = xyx \rangle \\
&= \langle x, y \ ; \ xy^2x = y^3, yx^2y = x^3, yxy = xyx \rangle.
\end{aligned}
$$

□

4.3 Main Theorem 5

Let A be a non-empty set. This set A is called an *alphabet* and the elements of A are called *letters*. We shall denote the free semigroup on A by A^+. The elements of A^+ are called *words*. Given a word $W \in A^+$, we shall denote its length by $\|W\|$, defined as the number of letters in W.

A *rewriting system* R over A is a set of rules $U \to V$, which are elements of $A^+ \times A^+$. A word $W_1 \in A^+$ is said to be rewritten to another word $W_2 \in A^+$ by a *one-step reduction* induced by R, if $W_1 = Z_1XZ_2$ and $W_2 = Z_1YZ_2$ for a rule $X \to Y$ in R. In this situation we write $W_1 \to_R W_2$. The reflexive transitive closure and the reflexive symmetric transitive closure of $\to_R$ are denoted by $\to_R^*$ and $\leftrightarrow_R^*$, respectively. The relation $\leftrightarrow_R^*$ is defined to be the congruence on A^+ generated by R and it defines the quotient semigroup $M = A^+/\leftrightarrow_R^*$. M is said to be presented by the *semigroup presentation* $[A; R]$. If both A and R are finite, we say the semigroup presentation is finitely presented. For $U \in A^+$, $[U]_R$ shall denote the class of U

modulo $\leftrightarrow_R^*$.

A word $W \in A^+$ is called an *irreducible word* if W does not contain any subword U in which $U \to V$ is a rule in R.

We say R is *Noetherian* if there is no infinite reduction sequence,

$$W_1 \to_R W_2 \to_R W_3 \to_R \cdots .$$

R is said to be *confluent* if whenever $U \to_R^* V$ and $U \to_R^* W$, then there is an $X \in A^+$ such that $V \to_R^* X$ and $W \to_R^* X$. If R is both Noetherian and confluent, we say that R is a *complete rewriting system* (see [8, 11, 12, 20, 19, 25, 34]).

Let $A = \{x, x^{-1}, y, y^{-1}, e\}$ and R be the following rules:

$$\begin{array}{ccc}
ee \to e, & ex \to x, & xe \to x, \\
ey \to y, & ye \to y, & ex^{-1} \to x^{-1}, \\
x^{-1}e \to x^{-1}, & y^{-1}e \to y^{-1}, & ey^{-1} \to y^{-1}, \\
xx^{-1} \to e, & x^{-1}x \to e, & yy^{-1} \to e, \\
y^{-1}y \to e, & yxy \to xyx, & xy^2x \to y^3, \\
yx^2y \to x^3. & &
\end{array}$$

By using the well known Knuth-Bendix rewriting completion algorithm (see [4, Chapter 7]) or by GAP [13], one can find a complete rewriting system R^c such that

$$M = [A; R] = [A; R^c].$$

Note that $M = [A; R]$ is a group and by Theorem 4.2.5,

$$M \cong \langle x, y \ ; \ xy^2x = y^3, yx^2y = x^3, yxy = xyx \rangle \cong G_5.$$

Recall that H_2 is the subgroup of S_{40}, generated by α_2, β_2, and it has exactly 120 elements. By Theorem 4.2.5, the mapping $\psi : G_5 \to H_2$ defined by $\psi(x) = \alpha_2$ and $\psi(y) = \beta_2$ is an epimorphism. For each $u \in H_2$, there exists a $v \in G_5$ such that $\psi(v) = u$. By using the complete rewriting system R^c, we may assume that v is irreducible. Therefore we have 120 irreducible words. Again, by using the complete rewriting system R^c, it can be shown that all these words are distinct in G_5. Let T be the set of all these words. Then $xT = T = yT$. Therefore T is a subgroup of G_5. Since G_5 is generated by x, y, we conclude that $T = G_5$. Thus $G_5 \cong H_2$, via ψ. Recall that A_5 is the subgroup of S_{40}, generated by α_1, β_1. Since $H_2/\langle \alpha_2^5 \rangle \cong A_5$, we conclude that $G_5/\langle x^5 \rangle \cong A_5$. Hence we have proved part (a) of the following theorem.

Main Theorem 5.

(a) $G_5 \cong \langle x, y\ ;\ xy^2x = y^3, yx^2y = x^3, yxy = xyx \rangle \cong H_2$

$G_5/\langle x^5 \rangle \cong \langle x, y\ ;\ xy^2x = y^3, yx^2y = x^3, yxy = xyx, x^5 \rangle \cong A_5.$

(b) *If N is a non-trivial proper normal subgroup of G_5, then $N = \langle x^5 \rangle = Z(G_5)$, where $Z(G_5)$ is the centre of G_5.*

(c) *Let G be a group and $h, g \in G$. If (h, g) is an n-Engel pair and $hgh = ghg$, then n must be a multiple of 5. Furthermore, the subgroup generated by $\{h, g\}$ is isomorphic to A_5 if the order of h is 5, and is isomorphic to H_2 if the order of h is 10.*

Proof. (b) By part (a) of Lemma 4.2.4, $x^5 \in Z(G_5)$. Since $N\langle x^5 \rangle/\langle x^5 \rangle$ is normal in $G_5/\langle x^5 \rangle \cong A_5$ and A_5 is simple, either $N\langle x^5 \rangle = G_5$ or $N\langle x^5 \rangle = \langle x^5 \rangle$. Suppose the latter holds. Then $|N||\langle x^5\rangle|/|N \cap \langle x^5\rangle| = |N\langle x^5\rangle| = |\langle x^5\rangle|$ and $|N| = |N \cap \langle x^5\rangle|$. Thus $N = \langle x^5\rangle$, for $|\langle x^5\rangle| = 2$.

Suppose the former holds. Then $|N||\langle x^5\rangle|/|N \cap \langle x^5\rangle| = |G_5| = |H_2| = 120 = 2^3 \cdot 3 \cdot 5$. If $|N \cap \langle x^5\rangle| \neq 1$, then $x^5 \in N$ and $N = G_5$. So, we may assume $|N \cap \langle x^5\rangle| = 1$. This implies that $|N| = 60$. Since 5 divides $|N|$, N contains all the Sylow 5-subgroups of G_5. In particular, $x^2 \in N$. Since $y = x^{-1}y^{-1}xyx$, we have $y^2 = x^{-1}y^{-1}x^2yx \in N$ and $(x^2y^2)^2 \in N$. Note that $(x^2y^2)^2 = x^2y^2x^2y^2 = x^2yx^3y$. So $x^2yx^3y(y^4) \in N$. By part (b) of Lemma 4.2.4, $x^2yx^{-2} = x^2yx^8 = x^2yx^3y^5 \in N$. Hence $y \in N$ and $N = G_5$. This completes the proof of part (b) of this theorem.

(c) Let U be the subgroup generated by h, g, and G_n be defined as in Theorem 4.2.5. Then $\phi : G_n \to U$ defined by $\phi(x) = h$ and $\phi(y) = g$ is an epimorphism.

If n is a not a multiple of 5, then by Theorem 4.2.5, G_n is the trivial group. This implies that U is the trivial group and $h = g = 1$, a contradiction, for (h, g) is an n-Engel pair. Hence n must be a multiple of 5.

Let $n = 5l$. By Theorem 4.2.5 and part (b) of this theorem, we conclude that $U \cong G_5 \cong H_2$ if h is of order 10, or $U \cong G_5/\langle x^5\rangle \cong H_1$ if h is of order 5. $\square$

4.4 A Generalization

In this section, we shall assume x, y are elements in a group G.

Lemma 4.4.1. *Suppose $x = [x,_n y]$ and $y = [y,_n x]$. If $yx^ty = xyx$ and $xy^tx = yxy$, then $xyx = yxy$.*

Proof. Note that the following are two equivalent variants of $xy^tx = yxy$.

$$y^{x^{-1}} = y^{-1}xy^t, \tag{4.4.1}$$

$$[x, y] = yx^{-1}y^{1-t}. \tag{4.4.2}$$

From these, we obtain

$$\begin{aligned}[x,_2 y] &= [yx^{-1}y^{1-t}, y] = [x^{-1}, y]^{y^{1-t}} = (y^{-x^{-1}}y)^{y^{1-t}} \\ &= ((y^{-1}xy^t)^{-1}y)^{y^{1-t}} = (yx^{-1}y^{1-t})^{y^2} = [x, y]^{y^2}. \end{aligned} \tag{4.4.3}$$

By induction on n and (4.4.3), we derive that

$$[x,_{n+1} y] = [x, y]^{y^{2n}}. \tag{4.4.4}$$

All identities have a variant where we swap x and y.

By (4.4.4), $[x, y] = [x,_{n+1} y] = [x, y]^{y^{2n}}$. It then follows from (4.4.2) that $x^{-1} = (x^{-1})^{y^{2n}}$. So, y^{2n} commutes with x. Then by using (4.4.3),

$$y^{-2}xy^2 = x^{y^2} = [x,_n y]^{y^2} = [[x, y]^{y^2},_{n-1} y] = [x,_{n+1} y] = [x, y]^{y^{2n}} = [x, y] = yx^{-1}y^{1-t},$$

which is equivalent to $x^{-1}y^{-1}x = y^{-3}xy^tx$. Therefore $x^{-1}y^{-1}x = y^{-3}(yxy) = y^{-2}xy$, i.e., $xy^{-2}x = y^{-1}xy^{-1}$.

By symmetry, $yx^{-2}y = x^{-1}yx^{-1}$, and multiplying these two identities together gives

$$xy^{-1}x^{-1} = (xy^{-2}x)(x^{-1}yx^{-1}) = (y^{-1}xy^{-1})(yx^{-2}y) = y^{-1}x^{-1}y,$$

which is equivalent to $xyx = yxy$. $\square$

Theorem 4.4.2. *Let*

$$G_{n,t} = \langle x, y \ ; \ x = [x,_n y], y = [y,_n x], yxy = xy^t x, xyx = yx^t y \rangle.$$

Then

(a) *$G_{n,t}$ is the trivial group if n or $t-1$ is not a multiple of 5.*

(b) *If $t \equiv 1 \mod 5$, then*

$$G_{5l,t} \cong \begin{cases} A_5 & \text{if } t \equiv 6 \mod 10 \\ H_2 & \text{if } t \equiv 1 \mod 10. \end{cases}$$

(c) *Let G be a group and $h, g \in G$. If (h, g) is an n-Engel pair, $hg^t h = ghg$, and $gh^t g = hgh$, then n and $t-1$ must be a multiple of 5. Furthermore, $hgh = ghg$.*

Proof. By Lemma 4.4.1,

$$G_{n,t} = \langle x, y \ ; \ x = [x,_n y], y = [y,_n x], yxy = xy^t x, yx^t y = xyx, yxy = xyx \rangle.$$

(a) Let G_n be defined as in Theorem 4.2.5. Then $G_{n,t}$ is an epimorphic image of G_n. If n is not a multiple of 5, then G_n and thus $G_{n,t}$ is the trivial group.

Suppose $\gcd(5, t-1) = 1$. Then there exist integers z_1, z_2 such that $5z_1 + (t-1)z_2 = 1$. From $yxy = xyx$ and $xy^t x = yxy$, we have $y^{t-1} = 1$. By Lemma 4.2.3 and part (a) of Lemma 4.2.4, $y^5 x = xy^5$. This implies that $y = y^{5z_1}$ commutes with x, and thus $y = [y,_n x] = 1$, $x = [x,_n y] = 1$, and $G_{n,t}$ is the trivial group.

(b) Note that $y^{t-1} = 1 = x^{t-1}$. Therefore

$$\begin{aligned} G_{5l,t} &= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xy^t x, yx^t y = xyx, yxy = xyx \rangle \\ &= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx, x^{t-1}, y^{t-1} \rangle. \end{aligned}$$

Furthermore, by Lemma 4.2.3 and part (b) of Lemma 4.2.4, $x^{10} = 1 = y^{10}$.

Now, either $t \equiv 1 \mod 10$ or $t \equiv 6 \mod 10$. Suppose $t \equiv 1 \mod 10$. By Theorem 4.2.5 and part (a) of Main Theorem 5,

$$\begin{aligned} G_{5l,t} &= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx, x^{t-1}, y^{t-1} \rangle \\ &= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx \rangle \\ &\cong H_2. \end{aligned}$$

Suppose $t \equiv 6 \mod 10$. Then $x^{t-1} = x^5 = 1 = y^5 = y^{t-1}$. By Lemma 4.2.3 and part (b) of Lemma 4.2.4, $y^5 = x^5$. So, by Theorem 4.2.5 and part (a) of Main Theorem 5,

$$\begin{aligned} G_{5l,t} &= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx, x^5, y^5 \rangle \\ &= \langle x, y \ ; \ x = [x,_{5l} y], y = [y,_{5l} x], yxy = xyx, x^5 \rangle \\ &\cong A_5. \end{aligned}$$

(c) Let U be the subgroup generated by h, g. Then $\phi : G_{n,t} \to U$ defined by $\phi(x) = h$ and $\phi(y) = g$ is an epimorphism. Since U cannot be the the trivial group, by part (a) of this theorem, n and $t - 1$ must be a multiple of 5. The identity $hgh = ghg$ follows from Lemma 4.4.1. □

Remark 1. Heineken [18, Theorem 1] showed that $SL(2, 5)$ is generated by a 5-Engel pair. In this chapter, we give a presentation of $SL(2, 5)$ in terms of a 5-Engel pair. We also characterize all groups generated by this Engel pair. The results in this chapter are therefore an extension of Theorem 1 in [18].

Chapter 5

Engel Pairs I

5.1 A Brief Introduction

Let $SL(n, q)$ and $PSL(n, q)$ be the special linear group and projective linear group, respectively, of order n over the field of order q. It was shown in Main Theorem 5 that if (h, g) is an n-Engel pair and $hgh = ghg$, then n must be a multiple of 5. Furthermore, the subgroup generated by $\{h, g\}$ is either isomorphic to A_5 ($PSL(2, 5)$) or is the central extension of the cyclic group of order 2 by A_5 ($SL(2, 5)$). This suggests us to look at Engel pairs in $SL(2, q)$. We find that most of the Engel pairs (h, g) in $SL(2, q)$ for all values of $q < 100$, satisfy either $hgh = ghg$, or $hgh^{-2}gh = ghg$ and $ghg^{-2}hg = hgh$. This motivates us to study Engel pairs satisfying the latter conditions.

Let n be a positive integer and

$$G_n = \langle x, y \ ; \ x = [x,_n y], y = [y,_n x], xyx^{-2}yx = yxy, yxy^{-2}xy = xyx\rangle.$$

We will show that $G_{2k} = \langle x, y \ ; \ y^k = x^k, xyx^{-2}yx = yxy, yxy^{-2}xy = xyx\rangle$ for all integers $k \geq 1$ (Lemma 5.2.2) and G_n is the trivial group when n is odd (Theorem 5.2.3). We apply these results to prove that if (h, g) is an n-Engel pair in a group H satisfying the conditions $hgh^{-2}gh = ghg$ and $ghg^{-2}hg = hgh$, then $n = 2k$ where

$k = 4$ or $k \geq 6$ (Main Theorem 6). Furthermore, the subgroup generated by $\{h, g\}$ is

(a) $SL(2,7)$ or $PSL(2,7)$ if $k = 4$,

(b) $SL(2,13)$ or $PSL(2,13)$ if $k = 6, 7$,

(c) an extension of an abelian group by $PSL(2,7)$ if $k = 8$.

The main results in this chapter have been published (see S. G. Quek, K. B. Wong, P. C. Wong, *On n-Engel pair satisfying certain conditions*, J. Algebra Appl. **13** (2014), #1350135).

5.2 Main Theorem 6

Lemma 5.2.1. *Let x, y be elements in a group G. If $xyx^{-2}yx = yxy$ and $yxy^{-2}xy = xyx$, then*

(a) $[x, y] = yx^{-1}y^{-1}x^2$ *and* $[x, y] = y^{-2}xyx^{-1}$,

(b) $[y, x] = xy^{-1}x^{-1}y^2$ *and* $[y, x] = x^{-2}yxy^{-1}$,

(c) $yxy^{-3}xy = x^3$,

(d) $yxy^{-1}xy = xyx^{-1}yx$,

(e) $yxy^{-1}x^r y^{-1}xy = xy^{r+1}x$ *and* $xyx^{-1}y^r x^{-1}yx = yx^{r+1}y$ *for* $r \geq 1$,

(f) $[x,_{2r} y] = x^{y^r}$ *and* $[y,_{2r} x] = y^{x^r}$ *for* $r \geq 1$.

Proof. (a) $[x, y] = yx^{-1}y^{-1}x^2$ is obtained from

$$(x^{-1}y^{-1})(xyx^{-2}yx)(x^{-1}y^{-1}x^2) = (x^{-1}y^{-1})(yxy)(x^{-1}y^{-1}x^2),$$

and $[x, y] = y^{-2}xyx^{-1}$ is obtained from

$$(x^{-1}y^{-1})(yxy^{-2}xy)(x^{-1}) = (x^{-1}y^{-1})(xyx)(x^{-1}).$$

(b) By swapping x and y in part (a).

(c) It follows from part (a), $yx^{-1}y^{-1}x^2 = [x, y] = y^{-2}xyx^{-1}$.

(d) It follows from

$$(yxy)(y^{-2}xy) = (xyx^{-2}yx)(y^{-2}xy) = (xyx^{-2})(yxy^{-2}xy) = (xyx^{-2})(xyx).$$

(e) It is sufficient to show that $yxy^{-1}x^r y^{-1}xy = xy^{r+1}x$. The second equation can be obtained similarly by swapping x and y.

By part (d),

$$\begin{aligned} yxy^{-1}xy^{-1}xy &= (yxy^{-2})(yxy^{-1}xy) \\ &= (yxy^{-2})(xyx^{-1}yx) \\ &= (yxy^{-2}xy)(x^{-1}yx) \\ &= (xyx)(x^{-1}yx) = xy^2x. \end{aligned}$$

Suppose $yxy^{-1}x^r y^{-1}xy = xy^{r+1}x$ for some r. Multiplying both sides on the right by $x^{-1}yx$, $(yxy^{-1}x^r y^{-1}xy)(x^{-1}yx) = (xy^{r+1}x)(x^{-1}yx) = xy^{r+2}x$. By part (d),

$$\begin{aligned} (yxy^{-1}x^r y^{-1}xy)(x^{-1}yx) &= (yxy^{-1}x^r y^{-1})(xyx^{-1}yx) \\ &= (yxy^{-1}x^r y^{-1})(yxy^{-1}xy) \\ &= yxy^{-1}x^{r+1}y^{-1}xy. \end{aligned}$$

Hence, $yxy^{-1}x^r y^{-1}xy = xy^{r+1}x$ for $r \geq 1$.

(f) It is sufficient to show that $[x,_{2r} y] = x^{y^r}$. The second equation can be obtained similarly by swapping x and y. Note that by part (a) and the identity $[uv, w] = [u, w]^v [u, w]$,

$$
\begin{aligned}
[x,_2 y] &= [[x, y], y] \\
&= [y^{-2}xyx^{-1}, y] \\
&= [xyx^{-1}, y] \\
&= xy^{-1}(x^{-1}y^{-1}xy)x^{-1}y \\
&= xy^{-1}([x, y])x^{-1}y \\
&= xy^{-1}(yx^{-1}y^{-1}x^2)x^{-1}y \\
&= x^y.
\end{aligned}
$$

Suppose $[x,_{2r} y] = x^{y^r}$ for some r. Then $[x,_{2(r+1)} y] = [[x,_2 y],_{2r} y] = [x^y,_{2r} y] = [x,_{2r} y]^y = (x^{y^r})^y = x^{y^{r+1}}$, where the second last equation follows by induction. Hence, $[x,_{2r} y] = x^{y^r}$ for $r \geq 1$. □

Lemma 5.2.2. *Let x, y be elements in a group G and l a positive integer. Then the following relations are equivalent:*

(a) $x = [x,_{2l} y]$, $y = [y,_{2l} x]$, $xyx^{-2}yx = yxy$ *and* $yxy^{-2}xy = xyx$*;*

(b) $y^l = x^l$, $xyx^{-2}yx = yxy$ *and* $yxy^{-2}xy = xyx$*.*

Proof. ((a) ⇒ (b)). By part (f) of Lemma 5.2.1, $x = x^{y^l} = y^{-l}xy^l$ and $y = y^{x^l} = x^{-l}yx^l$. By part (e) of Lemma 5.2.1,

$$x^l(yxy^{-2}xy) = yxy^{-1}x^ly^{-1}xy = xy^{l+1}x = y^l(xyx).$$

Since $yxy^{-2}xy = xyx$, $y^l = x^l$.

((b) $\Rightarrow$ (a)). It follows from part (f) of Lemma 5.2.1. $\square$

Let n be a positive integer and

$$G_n = \langle x, y \ ; \ x = [x,_n y], y = [y,_n x], xyx^{-2}yx = yxy, yxy^{-2}xy = xyx \rangle.$$

Theorem 5.2.3. *G_n is the trivial group when n is odd.*

Proof. Let $n = 2k + 1$. Note that $x = [x,_n y] = [[x,_n y],_n y] = [x,_{2n} y]$. Similarly, $y = [y,_{2n} x]$. By Lemma 5.2.2, $y^n = x^n$.

By part (f) of Lemma 5.2.1, $x = [x,_{2k+1} y] = [[x,_{2k} y], y] = [x^{y^k}, y] = [x, y]^{y^k}$. Therefore $y^k x y^{-k} = x^{-1} y^{-1} x y$ and $x y^k x = y^{-1} x y^{k+1}$. Multiplying the last equation on the right by y^k, we obtain $x y^k x y^k = y^{-1} x y^n$. Similarly, $y x^k y x^k = x^{-1} y x^n$. This implies that

$$\begin{aligned} x^{-k} y^{-1} x^{-k} y^{-1} &= x^{-n} y^{-1} x \\ &= y^{-n} y^{-1} x \\ &= y^{-1} x y^n (y^{-2n}) \\ &= x y^k x y^k (y^{-2n}). \end{aligned}$$

Therefore $y^{-1} x^{-k} y^{-1} = x^{k+1} y^k x y^k (y^{-2n})$. Multiplying the equation by x^n,

$$x^{k+1} y^k x y^k (y^{-n}) = y^{-1} x^{n-k} y^{-1} = y^{-1} x^{k+1} y^{-1}.$$

By part (e) of Lemma 5.2.1,

$$\begin{aligned} x y^{k+2} x &= y x (y^{-1} x^{k+1} y^{-1}) x y \\ &= y x (x^{k+1} y^k x y^k (y^{-n})) x y \\ &= y x^{k+2} y^k x y^k x y (y^{-n}). \end{aligned}$$

Similarly, $yx^{k+2}y = xy^{k+2}x^kyx^kyx(x^{-n})$. Therefore

$$\begin{aligned} xy^{k+2}x &= yx^{k+2}y^kxy^kxy(y^{-n}) \\ &= (yx^{k+2}y)(y^{k-1}xy^kxy(y^{-n})) \\ &= (xy^{k+2}x^kyx^kyx(x^{-n}))(y^{k-1}xy^kxy(y^{-n})) \\ &= xy^{k+2}x^kyx^kyxy^{k-1}xy^kxy(y^{-2n}), \end{aligned}$$

and $x^{k-1}yx^kyxy^{k-1}xy^kxy(y^{-2n}) = 1$.

The equations $y^kxy^kx = x^{-1}y^{-1}x^2y^n$ and $yxy^{-1}x^{-1} = x^2y^{-1}x^{-1}y$ are obtained from $xy^kxy^k = y^{-1}xy^n$ and $yxy = xyx^{-2}yx$, respectively. We will use these equations together with $yx^kyx^k = x^{-1}yx^n$ to simplify the term $x^{k-1}yx^kyxy^{k-1}xy^kxy(y^{-2n})$,

$$\begin{aligned} 1 = x^{k-1}yx^kyxy^{k-1}xy^kxy(y^{-2n}) &= x^{k-1}yx^kyxy^{-1}(y^kxy^kx)y(y^{-2n}) \\ &= x^{k-1}yx^kyxy^{-1}(x^{-1}y^{-1}x^2y^n)y(y^{-2n}) \\ &= x^{k-1}yx^k(yxy^{-1}x^{-1})y^{-1}x^2y(y^{-n}) \\ &= x^{k-1}yx^k(x^2y^{-1}x^{-1}y)y^{-1}x^2y(y^{-n}) \\ &= x^{k-1}yx^{k+2}y^{-1}xy(y^{-n}) \\ &= x^{k-1}yx^{k+2}(x^{-1}yx^n)^{-1}y \\ &= x^{k-1}yx^{k+2}(yx^kyx^k)^{-1}y \\ &= x^{k-1}yx^{k+2}(x^{-k}y^{-1}x^{-k}y^{-1})y \\ &= x^{k-1}yx^2y^{-1}x^{-k}. \end{aligned}$$

Therefore $yx^2y^{-1} = x$ and $yx^2 = xy$. Similarly, $xy^2 = yx$. This implies that $xy^2x = yx^2 = xy$ and $yx = 1$, i.e., $x = y^{-1}$. So $x = [x,_n y] = [y^{-1},_n y] = 1$ and $y = 1$. Hence G_n is the trivial group. $\square$

Main Theorem 6. *If (h, g) is an n-Engel pair in a group H satisfying the conditions $hgh^{-2}gh = ghg$ and $ghg^{-2}hg = hgh$, then $n = 2k$ where $k = 4$ or $k \geq 6$.*

Proof. Note that the subgroup generated by $\{h, g\}$ in H, $\langle h, g\rangle$ is the epimorphic image of G_n via the epimorphism $x \to h$ and $y \to g$. Since $h \neq 1$, G_n cannot be the trivial group. Therefore by Theorem 5.2.3, $n = 2k$. It is sufficient to show that G_{2k} is the trivial group for $k = 1, 2, 3$ and 5.

By Lemma 5.2.2,

$$G_{2k} = \langle x, y \ ; \ y^k = x^k, xyx^{-2}yx = yxy, yxy^{-2}xy = xyx\rangle.$$

If $k = 1$, then $y = x$. This implies that $x = [x,_2 y] = 1$ and $y = 1$. If $k = 2$, then $y^2 = x^2$. This implies that $yxy = xyx^{-2}yx = xy(y^{-2})yx = x^2 = y^2$. So $x = 1$ and $y = 1$. If $k = 3$, then $y^3 = x^3$. By part (c) of Lemma 5.2.1, $y^3 = x^3 = yxy^{-3}xy = yx(x^{-3})xy = yx^{-1}y$. So, $y = x^{-1}$, $x = [x,_2 y] = 1$ and $y = 1$.

Suppose $k = 5$. Then $y^5 = x^5$. By part (c) of Lemma 5.2.1, $x^3 = yxy^{-3}xy = yxy^2xy(y^{-5})$. By part (e) of Lemma 5.2.1, $xyx^{-1}y^4x^{-1}yx = yx^5y = y^7$. So,

$$\begin{aligned} x^3 &= yxy^2xy(y^{-5}) \\ &= yx(y^7)xy(y^{-10}) \\ &= yx(xyx^{-1}y^4x^{-1}yx)xy(y^{-10}) \\ &= yx^2yx^{-1}y^4x^{-1}yx^2y(y^{-10}), \end{aligned}$$

i.e., $1 = yx^2yx^{-1}y^4x^{-1}yx^2yx^{-3}(y^{-10})$.

By part (c) of Lemma 5.2.1, $y^3 = xyx^{-3}yx = xyx^2yx(y^{-5})$. Therefore $yx^2y =$

$x^{-1}y^3x^{-1}y^5$ and

$$\begin{aligned}1 = yx^2yx^{-1}y^4x^{-1}yx^2yx^{-3}(y^{-10}) &= (yx^2y)(x^{-1}y^4x^{-1})(yx^2y)x^{-3}(y^{-10})\\ &= (x^{-1}y^3x^{-1}y^5)(x^{-1}y^4x^{-1})(x^{-1}y^3x^{-1}y^5)x^{-3}(y^{-10})\\ &= x^{-1}y^3x^{-2}y^4x^{-2}y^3x^{-4}.\end{aligned}$$

This implies that $x^5 = y^3x^{-2}y^4x^{-2}y^3$ and

$$1 = x^5y^{-5} = yx^{-2}y^4x^{-2} = y^6x^{-2}y^{-1}x^{-2}.$$

So, $y^{-1} = y^5x^{-2}y^{-1}x^{-2}$ and $y^{-1} = y^5x^{-2}(y^5x^{-2}y^{-1}x^{-2})x^{-2} = (y^{10})x^{-4}y^{-1}x^{-4} = xy^{-1}x$. This implies that $y^{-1}x^{-1}yx = x^2$ and $y = [y,_{10} x] = [[y,x],_9 x] = [x^2,_9 x] = 1$.

Hence G_{2k} is the trivial group when $k = 1, 2, 3$ and 5. $\square$

By using GAP [13], the sizes of G_8, G_{12} and G_{14} are determined to be 336, 2184 and 2184, respectively. Note that GAP uses the Todd-Coxeter procedure (coset enumeration). Coset enumeration is one of the fundamental tools for the examination of finitely presented groups (see [7, 17, 26, 30, 32]).

Let $SL(n, q)$ and $PL(n, q)$ be the special linear group and projective linear group, respectively, of order n over the field of order q. Note that $PL(n, q)$ is a simple group except when $(n, q) = (2, 2)$ or $(2, 3)$ [21, Theorem 6.14 on p. 380].

Let $\alpha_1 = \begin{pmatrix} 5 & 5 \\ 2 & 5 \end{pmatrix}$ and $\beta_1 = \begin{pmatrix} 4 & 5 \\ 6 & 6 \end{pmatrix}$ be elements in $SL(2, 7)$ and $\alpha_2 = \begin{pmatrix} 7 & 7 \\ 0 & 2 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 7 & 0 \\ 12 & 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 7 & 7 \\ 12 & 1 \end{pmatrix}$ and $\beta_3 = \begin{pmatrix} 7 & 5 \\ 9 & 1 \end{pmatrix}$ be elements in $SL(2, 13)$. Note that $\alpha_i\beta_i\alpha_i^{-2}\beta_i\alpha_i = \beta_i\alpha_i\beta_i$ and $\beta_i\alpha_i\beta^{-2}\alpha_i\beta_i = \alpha_i\beta_i\alpha_i$. Furthermore, $\alpha_1^4 = \beta_1^4$, $\alpha_2^6 = \beta_2^6$, $\alpha_2^7 = \beta_2^7$, $\langle\alpha_1, \beta_1\rangle = SL(2, 7)$ and $\langle\alpha_2, \beta_2\rangle = \langle\alpha_3, \beta_3\rangle = SL(2, 13)$. Now, $SL(2, 7)$ is an epimorphic image of G_8 via the epimorphism $x \to \alpha_1$ and $y \to \beta_1$. Since both

$SL(2,7)$ and G_8 have size 336, the epimorphism is an isomorphism. Hence

$$G_8 \cong SL(2,7).$$

Similarly,

$$G_{12} \cong SL(2,13) \cong G_{14}.$$

Let us denote the centre of a group G by $Z(G)$, the derived subgroup of G by G', and the normal closure of a subgroup S of G by S^G.

Note that both $Z(SL(2,7))$ and $Z(SL(2,13))$ have size 2. By Lemma 5.2.2, x^k is in the centre of G_{2k}. Therefore

$$\begin{aligned} G_8/\langle x^4\rangle &\cong SL(2,7)/Z(SL(2,7)) = PL(2,7), \\ G_{12}/\langle x^6\rangle &\cong SL(2,13)/Z(SL(2,13)) = PL(2,13) \cong G_{14}/\langle x^7\rangle. \end{aligned} \tag{1}$$

5.3 Main Theorem 7

By using GAP, the subgroup $\langle x^4, y^4\rangle^{G_{16}}$ is generated by

$$A = \{x^4, y^4, xy^4x^{-1}, x^{-1}y^4x, y^{-1}x^4y, xy^{-1}x^4yx^{-1}, (yx)^3\}.$$

All the elements in A commute with each other. Furthermore, all elements in A are of order 4 except for $(yx)^3$ which is of order 2. Let $a_1 = x^4$, $a_2 = x^4y^4$, $a_3 = x^5y^4x^{-1}$, $a_4 = x^3y^4x$, $a_5 = x^4y^{-1}x^4y$, $a_6 = x^5y^{-1}x^4yx^{-1}$ and $a_7 = (yx)^3$. Then

$$\begin{aligned} \langle x^4, y^4\rangle^{G_{16}} &= \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle \times \langle a_4\rangle \times \langle a_5\rangle \times \langle a_6\rangle \times \langle a_7\rangle \\ &= \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \end{aligned} \tag{2}$$

Since

$$G_{16}/\langle x^4, y^4\rangle^{G_{16}} \cong G_8/\langle x^4\rangle \cong PL(2,7),$$

G_{16} is an extension of $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by $PL(2,7)$.

In fact, one can use the Knuth-Bendix rewriting completion algorithm to get a complete rewriting system (see [4, 8, 11, 12, 20, 19, 25, 34]). By using the complete rewriting system for G_8, a multiplication table can be obtained. So, the size of G_8 will be known. Similarly, the size of G_{12} and G_{14}, and the structure of $\langle x^4, y^4 \rangle^{G_{16}}$ will also be known.

Lemma 5.3.1. *Let S be a normal subgroup of a group G. Suppose $G = G'$, G/S is simple and S is abelian. If N is a proper normal subgroup of G, then N is a subgroup of S.*

Proof. Now NS/S is a normal subgroup of G/S implies that $NS = G$ or $NS = S$. If the latter holds, we are done. Suppose $NS = G$. Then $G/N = NS/N \cong S/(N \cap S)$ is abelian. This implies that $G = G' \subseteq N$, a contradiction. $\square$

Main Theorem 7. *Let (h, g) be a $2k$-Engel pair in a group H satisfying the conditions $hgh^{-2}gh = ghg$ and $ghg^{-2}hg = hgh$. Then*

(a) *$\langle h, g \rangle = SL(2,7)$ or $PL(2,7)$ if $k = 4$;*

(b) *$\langle h, g \rangle = SL(2,13)$ or $PL(2,13)$ if $k = 6, 7$;*

(c) *$\langle h, g \rangle$ is an extension of an abelian group by $PL(2,7)$ if $k = 8$.*

Proof. Since $\langle h, g \rangle$ is the epimorphic image of G_{2k} via the epimorphism $x \to h$ and $y \to g$, the theorem follows from Lemma 5.3.1, equations (1) and (2). $\square$

Remark 2. Heineken [18, Theorem 2] showed that $SL(2,7)$ and $SL(2,13)$ are generated by an 8-Engel pair and a 12-Engel pair, respectively. In this chapter, we give a presentation of $SL(2,7)$ and $SL(2,13)$ in terms of an 8-Engel pair and a 12-Engel pair, respectively. We obtain the presentations for $SL(2,7)$ and $SL(2,13)$ by studying the group G_{2k}. The results here are therefore different from that of Heineken. Furthermore, when $k = 9$, the group G_{2k} is shown to be infinite.

Chapter 6

Engel Pairs II

6.1 A Brief Introduction

Let L be a field. A field F is called a *field extension* of L if $L \subseteq F$. Note that F can be considered as a vector space over the field L. The dimension of F as a vector space over the field L is denoted by $[F : L]$. Let $SL(2, F)$ be the special linear group of order 2 over the field F. If $F = \mathbb{Z}_p$ for a primes p, then we shall write $SL(2, p)$ instead of $SL(2, F)$.

Problem 6.1.1. Given a field F, can we tell whether $SL(2, F)$ has an Engel pair or not?

Problem 6.1.2. Determine the prime p, so that $SL(2, p)$ has an Engel pair.

Lemma 6.1.3. *A solvable group does not have Engel pairs.*

Proof. Let G be a solvable group and (h, g) be an n-Engel pair in G. Then $h \neq 1$. Let the mth derived subgroup of G be denoted by $G^{(m)}$. Since G is solvable, $G^{(m_0)} = \{1\}$ for some positive integer m_0. Now, $h = [h,_n g]$ and $g = [g,_n h]$ imply that $h, g \in G^{(1)}$. In fact, by induction, $h, g \in G^{(m)}$ for every positive integer m. Thus, $h \in G^{(m_0)} = \{1\}$, a contradiction. Hence, the lemma holds. $\square$

Theorem 6.1.4. *$SL(2, 2)$ and $SL(2, 3)$ do not have Engel pairs.*

Proof. It follows from Lemma 6.1.3 and the fact that $SL(2,2)$ and $SL(2,3)$ are solvable. □

In this chapter we will prove the following two theorems.

Main Theorem 8. *Given any field L, there is a field extension F of L with $[F : L] \leq 6$ such that $SL(2,F)$ has an n-Engel pair for some integer $n \geq 4$.*

Main Theorem 9. *Given any field F of characteristic $p \equiv \pm 1 \mod 5$, $SL(2,F)$ has a 5-Engel pair.*

6.2 Main Theorem 8

Lemma 6.2.1. *Let x, y be elements in a group G. If $xyx^{-2}yx = yxy$ and $yxy^{-2}xy = xyx$, then (x,y) cannot be an n-Engel pair for $n = 2$ or 3.*

Proof. Suppose (x,y) is a 2-Engel pair. Then $x = [x,_2 y]$, $y = [y,_2 x]$ and $x \neq 1$. By part (f) of Lemma 5.2.1, $[x,_2 y] = y^{-1}xy$ and $[y,_2 x] = x^{-1}yx$. Therefore, $y^{-1}xy = x$, i.e., $[x,y] = 1$. Thus, $x = [[x,y],y] = 1$, a contradiction.

Suppose (x,y) is a 3-Engel pair. Then $x = [x,_3 y]$, $y = [y,_3 x]$ and $x \neq 1$. Note that $[x,_6 y] = [[x,_3 y],_3 y] = [x,_3 y] = x$ and $[y,_6 x] = y$. By part (f) of Lemma 5.2.1,

$$x = [x,_3 y] = [[x,_2 y], y] = [y^{-1}xy, y] = y^{-1}[x,y]y = y^{-1}x^{-1}y^{-1}xy^2,$$

$$x = [x,_6 y] = y^{-3}xy^3.$$

Therefore $xy^2 = yxyx$ and $xy^3 = y^3x$. Similarly, $yx^2 = xyxy$ and $yx^3 = x^3y$. Thus,

$$y^3x = xy^3 = xy^2y = (yxyx)y = y(xyxy) = y(yx^2) = y^2x^2,$$

which may be equal to $x = y$. So, $[x, y] = 1$, and $x = [x,_3 y] = [[x, y],_2 y] = 1$, a contradiction.

This completes the proof of the lemma. $\square$

Lemma 6.2.2. *Let x, y be non-identity elements in a finite group G. If $xyx^{-2}yx = yxy$ and $yxy^{-2}xy = xyx$, then (x, y) is an n-Engel pair for some integer $n \geq 4$.*

Proof. Since G is a finite group, x and y have finite order. Let m_1 and m_2 be the orders of x and y, respectively. Let m be the least common multiple of m_1 and m_2. Then $x^m = 1 = y^m$. By Lemma 5.2.2, $x = [x,_{2m} y]$ and $y = [y,_{2m} x]$. Since $x \neq 1$, (x, y) is a $2m$-Engel pair. It then follows from Lemma 6.2.1 that $2m \geq 4$. $\square$

Proof of Main Theorem 8. Consider the following equation

$$x^6 + 4x^4 + 3x^2 - 1 = 0$$

over the field L. Let F be a field extension of L that contains a root of the above equation. Let the root be denoted by a. Note that $[F : L] \leq 6$ and F may be equal to L.

Let $A = \begin{pmatrix} 1 & a \\ a & 1+a^2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 2+a^2 \end{pmatrix}$. Since $|A| = |B| = 1$, $A, B \in SL(2, F)$. It can be verified that

$$A^{-1} = \begin{pmatrix} 1+a^2 & -a \\ -a & 1 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 2+a^2 & -1 \\ 1 & 0 \end{pmatrix},$$

$$AB = \begin{pmatrix} -a & 1+2a+a^3 \\ -1-a^2 & 2+a+3a^2+a^4 \end{pmatrix},$$
$$BA = \begin{pmatrix} a & 1+a^2 \\ -1+2a+a^3 & 2-a+3a^2+a^4 \end{pmatrix},$$
$$A^{-2} = \begin{pmatrix} 1+3a^2+a^4 & -2a-a^3 \\ -2a-a^3 & 1+a^2 \end{pmatrix},$$
$$B^{-2} = \begin{pmatrix} 3+4a^2+a^4 & -2-a^2 \\ 2+a^2 & -1 \end{pmatrix}.$$

Therefore,

$$(AB)A^{-2}(BA) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where

$$c_{11} = -1-2a^2+3a^4+4a^6+a^8,$$

$$c_{12} = 2+3a^2+2a^3+a^4+7a^5+5a^7+a^9,$$

$$c_{21} = -2-3a^2+2a^3-a^4+7a^5+5a^7+a^9,$$

$$c_{22} = 3+6a^2+10a^4+12a^6+6a^8+a^{10}.$$

Since $a^6+4a^4+3a^2-1=0$,

$$c_{11} = (a^2)(-1+3a^2+4a^4+a^6)+(-1-a^2) = -1-a^2,$$

$$c_{12} = (a+a^3)(-1+3a^2+4a^4+a^6)+(2+a+3a^2+a^4) = 2+a+3a^2+a^4,$$

$$c_{21} = (a+a^3)(-1+3a^2+4a^4+a^6)+(-2+a-3a^2-a^4) = -2+a-3a^2-a^4,$$

$$c_{22} = (1+2a^2+a^4)(-1+3a^2+4a^4+a^6)+(4+5a^2+a^4) = 4+5a^2+a^4$$

$$= 4+5a^2+a^4+(-1+3a^2+4a^4+a^6) = a^6+5a^4+8a^2+3.$$

Thus,

$$(AB)A^{-2}(BA) = \begin{pmatrix} -1-a^2 & 2+a+3a^2+a^4 \\ -2+a-3a^2-a^4 & a^6+5a^4+8a^2+3 \end{pmatrix}$$

$$= BAB.$$

Next,

$$(BA)B^{-2}(AB) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

where

$$d_{11} = 1 - a^2 - 3a^4 - a^6,$$

$$d_{12} = 2a + 2a^2 + 3a^3 + 7a^4 + a^5 + 5a^6 + a^8,$$

$$d_{21} = 2a - 2a^2 + 3a^3 - 7a^4 + a^5 - 5a^6 - a^8,$$

$$d_{22} = 1 + 5a^2 + 16a^4 + 17a^6 + 7a^8 + a^{10}.$$

Again from $a^6 + 4a^4 + 3a^2 - 1 = 0$,

$$\begin{aligned}
d_{11} &= (-1)(-1 + 3a^2 + 4a^4 + a^6) + (2a^2 + a^4) \\
&= 2a^2 + a^4, \\
d_{12} &= (1 + a^2)(-1 + 3a^2 + 4a^4 + a^6) + (1 + 2a + 3a^3 + a^5) \\
&= 1 + 2a + 3a^3 + a^5, \\
d_{21} &= (-1 - a^2)(-1 + 3a^2 + 4a^4 + a^6) + (-1 + 2a + 3a^3 + a^5) \\
&= -1 + 2a + 3a^3 + a^5, \\
d_{22} &= (2 + 3a^2 + a^4)(-1 + 3a^2 + 4a^4 + a^6) + (3 + 2a^2) \\
&= 3 + 2a^2 \\
&= 3 + 2a^2 + (-1 + 3a^2 + 4a^4 + a^6) \\
&= a^6 + 4a^4 + 5a^2 + 2.
\end{aligned}$$

Thus,

$$\begin{aligned}
(BA)B^{-2}(AB) &= \begin{pmatrix} 2a^2 + a^4 & 1 + 2a + 3a^3 + a^5 \\ -1 + 2a + 3a^3 + a^5 & a^6 + 4a^4 + 5a^2 + 2 \end{pmatrix} \\
&= ABA.
\end{aligned}$$

Now, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $a = 0$ and $0 = a^6 + 4a^4 + 3a^2 - 1 = -1$, a contradiction. If $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $1 = 0$, a contradiction. Hence, A and B are non-identity, and by Lemma 6.2.2, (A, B) is an n-Engel pair for some integer $n \geq 4$.

This completes the proof of Main Theorem 8. $\square$

6.3 Main Theorem 9

Lemma 6.3.1. *Let $p \geq 2$ be a prime and $a, c \in \mathbb{Z}_p$ with $a \neq 0$. Let $U = \{y^2 \ : \ y \in \mathbb{Z}_p\}$ and $V = \{ax^2 + c \ : \ x \in \mathbb{Z}_p\}$. Then $U \cap V \neq \varnothing$.*

Proof. Note that $0 \in U$ and for each $y_0 \in \mathbb{Z}_p \setminus \{0\}$, $y_0^2 = (-y_0)^2 \in U \setminus \{0\}$. Furthermore, $x^2 = y_0^2$ implies that $x = \pm y_0$. Therefore, we conclude that $|U| = \frac{p+1}{2}$. Similarly, $|V| = \frac{p+1}{2}$. If $U \cap V = \varnothing$, then

$$\begin{aligned} p = |Z_p| &\geq |U \cup V| \\ &= |U| + |V| \\ &= \frac{p+1}{2} + \frac{p+1}{2} = p + 1, \end{aligned}$$

a contradiction. Hence, $U \cap V \neq \varnothing$. $\square$

Lemma 6.3.2. *Let $p \geq 2$ be a prime and $a, b, c \in \mathbb{Z}_p$ with $(a, b) \neq (0, 0)$. Then there exist $x, y \in \mathbb{Z}_p$ with $y^2 = ax^2 + bx + c$.*

Proof. Suppose $a = 0$. Then $b \neq 0$. Note that $\{bx + c \ : \ x \in \mathbb{Z}_p\} = \mathbb{Z}_p$. So, by choosing any $y \in \mathbb{Z}_p$, there exists an $x \in \mathbb{Z}_p$ with $y^2 = bx + c$.

Suppose $a \neq 0$. Then $ax^2 + bx + c = a(x + 2^{-1}ba^{-1})^2 + a(ca^{-1} - (2^{-1}ba^{-1})^2)$ where 2^{-1} and a^{-1} are inverses of 2 and a, respectively, in $\mathbb{Z}_p$. By Lemma 6.3.1, there exist $X, y \in \mathbb{Z}_p$ with $y^2 = aX^2 + a(ca^{-1} - (2^{-1}ba^{-1})^2)$. The lemma follows by taking $x = X - 2^{-1}ba^{-1}$. $\square$

Lemma 6.3.3. *Let $p \geq 3$ be a prime. Then* $\{(2s-10, 4s-4) \ : \ s \in \mathbb{Z}_p\} \neq \{(0,0)\}$.

Proof. Let $s \in \mathbb{Z}_p$ be such that $2s - 10 = 0 = 4s - 4$. Since $p \geq 3$, $4 = 4s$ implies that $s = 1$. So, $-8 = 2s - 10 = 0$, a contradiction. Hence, the lemma holds. $\square$

Proof of Main Theorem 9. Since F is of characteristic p, $\mathbb{Z}_p \subseteq F$. By the Quadratic Reciprocity Law, $x^2 \equiv 5 \bmod p$ is solvable for all $p \equiv \pm 1 \bmod 5$. Therefore, there exists an $s \in \mathbb{Z}_p$ with

$$s^2 = 5. \tag{6.3.1}$$

By Lemma 6.3.3, $(2s - 10, 4s - 4) \neq (0, 0)$. It then follows from Lemma 6.3.2 that

$$y^2 = (2s - 10)x^2 + (4s - 4)x + (2s - 10), \tag{6.3.2}$$

for some $x, y \in \mathbb{Z}_p$.

Therefore,

$$\begin{aligned} y^2 + sy^2 = ((2s - 10)x^2 + (4s - 4)x + (2s - 10))+ \qquad\qquad & \qquad (6.3.3) \\ s((2s - 10)x^2 + (4s - 4)x + (2s - 10)) & \\ = -10 - 8s + 2s^2 - 4x + 4s^2x - 10x^2 - 8sx^2 + 2s^2x^2 & \\ = -10 - 8s + 2(5) - 4x + 4(5)x - 10x^2 - 8sx^2 + 2(5)x^2 & \\ = -8s + 16x - 8sx^2. \qquad\qquad & \qquad (6.3.4) \end{aligned}$$

Let

$$A = 4^{-1} \begin{pmatrix} 4x & (1-x)(1+s) - y \\ (x-1)(1+s) - y & 2(1 - 2x + s) \end{pmatrix}$$

and

$$B = 2^{-1}\begin{pmatrix} 0 & 2 \\ -2 & 1+s \end{pmatrix}.$$

Note that by equations (6.3.1) and (6.3.2),

$$\begin{aligned} |A| &= 16^{-1}(s^2x^2 - 2s^2x + 2sx^2 + s^2 + 4sx - 15x^2 - y^2 + 2s + 6x + 1) \\ &= 16^{-1}(5x^2 - 2(5)x + 2sx^2 + 5 + 4sx - 15x^2 \\ &\qquad - ((2s-10)x^2 + (4s-4)x + (2s-10)) + 2s + 6x + 1) \\ &= 1, \end{aligned}$$

and $|B| = 1$. Thus, $A, B \in SL(2, F)$.

Now,

$$A^2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= 16^{-1}(-1 - 2s - s^2 + 2x + 4sx + 2s^2x + 15x^2 - 2sx^2 - s^2x^2 + y^2) \\ a_{12} &= 16^{-1}(2 + 4s + 2s^2 - 2x - 4sx - 2s^2x - 2y - 2sy) \\ a_{21} &= 16^{-1}(-2 - 4s - 2s^2 + 2x + 4sx + 2s^2x - 2y - 2sy) \\ a_{22} &= 16^{-1}(3 + 6s + 3s^2 - 14x - 12sx + 2s^2x + 15x^2 - 2sx^2 - s^2x^2 + y^2). \end{aligned}$$

By equations (6.3.1) and (6.3.2),

$$\begin{aligned} a_{11} &= 8^{-1}(-8 + 4x + 4sx) \\ a_{12} &= 8^{-1}(6 + 2s - 6x - 2sx - y - sy) \\ a_{21} &= 8^{-1}(-6 - 2s + 6x + 2sx - y - sy) \\ a_{22} &= 8^{-1}(4 + 4s - 4x - 4sx). \end{aligned}$$

So,

$$A^3 = A^2(A) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

where

$$
\begin{aligned}
b_{11} &= 32^{-1}(-2s^2x^2 - s^2xy + 4s^2x + s^2y + 8sx^2 + (y^2 + sy^2) - 2s^2 + 16sx + 10x^2 + 5xy - 8s - 20x - 5y - 6) \\
b_{12} &= 32^{-1}(-4s^2x^2 - 2s^2y + 4s^2 - 8sx - 4sy + 20x^2 + 8s - 24x + 6y + 4) \\
b_{21} &= 32^{-1}(-4s^2x^2 + 8s^2x - 4s^2 + 8sx - 4sy + 20x^2 - 8s - 16x - 4y - 4) \\
b_{22} &= 32^{-1}(-2s^2x^2 + s^2xy - 4s^2x - s^2y + 8sx^2 + (y^2 + sy^2) + 6s^2 - 16sx + 10x^2 - 5xy + 8s - 12x + 5y + 2).
\end{aligned}
$$

By equations (6.3.1) and (6.3.4),

$$b_{11} = 8^{-1}(-4 + 4x - 4s + 4sx)$$

$$b_{12} = 8^{-1}(6 - y - 6x + 2s - sy - 2sy)$$

$$b_{21} = 8^{-1}(-6 - y + 6x - 2s - sy + 2sy)$$

$$b_{22} = 8^{-1}(8 - 4x - 4sx).$$

Next,

$$
\begin{aligned}
B^3 &= 8^{-1}\begin{pmatrix} -4 & 2+2s \\ -2-2s & -3+2s+s^2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 1+s \end{pmatrix} \\
&= 8^{-1}\begin{pmatrix} -4 & 2+2s \\ -2-2s & -3+2s+5 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 1+s \end{pmatrix} \\
&= 4^{-1}\begin{pmatrix} -2 & 1+s \\ -1-s & 1+s \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 1+s \end{pmatrix} \\
&= 4^{-1}\begin{pmatrix} -2s-2 & s^2+2s-3 \\ -2s-2 & s^2-1 \end{pmatrix} \\
&= 4^{-1}\begin{pmatrix} -2s-2 & 5+2s-3 \\ -2s-2 & 5-1 \end{pmatrix} \\
&= 2^{-1}\begin{pmatrix} -1-s & 1+s \\ -1-s & 2 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
AB &= 8^{-1}\begin{pmatrix} -2-2s+2x+2sx+2y & 1+2s+s^2+7x-2sx-s^2x-y-sy \\ -4-4s+8x & 2s+2s^2-2x-2sx-2y \end{pmatrix} \\
&= 8^{-1}\begin{pmatrix} -2-2s+2x+2sx+2y & 1+2s+5+7x-2sx-5x-y-sy \\ -4-4s+8x & 2s+2(5)-2x-2sx-2y \end{pmatrix} \\
&= 8^{-1}\begin{pmatrix} -2-2s+2x+2sx+2y & 6+2s+2x-2sx-y-sy \\ -4-4s+8x & 10+2s-2x-2sx-2y \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
BA &= 8^{-1}\begin{pmatrix} -2-2y+2x-2s+2sx & 4-8x+4s \\ -1-y-7x-2s-sy+2sx-s^2+s^2x & 2y-2x+2s-2sx+2s^2 \end{pmatrix} \\
&= 8^{-1}\begin{pmatrix} -2-2y+2x-2s+2sx & 4-8x+4s \\ -1-y-7x-2s-sy+2sx-5+5x & 2y-2x+2s-2sx+2(5) \end{pmatrix} \\
&= 8^{-1}\begin{pmatrix} -2-2y+2x-2s+2sx & 4-8x+4s \\ -6-y-2x-2s-sy+2sx & 10+2y-2x+2s-2sx \end{pmatrix}.
\end{aligned}
$$

So,

$$
(AB)A = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},
$$

where

$$
\begin{aligned}
c_{11} &= 32^{-1}(-6-8s-2s^2-4x+4s^2x+10x^2+8sx^2-2s^2x^2-5y+s^2y+5xy-s^2xy+(y^2+sy^2)) \\
c_{12} &= 32^{-1}(10+12s+2s^2-16x-10x^2+4sx^2-2s^2x^2+2y-2s^2y-2y^2) \\
c_{21} &= 32^{-1}(-10-12s-2s^2-4x+4s^2x+30x^2-4sx^2-2s^2x^2-8y+2y^2) \\
c_{22} &= 32^{-1}(16+16s-32x).
\end{aligned}
$$

By equations (6.3.1), (6.3.2) and (6.3.4),

$$
c_{11} = 16^{-1}(-8-8s+16x)
$$

$$
c_{12} = 16^{-1}(20+4s-4x-4sx-4y)
$$

$$
c_{21} = 16^{-1}(-4s-20+4x+4sx-4y)
$$

$$
c_{22} = 16^{-1}(8+8s-16x).
$$

On the other hand,

$$
\begin{aligned}
B(AB) &= 16^{-1}\begin{pmatrix} -8-8s+16x & 20+4s-4x-4sx-4y \\ -4s-4s^2+4x+4sx-4y & -2+8s+2s^2-6x-2s^2x \end{pmatrix} \\
&= 16^{-1}\begin{pmatrix} -8-8s+16x & 20+4s-4x-4sx-4y \\ -4s-4(5)+4x+4sx-4y & -2+8s+2(5)-6x-2(5)x \end{pmatrix} \\
&= ABA.
\end{aligned}
$$

Now,

$$
AB^2A = (AB)(BA) = 64^{-1}\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},
$$

where

$$
\begin{aligned}
d_{11} &= -32 - 32x - 16s + 16sx^2 + ((s^2-3)y^2 + 2sy^2) \\
d_{12} &= 52 + 10y + 32x + 16s - 2(y^2 + sy^2) - 10xy - 20x^2 - 4s^2 - 16sx^2 - 2s^2y + 2s^2xy + 4s^2x^2 \\
d_{21} &= -52 + 10y - 32x - 16s + 2(y^2 + sy^2) - 10xy + 20x^2 + 4s^2 + 16sx^2 - 2s^2y + 2s^2xy - 4s^2x^2 \\
d_{22} &= 84 + 24x + 8s - 4y^2 - 60x^2 + 16sx - 12s^2 + 8sx^2 - 8s^2x + 4s^2x^2.
\end{aligned}
$$

By equations (6.3.1), (6.3.2) and (6.3.4),

$$d_{11} = -32 - 32s$$

$$d_{12} = 32 + 32s$$

$$d_{21} = -32 - 32s$$

$$d_{22} = 64.$$

Hence, $AB^2A = B^3$.

Next,

$$BA^2B = (BA)(AB) = 64^{-1} \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix},$$

where

$$
\begin{aligned}
e_{11} &= -12 + 56x - 24s - 4y^2 - 60x^2 + 48sx - 12s^2 + 8sx^2 - 8s^2x + 4s^2x^2 \\
e_{12} &= 28 - 18y - 80x + 32s + 10xy + 20x^2 - 8sy - 16sx + 4s^2 + 2(sy^2 + y^2) + 16sx^2 + 2s^2y - 2s^2xy - 4s^2x^2 \\
e_{21} &= -28 - 18y + 80x - 32s + 10xy - 20x^2 - 8sy + 16sx - 4s^2 - 2(sy^2 + y^2) - 16sx^2 + 2s^2y - 2s^2xy + 4s^2x^2 \\
e_{22} &= 64 - 64x + 16s - 32sx + 16sx^2 + (s^2-3)y^2 + 2sy^2.
\end{aligned}
$$

By equations (6.3.1), (6.3.2) and (6.3.4),

$$e_{11} = -32 - 32s + 32x + 32sx$$

$$e_{12} = 48 - 8y - 48x + 16s - 8sy - 16sx$$

$$e_{21} = -48 - 8y + 48x - 16s - 8sy + 16sx$$

$$e_{22} = 64 - 32x - 32sx.$$

Hence, $BA^2B = A^3$.

By part (b) of Lemma 4.2.4, $A = [A,_5 B]$, and $B = [B,_5 A]$. Now, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $x = 1$ and $1 = 2^{-1}(1 - 2x + s) = 2^{-1}(s - 1)$, i.e., $s = 3$. By equation (6.3.1), $9 = s^2 = 5$. This implies that $4 = 0$, which is impossible, for $p \geq 3$. So, A is not the identity. Hence, (A, B) is a 5-Engel pair in $SL(2, F)$.

This completes the proof of Main Theorem 9. □

We also get the following corollary.

Corollary 6.3.4. *If* $p \equiv \pm 1 \mod 5$ *is a prime, then* $SL(2, p)$ *has a* 5*-Engel pair.*

Remark 3. Heineken [18, Theorem 2] showed that $SL(2, p^3)$ is generated by an n-Engel pair. This overlaps with Main Theorem 8. However, the proof for Main Theorem 8 is different from that of Heineken.

Remark 4. It can be seen by GAP that $H_2 = SL(2, 5)$. This means that $SL(2, 5)$ has a 5-Engel pair. By page 411 of [31], it is known that $SL(2, 5)$ is isomorphic to a subgroup of $SL(2, p)$ if and only if p is odd and 5 divides $p(p^2 - 1)$. This gives another proof for Main Theorem 9.

6.4 Computer Codes

The following programming script was used to search for Engel pairs in the special linear group $SL(2, p)$ with Mathematica. No complex symbolic computations is involved and therefore it can be translated to work on other programs, such as the open source mathematical system SAGE (www.sagemath.org).

```
Set1 := { };
Set2 := { };
X1 := { {1, 0} {0, 1} };
Y1 := { {1, 0} {0, 1} };
```

```
EX1 := { {1, 0} {0, 1} };
EY1 := { {1, 0} {0, 1} };
invX1 := { {1, 0} {0, 1} };
invY1 := { {1, 0} {0, 1} };
invEX1 := { {1, 0} {0, 1} };
invEY1 := { {1, 0} {0, 1} };
ClX := { };
ClY := { };
CIE := { };
PrmD := 1;
Engl := 0;

For[jv = 1, jv < 10, jv++,
 {

  PrmD = Prime[jv],
  Print[PrmD],
  Set1 = { },
  Set2 = { },
  CIE = { },
  For[ja = 0, ja < PrmD, ja++,
   {
    For[jb = 0, jb < PrmD, jb++,
     {
      For[jc = 0, jc < PrmD, jc++,
       {
        For[jd = 0, jd < PrmD, jd++,
         {
          If[Mod[ja*jd - jb*jc, PrmD] == 1,
           Set1 = Append[Set1, { {ja, jb}, {jc, jd} }]],
          If[ja == 0 && jb == 1 && jc == PrmD - 1,
           Set2 = Append[Set2, { {ja, jb}, {jc, jd} }]],
         }
        ]
       }
      ]
     }
    ]
   }
  ],

  For[je = 1, je < Length[Set2] + 1, je++,
   {
    For[jf = 1, jf < Length[Set1] + 1, jf++,
     {
      Engl = 0,
      ClX = { },
      ClY = { },
      X1 = Set2[[je]],
      Y1 = Set1[[jf]],
      EX1 = X1,
      EY1 = Y1,
      invX1 = { {X1[[2, 2]], PrmD - X1[[1, 2]]},
                {PrmD - X1[[2, 1]], X1[[1, 1]]} },
      invY1 = { {Y1[[2, 2]], PrmD - Y1[[1, 2]]},
                {PrmD - Y1[[2, 1]], Y1[[1, 1]]} },
      For[jg = 1, jg < Length[Set1] + 1, jg++,
       {
        Engl = Engl + 1,
        invEX1 = { {EX1[[2, 2]], PrmD - EX1[[1, 2]]},
```

```
                {PrmD - EX1[[2, 1]], EX1[[1, 1]]} },
        invEY1 = { {EY1[[2, 2]], PrmD - EY1[[1, 2]]},
                {PrmD - EY1[[2, 1]], EY1[[1, 1]]} },
        EX1 = invEX1.invY1.EX1.Y1,
        EY1 = invEY1.invX1.EY1.X1,
        EX1 = { {Mod[EX1[[1, 1]], PrmD], Mod[EX1[[1, 2]], PrmD]},
              {Mod[EX1[[2, 1]], PrmD], Mod[EX1[[2, 2]], PrmD]} },
        EY1 = { {Mod[EY1[[1, 1]], PrmD], Mod[EY1[[1, 2]], PrmD]},
              {Mod[EY1[[2, 1]], PrmD], Mod[EY1[[2, 2]], PrmD]} },
        If[
         EX1 == X1 && EY1 == Y1 && Engl > 2, {
          If[ MemberQ[CIE, Engl] == False, {Print[{X1, Y1, Engl}],
           CIE = Append[CIE, Engl]} ] , Break[]}],
        If[MemberQ[ClX, EX1] == True, Break[]],
        If[MemberQ[ClY, EY1] == True, Break[]],
        ClX = Append[ClX, EX1],
        ClY = Append[ClY, EY1]
       }
      ]
     }
    ]
   }
  ]

 }
]
```

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